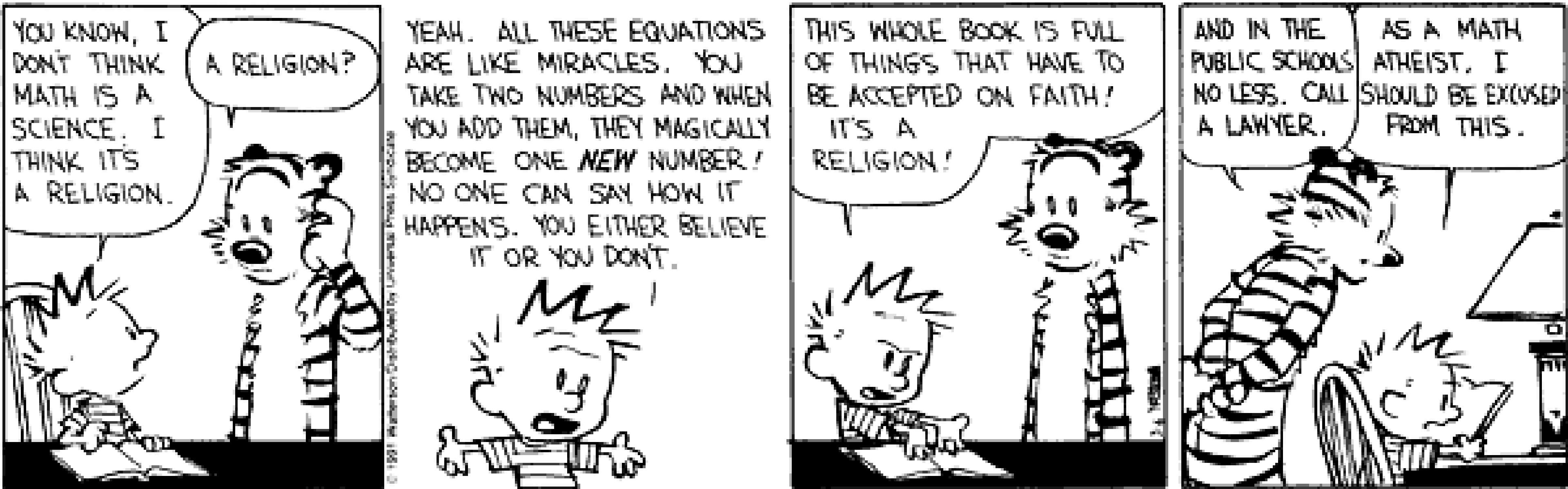
More Lambda Calculus *and* Intro to Type Systems


YOU KNOW, I DON'T THINK MATH IS A SCIENCE. I THINK IT'S A RELIGION.
A RELIGION?
YEAH. ALL THESE EQUATIONS ARE LIKE MIRACLES. YOU TAKE TWO NUMBERS AND WHEN YOU ADD THEM, THEY MAGICALLY BECOME ONE *NEW* NUMBER! NO ONE CAN SAY HOW IT HAPPENS. YOU EITHER BELIEVE IT OR YOU DON'T.
THIS WHOLE BOOK IS FULL OF THINGS THAT HAVE TO BE ACCEPTED ON FAITH! IT'S A RELIGION!
AND IN THE PUBLIC SCHOOLS NO LESS. CALL A LAWYER.
AS A MATH ATHEIST, I SHOULD BE EXCUSED FROM THIS.

Plan

- Heavy Class Participation
  - Thus, wake up! *(not actually kidding)*
- Lambda Calculus
  - How is it related to real life?
  - Encodings
  - Fixed points
- Type Systems
  - Overview
  - Static, Dyamic
  - Safety, Judgments, Derivations, Soundness

Lambda Review

- λ-calculus is a calculus of functions

  $$e := x \mid \lambda x.\, e \mid e_1\, e_2$$

- Several evaluation strategies exist based on β-reduction

  $$(\lambda x.e)\, e' \rightarrow_\beta [e'/x]\, e$$

- How does this simple calculus relate to real programming languages?

