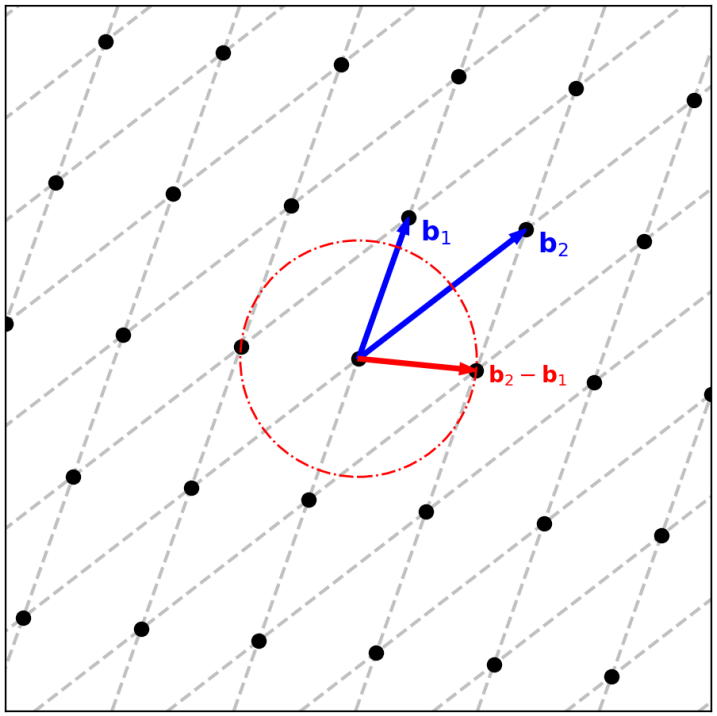
Improved Hardness of BDD and SVP under Gap-(S)ETH

Yi Tang
Joint work with Huck Bennett and Chris Peikert

September 16, 2021
(last updated on January 31, 2022)

Preliminaries: Lattices

Lattice $\mathcal{L} \subset \mathbb{R}^d$: set of all integer linear combinations of a basis.
Basis $\boldsymbol{B} \in \mathbb{R}^{d \times n}$: rank n, dimension d, $\mathcal{L} = \boldsymbol{B} \cdot \mathbb{Z}^n$.



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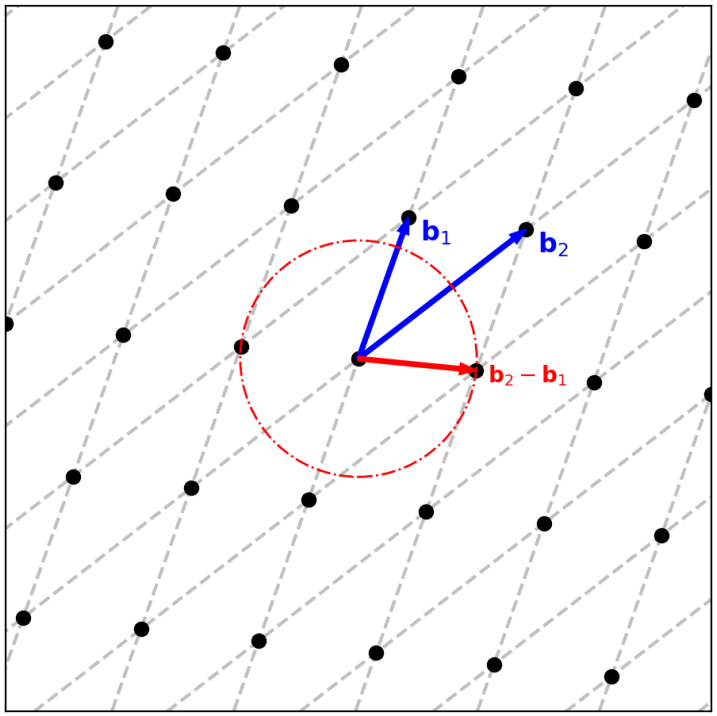

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Lattice Problems and post-quantum cryptography:

- Cryptography based on number theory would be broken by attacks with quantum.
- People believe lattice problems have no quantum solution, and thus lattice-based cryptosystems are quantum-secure.

Desired hardness of lattice problems:

- Good news: *worst-case* hardness of lattice problems leads to *average-case* security of the cryptosystems.
- Need precise fine-grained hardness of lattice problems for setting parameters of the cryptosystems confidently.
- Cryptosystems are based on problems unlikely to be NP-hard, while state-of-the-art attacks reduce to problems where we can show NP-hardness / fine-grained hardness.

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γ-approximate Shortest Vector Problem in ℓ_p ($\mathrm{SVP}_{p,\gamma}$)

Instance: lattice $\mathcal{L}$.
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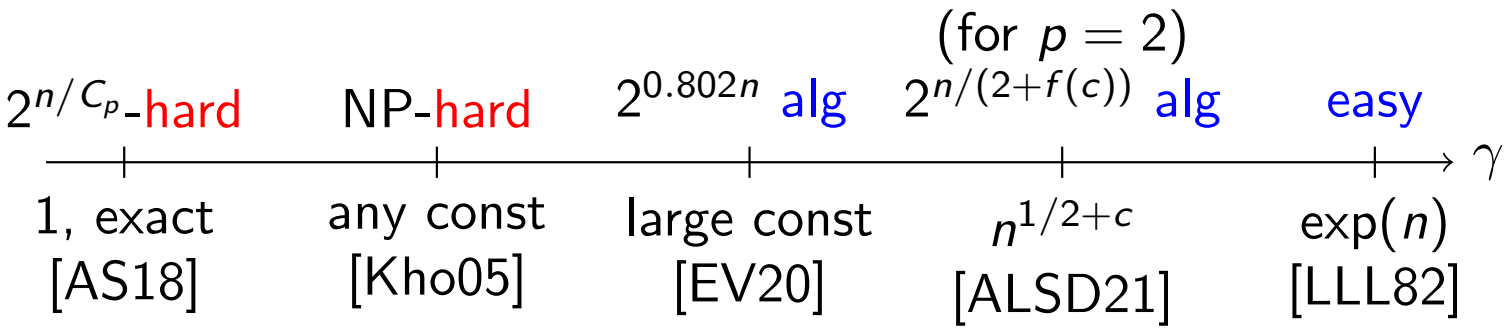

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Bounded Distance Decoding in ℓ_p with relative distance α ($\mathrm{BDD}_{p,\alpha}$)

Instance: lattice $\mathcal{L} \subset \mathbb{R}^d$ and target $\boldsymbol{t} \in \mathbb{R}^d$
satisfying $\mathrm{dist}_p(\boldsymbol{t}, \mathcal{L}) \leq \alpha \cdot \lambda_1^{(p)}(\mathcal{L})$.

