EECS 598-008 & EECS 498-008: Intelligent Programming Systems

Lecture 5

Announcements

- A1 due **midnight Tuesday September 14** (today)
- A2 out today (due **midnight Monday September 27**)
  - More challenging! Start early!
- **Remote OH** 3-4pm Friday September 17
  - Z3 setup and tutorial (video recording released by Thursday), A2 (briefly)

Propositional Logic Review

- Syntax
  - propositional variables, logical connectives
- Semantics
  - Evaluated under an interpretation
- Satisfiability and validity
  - Duality between satisfiability and validity
- Deciding satisfiability and validity
  - Truth table method, semantic argument method
  - Automated solvers such as Microsoft Z3

Agenda

- Propositional Logic
- **First-Order Logic**
- First-Order Theories

First-Order Logic

- E.g., $\forall x \,.\, P(x) \wedge Q(x)$
- FOL is more expressive than propositional logic:
  - More constants beyond only True and False, e.g., *Jack*, *Apple*, *Blue*, …
  - Functions, e.g., *MotherOf, ColorOf, …*
  - Predicates, e.g., *Loves*, *BiggerThan*, …
  - Quantifiers, e.g., “for all”, “there exists”
  - Variables

First-Order Logic Syntax

- Basic building blocks
  - *Object constants (a, b, c, …)*
    - E.g., people {*Jack, Smith,* …}, numbers {…, *-1, 0, 1,* …}

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  - *Relation constants (p, q, r, …)*
    - Relations between objects, or properties of objects, also called predicates
    - E.g., *Loves, IsBiggerThan*
    - Uninterpreted

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    - Relations between objects, or properties of objects, also called predicates
    - E.g., *Loves*, *IsBiggerThan*
    - Uninterpreted
  - *Variables (x, y, z, …)*
    - These are “object variables”. They cannot refer to functions.

First-Order Logic Syntax

- Building blocks:
  - Object constants
  - Function constants
  - Relation constants
  - Variables (x, y, z, …)

- First, use building blocks to create **terms**:
  - Basic terms: Any object constant or a variable, e.g., Jack, Apple, x, y
  - Compound terms: Function constants applied to terms, e.g., MotherOf(Jack)

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- Then, build **formulas**:
  - Base case: Relation constant applied to terms, e.g., isOlder(motherOf(Jack), Jack)
  - Inductive case:
    - If F_1, F_2 are formulas, then $F_1 \star F_2$ is also formula ($\star \in \{ \land , \lor , \to , \leftrightarrow \}$)
    - If F is formula, then $(F), \neg F$ are also formulas
    - If F is formula and x is variable, then $\forall x . F, \exists x . F$ are also formulas

First-Order Logic Syntax

- Example: $\forall x \,.\, p(a, f(b)) \wedge q(x)$
  - Object constants?
  - Function constants?
  - Relation constants?
  - Variables?

First-Order Logic Syntax

- Express the following sentence in FOL using function constant *size*, relation constant *biggerThan.*

  "For any x, y, z, if x is bigger than y and y is bigger than z, then x is bigger than z."

First-Order Logic Semantics

- What truth value does a FOL formula evaluate to?

First-Order Logic Semantics

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- Similar to propositional logic, need an **interpretation**
- In addition, also need **universe of discourse** (i.e., universe, domain)

- Universe of discourse U
  - Non-empty set of objects
  - E.g., set of positive integers, all real numbers, all students in this class
  - Object constants refer to objects in U
  - Functions/predicates are defined over U

First-Order Logic Semantics

- First-order interpretation
  - I mapping from object, function, relation constants to objects in universe U
  - E.g., consider:
    - $U = \{1,2,3,4\}$
    - Object constants: $a, b, c \in U$
    - Unary function constants: $f : U \to U$
    - Binary relation constant: $p \subseteq U^2$

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    - $U = \{1,2,3,4\}$
    - Object constants: $a, b, c \in U$
    - Unary function constants: $f : U \rightarrow U$
    - Binary relation constant: $p \subseteq U^2$
    - A possible interpretation:

$$I(a) = 1, I(b) = 2, I(c) = 3 \qquad I(f) = \{1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 1\}$$

$$I(p) = \{\langle 1,2 \rangle, \langle 3,4 \rangle\}$$

First-Order Logic Semantics

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- $\models$ is defined inductively

First-Order Logic Semantics

- $U, I \models F$: F evaluates to $\top$ under U, I $\quad$ $U, I \not\models F$: F evaluates to $\bot$ under U, I
- $\models$ is defined inductively

- Base cases: predicates
- Inductive cases: logical operators/quantifiers over predicates

First-Order Logic Semantics

- $U, I \vDash F$: F evaluates to $\top$ under U, I $U, I \nvDash F$: F evaluates to $\bot$ under U, I
- $\vDash$ is defined inductively
- Base cases
  - $U, I \vDash \top$ $U, I \nvDash \bot$
  - $U, I \vDash p(t_1, \ldots, t_n)$ *iff* predicate p holds for $\langle I \rangle(t_1), \ldots, \langle I \rangle(t_n)$

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  - $U, I \vDash \top$ $U, I \nvDash \bot$
  - $U, I \vDash p(t_1, \ldots, t_n)$ *iff* predicate p holds for $\langle I \rangle(t_1), \ldots, \langle I \rangle(t_n)$
    - Evaluating terms
      - Base cases: $\langle I \rangle(a) = I(a)$
      - Inductive case: $\langle I \rangle\big(f(t_1, \ldots, t_n)\big) = I(f)\big(\langle I \rangle(t_1), \ldots, \langle I \rangle(t_n)\big)$

