Proof Techniques for Operational Semantics

Small-Step Contextual Semantics

- In small-step contextual semantics, derivations are not tree-structured
- A contextual semantics derivation is a sequence (or list) of atomic rewrites:

$$\langle x+(7-3),\sigma\rangle \to \langle x+(4),\sigma\rangle \to \langle 5+4,\sigma\rangle \to \langle 9,\sigma\rangle$$

$\sigma(x)=5$

If $\langle r, \sigma\rangle \to \langle e, \sigma'\rangle$

then $\langle H[r], \sigma\rangle \to \langle H[e], \sigma'\rangle$

r = redex
H = context (has hole)

Context Decomposition

- Decomposition theorem:

  If c is not "skip" then there exist unique H and r such that c is H[r]

  - "Exist" means progress
  - "Unique" means determinism

Short-Circuit Evaluation

- What if we want to express short-circuit evaluation of $\wedge$?
  - Define the following contexts, redexes and local reduction rules

$$H ::= \ldots \mid H \wedge b_2$$

$$r ::= \ldots \mid \text{true} \wedge b \mid \text{false} \wedge b$$

$$\langle \text{true} \wedge b, \sigma \rangle \to \langle b, \sigma \rangle$$

$$\langle \text{false} \wedge b, \sigma \rangle \to \langle \text{false}, \sigma \rangle$$

  - the local reduction kicks in before b_2 is evaluated

Contextual Semantics Summary

- Can view • as representing the program counter
- Contextual semantics is inefficient to implement directly

- The major advantage of contextual semantics: it allows a mix of local and global reduction rules
  - For IMP we have only local reduction rules: only the redex is reduced
  - Sometimes it is useful to work on the context too
  - We'll do that when we study memory allocation, etc.

Cunning Plan for Proof Techniques

- Why Bother?
- Mathematical Induction
- Well-Founded Induction
- Structural Induction
  - "Induction On The Structure Of The Derivation"

One-Slide Summary

- **Mathematical Induction** is a proof technique: If you can prove P(0) and you can prove that P(n) implies P(n+1), then you can conclude that for all natural numbers n, P(n) holds.
- Induction works because the natural numbers are **well-founded**: there are no **infinite descending chains** n > n-1 > n-2 > ... > ... .
- **Structural induction** is induction on a formal structure, like an AST. The base cases use the leaves, the inductive steps use the inner nodes.
- **Induction on a derivation** is structural induction applied to a derivation D (e.g., D::$\langle c, \sigma\rangle \Downarrow \sigma'$).

Why Bother?

- I am loathe to teach you anything that I think is a waste of your time.
- Thus I must convince you that inductive opsem proof techniques are useful.
  - Recall class goals: understand PL research techniques and apply them to your research
- This motivation should also highlight where you might use such techniques in your own research.

Never Underestimate

"Any counter-example posed by the Reviewers against this proof would be a useless gesture, no matter what technical data they have obtained. **Structural Induction** is now the ultimate proof technique in the universe. I suggest we use it." --- Admiral Motti, *A New Hope*

Classic Example (Schema)

- "A well-typed program cannot go wrong."
  - Robin Milner
- When you design a new type system, you must show that it is safe (= that the type system is sound with respect to the operational semantics).
- *A Syntactic Approach to Type Soundness.* Andrew K. Wright, Matthias Felleisen, 1992.
  - Type preservation: "if you have a well-typed program and apply an opsem rule, the result is well-typed."
  - Progress: "a well-typed program will never get stuck in a state with no applicable opsem rules"
- Done for real languages: SML/NJ, SPARK ADA, Java
  - PL/I, plus basically every toy PL research language ever.

Classic Examples

- **CCured Project (Berkeley)**
  - A program that is instrumented with CCured run-time checks (= "adheres to the CCured type system") will not segfault (= "the x86 opsem rules will never get stuck").
- **Vault Language (Microsoft Research)**
  - A well-typed Vault program does not leak any tracked resources and invokes tracked APIs correctly (e.g., handles IRQL correctly in asynchronous Windows device drivers, cf. Capability Calculus)
- **RC - Reference-Counted Regions For C (Intel Research)**
  - A well-typed RC program gains the speed and convenience of region-based memory management but need never worry about freeing a region too early (run-time checks).
- **Typed Assembly Language (Cornell)**
  - Reasonable C programs (e.g., device drivers) can be translated to TALx86. Well-typed TALx86 programs are type- and memory-safe.
- **Secure Information Flow (Many, e.g,. Volpano et al. '96)**
  - Lattice model of secure flow analysis is phrased as a type system, so type soundness = noninterference.

