Motivation: Post-Quantum Cryptography

Problem: Attacker with quantum computation can break number theoretical cryptography that are widely used, such as RSA. **Solution**: Use *lattice-based* cryptography!

Fact: State-of-the-art attacks are based on solving exact or low-approximation-factor lattice problems (e.g. SVP).

Problem: Can attacker solve these problems in 2^n vs. $2^{n/10}$ vs. $2^{\sqrt{n}}$ time? It has a huge impact on security. **Our work**: Address this by showing *fine-grained* hardness results for lattice problems, under variants of ETH.

Lattices

Lattice: regular grid of points in space. Formally, lattice \mathcal{L} of rank *n*: set of all *integer* linear combinations of a basis $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n).$



Smaller α corresponds to stronger promise and easier problem.

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Our work exploits the power of different ETH variants, showing stronger hardness results for BDD/SVP under stronger variants.

More specifically, we reduce SAT on *n* variables to lattice problems in rank $C \cdot n$ for constant C > 0 to show fine-grained hardness results.

Line of research in fine-grained hardness of lattice problems: CVP [BGS17, ABGS21], SVP [AS18], BDD [BP20], SIVP [AC20].

Improved Hardness of BDD and SVP under Gap-(S)ETH

Yi Tang @ U-M CSE Prospective Student Visit Days 2022 Joint work with Huck Bennett and Chris Peikert, ITCS 2022

Lattice Problems: Shortest Vector Problem (SVP) Shortest ℓ_p norm of nonzero vector in lattice \mathcal{L} : $\lambda_1^{(p)}(\mathcal{L})$.



 γ -approximate SVP in ℓ_p (SVP_{p,γ}) **Instance**: Basis **B** of lattice \mathcal{L} . **Goal**: Decide whether $\lambda_1^{(p)}(\mathcal{L}) \leq 1$ or $\lambda_1^{(p)}(\mathcal{L}) > \gamma$.

Lattice Problems: Bounded Distance Decoding (BDD)

BDD in ℓ_p with relative distance α (BDD_{p, α})

Instance: Lattice \mathcal{L} and target **t** with dist_ $\rho(\mathbf{t}, \mathcal{L}) \leq \alpha \cdot \lambda_1^{(\rho)}(\mathcal{L})$. **Goal**: Find closest lattice vector to \mathbf{t} in \mathcal{L} .



Exponential Time Hypothesis (ETH)

ETH variants:

- **ETH**: 3-SAT cannot be solved in $2^{o(n)}$ time.
- Strong ETH (SETH): k-SAT cannot be solved in $2^{(1-\varepsilon)n}$ time.
- ► Gap-(S)ETH: Gap-3-SAT_{1- δ ,1 & Gap-*k*-SAT_{1- δ (*k*),1.}}
- ► Randomized/non-uniform variants: randomized/non-uniform time.

Assumption strength:

- \blacktriangleright ETH \leq SETH;
- \blacktriangleright plain \leq gap;
- \blacktriangleright plain \leq randomized \leq non-uniform.

0.95

2.2 ⊦ 2.0 1.8 1.6 ⊢`

 $2^{n/2}$



 $SVP_{p,\gamma}$ cannot be solved in $2^{n/C}$ time for some constant $\gamma > 1$, under randomized Gap-SETH. ($C_p \rightarrow 1$ for $p \rightarrow \infty$.)



^{/Cp} -hard	$2^{n/C_p}$ -hard	NP- <mark>hard</mark>	2 ^{0.802n} alg	(for $p = 2$) $2^{n/(2+f(c))}$ alg	easy γ	
, exact AS18]	1+arepsilon [this work]	any const [Kho05]	large const [EV20]	n ^{1/2+c} [ALSD21]	exp(<i>n</i>) [LLL82]	

Derandomize the reductions? Randomness is used in gadgets and in main theorem.

Core Proof Technique: Locally Dense Gadgets



Main Theorem for BDD

If there exist locally dense gadgets with parameters α_{close} and ν , then $BDD_{p,\alpha}$: ► cannot be solved in $2^{o(n)}$ time for any $\alpha > \alpha_{close}$, under Gap-ETH variants;

Instantiating the Main Theorem

[VIă19]: There exist lattices \mathcal{L}^{\dagger} with exponential kissing number: $2^{c_{kn}n-o(n)}$



► **Gadgets**: exponential kissing number lattice \mathcal{L}^{\dagger} with $\mathbf{t}^{\dagger} = \mathbf{0}$. **> Parameters**: $\alpha_{close} = 1$, $\nu = 2^{c_{kn}}$.

► Result 3: Immediately get $\alpha_{p,C}^{\dagger} := 1 + \varepsilon_p(2^{c_{kn}(C-1)})$ by main theorem. ▶ Result 1: Get $\alpha_{kn} := 2^{-c_{kn}}$ by perturbing \mathbf{t}^{\dagger} away from **0**.

Result 2 and 4: Use gadgets from integer lattices: ► Gadgets: $\mathcal{L}^{\dagger} = \mathbb{Z}^{n}$, $\mathbf{t}^{\dagger} = t \cdot \mathbf{1}_{n}$.

Result 2: Minimize α_{close} subject to $\nu > 1$, and get α_p^{\dagger} as the optimum. ► Result 4: Similar theorem for SVP based on gadgets, and same gadgets.

Open Questions

Construct locally "denser" gadgets?

E.g. better bound on kissing number immediately leads to better quantities in Result 1 and 3 (α_{kn} and $\alpha_{n,C}^{\dagger}$).