Functional Programming

- The λ-calculus is a prototypical functional language with:
  - no side effects
  - several evaluation strategies
  - lots of functions
  - nothing but functions (pure λ-calculus does not have any other data type)

- How can we program with functions?
- How can we program with *only* functions?

Programming With Functions

- Functional programming is a programming style that relies on lots of functions
- A typical functional paradigm is *using functions as arguments or results of other functions*
  - Called “higher-order programming”
- Some “impure” functional languages permit side-effects (e.g., Lisp, Scheme, ML, Python)
  - references (pointers), in-place update, arrays, exceptions
  - Others (and by “others” we mean “Haskell”) use monads to model state updates

Variables in Functional Languages

- We can introduce new variables:

  let x = e_1 in e_2

  - x is bound by let
  - x is statically scoped in (a subset of) e_2
- This is pretty much like $(\lambda x.\ e_2)\ e_1$
- In a functional language, variables are never updated
  - they are just *names for expressions or values*
  - e.g., x is a name for the value denoted by e_1 in e_2
- This models the meaning of “let” in math (proofs)

Referential Transparency

- In “pure” functional programs, we can reason equationally, by substitution
  - Called “referential transparency”

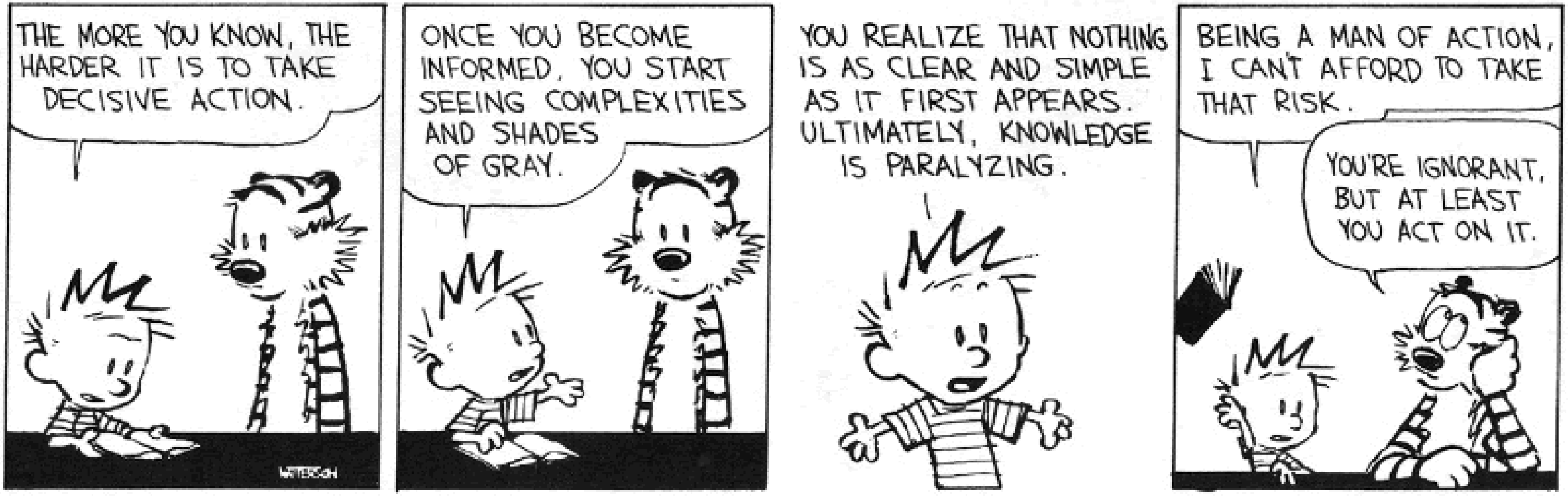
$$\text{let } x = e_1 \text{ in } e_2 \quad === \quad [e_1/x]e_2$$

- In an imperative language a side-effect in e_1 might invalidate the above equation
- The behavior of a function in a “pure” functional language depends only on the actual arguments
  - Just like a function in math
  - This makes it easier to understand and to reason about functional programs