Examples from POPL 2021

- Mechanized Logical Relations for Termination-Insensitive Noninterference

  4 THE FUNDAMENTAL THEOREMS AND SOUNDNESS

  It is straightforward to show the unary fundamental theorem by structural induction on the typing derivation. All proofs are carried out at an abstraction level similar to the structural rules shown in

- Transfinite Step-Indexing for Termination

  have to prove the syntactic typing system $\Gamma \vdash e : A$ sound:

  THEOREM 4.5. *If* $\Gamma \vdash e : A$, *then* $\Gamma \vDash e : A$.

  As for traditional logical relations, the proof proceeds by induction on the typing derivation. For

- Modeling and Analyzing Evaluation Cost of CUDA Kernels

  All proofs are by induction on the derivation in the logic and are formalized in Coq. The case for while loops also includes an inner induction on the evaluation of the loop.

- Automatic Differentiation in PCF

  LEMMA 29. *Let* $M \lhd M'$ *and* $M \to N$, *then there exists* N' *such that* $M' \to^* N'$ *and* $N \lhd N'$.

  PROOF SKETCH. Let (C, R, P) be the reduction step $M \to N$. The proof is an induction on C. The

Examples From the Recent Past

- "We prove soundness (Theorem 6.8) by mutual **induction on the derivations** of ..."
  - An Operational and Axiomatic Semantics for Non-determinism and Sequence Points in C, POPL 2014
- "The proof goes by **induction on the structure** of *p*."
  - NetKAT: Semantic Foundations of Networks, POPL 2014
- "The operational semantics is given as a big-step relation, on which our compiler correctness proofs can all proceed by **induction** ..."
  - CakeML: A Verified Implementation of ML, POPL 2014
- Method: Chose 4 papers from POPL 2014, 3 of them use structural induction.

Another Year of Examples

- "The proof proceeds by <u>rule induction</u> over the target term producing translation rules."
  - Chakravarty et al. '05
- "Type preservation can be proved by standard <u>induction on the derivation</u> of the evaluation relation."
  - Hosoya et al. '05
- "Proof: By <u>induction on the derivation</u> of $N \Downarrow W$."
  - Sumi and Pierce '05
- Method: chose four POPL 2005 papers at random, the three above mentioned structural induction.

(emphasis mine)

Induction

- Most important technique for studying the formal semantics of prog languages
  - If you want to perform or understand PL research, you must grok this!

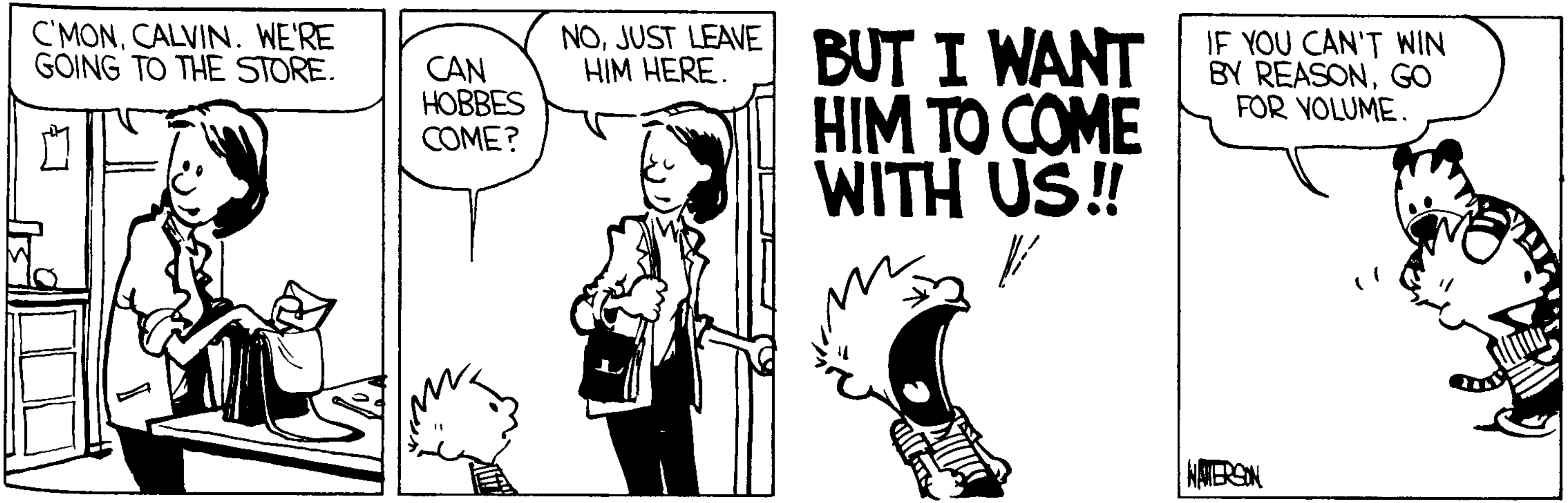
- Mathematical Induction (simple)
- Well-Founded Induction (general)
- Structural Induction (widely used in PL)