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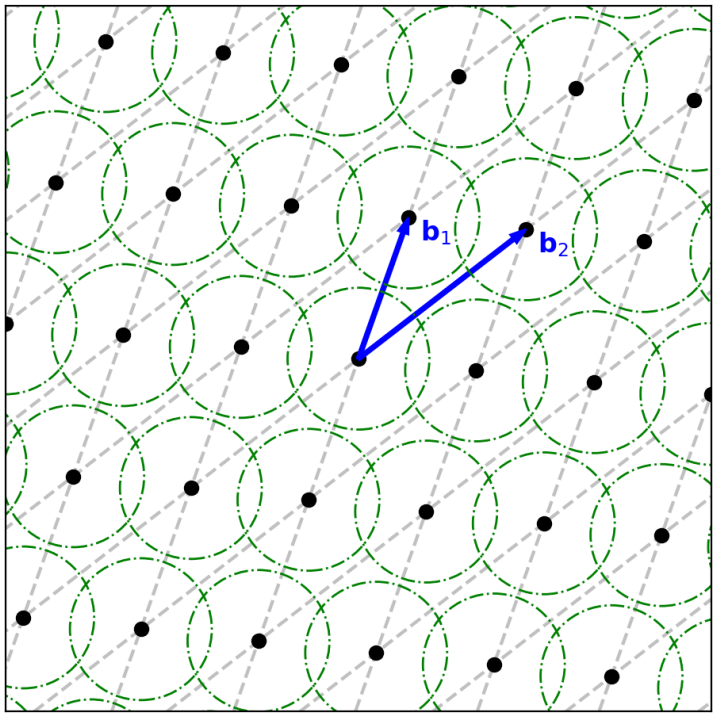

($p = 2$, $\alpha = 0.6$)

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$\mathrm{BDD}_{p,\alpha}$

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Smaller α corresponds to stronger promise and easier problem.

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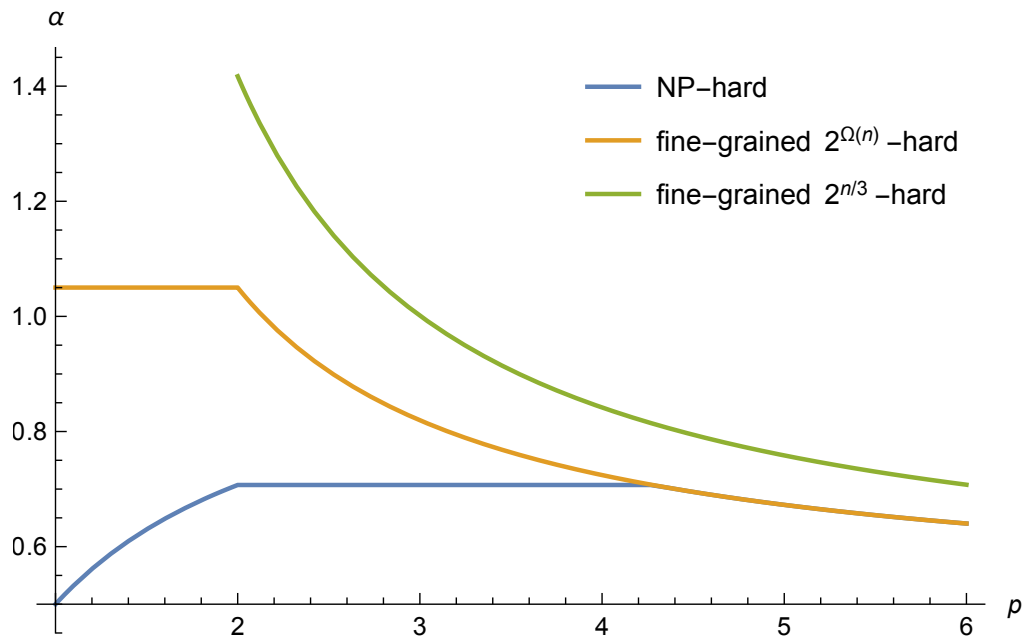

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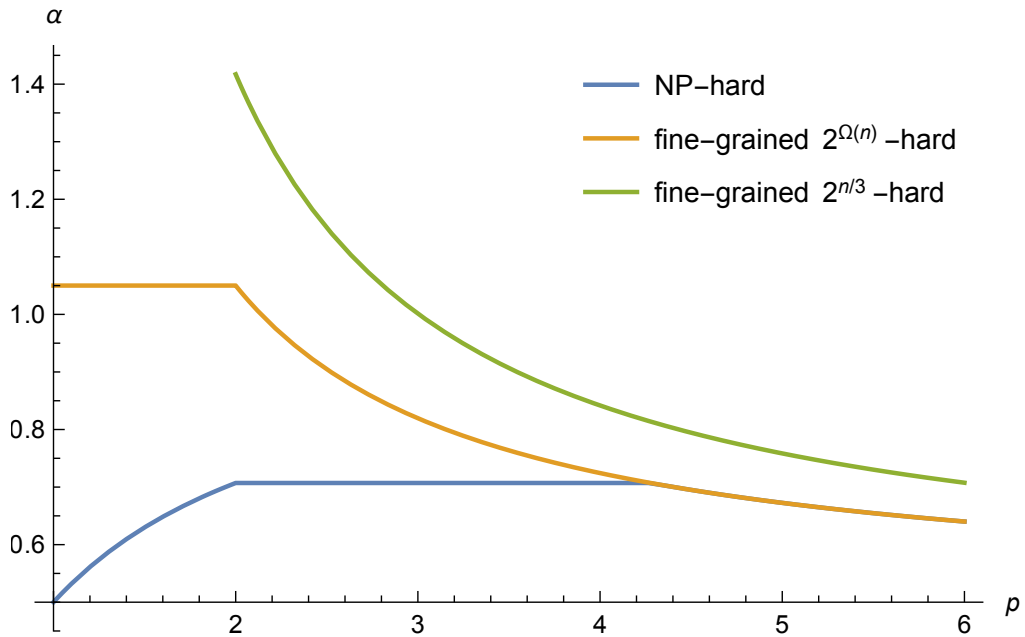

