Nearly Optimal Embeddings of Flat Tori

Ishan Agarwal  Oded Regev  Yi Tang

APPROX 2020
Lattice $\mathcal{L}$: set of all integer linear combinations of a basis in $\mathbb{R}^n$.

Figure: Example of a 2D lattice, generated by basis $\{b_1, b_2\}$. 
$\lambda_1(\mathcal{L})$: shortest nonzero length in $\mathcal{L}$.

**Figure**: $\lambda_1$ for the 2D example.

We often refer to $\lambda_1(\mathcal{L})$ as the “scale” of $\mathcal{L}$.
Preliminaries: Flat Tori

Flat torus $\mathbb{R}^n/\mathcal{L}$: quotient space of Euclidean space by lattice; elements: cosets of the form $x + \mathcal{L}$.

Generalizes the standard 2D “torus” $\mathbb{R}^2/\mathbb{Z}^2$:

Figure: Transition between the 3D and the quotient representations of $\mathbb{R}^2/\mathbb{Z}^2$. 
Flat torus $\mathbb{R}^n / \mathcal{L}$: quotient space of Euclidean space by lattice; elements: cosets of the form $\mathbf{x} + \mathcal{L}$.
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Figure: Transition between the 3D and the quotient representations of $\mathbb{R}^2 / \mathbb{Z}^2$. 
Preliminaries: Distances in Flat Tori

Standard quotient metric on $\mathbb{R}^n/\mathcal{L}$:

$$\text{dist}_{\mathbb{R}^n/\mathcal{L}}(x + \mathcal{L}, y + \mathcal{L}) = \min \text{ distance between } x + \mathcal{L} \text{ and } y + \mathcal{L}.$$ 

1D Example: What is the distance between 0.2 and 0.8 in $\mathbb{R}/\mathbb{Z}$?

$$\text{dist}_{\mathbb{R}/\mathbb{Z}}(0.2 + \mathbb{Z}, 0.8 + \mathbb{Z}) = 0.4,$$ the min distance between the cosets:
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\begin{align*}
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-0.2 & \quad 0.2 & \quad 0.8 & \quad 1.2 & \quad 1.8 & \quad 2.2 \\
0 & \quad 1 & \quad 2
\end{align*}
\]
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2D Example: What is the distance between the colored points in the following torus (dashed)?

Similarly the distance is not the one within parallelogram, but again the min distance between corresponding cosets.
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Preliminaries: Distortion

Distortion of metric embedding $f : M_1 \rightarrow M_2$: the factor by which $f$ changes the distance between two points;

definition: \[
\frac{\text{expansion factor}}{\text{contraction factor}}.
\]

Example: embed $\mathbb{R}/\mathbb{Z}$ into Euclidean space $\mathbb{R}^2$.

Figure: Embedding $f : x + \mathbb{Z} \mapsto (\cos(2\pi x), \sin(2\pi x))$, whose distortion is $\pi/2$. 
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Figure: Embedding $f : x + \mathbb{Z} \mapsto (\cos(2\pi x), \sin(2\pi x))$, whose distortion is $\pi/2$. 
Previous Literature

- **Question (Khot and Naor, 2005):** How to embed flat tori into Hilbert space with low distortion?
  - Lower bound (KN05): worst case $\Omega(\sqrt{n})$.
  - Upper bound (KN05): embedding with distortion $O(n^{3n/2})$.
  - Upper bound (Haviv and Regev, 2010):
    - embedding with distortion $O(n\sqrt{\log n})$.
    - embedding with distortion $O(\sqrt{n \log n})$ (under certain condition).

Our Contribution

For any torus $\mathbb{R}^n/L$, we construct a metric embedding of $\mathbb{R}^n/L$ into Hilbert space with distortion $O(\sqrt{n \log n})$. 
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Embed into Hilbert Space using Gaussians

Observation: for embedding $f : \mathbb{R}^n \to L_2(\mathbb{R}^n)$, $x \mapsto$ Gaussian centered at $x$, the $L_2$ distance between two Gaussians $f(x)$ and $f(y)$

- “saturates” at certain distance, while
- approx. $\propto$ distance between $x$ and $y$ before saturation.

Figure: $L_2$ distance between Gaussians as function of difference of points in 1D.
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The Gaussian Embedding in HR10

The HR10 embedding with distortion $O(\sqrt{n \log n})$:

- Uses Gaussians in the construction.
- Only works if distances to embed never exceed $\text{poly}(n) \cdot \lambda_1$.
- “Saturates” at $\text{poly}(n) \cdot \lambda_1$, like single Gaussian.