Mathematical Induction

- Goal: prove $\forall n \in \mathbb{N}.\ P(n)$
- Base Case: prove $P(0)$
- Inductive Step:
  - Prove $\forall\ n>0.\ P(n) \Rightarrow P(n+1)$
  - “Pick arbitrary n, assume P(n), prove P(n+1)”
- Why does induction work?

Why Does It Work?

- There are no [infinite descending chains]() of natural numbers
- For any *n*, P(*n*) can be obtained by starting from the base case and applying *n* instances of the inductive step



Well-Founded Induction

- A relation $\preceq \subseteq A \times A$ is well-founded if there are no infinite descending chains in A
  - Example: $<_1 = \{ (x, x+1) \mid x \in \mathbb{N} \}$
    - aka the predecessor relation
  - Example: $< \; = \{ (x, y) \mid x, y \in \mathbb{N}$ and $x < y \}$
- Well-founded induction:
  - To prove $\forall x \in A.\ P(x)$ it is enough to prove

    $\forall x \in A.\ [\forall y \preceq x \Rightarrow P(y)] \Rightarrow P(x)$
- If $\preceq$ is $<_1$ then we obtain mathematical induction as a special case

Structural Induction

- Recall $e ::= n \mid e_1 + e_2 \mid e_1 * e_2 \mid x$
- Define $\preceq \subseteq \text{Aexp} \times \text{Aexp}$ such that

  $e_1 \preceq e_1 + e_2 \qquad e_2 \preceq e_1 + e_2$

  $e_1 \preceq e_1 * e_2 \qquad e_2 \preceq e_1 * e_2$

  - no other elements of $\text{Aexp} \times \text{Aexp}$ are $\preceq$-related
- To prove $\forall e \in \text{Aexp}.\ P(e)$
  - $\vdash \forall n \in Z.\ P(n)$
  - $\vdash \forall x \in L.\ P(x)$
  - $\vdash \forall e_1, e_2 \in \text{Aexp}.\ P(e_1) \wedge P(e_2) \Rightarrow P(e_1 + e_2)$
  - $\vdash \forall e_1, e_2 \in \text{Aexp}.\ P(e_1) \wedge P(e_2) \Rightarrow P(e_1 * e_2)$

Notes on Structural Induction

- Called structural induction because the proof is guided by the structure of the expression
- One proof case per form of expression
  - Atomic expressions (with no subexpressions) are all base cases
  - Composite expressions are the inductive case
- This is the *most useful form of induction* in the study of PL

Example of Induction on Structure of Expressions

- Let
  - $L(e)$ be the # of literals and variable occurrences in e
  - $O(e)$ be the # of operators in e
- Prove that $\forall e \in \text{Aexp}.\ L(e) = O(e) + 1$
- Proof: by induction on the structure of e
  - Case $e = n$. $L(e) = 1$ and $O(e) = 0$
  - Case $e = x$. $L(e) = 1$ and $O(e) = 0$
  - Case $e = e_1 + e_2$.
    - $L(e) = L(e_1) + L(e_2)$ and $O(e) = O(e_1) + O(e_2) + 1$
    - By induction hypothesis $L(e_1) = O(e_1) + 1$ and $L(e_2) = O(e_2) + 1$
    - Thus $L(e) = O(e_1) + O(e_2) + 2 = O(e) + 1$
  - Case $e = e_1 * e_2$. Same as the case for +

Other Proofs by Structural Induction on Expressions

- Most proofs for Aexp sublanguage of IMP
- Small-step and natural semantics obtain equivalent results:

$$\forall e \in \text{Exp}.\ \forall n \in \mathbb{N}.\quad e \rightarrow^* n \Leftrightarrow e \Downarrow n$$

- Structural induction on expressions works here because all of the semantics are syntax directed

Stating The Obvious (With a Sense of Discovery)

- You are given a concrete state σ.
- You have $\vdash <x + 1, \sigma> \Downarrow 5$
- You also have $\vdash <x + 1, \sigma> \Downarrow 88$
- Is this possible?

