More Lambda Calculus

and

Intro to Type Systems
One Slide Summary

• The **lambda calculus** is a model of computation or a programming language that is as expressive as a Turing machine.

• The lambda calculus centers on **function definition** and **function application**. The meaning of function application is given by substitution (**beta reduction**).

• We can **encode** the **booleans** (and, or, not, if) and the **numbers** (zero, successor, add, multiply, equality, looping) via lambdas.
It May Seem Difficulty, But ... 

YOU CAN SUCCEED 

IF YOU APPLY YOURSELF

(Jiawei Chen memorial meme)
Plan

• Heavy Class Participation
  - Thus, wake up! *(not actually kidding)*

• Lambda Calculus
  - How is it related to real life?
  - Encodings
  - Fixed points

• Type Systems
  - Overview
  - Static, Dynamic
  - Safety, Judgments, Derivations, Soundness
Lambda Review

• \( \lambda \)-calculus is a calculus of **functions**
  \[
e := x \mid \lambda x. \ e \mid e_1 \ e_2
  \]

• Several evaluation strategies exist based on **\( \beta \)-reduction**
  \[
  (\lambda x.e) \ e' \rightarrow_\beta [e'/x] \ e
  \]

• How does this simple calculus relate to real programming languages?
Functional Programming

• The $\lambda$-calculus is a prototypical functional language with:
  - no side effects
  - several evaluation strategies
  - lots of functions
  - nothing but functions (pure $\lambda$-calculus does not have any other data type)

• How can we program with functions?
• How can we program with only functions?
Programming With Functions

- **Functional programming** is a programming style that relies on lots of functions
- A typical functional paradigm is *using functions as arguments or results of other functions*
  - Called “**higher-order programming**”
- Some “impure” functional languages permit side-effects (e.g., Lisp, Scheme, ML, Python)
  - references (pointers), in-place update, arrays, exceptions
  - Others (and by “others” we mean “Haskell”) use monads to model state updates
Variables in Functional Languages

- We can introduce new variables:
  \[
  \text{let } x = e_1 \text{ in } e_2
  \]
  - \(x\) is \textit{bound} by let
  - \(x\) is \textit{statically scoped} in (a subset of) \(e_2\)

- This is pretty much like \((\lambda x. e_2) e_1\)

- In a functional language, variables are never updated
  - they are just \textit{names for expressions or values}
  - e.g., \(x\) is a name for the value denoted by \(e_1\) in \(e_2\)

- This models the meaning of “let” in math (proofs)
Referential Transparency

- In “pure” functional programs, we can reason equationally, by substitution
  - Called “referential transparency”
    
    \[\text{let } x = e_1 \text{ in } e_2 \quad \Longrightarrow \quad [e_1/x]e_2\]

- In an imperative language a side-effect in \(e_1\) might invalidate the above equation

- The behavior of a function in a “pure” functional language depends only on the actual arguments
  - Just like a function in math
  - This makes it easier to understand and to reason about functional programs
How Tough Is Lambda?

- Given $e_1$ and $e_2$, how complex (a la CS theory) is it to determine if:

$$e_1 \rightarrow^* e \quad \text{and} \quad e_2 \rightarrow^* e$$
Expressiveness of $\lambda$-Calculus

- The $\lambda$-calculus is a minimal system but can express:
  - data types (integers, booleans, lists, trees, etc.)
  - branching
  - recursion
- This is enough to encode Turing machines
  - We say the lambda calculus is Turing-complete
- Corollary: $e_1 =_\beta e_2$ is undecidable
- Still, how do we encode all these constructs using only functions?
- Idea: *encode the “behavior” of values* and not their structure
Encoding Booleans in \( \lambda \)-Calculus

- What can we \textit{do} with a boolean?
  - we can make a binary choice (= “if” statement)

- A boolean is a function that, given two choices, selects one of them:
  - \texttt{true} = \texttt{def} \( \lambda x. \lambda y. x \)
  - \texttt{false} = \texttt{def} \( \lambda x. \lambda y. y \)
  - if \( E_1 \) then \( E_2 \) else \( E_3 \) = \texttt{def} \( E_1 E_2 E_3 \)