Preliminaries: Exponential Time Hypothesis

Standard approach to fine-grained hardness: Exponential Time Hypothesis (ETH).

ETH variants:

- ETH: 3-SAT cannot be solved in $2^{o(n)}$ time.
- Strong ETH (SETH): k-SAT cannot be solved in $2^{(1-\varepsilon)n}$ time.
- Gap-ETH & Gap-SETH: Gap-3-$\mathrm{SAT}_{1-\delta,1}$ & Gap-k-$\mathrm{SAT}_{1-\delta,1}$.
- Randomized/non-uniform variants: rand/non-unif time.

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1. $\mathrm{BDD}_{p,\alpha}$ cannot be solved in $2^{o(n)}$ time for any $p \in [1, \infty)$ and $\alpha > \alpha_{\mathsf{kn}} \approx 0.98491$, under non-unif Gap-ETH.
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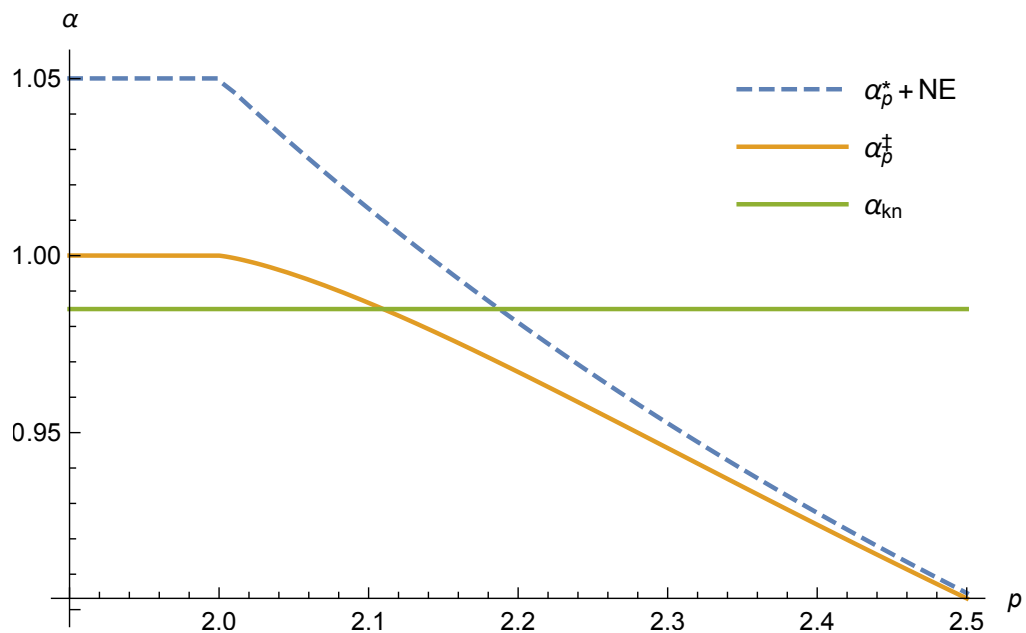

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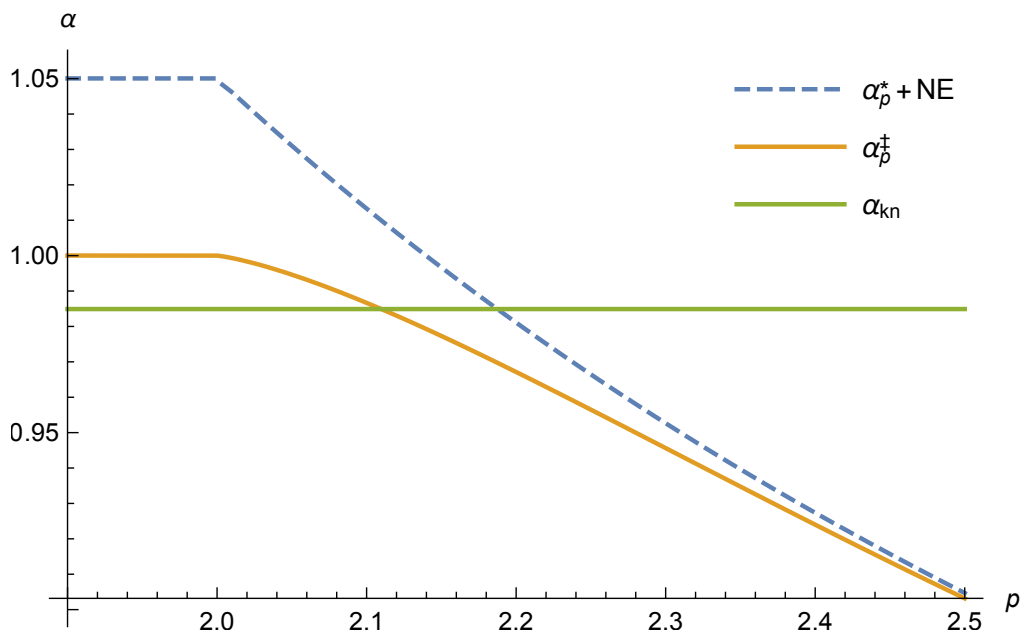