Why That Is Not Possible

- Prove that IMP is deterministic

  $\forall e \in \text{Aexp}.\ \forall \sigma \in \Sigma.\ \forall n, n' \in \mathbb{N}.\ \ \langle e, \sigma\rangle \Downarrow n \ \wedge\ \langle e, \sigma\rangle \Downarrow n' \ \Rightarrow\ n = n'$

  $\forall b \in \text{Bexp}.\ \forall \sigma \in \Sigma.\ \forall t, t' \in \mathbb{B}.\ \ \langle b, \sigma\rangle \Downarrow t \ \wedge\ \langle b, \sigma\rangle \Downarrow t' \ \Rightarrow\ t = t'$

  $\forall c \in \text{Comm}.\ \forall \sigma, \sigma', \sigma'' \in \Sigma.\ \ \langle c, \sigma\rangle \Downarrow \sigma' \ \wedge\ \langle c, \sigma\rangle \Downarrow \sigma'' \ \Rightarrow\ \sigma' = \sigma''$

- No immediate way to use *mathematical* induction
- For commands we cannot use *induction on the structure of the command*
  - `while`'s evaluation does ***not*** depend only on the evaluation of its strict subexpressions

$$\frac{\langle b, \sigma\rangle \Downarrow \text{true} \qquad \langle c, \sigma\rangle \Downarrow \sigma' \qquad \langle \text{while } b \text{ do } c, \sigma'\rangle \Downarrow \sigma''}{\langle \text{while } b \text{ do } c, \sigma\rangle \Downarrow \sigma''}$$

French Literature and Feminism

- This French existentialist author is known for her novels, essays and treatment of feminist and social issues. Her work, Le Deuxième Sexe, is often regarded as the start of second-wave feminism. Her argument, "On ne naît pas femme, on le devient" (one is not born a woman, one becomes a woman), is viewed as one of the first articulations of a distinction between biological sex and socially-constructed gender.

Q: Movies (292 / 842)

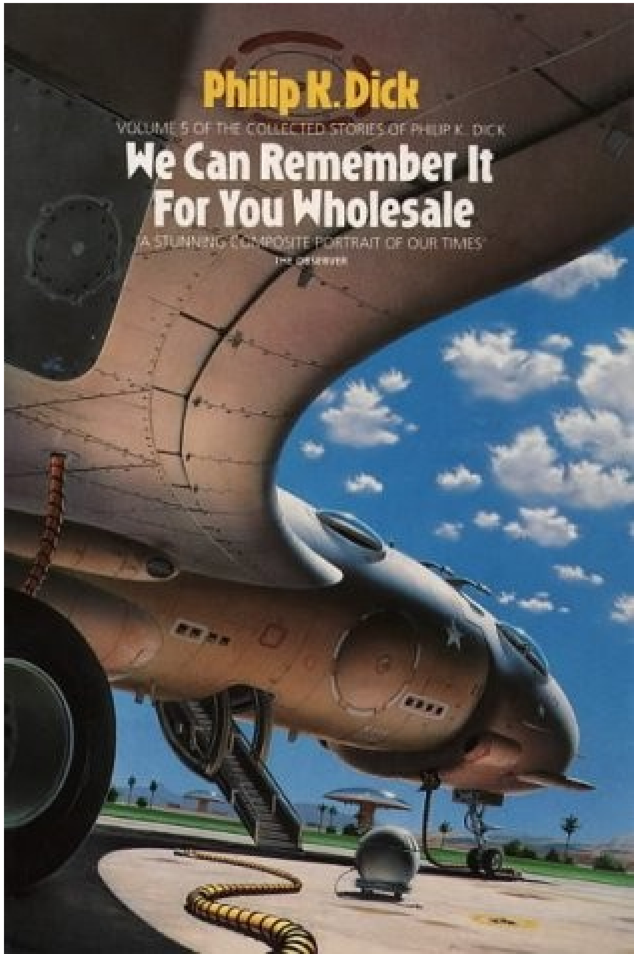
- From the 1981 movie **Raiders of the Lost Ark**, give either the protagonist's phobia xor the composer of the musical score.

Computer Science

- This Dutch Turing-award winner is famous for the semaphore, “THE” operating system, the Banker's algorithm, and a shortest path algorithm. He favored structured programming, as laid out in the 1968 article *Go To Statement Considered Harmful*. He was a strong proponent of formal verification and correctness by construction. He also penned *On The Cruelty of Really Teaching Computer Science*, which argues that CS is a branch of math and relates provability to correctness.