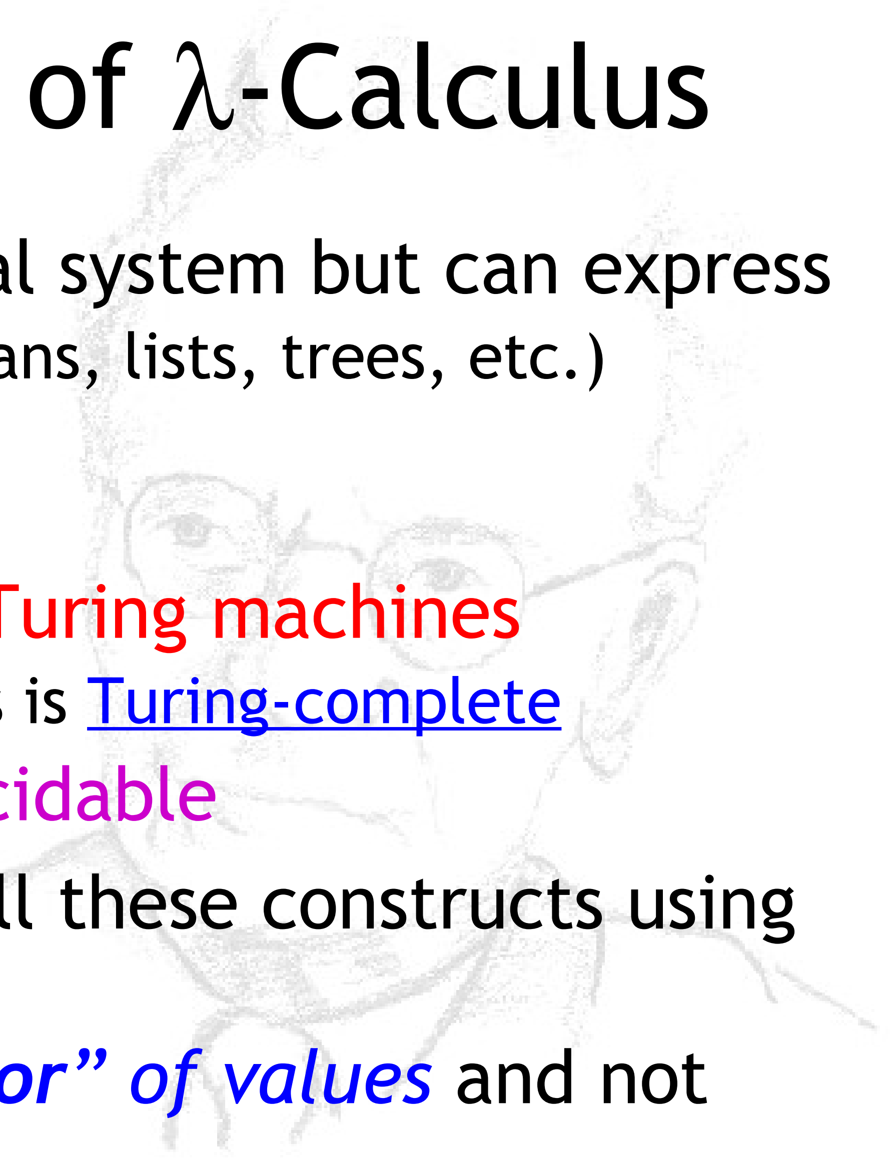
How Tough Is Lambda?

- Given e_1 and e_2, how complex (a la CS theory) is it to determine if:

$$e_1 \rightarrow_\beta^* e \quad \textit{and} \quad e_2 \rightarrow_\beta^* e$$


THE MORE YOU KNOW, THE HARDER IT IS TO TAKE DECISIVE ACTION.
ONCE YOU BECOME INFORMED, YOU START SEEING COMPLEXITIES AND SHADES OF GRAY.
YOU REALIZE THAT NOTHING IS AS CLEAR AND SIMPLE AS IT FIRST APPEARS. ULTIMATELY, KNOWLEDGE IS PARALYZING.
BEING A MAN OF ACTION, I CAN'T AFFORD TO TAKE THAT RISK.
YOU'RE IGNORANT, BUT AT LEAST YOU ACT ON IT.
WATTERSON