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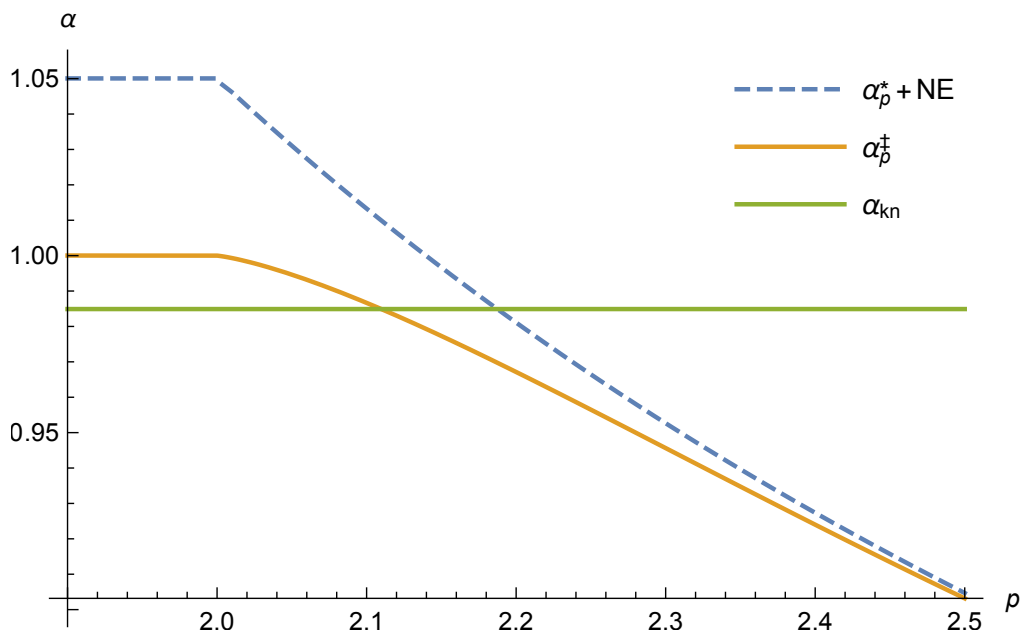

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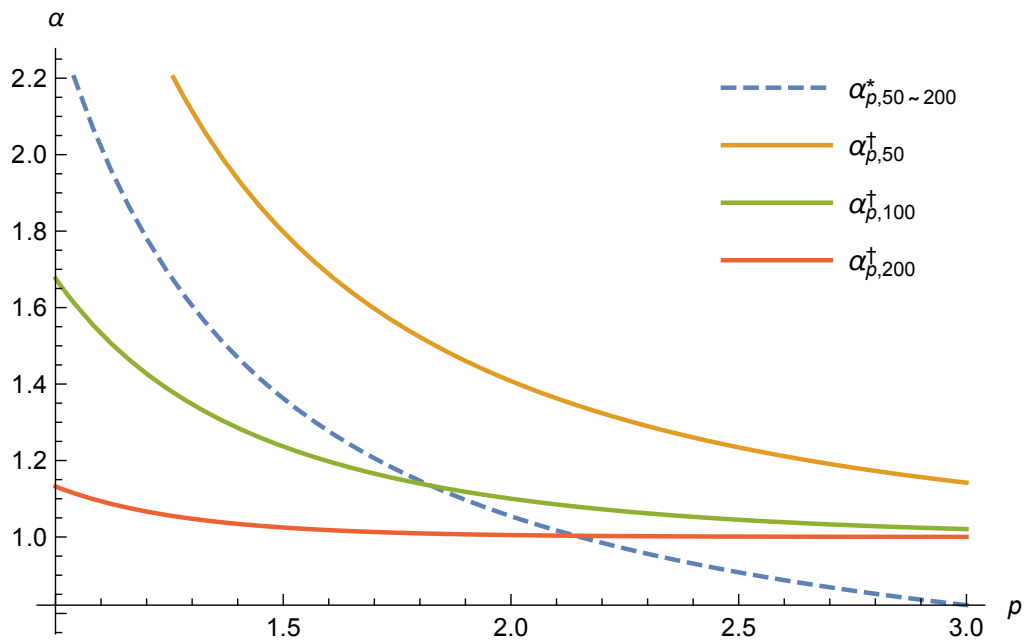

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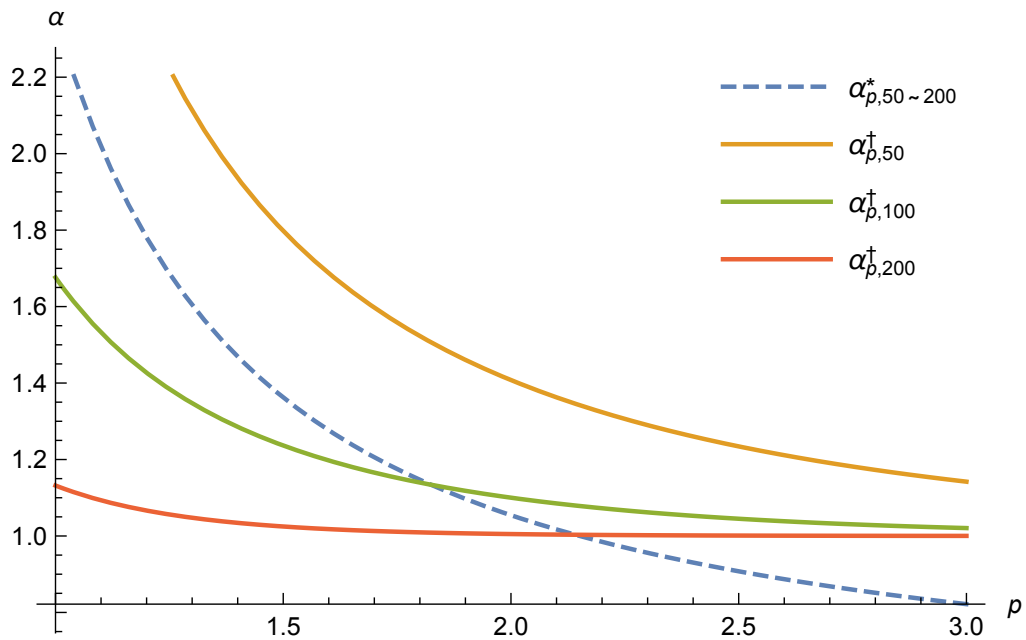