Recall Opsem

- Operational semantics assigns meanings to programs by listing rules of inference that allow you to prove judgments by making derivations.
- A derivation is a tree-structured object made up of valid instances of inference rules.



We Need Something New

- Some more powerful form of induction …
- With all the bells and whistles!

Induction on the Structure of Derivations

- Key idea: The hypothesis does not just assume a $c \in$ Comm but the **existence of a derivation of $<c, \sigma> \Downarrow \sigma'$**
- Derivation trees are also defined inductively, just like expression trees
- A derivation is built of <u>subderivations</u>:

$$
\dfrac{
\dfrac{\langle x, \sigma_{i+1}\rangle \Downarrow 5 - i \qquad 5 - i \leq 5}{\langle x \leq 5, \sigma_{i+1}\rangle \Downarrow \text{true}}
\qquad
\dfrac{
\dfrac{\langle x + 1, \sigma_{i+1}\rangle \Downarrow 6 - i}{\langle x{:=}x{+}1, \sigma_{i+1}\rangle \Downarrow \sigma_i}
\qquad
\langle W, \sigma_i\rangle \Downarrow \sigma_0
}{\langle x{:=}x{+}1;\ W, \sigma_{i+1}\rangle \Downarrow \sigma_0}
}{\langle \text{while } x \leq 5 \text{ do } x := x + 1, \sigma_{i+1}\rangle \Downarrow \sigma_0}
$$

- Adapt the structural induction principle to work on the structure of derivations

Induction on Derivations

- To prove that for all derivations D of a judgment, property P holds
- For each derivation rule of the form

$$\frac{H_1 \;\ldots\; H_n}{C}$$

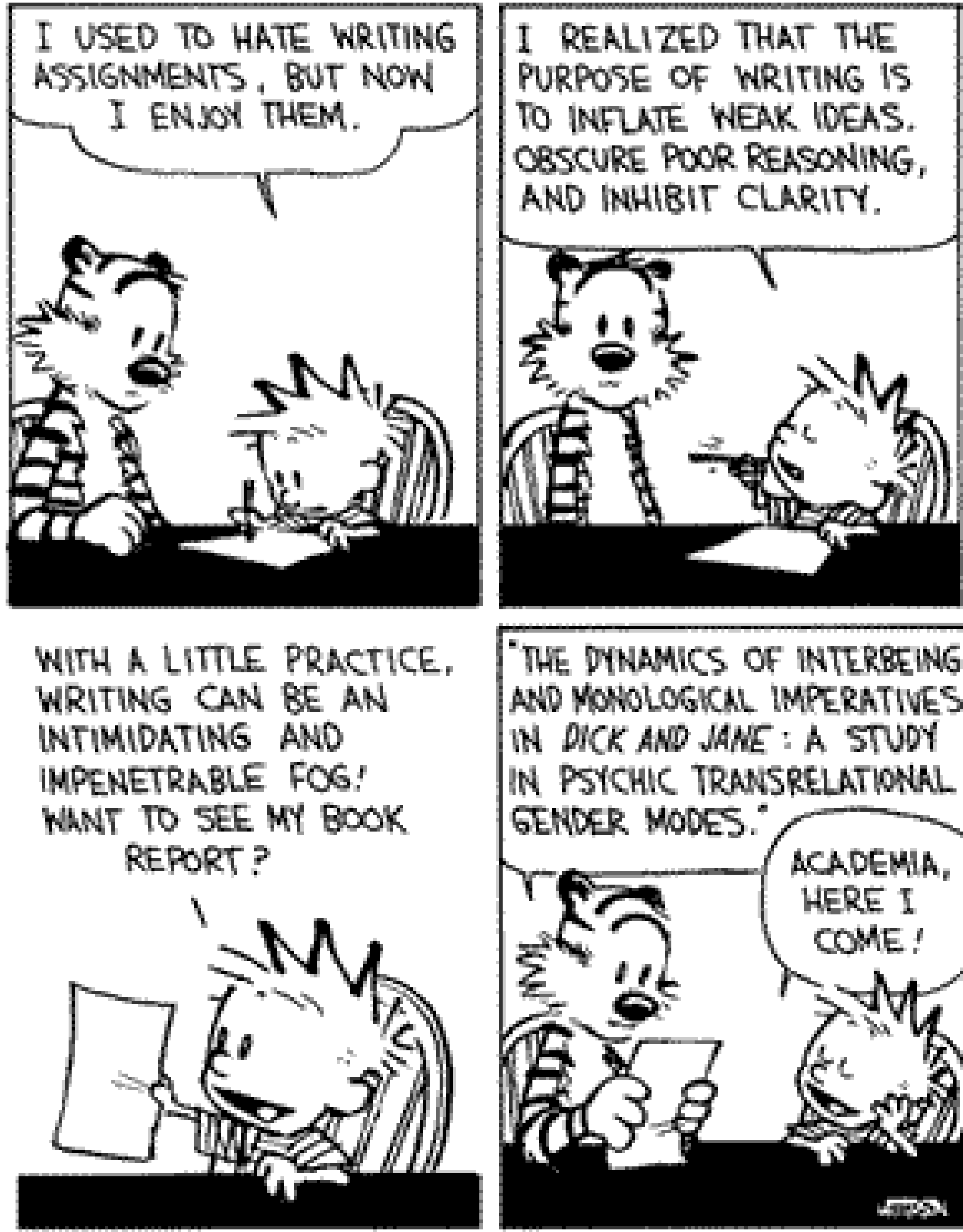
- Assume P holds for derivations of H_i $(i = 1..n)$
- Prove the the property holds for the derivation obtained from the derivations of H_i using the given rule