Idea about next step:

- first partition torus into direct sum of tori, each representing a different scale;
- then apply the HR10 embedding to each scale separately;
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2D Example of Saturation

Consider two points in the following torus:

Note that $\lambda_1$ is the scale in $x$-direction, while imagine the scale in $y$-direction could be arbitrarily large. Saturation happens and distances in $y$-direction are not captured.
Besides the original torus, also consider its projection on the $y$-axis:

$\lambda_1$ of the projection is the scale in $y$-direction, and HR10 embedding of the projection would capture distances in $y$-direction.
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Partition of Scales in General

General technique:
- Take a chain of sublattices \( \{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m = \mathcal{L} \);
- Take projections of \( \mathbb{R}^n/\mathcal{L} \) orthogonal to \( \mathcal{L}_i \) for all \( i = 0, 1, \ldots, m-1 \);
- Proper choice of sublattices can partition the scales well.

For the 2D example:
- \( m = 2, \mathcal{L}_1 = \mathcal{L} \cap x\text{-axis}. \)
- To project orthogonally to \( \mathcal{L}_1 \) is to project onto \( y\text{-axis}. \)
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HR10 indeed uses the orthogonal projections.

They only get $O(n\sqrt{\log n})$, losing a factor of $\sqrt{n}$.

Issue: distances in projected tori with large $i$ get counted repeatedly. For the last one the repetition is $m$, which at worst could be $n$.

For the 2D example, (small) distances in the $y$-direction get counted twice — both in the original torus and the projection.
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Pitfall of HR10

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Distances in projected tori with large $i$ get counted repeatedly:

- Consider the subspaces spanned by projected tori — they have reversed subspace relationship from the end.

- Orthogonally decompose the entire space according to the subspace relationship, then the repeated counting is like:

  \[
  \begin{array}{cccccc}
  1 & 1 & 1 & \ldots & 1 \\
  1 & 1 & \ldots & 1 \\
  1 & \ldots & 1 \\
   \vdots & \ddots & \ddots \\
   \end{array}
  \]

  (columns: orthogonal decomposition of the entire space, rows: subspaces spanned by projected tori)
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Impossible to Isolate Block

Ideally want projection of $\mathcal{L}$ in each block (orthogonal component) of the entire space separately to avoid repeated counting.

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However projecting into arbitrary block is invalid for general lattices.

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\begin{array}{c|c|c|c|c}
& \cdots & \text{green} & \cdots & \\
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Novel Contribution: Compressed Projections

- Use *compressed projections* that compress the projected tori recursively by a factor $0 < \alpha < 1$.
- The repeated counting becomes:

\[
\begin{array}{cccc}
1 & \alpha & \alpha^2 & \ldots & \alpha^{m-1} \\
1 & \alpha & \alpha^2 & \ldots & \alpha^{m-2}
\end{array}
\]

instead of

\[
\begin{array}{cccc}
1 & 1 & 1 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 1
\end{array}
\]

- The geometric series suppresses the $\sqrt{n}$ extra factor into constant.
Using Compressed Projection in the 2D Example
Proof Overview

- **Our entire embedding**: first partition the scales using compressed projections, and then apply the HR10 embedding to each scale.

- Main technical lemma: the partition step has constant distortion.
- Recall the HR10 embedding has distortion $O(\sqrt{n \log n})$.
- Hence our entire embedding has distortion $O(1) \cdot O(\sqrt{n \log n})$. 

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Hence our entire embedding has distortion $O(1) \cdot O(\sqrt{n \log n})$. 
Main technical lemma: the partition step has constant distortion (even after taking care of the saturation issue of the HR10 embedding afterwards).

- Proving constant expansion of partition step: easy part, due to the geometric series in compressed projections.
- Proving constant contraction of partition step:
  - the main technical part, nontrivial while still elementary;
  - difficulty 1: compressed projections worsen the contraction — the geometric series introduces some exponentially small factors;
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Our result: \( O(\sqrt{n \log n}) \), lower bound: worst case \( \Omega(\sqrt{n}) \).

KN05, HR10 both give lattice-specific lower bounds as well.

Tighter bounds for every lattice in terms of lattice parameters?
Thank you!