- Example: “\texttt{if true then u else v}” is
  \( (\lambda x. \lambda y. x) \ u \ v \rightarrow_\beta (\lambda y. \ u) \ v \rightarrow_\beta \ u \)
More Boolean Encodings

- Let’s try to do boolean **or** together
- Recall:
  - true = \( \lambda x. \lambda y. x \)
  - false = \( \lambda x. \lambda y. y \)
  - if \( E_1 \) then \( E_2 \) else \( E_3 \) = \( E_1 \ E_2 \ E_3 \)

- We want **or** to take in two booleans and yield a result that is a boolean
- How can we do this?
A *Trying Ordeal*

• **Recall:**
  - true
  - false
  - if $E_1$ then $E_2$ else $E_3$

  $$
  \text{true} = \text{def} \lambda x. \lambda y. x \\
  \text{false} = \text{def} \lambda x. \lambda y. y \\
  \text{if } E_1 \text{ then } E_2 \text{ else } E_3 = \text{def} E_1 \ E_2 \ E_3
  $$

• **Intuition:**
  - **or** a b

  $$
  \text{or a b} = \text{if a then true else b}
  $$

• **Either of these will work:**
  - **or**
    $$
    \text{or} = \text{def} \lambda a. \lambda b. a \ \text{true} \ b
    $$
  - **or**
    $$
    \text{or} = \text{def} \lambda a. \lambda b. \lambda x. \lambda y. \ a \ x \ (b \ x \ y)
    $$
Final Boolean Encodings

- Think about how to do **and** and **not**
- Without peeking! Now come up and do it!
Another Demand

• How to do **and** and **not**

• **and** a b = if a then b else false
  - **and**
    = _def_ \( \lambda a. \lambda b. \ a \ b \ false \)
  - **and**
    = _def_ \( \lambda a. \lambda b. \lambda x. \lambda y. \ a \ (b \ x \ y) \ y \)

• **not** a = if a then false else true
  - **not**
    = _def_ \( \lambda a. \ a \ false \ false \ true \)
  - **not**
    = _def_ \( \lambda a. \lambda x. \lambda y. \ a \ y \ x \)
Encoding Pairs in \( \lambda \)-Calculus

- What can we *do* with a pair?
  - we can access one of its elements (= “.field access”)

- A pair is a function that, given a boolean, returns the first or second element

  \[
  \text{mkpair } x \ y \quad =_{\text{def}} \quad \lambda b. \ b \ x \ y \\
  \text{fst } p \quad =_{\text{def}} \quad p \ true \\
  \text{snd } p \quad =_{\text{def}} \quad p \ false
  \]

- \( \text{fst} \ (\text{mkpair } x \ y) \rightarrow_{\beta} \ (\text{mkpair } x \ y) \ true \rightarrow_{\beta} \ true \ x \)

- \( \text{snd} \ (\text{mkpair } x \ y) \rightarrow_{\beta} \ x \)
Encoding Numbers in $\lambda$–Calculus

- What can we do with a natural number?
  - What do you, the viewers at home, think?
Encoding Numbers $\lambda$-Calculus

• What can we *do* with a natural number?
  - we can *iterate a number of times* over some function
    (= “for loop”)

• A natural number is a function that given an operation $f$ and a starting value $s$, applies $f$ a number of times to $s$:

\[
\begin{align*}
0 &= \text{def } \lambda f. \lambda s. s \\
1 &= \text{def } \lambda f. \lambda s. f \ s \\
2 &= \text{def } \lambda f. \lambda s. f \ (f \ s)
\end{align*}
\]
  - Very similar to List.fold_left and friends

• These are numerals in a unary representation

• Called *Church numerals*
Test Time!