New Notation

- Write D :: Judgment to mean "D is the derivation that proves Judgment"

- Example:

$D :: \langle x+1, \sigma \rangle \Downarrow 2$



Induction on Derivations (2)

- Prove that evaluation of commands is deterministic:

  $<c, \sigma> \Downarrow \sigma' \Rightarrow \forall \sigma'' \in \Sigma.\ <c, \sigma> \Downarrow \sigma'' \Rightarrow \sigma' = \sigma''$
- Pick arbitrary c, σ, σ' and D :: $<c, \sigma> \Downarrow \sigma'$
- To prove: $\forall \sigma'' \in \Sigma.\ <c, \sigma> \Downarrow \sigma'' \Rightarrow \sigma' = \sigma''$
  - Proof: by induction on the structure of the derivation D
- Case: last rule used in D was the one for skip

$$D :: \quad \frac{}{<\mathrm{skip}, \sigma> \Downarrow \sigma}$$

  - This means that c = skip, and $\sigma' = \sigma$
  - By inversion $<c, \sigma> \Downarrow \sigma''$ uses the rule for skip
  - Thus $\sigma'' = \sigma$
  - This is a base case in the induction

Induction on Derivations (3)

- Case: the last rule used in D was the one for sequencing

$$D :: \frac{D_1 :: \langle c_1, \sigma\rangle \Downarrow \sigma_1 \qquad D_2 :: \langle c_2, \sigma_1\rangle \Downarrow \sigma'}{\langle c_1; c_2, \sigma\rangle \Downarrow \sigma'}$$

- Pick arbitrary σ'' such that $D'' :: \langle c_1; c_2, \sigma\rangle \Downarrow \sigma''$.
  - by inversion D'' uses the rule for sequencing
  - and has subderivations $D''_1 :: \langle c_1, \sigma\rangle \Downarrow \sigma''_1$ and $D''_2 :: \langle c_2, \sigma''_1\rangle \Downarrow \sigma''$
- By induction hypothesis on D_1 (with D''_1): $\sigma_1 = \sigma''_1$
  - Now $D''_2 :: \langle c_2, \sigma_1\rangle \Downarrow \sigma''$
- By induction hypothesis on D_2 (with D''_2): $\sigma'' = \sigma'$
- This is a simple inductive case

Induction on Derivations (4)

- Case: the last rule used in D was `while true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow \text{true} \quad D_2 :: \langle c, \sigma\rangle \Downarrow \sigma_1 \quad D_3 :: \langle \text{while } b \text{ do } c, \sigma_1\rangle \Downarrow \sigma'}{\langle \text{while } b \text{ do } c, \sigma\rangle \Downarrow \sigma'}$$

- Pick arbitrary σ'' s.t. $D'' :: \langle \text{while } b \text{ do } c, \sigma\rangle \Downarrow \sigma''$
  - by inversion and determinism of boolean expressions, D'' also uses the rule for `while true`
  - and has subderivations $D''_2 :: \langle c, \sigma\rangle \Downarrow \sigma''_1$ and $D''_3 :: \langle W, \sigma''_1\rangle \Downarrow \sigma''$
- By induction hypothesis on D_2 (with D''_2): $\sigma_1 = \sigma''_1$
  - Now $D''_3 :: \langle \text{while } b \text{ do } c, \sigma_1\rangle \Downarrow \sigma''$
- By induction hypothesis on D_3 (with D''_3): $\sigma'' = \sigma'$

What Do You, The Viewers At Home, Think?

