Nearly Optimal Embeddings of Flat Tori

Ishan Agarwal Oded Regev Yi Tang

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Lattice \mathcal{L} : set of all *integer* linear combinations of a basis in \mathbb{R}^n .



Figure: Example of a 2D lattice, generated by basis $\{\mathbf{b}_1, \mathbf{b}_2\}$.

 $\lambda_1(\mathcal{L})$: shortest nonzero length in \mathcal{L} .



Figure: λ_1 for the 2D example.

We often refer to $\lambda_1(\mathcal{L})$ as the "scale" of \mathcal{L} .

Preliminaries: Flat Tori

Flat torus \mathbb{R}^n/\mathcal{L} : quotient space of Euclidean space by lattice; elements: cosets of the form $\mathbf{x} + \mathcal{L}$.

Generalizes the standard 2D "torus" $\mathbb{R}^2/\mathbb{Z}^2$:



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Standard quotient metric on \mathbb{R}^n/\mathcal{L} : dist $_{\mathbb{R}^n/\mathcal{L}}(\mathbf{x} + \mathcal{L}, \mathbf{y} + \mathcal{L}) = \text{min distance between } \mathbf{x} + \mathcal{L} \text{ and } \mathbf{y} + \mathcal{L}$.



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Preliminaries: Distances in Flat Tori cont.

2D Example: What is the distance between the colored points in the following torus (dashed)?



Similarly the distance is not the one within parallelogram, but again the min distance between corresponding cosets.



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Preliminaries: Distortion



Figure: Embedding $f : x + \mathbb{Z} \mapsto (\cos(2\pi x), \sin(2\pi x))$, whose distortion is $\pi/2$.

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- Question (Khot and Naor, 2005): How to embed flat tori into Hilbert space with low distortion?
- Lower bound (KN05): worst case $\Omega(\sqrt{n})$.
- Upper bound (KN05): embedding with distortion $O(n^{3n/2})$.
- Upper bound (Haviv and Regev, 2010):
 - embedding with distortion $O(n\sqrt{\log n})$.
 - embedding with distortion $O(\sqrt{n \log n})$ (under certain condition).

For any torus \mathbb{R}^n/\mathcal{L} , we construct a metric embedding of \mathbb{R}^n/\mathcal{L} into Hilbert space with distortion $O(\sqrt{n \log n})$.

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Embed into Hilbert Space using Gaussians

Observation: for embedding $f : \mathbb{R}^n \to L_2(\mathbb{R}^n), \mathbf{x} \mapsto \text{Gaussian centered at } \mathbf{x}$, the L_2 distance between two Gaussians $f(\mathbf{x})$ and $f(\mathbf{y})$

- "saturates" at certain distance, while
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The HR10 embedding with distortion $O(\sqrt{n \log n})$:

- Uses Gaussians in the construction.
- Only works if distances to embed never exceed $poly(n) \cdot \lambda_1$.
- "Saturates" at $poly(n) \cdot \lambda_1$, like single Gaussian.

Idea about next step:

- first partition torus into direct sum of tori, each representing a different scale;
- then apply the HR10 embedding to each scale separately;
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Consider two points in the following torus:



Note that λ_1 is the scale in *x*-direction, while imagine the scale in *y*-direction could be arbitrarily large.

Saturation happens and distances in y-direction are not captured.

Besides the original torus, also consider its projection on the y-axis:



 λ_1 of the projection is the scale in y-direction, and HR10 embedding of the projection would capture distances in y-direction.

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 λ_1 of the projection is the scale in *y*-direction, and HR10 embedding of the projection would capture distances in *y*-direction.

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- Take a chain of sublattices $\{\mathbf{0}\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m = \mathcal{L};$
- Take projections of \mathbb{R}^n/\mathcal{L} orthogonal to \mathcal{L}_i for all $i=0,1,\ldots,m-1;$
- Proper choice of sublattices can partition the scales well.

- m = 2, $\mathcal{L}_1 = \mathcal{L} \cap x$ -axis.
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- HR10 indeed uses the orthogonal projections.
- They only get $O(n\sqrt{\log n})$, losing a factor of \sqrt{n} .
- Issue: distances in projected tori with large *i* get counted repeatedly. For the last one the repetition is *m*, which at worst could be *n*.
- For the 2D example, (small) distances in the *y*-direction get counted twice both in the original torus and the projection.

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Pitfall of HR10 cont.

Distances in projected tori with large *i* get counted repeatedly:

- Consider the subspaces spanned by projected tori they have reversed subspace relationship from the end.
- Orthogonally decompose the entire space according to the subspace relationship, then the repeated counting is like:



(columns: orthogonal decomposition of the entire space, rows: subspaces spanned by projected tori)

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Novel Contribution: Compressed Projections

- Use compressed projections that compress the projected tori recursively by a factor 0 < α < 1.
- The repeated counting becomes:



• The geometric series suppresses the \sqrt{n} extra factor into constant.

Using Compressed Projection in the 2D Example



- Our entire embedding: first partition the scales using compressed projections, and then apply the HR10 embedding to each scale.
- Main technical lemma: the partition step has constant distortion.
- Recall the HR10 embedding has distortion $O(\sqrt{n \log n})$.
- Hence our entire embedding has distortion $O(1) \cdot O(\sqrt{n \log n})$.

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- Proving constant expansion of partition step: easy part, due to the geometric series in compressed projections.
- Proving constant contraction of partition step:
 - the main technical part, nontrivial while still elementary;
 - difficulty 1: compressed projections worsen the contraction the geometric series introduces some exponentially small factors;
 - difficulty 2: need to prove stronger contraction property that takes care of the saturation issue so that the HR10 embedding works afterwards.

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- Our result: $O(\sqrt{n \log n})$, lower bound: worst case $\Omega(\sqrt{n})$.
- KN05, HR10 both give lattice-specific lower bounds as well.
- Tighter bounds for every lattice in terms of lattice parameters?

Thank you!

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