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Restriction $\mathrm{CVP}'_{p,\gamma}$: further require $\mathrm{dist}_p(\boldsymbol{t}, \boldsymbol{B} \cdot \{0,1\}^n) \leq 1$ for the case $\mathrm{dist}_p(\boldsymbol{t}, \mathcal{L}) \leq 1$.

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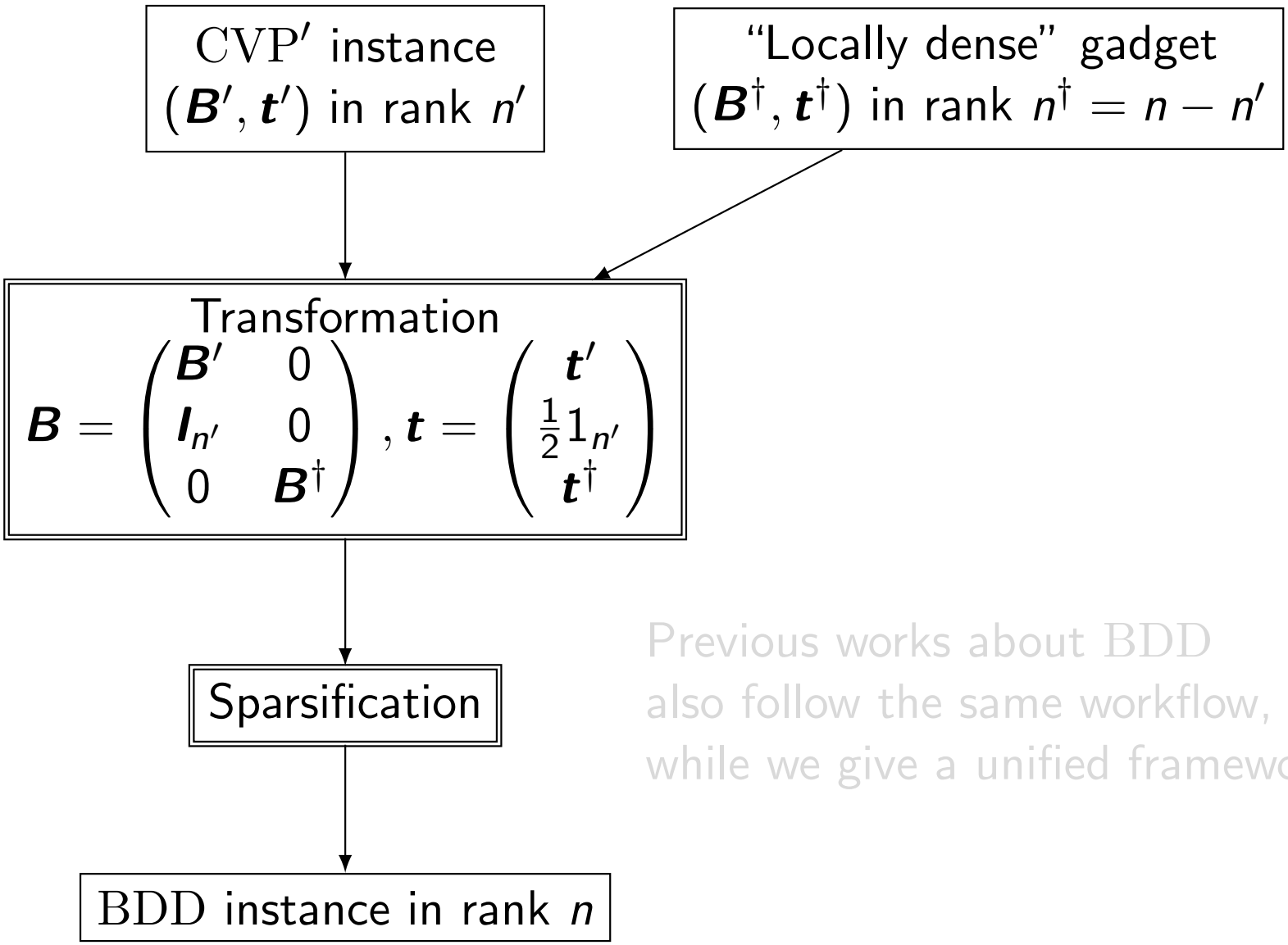