• How would you encode the successor function (\texttt{succ} \( x \Longrightarrow x+1 \))?
• How would you encode more general addition?
• Recall: \( 4 =_{\text{def}} \lambda f. \lambda s. f f f (f s) \)
Computing with Natural Numbers

- The successor function
  \[ \text{succ} \ n = \ \lambda f. \ \lambda s. \ f \ (n \ f \ s) \]
  or
  \[ \text{succ} \ n = \ \lambda f. \ \lambda s. \ n \ f \ (f \ s) \]

- Addition
  \[ \text{add} \ n_1 \ n_2 = \ \text{succ} \ n_1 \ n_2 \]

- Multiplication
  \[ \text{mult} \ n_1 \ n_2 = \ n_1 \ (\text{add} \ n_2) \ 0 \]

- Testing equality with 0
  \[ \text{iszero} \ n = \ (\lambda b. \ \text{false}) \ \text{true} \]

- Subtraction
  - Is not instructive, but makes a fun exercise...
Computation Example

• What is the result of the application `add 0`?

\[(\lambda n_1. \lambda n_2. n_1 \text{ succ } n_2) \ 0 \rightarrow^\beta \]

\[\lambda n_2. 0 \text{ succ } n_2 =\]

\[\lambda n_2. (\lambda f. \lambda s. s) \text{ succ } n_2 \rightarrow^\beta \]

\[\lambda n_2. n_2 =\]

\[\lambda x. x\]

• By computing with functions we can express some optimizations
  - But we need to reduce under the lambda
  - Thus this “never” happens in practice
Toward Recursion

- Given a predicate $P$, encode the function “find” such that “find $P \ n$” is the smallest natural number which is larger than $n$ and satisfies $P$.
- Claim: with find we can encode all recursion.

Intuitively, why is this true?
Encoding Recursion

- Given a predicate $P$ encode the function “find” such that “find $P$ n” is the smallest natural number which is larger than $n$ and satisfies $P$
- $\text{find}$ satisfies the equation
  \[
  \text{find } p \ n = \text{if } p \ n \text{ then } n \text{ else find } p \ (\text{succ } n)
  \]
- Define
  \[
  F = \lambda f. \lambda p. \lambda n. (p \ n) \ n \ (f \ p \ (\text{succ } n))
  \]
- We need a fixed point of $F$
  \[
  \text{find} = F \ \text{find}
  \]
  or
  \[
  \text{find } p \ n = F \ \text{find } p \ n
  \]
The Fixed-Point Combinator $\text{Y}$

- Let \( Y = \lambda F. \ (\lambda y. F(y \ y)) \ (\lambda x. F(x \ x)) \)
  - This is called the **fixed-point combinator**
  - Verify that \( Y \ F \) is a fixed point of \( F \)
    \[
    Y \ F \rightarrow^\beta (\lambda y. F (y \ y)) \ (\lambda x. F (x \ x)) \rightarrow^\beta F \ (Y \ F)
    \]
  - Thus \( Y \ F =^\beta F \ (Y \ F) \)
- Given any function in $\lambda$-calculus we can define its fixed-point (w00t! why do we *not* win here?)
- Thus we can define “find” as the fixed-point of the function \( F \) from the previous slide
- Essence of recursion is the self-application “\( y \ y \)”
Expressiveness of Lambda Calculus

• Encodings are fun
  - Yes! Yes they are!

• But programming in pure $\lambda$-calculus is painful

• So we will add constants (0, 1, 2, ..., true, false, if-then-else, etc.)

• Next we will add types
Still Going!

- One minute break
- Stretch!
In this 1943 Antoine de Saint-Exupery novel the title character lives on an asteroid with a rose but eventually travels to Earth.
Cuisine

• This mixed-rice dish is generally made with Indian spices, rice, and meat. It is popular in the Indian subcontinent as well as in Afghanistan, Pakistan, Iran, and Iraq. It appeared on a 2017 Indian postage stamp.
Q: Computer Science (姚期智)

- This Shanghai-born Turing Award winner is known for contributions to the theory of computation. He formulated the Millionaire's Problem and stated this minimax principle: “the expected cost of any randomized algorithm for solving a given problem, on the worst case input for that algorithm, can be no better than the expected cost, for a worst-case random probability distribution on the inputs, of the deterministic algorithm that performs best against that distribution.”
Types

• A program variable can assume a range of values during the execution of a program.