First-Order Logic Semantics

- Base cases: predicates
- Inductive cases:
  - $U, I \models \neg F$ iff $U, I \not\models F$
  - $U, I \models F_1 \wedge F_2$ iff $U, I \models F_1$ and $U, I \models F_2$
  - $U, I \models F_1 \vee F_2$ iff $U, I \models F_1$ or $U, I \models F_2$
  - $U, I \models \forall x . F$ iff **for all** $o \in U : U, I \models F[x \mapsto o]$
  - $U, I \models \exists x . F$ iff **there exists** $o \in U,$ such that $U, I \models F[x \mapsto o]$

First-Order Logic Semantics

- Consider $U = \{ \star , \bullet \}$ and I :

$$I(a) = \bullet \, , I(b) = \star$$

$$I(f) = \{ \star \mapsto \bullet \, , \bullet \mapsto \star \}$$

$$I(p) = \{\langle \bullet , \bullet \rangle, \langle \star , \bullet \rangle\}$$

- Given $U, I,$ what do these formulas evaluate to?
  - $\forall x \, . \, p(a, x)$
  - $\forall x \, . \, p(x, a)$
  - $\exists x \, . \, p(a, x)$
  - $\exists x \, . \, p(f(x), f(a))$

Satisfiability and Validity

- A FOL formula F is **satisfiable** iff **there exists** a universe U and an interpretation I such that $U, I \models F$
  - Otherwise, unsatisfiable
- F is **valid** iff **for all** universes U and interpretations I, we have $U, I \models F$
  - Otherwise, not valid

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- Is $\forall x . \exists y . p(x, y)$ satisfiable and/or valid?
- Is $(\forall x . p(x, x)) \rightarrow (\exists y . p(y, y))$ satisfiable and/or valid?

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- How about *Equal(Plus(a, b), Plus(b, a))*?

Deciding Satisfiability and Validity

- Truth table method?
  - No! because universe may be infinite

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Deciding Satisfiability and Validity

- Truth table method?
  - No! because universe may be infinite
- Semantic argument method
  - Yes, but it is undecidable (for both satisfiability and validity)
- Automated solvers (e.g., Microsoft Z3, CVC4) work pretty well in practice!

Microsoft Z3 Demo

- https://compsys-tools.ens-lyon.fr/z3/index.php
- Use SMT-LIB to express formulas
  - https://compsys-tools.ens-lyon.fr/z3/smt-lib-reference-v2.5-r2015-06-28.pdf
  - https://link.springer.com/content/pdf/bbm%3A978-3-662-50497-0%2F1.pdf

Microsoft Z3 Demo

- Prove $F : \big(\forall x \,.\, p(x)\big) \rightarrow \big(\forall y \,.\, p(y)\big)$ is valid

```
; declarations
(declare-fun p (Int) Bool)

; constraints
(assert (=> (forall ((x Int)) (p x)) (forall ((y Int)) (p y))))

; solve
(check-sat)
;(get-model)
```

Microsoft Z3 Demo

- Prove $F : \big(\forall x\,.\,(p(x) \vee q(x))\big) \rightarrow \big(\exists x\,.\,p(x) \vee \forall x\,.\,q(x)\big)$ is valid

```
; declarations
(declare-fun p (Int) Bool)
(declare-fun q (Int) Bool)

; constraints
(assert (=> (forall ((x Int)) (or (p x) (q x))) (or (forall ((x Int)) (q x)) (exists ((x Int)) (p x)))))

; solve
(check-sat)
;(get-model)
```

Microsoft Z3 Demo

- Is $\forall x \,.\, x + 1 = 1 + x$ valid?

(assert (forall ((x Int)) (= (+ x 1) (+ 1 x))))

(check-sat)

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- **First-Order Theories**

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  - Propositional logic is limited in expressiveness
  - FOL is more expressive, but functions are uninterpreted (can assign any meaning)

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  - Propositional logic is limited in expressiveness
  - FOL is more expressive, but functions are uninterpreted (can assign any meaning)

- In many cases, we want functions to have certain meanings (e.g., +, =, >)
- **Theories assign meanings to symbols**

First-Order Theories Syntax

- A first-order theory has
  - object/function/relation constants, variables, quantifiers, logical connectives (FOL)
  - **axioms (new!)**

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- E.g., let's make up a first-order theory — theory of heights T_H
  - T_H has only one relation constant called $taller$ and no other constants
  - T_H has one axiom $\forall x, y \,.\, (taller(x, y) \rightarrow \neg taller(y, x))$

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  - Is $\forall x\,.\,\exists y\,.\,taller(y, x)$ in T_H?

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  - T_H has one axiom $\forall x, y \,.\, (taller(x, y) \rightarrow \neg taller(y, x))$
  - Is $\forall x \,.\, \exists y \,.\, taller(y, x)$ in T_H?
  - Is $\forall x \,.\, taller(Jack, x)$ in T_H?

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  - **We are only interested in those interpretations that are consistent!**
- Given U, I, formula F can be evaluated in the same way as in FOL, but we only consider interpretations that are consistent with axioms
  - … which means some formulas not valid in FOL may be valid in first-order theories

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- Satisfiability Modulo Theory (SMT) solvers: Microsoft z3, CVC4, …

Satisfiability and Validity Modulo Theory T

- If F is valid in FOL, is it also valid modulo T ?
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