Expressiveness of λ-Calculus



- The λ-calculus is a minimal system but can express
  - data types (integers, booleans, lists, trees, etc.)
  - branching
  - recursion
- This is enough to encode Turing machines
  - We say the lambda calculus is Turing-complete
- Corollary: $e_1 =_\beta e_2$ is undecidable
- Still, how do we encode all these constructs using only functions?
- Idea: *encode the "**behavior**" of values* and not their structure

Encoding Booleans in λ-Calculus

- What can we *do* with a boolean?
  - we can make a binary choice (= “if” statement)
- A boolean is a function that, given two choices, selects one of them:
  - **true** $=_{def}$ $\lambda x.\ \lambda y.\ x$
  - **false** $=_{def}$ $\lambda x.\ \lambda y.\ y$
  - **if E_1 then E_2 else E_3** $=_{def}$ $E_1\ E_2\ E_3$
- Example: “if true then u else v” is

$$(\lambda x.\ \lambda y.\ x)\ u\ v \rightarrow_\beta (\lambda y.\ u)\ v \rightarrow_\beta u$$

More Boolean Encodings

- Let's try to do boolean or together
- Recall:
  - true $=_{def}$ $\lambda x.\ \lambda y.\ x$
  - false $=_{def}$ $\lambda x.\ \lambda y.\ y$
  - if E_1 then E_2 else E_3 $=_{def}$ $E_1\ E_2\ E_3$
- We want or to take in two booleans and yield a result that is a boolean
- How can we do this?

A *Try*ing *Or*deal

- Recall:
  - true $=_{def}$ $\lambda x.\ \lambda y.\ x$
  - false $=_{def}$ $\lambda x.\ \lambda y.\ y$
  - if E_1 then E_2 else E_3 $=_{def}$ $E_1\ E_2\ E_3$
- Intution:
  - **or a b** **= if a then true else b**
- Either of these will work:
  - **or** $=_{def}$ **λa. λb. a true b**
  - **or** $=_{def}$ **λa. λb. λx. λy. a x (b x y)**

Final Boolean Encodings

- Think about how to do **and** and **not**
- Without peeking! Now come up and do it!

A***not***her Dem***and***

- How to do **and** and **not**
- **and** a b = if a then b else false
  - **and** $=_{\text{def}}$ λa. λb. a b false
  - **and** $=_{\text{def}}$ λa. λb. λx. λy. a (b x y) y

- **not** a = if a then false else true
  - **not** $=_{\text{def}}$ λa. a false true
  - **not** $=_{\text{def}}$ λa. λx. λy. a y x

Encoding Pairs in λ-Calculus

- What can we *do* with a pair?
  - we can access one of its elements (= ".field access")
- A pair is a function that, given a boolean, returns the first or second element