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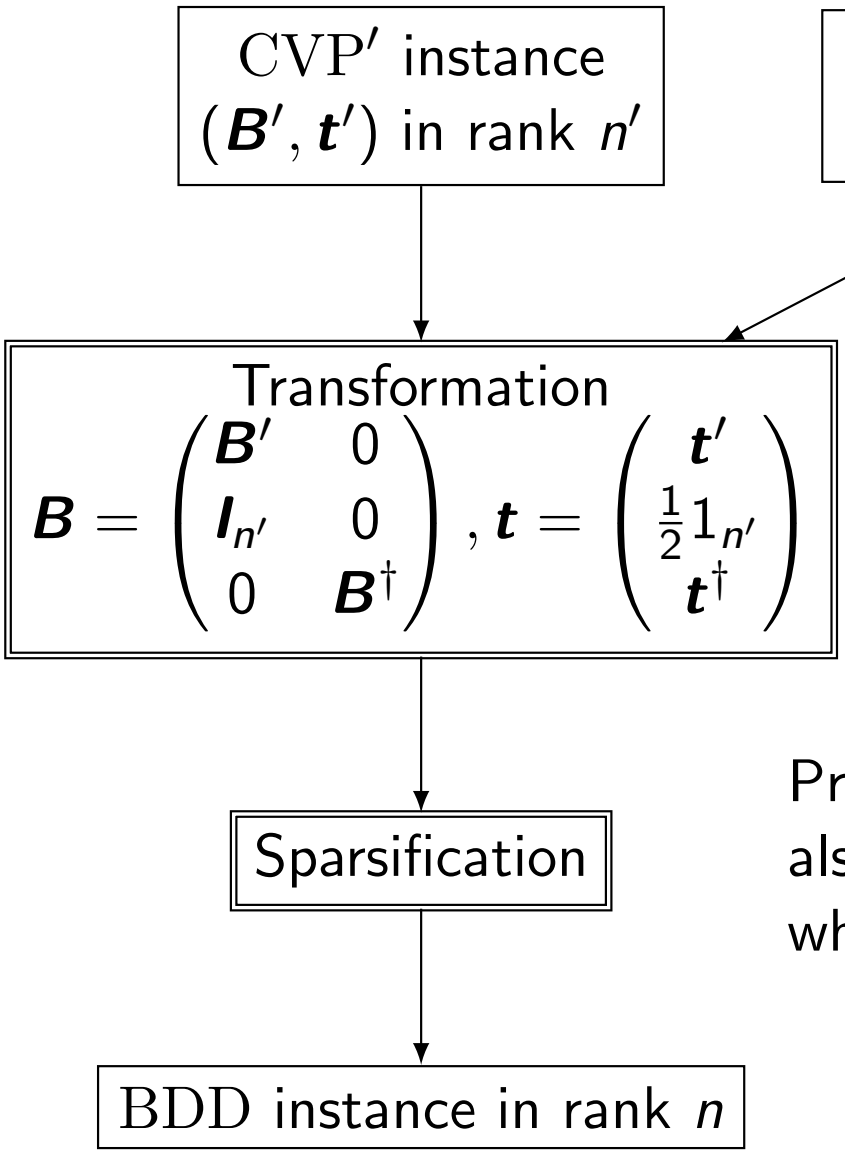

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Rephrasing BDD with Point-Counting

Recall (search) $\mathrm{BDD}_{p,\alpha}$: given $\mathcal{L}, \boldsymbol{t}$ with $\mathrm{dist}_p(\boldsymbol{t}, \mathcal{L}) \leq \alpha \cdot \lambda_1^{(p)}(\mathcal{L})$, find closest lattice vector to $\boldsymbol{t}$ in $\mathcal{L}$.

Decisional $\mathrm{BDD}_{p,\alpha}$: given $\mathcal{L}, \boldsymbol{t}$ and distance r, decide whether

- $\mathrm{dist}_p(\boldsymbol{t}, \mathcal{L}) \leq r$ and $\lambda_1^{(p)}(\mathcal{L}) \geq r/\alpha$, or
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In terms of point-counting: decide whether

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Relaxation (A, G)-$\mathrm{BDD}_{p,\alpha}$: decide whether

- "(good) close" count $\geq G$ and "short" count $\leq A$, or
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(Decisional BDD is just $(0, 1)$-BDD.)

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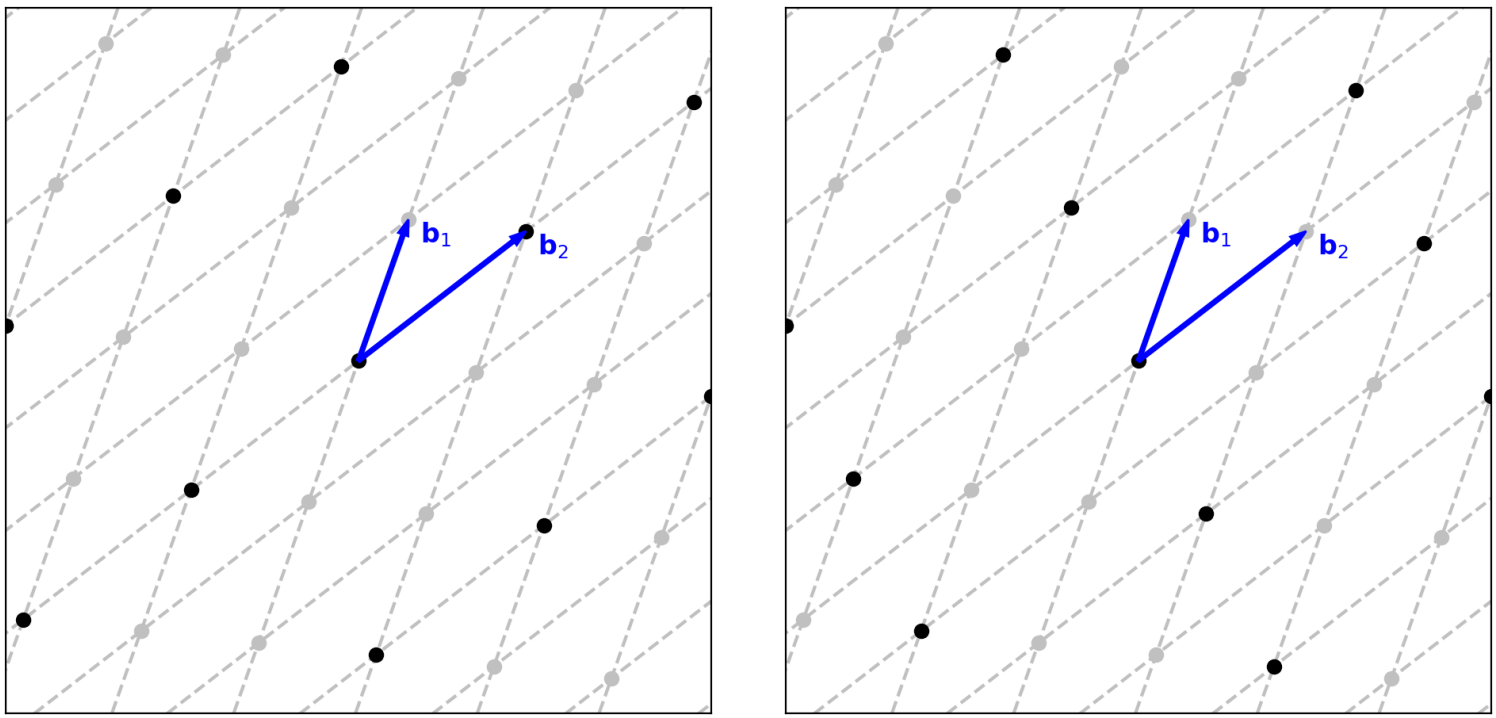
Lattice Sparsification