- Let's do `if true` together!
- Case: the last rule in D was `if true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow \text{true} \qquad D_2 :: \langle c1, \sigma\rangle \Downarrow \sigma_1}{\langle \text{if } b \text{ do } c1 \text{ else } c2, \sigma\rangle \Downarrow \sigma_1}$$

- Try to do this on a piece of paper. In a few minutes I'll have some lucky winners come on down.

Induction on Derivations (5)

- Case: the last rule in D was `if true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow \text{true} \qquad\qquad D_2 :: \langle c1, \sigma\rangle \Downarrow \sigma'}{\langle \text{if } b \text{ do } c1 \text{ else } c2, \sigma\rangle \Downarrow \sigma'}$$

- Pick arbitrary σ″ such that
  D″ :: <if b do c1 else c2, σ> ⇓ σ″
  - By inversion and determinism, D″ also uses `if true`
  - And has subderivations D''_1 :: <b, σ> ⇓ true and D''_2 :: <c1, σ> ⇓ σ″
- By induction hypothesis on D_2 (with D''_2): σ′ = σ″

Induction on Derivations Summary

- If you must prove $\forall x \in A.\ P(x) \Rightarrow Q(x)$
  - with A inductively defined and P(x) rule-defined
  - we pick arbitrary $x \in A$ and $D :: P(x)$
  - we could do induction on both facts
    - $x \in A$ leads to induction on the structure of x
    - $D :: P(x)$ leads to induction on the structure of D
  - Generally, the induction on the structure of the derivation is more powerful and a safer bet
- Sometimes there are many choices for induction
  - choosing the right one is a trial-and-error process
  - a bit of practice can help a lot

Equivalence

Starting here, the material is not covered in lecture but is referenced in the HW.

- Two expressions (commands) are **equivalent** if they yield the same result from all states

$$e_1 \approx e_2 \text{ iff}$$

$$\forall \sigma \in \Sigma.\ \forall n \in \mathbb{N}.$$

$$\langle e_1, \sigma\rangle \Downarrow n \text{ iff } \langle e_2, \sigma\rangle \Downarrow n$$

and for commands

$$c_1 \approx c_2 \text{ iff}$$

$$\forall \sigma, \sigma' \in \Sigma.$$

$$\langle c_1, \sigma\rangle \Downarrow \sigma' \text{ iff } \langle c_2, \sigma\rangle \Downarrow \sigma'$$

Notes on Equivalence

- Equivalence is like logical validity
  - It must hold in all states (= all valuations)
  - $2 \approx 1 + 1$ is like "$2 = 1 + 1$ is valid"
  - $2 \approx 1 + x$ might or might not hold.
    - So, 2 is not equivalent to $1 + x$
- Equivalence (for IMP) is **undecidable**
  - If it were decidable we could solve the halting problem for IMP. *How?*
- Equivalence justifies code transformations
  - compiler optimizations
  - code instrumentation
  - abstract modeling
- Semantics is the basis for proving equivalence

Equivalence Examples

- skip; c $\approx$ c
- while b do c $\approx$
  if b then c; while b do c else skip
- If $e_1 \approx e_2$ then x := $e_1 \approx$ x := e_2
- while true do skip $\approx$ while true do x := x + 1
- Let c be