  mkpair x y $=_{\text{def}}$ **λb. b x y**

  fst p $=_{\text{def}}$ **p true**

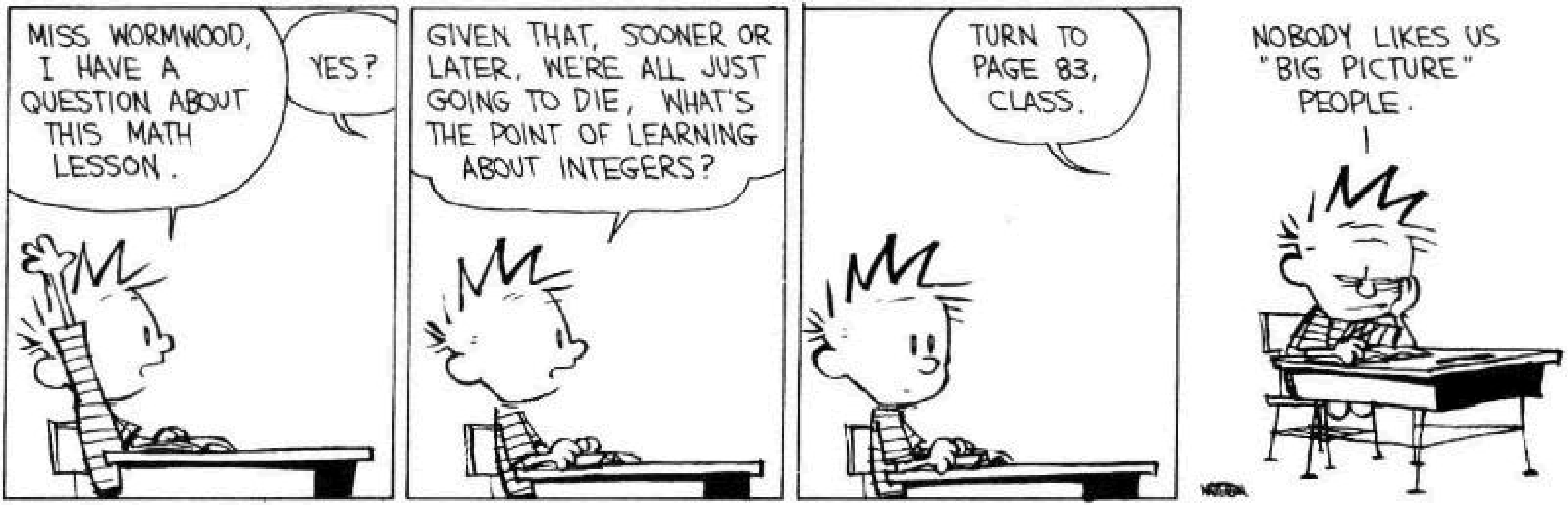
  snd p $=_{\text{def}}$ **p false**

- fst (mkpair x y) $\rightarrow_\beta$ (mkpair x y) true

  $\rightarrow_\beta$ true x y $\rightarrow_\beta$ x

Encoding Numbers in λ–Calculus

- What can we *do* with a natural number?
  - What do you, the viewers at home, think?


MISS WORMWOOD, I HAVE A QUESTION ABOUT THIS MATH LESSON.
YES?
GIVEN THAT, SOONER OR LATER, WE'RE ALL JUST GOING TO DIE, WHAT'S THE POINT OF LEARNING ABOUT INTEGERS?
TURN TO PAGE 83, CLASS.
NOBODY LIKES US "BIG PICTURE" PEOPLE.
WATTERSON

Encoding Numbers λ-Calculus

- What can we *do* with a natural number?
  - we can iterate a number of times over some function (= “for loop”)
- A natural number is a function that given an operation f and a starting value s, applies f a number of times to s:

  0 $=_{def}$ λf. λs. s

  1 $=_{def}$ λf. λs. f s

  2 $=_{def}$ λf. λs. f (f s)

  - Very similar to List.fold_left and friends
- These are numerals in a unary representation
- Called Church numerals

Test Time!

- How would you encode the successor function (succ x === x+1)?
- How would you encode more general addition?
- Recall: $4 =_{\text{def}} \lambda f.\ \lambda s.\ f\ f\ f\ (f\ s)$

SEE, HOBBES, WE SHOULDN'T NEED ACCOMPLISHMENTS TO FEEL GOOD ABOUT OURSELVES. SELF-ESTEEM SHOULDN'T BE CONDITIONAL.
THAT'S WHY I'VE STOPPED DOING HOMEWORK. I DON'T NEED TO LEARN THINGS TO LIKE MYSELF. I'M FINE THE WAY I AM.
SO THE SECRET TO GOOD SELF-ESTEEM IS TO LOWER YOUR EXPECTATIONS TO THE POINT WHERE THEY'RE ALREADY MET?
RIGHT. WE SHOULD TAKE PRIDE IN OUR MEDIOCRITY.
REMIND ME TO INVEST OVERSEAS.
I THINK THIS SNOWMAN IS GOOD ENOUGH, DON'T YOU?
WATTERSON

Computing with Natural Numbers

- The successor function

  succ n $=_{def}$ $\lambda f.\ \lambda s.\ f\ (n\ f\ s)$

  or succ n $=_{def}$ $\lambda f.\ \lambda s.\ n\ f\ (f\ s)$

- Addition

  add n_1 n_2 $=_{def}$ n_1 succ n_2

- Multiplication

  mult n_1 n_2 $=_{def}$ n_1 (add n_2) 0

- Testing equality with 0

  iszero n $=_{def}$ n (λb. false) true

- Subtraction
  - Is not instructive, but makes a fun exercise …

Computation Example

- What is the result of the application add 0?