• An upper bound of such a range is called a type of the variable.
  - A variable of type “bool” is supposed to assume only boolean values.
  - If x has type “bool” then the boolean expression “not(x)” has a sensible meaning during every run of the program.
Typed and Untyped Languages

- **Untyped languages**
  - Do *not* restrict the range of values for a given variable
  - Operations might be applied to inappropriate arguments. The behavior in such cases might be unspecified
  - The pure $\lambda$-calculus is an extreme case of an untyped language (however, its behavior is completely specified)

- **(Statically) Typed languages**
  - Variables are assigned (non-trivial) types
  - A type system keeps track of types
  - Types might or might not appear in the program itself
  - Languages can be explicitly typed or implicitly typed
The Purpose Of Types

- The foremost purpose of types is to prevent certain types of run-time execution errors

- Traditional trapped execution errors
  - Cause the computation to stop immediately
  - And are thus well-specified behavior
  - Usually enforced by hardware
  - e.g., Division by zero, floating point op with a NaN
  - e.g., Dereferencing the address 0 (on most systems)

- Untrapped execution errors
  - Behavior is unspecified (depends on the state of the machine = this is very bad!)
  - e.g., accessing past the end of an array
  - e.g., jumping to an address in the data segment
Execution Errors

- A program is deemed **safe** if it does *not* cause untrapped errors
  - Languages in which all programs are safe are **safe languages**
- For a given language we can designate a set of forbidden errors
  - A superset of the untrapped errors, usually including some trapped errors as well
    - e.g., null pointer dereference
- **Modern Type System Powers:**
  - prevent race conditions (e.g., Flanagan TLDI ‘05)
  - prevent insecure information flow (e.g., Li POPL ’05)
  - prevent resource leaks (e.g., Vault, Weimer)
  - help with generic programming, probabilistic languages, ...
  - ... are often combined with dynamic analyses (e.g., CCured)
Preventing Forbidden Errors - Static Checking

- Forbidden errors can be caught by a combination of static and run-time checking

- Static checking
  - Detects errors early, *before testing*
  - Types provide the necessary static information for static checking
  - e.g., ML, Modula-3, Java
  - Detecting certain errors statically is *undecidable* in most languages
Preventing Forbidden Errors - Dynamic Checking

• Required when static checking is undecidable
  - e.g., array-bounds checking

• Run-time encodings of types are still used (e.g. Lisp)

• Should be limited since it delays the manifestation of errors

• Can be done in hardware (e.g. null-pointer)
Safe Languages

- There are typed languages that are not safe ("weakly typed languages")
- All safe languages use types (static or dynamic)

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<thead>
<tr>
<th></th>
<th>Typed</th>
<th>Untyped</th>
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<tbody>
<tr>
<td>Static</td>
<td></td>
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<tr>
<td>Safe</td>
<td>ML, Java, Ada, C#, Haskell, ...</td>
<td>Lisp, Scheme, Ruby, Perl, Smalltalk, PHP, Python, ...</td>
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<td>Dynamic</td>
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<tr>
<td>Unsafe</td>
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<td>Assembly</td>
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<td>C, C++, Pascal, ...</td>
<td>?</td>
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- We focus on statically typed languages
Why Typed Languages?

• Development
  - *Type checking catches early many mistakes*
  - Reduced debugging time
  - Typed signatures are a powerful basis for design
  - Typed signatures enable separate compilation

• Maintenance
  - Types act as checked specifications
  - Types can enforce abstraction

• Execution
  - Static checking reduces the need for dynamic checking
  - *Safe languages are easier to analyze statically*
    - the compiler can generate better code
Homework

- Read Cardelli article
- Read Wright & Matthias article
- Homework Due Soon