  while x $\neq$ y do

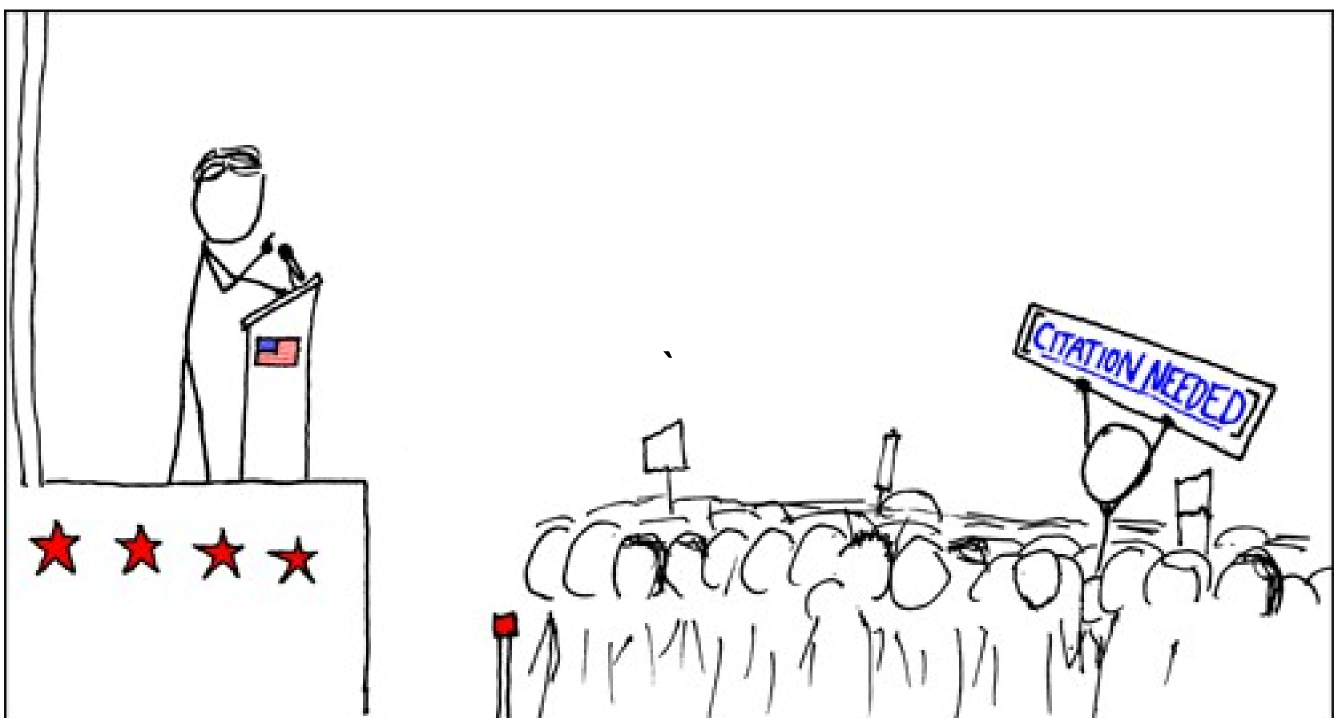
  if x $\geq$ y then x := x - y else y := y - x

  then

  (x := 221; y := 527; c) $\approx$ (x := 17; y := 17)

Potential Equivalence

- $(x := e_1; x := e_2) \approx x := e_2$
- Is this a valid equivalence?



Not An Equivalence

- $(x := e_1;\ x := e_2) \not\approx x := e_2$
- Iie. Chigau yo. Dame desu!
- Not a valid equivalence for all e_1, e_2.
- Consider:
  - $(x := x+1;\ x := x+2) \not\approx x := x+2$
- But for n_1, n_2 it's fine:
  - $(x := n_1;\ x := n_2) \approx x := n_2$

Proving An Equivalence

- Prove that "skip; c $\approx$ c" for all c
- Assume that D :: <skip; c, σ> $\Downarrow$ σ'
- By inversion (twice) we have that

$$D :: \dfrac{\dfrac{}{\langle \text{skip}, \sigma\rangle \Downarrow \sigma} \quad D_1 :: \langle c, \sigma\rangle \Downarrow \sigma'}{\langle \text{skip}; c, \sigma\rangle \Downarrow \sigma'}$$

- Thus, we have D_1 :: <c,σ> $\Downarrow$ σ'
- The other direction is similar

Proving An Inequivalence

- Prove that $x := y \not\sim x := z$ when $y \neq z$
- It suffices to exhibit a σ in which the two commands yield different results

- Let $\sigma(y) = 0$ and $\sigma(z) = 1$
- Then

$$\langle x := y, \sigma \rangle \Downarrow \sigma[x := 0]$$

$$\langle x := z, \sigma \rangle \Downarrow \sigma[x := 1]$$

Special Material Ends

Summary of Operational Semantics

- Precise specification of dynamic semantics
  - order of evaluation (or that it doesn't matter)
  - error conditions (sometimes implicitly, by rule applicability; "no applicable rule" = "get stuck")
- Simple and abstract (vs. implementations)
  - no low-level details such as stack and memory management, data layout, etc.
- Often not compositional (see while)
- Basis for many proofs about a language
  - Especially when combined with type systems!
- Basis for much reasoning about programs
- Point of reference for other semantics

Homework

- Don't Neglect Your Homework
- Read DPLL(T) and Simplex
- Peer Review for HW1