Sparsification algorithm: given lattice $\mathcal{L}$ and prime index q, sample sublattice $\mathcal{L}' \subset \mathcal{L}$ such that for any[1] finite set $S \subset \mathcal{L}$, $|S \cap \mathcal{L}'|$ concentrates around $|S|/q$.



$(q = 3)$

[1] S needs to satisfy certain technical conditions.

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If $G \gg A$, say $G \geq 400A$, then (A, G)-$\mathrm{BDD}_{p,\alpha}$ reduces to decisional $\mathrm{BDD}_{p,\alpha}$ by sparsification with index $q \approx 20A$.

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Transforming CVP' Instances

The transformation takes as input $\mathrm{CVP}'_{p,\gamma}$ instance $(\boldsymbol{B}', \boldsymbol{t}')$ and parameters $\boldsymbol{B}^\dagger, \boldsymbol{t}^\dagger, r, s$, and outputs (A, G)-$\mathrm{BDD}_{p,\alpha}$ instance:

$$\boldsymbol{B} = \begin{pmatrix} s\boldsymbol{B}' & 0 \\ \boldsymbol{I}_{n'} & 0 \\ 0 & \boldsymbol{B}^\dagger \end{pmatrix}, \qquad \boldsymbol{t} = \begin{pmatrix} s\boldsymbol{t}' \\ \frac{1}{2}1_{n'} \\ \boldsymbol{t}^\dagger \end{pmatrix}, \qquad r\ .$$

For CVP' YES instance:

- Promise: $\mathrm{dist}_p(\boldsymbol{t}', \boldsymbol{B}'\boldsymbol{x}) \leq 1$ for some $\boldsymbol{x} \in \{0,1\}^{n'}$.
- "Short" count: $|\mathcal{B}_p^\circ(r/\alpha) \cap \mathcal{L}| \leq |\mathcal{B}_p^\circ(r/\alpha) \cap (\mathbb{Z}^{n'} \oplus \mathcal{L}^\dagger)|$.
- "Close" count: $|\mathcal{B}_p(r; \boldsymbol{t}) \cap \mathcal{L}| \geq |\mathcal{B}_p(r - s - n'/2; \boldsymbol{t}^\dagger) \cap \mathcal{L}^\dagger|$.[2]

[2] The arithmetic of the distances here is showcased for ℓ_1, and should be $(r^p - s^p - n'/2^p)^{1/p}$ for general ℓ_p. We will continue to simplify this way in the remaining slides.

Transforming CVP' Instances

The transformation takes as input $\mathrm{CVP}'_{p,\gamma}$ instance $(\boldsymbol{B}', \boldsymbol{t}')$ and parameters $\boldsymbol{B}^\dagger, \boldsymbol{t}^\dagger, r, s$, and outputs (A, G)-$\mathrm{BDD}_{p,\alpha}$ instance:

$$\boldsymbol{B} = \begin{pmatrix} s\boldsymbol{B}' & 0 \\ \boldsymbol{I}_{n'} & 0 \\ 0 & \boldsymbol{B}^\dagger \end{pmatrix}, \qquad \boldsymbol{t} = \begin{pmatrix} s\boldsymbol{t}' \\ \frac{1}{2}1_{n'} \\ \boldsymbol{t}^\dagger \end{pmatrix}, \qquad r\ .$$

For CVP' YES instance:

- Promise: $\mathrm{dist}_p(\boldsymbol{t}', \boldsymbol{B}'\boldsymbol{x}) \leq 1$ for some $\boldsymbol{x} \in \{0,1\}^{n'}$.
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Putting together, for $G \gg A$, we want:

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Locally Dense Gadgets

Desired property (first consider the "short" term):

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Observations:

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- $|\mathcal{B}_p^\circ(\rho) \cap \mathbb{Z}^{n'}|$ is exponential in n' (for sufficiently large ρ).

Hence we want the gadget to be *locally dense*, i.e., to have exponentially more "close" than "short" lattice vectors:

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Main Theorem for BDD

Main theorem for BDD, informal & simplified

If there exist locally dense gadgets $(\boldsymbol{B}^\dagger, \boldsymbol{t}^\dagger)$ satisfying[3]

$$|\mathcal{B}_p(\alpha_G; \boldsymbol{t}^\dagger) \cap \mathcal{L}^\dagger| \geq \nu^{n^\dagger} |\mathcal{B}_p^\circ(1) \cap \mathcal{L}^\dagger| \ ,$$

then for $\mathrm{BDD}_{p,\alpha}$:
under Gap-ETH,[4] it cannot be solved in $2^{o(n)}$ time for all $\alpha > \alpha_G$;
under Gap-SETH, it cannot be solved in $2^{n/C}$ time for all

$$\alpha > \alpha_G + \frac{1}{f_p(\nu^{C-1})} \ .$$

(Here $f_p(\cdot)$ is increasing and has $\lim_{x\to 1} f_p(x) = 0$, $\lim_{x\to\infty} f_p(x) = \infty$.)

[3] The locally dense gadget needs to satisfy another similar property involving "annoying close" count, which contains similar parameters α_A, ν' and they also (substantially) affect the bounds on α.

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Instantiating the Main Theorem: Result 3

Lattice *kissing number* τ_n^{L}: $\max_{\mathcal{L}} |\mathcal{B}_p(1) \cap (\mathcal{L} \setminus \{0\})|$ for rank-n lattice $\mathcal{L}$ with $\lambda_1^{(p)}(\mathcal{L}) = 1$.

[Vlă19]: for $p = 2$, $\tau_n^{\mathsf{L}} \geq 2^{c_{\mathsf{kn}} n - o(n)}$, where $c_{\mathsf{kn}} \geq 0.02194$.

Gadgets (in ℓ_2): exponential kissing number lattice $\mathcal{L}^\dagger$, $\boldsymbol{t}^\dagger = 0$. Parameters: $\alpha_G = 1$, $\nu = 2^{c_{\mathsf{kn}}}$.

Using norm embeddings, we also get gadgets in all ℓ_p in cost of slightly larger $\alpha_G = 1 + o(1)$. Then we have our Result 3: $\mathrm{BDD}_{p,\alpha}$ cannot be solved in $2^{n/C}$ time for all

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Instantiating the Main Theorem: Result 1

To decrease α_G for the exponential kissing number gadgets:

- Move $\boldsymbol{t}^\dagger$ away from 0 by δ in random direction.
- Set $\alpha_G = 1 - \varepsilon$ for $\varepsilon < \delta$.
- Nevertheless this decreases the "close" count as well, by an expected factor of $\mathrm{area}(S^{n-1} \cap \mathcal{B}_p(1-\varepsilon; \boldsymbol{t}^\dagger))/\mathrm{area}(S^{n-1})$, where S^{n-1} is the unit sphere.
- ([AS18] also uses this idea while we have tighter loss factor.)

Taking care of the tradeoff between the "close" count and δ, ε, we manage to get α_G approaching $2^{-c_{\mathsf{kn}}}$, which gives our Result 1: $\mathrm{BDD}_{p,\alpha}$ cannot be solved in $2^{o(n)}$ time for all $\alpha > \alpha_{\mathsf{kn}} := 2^{-c_{\mathsf{kn}}}$.

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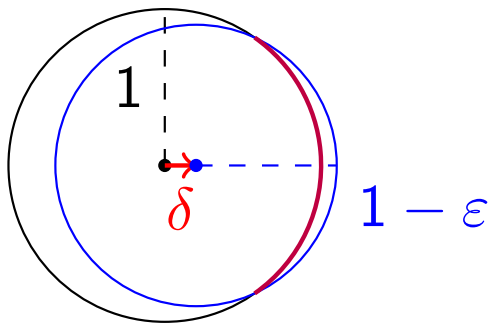

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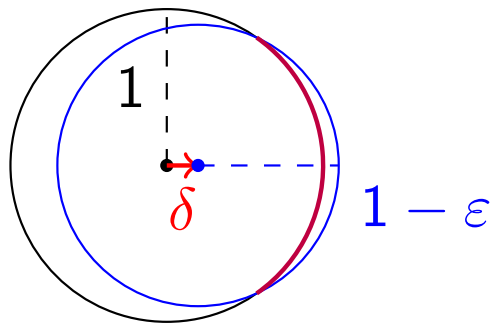

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Instantiating the Main Theorem: Result 2

Gadgets from integer lattices: $\mathcal{L}^\dagger = \mathbb{Z}^n/\rho$, $\boldsymbol{t}^\dagger = (t/\rho) \cdot 1_n$.
Minimize α_G over ρ, t subject to

$$|\mathcal{B}_p(\alpha_G \rho; t \cdot 1_n) \cap \mathbb{Z}^n| > |\mathcal{B}_p^\circ(\rho) \cap \mathbb{Z}^n| \ .$$

Suppose $\alpha_p^\ddagger$ is the optimum. Then we have our Result 2: $\mathrm{BDD}_{p,\alpha}$ cannot be solved in $2^{o(n)}$ time for all $\alpha > \alpha_p^\ddagger$.

- $|\mathcal{B}_p(a \cdot n; t \cdot 1_n) \cap \mathbb{Z}^n|$ can be approximated by a numerical function $\beta_{p,t}(a)^n$ to within a $2^{o(n)}$ factor.
- We find that empirically the optimizer for t is always $1/2$.
- [BP20] does no optimization and fix $t = 1/2$, $\rho = n/(2\alpha_G)$. As a result, our Result 2 is always no weaker than [BP20].

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$$|\mathcal{B}_p(\alpha_G \rho; t \cdot 1_n) \cap \mathbb{Z}^n| > |\mathcal{B}_p^\circ(\rho) \cap \mathbb{Z}^n| \ .$$

Suppose $\alpha_p^\ddagger$ is the optimum. Then we have our Result 2: $\mathrm{BDD}_{p,\alpha}$ cannot be solved in $2^{o(n)}$ time for all $\alpha > \alpha_p^\ddagger$.

- $|\mathcal{B}_p(a \cdot n; t \cdot 1_n) \cap \mathbb{Z}^n|$ can be approximated by a numerical function $\beta_{p,t}(a)^n$ to within a $2^{o(n)}$ factor.
- We find that empirically the optimizer for t is always $1/2$.
- [BP20] does no optimization and fix $t = 1/2$, $\rho = n/(2\alpha_G)$. As a result, our Result 2 is always no weaker than [BP20].

Instantiating the Main Theorem: Result 2

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Reduction to SVP

Overview:

- Similar to the case of BDD, the reduction consists of the (same!) transformation and the sparsification, as well as a standard technique, Kannan's embedding, at the end.
- The transformation maps $\mathrm{CVP}'_{p,\gamma}$ instances to instances of a similar intermediate problem $(A, G)\text{-}\mathrm{CVP}_{p,\gamma'}$.
- [AS18] has the same workflow, while we have a more general transformation with a larger parameter space, and we can set parameters working for $\mathrm{CVP}'_{p,\gamma}$ other than $\mathrm{CVP}'_{p,1}$.
- The same gadgets from integer lattices as Result 2 are used.

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