  $(\lambda n_1.\ \lambda n_2.\ n_1 \text{ succ } n_2)\ 0 \ \to_\beta$

  $\lambda n_2.\ 0 \text{ succ } n_2 =$

  $\lambda n_2.\ (\lambda f.\ \lambda s.\ s) \text{ succ } n_2 \to_\beta$

  $\lambda n_2.\ n_2 =$

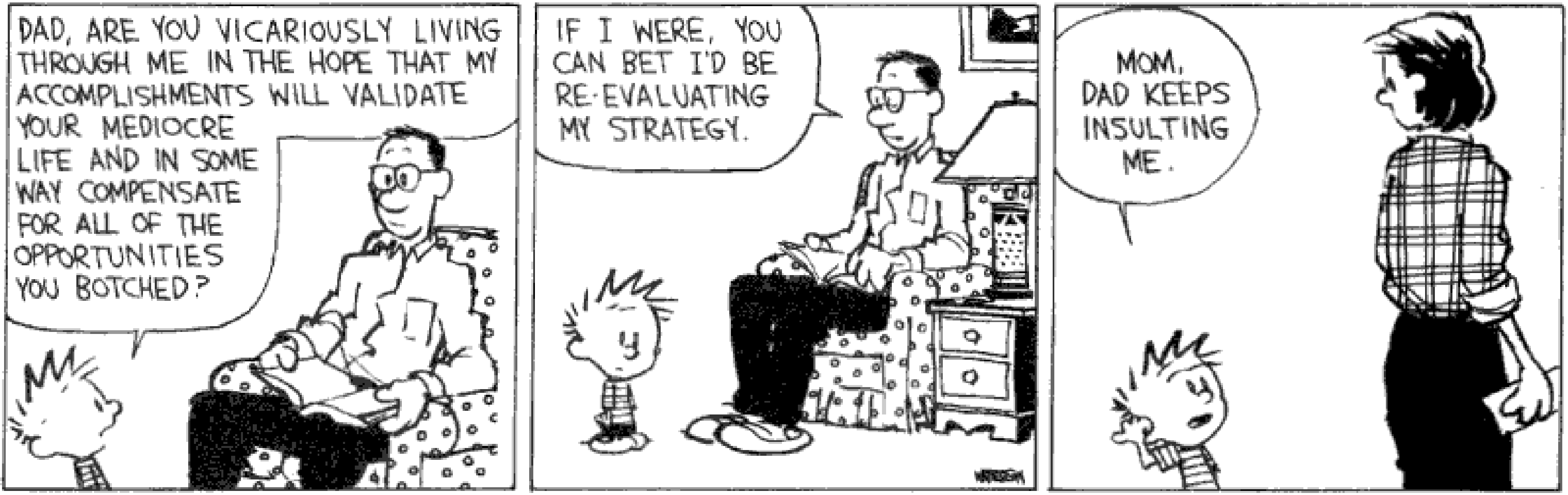
  $\lambda x.\ x$

- By computing with functions we can express some optimizations
  - But we need to reduce under the lambda
  - Thus this “never” happens in practice

Toward Recursion

- Given a predicate P, encode the function "find" such that "find P n" is the smallest natural number which is larger than n and satisfies P
- Claim: with find we can encode all recursion

Intuitively, why is this true?


DAD, ARE YOU VICARIOUSLY LIVING THROUGH ME IN THE HOPE THAT MY ACCOMPLISHMENTS WILL VALIDATE YOUR MEDIOCRE LIFE AND IN SOME WAY COMPENSATE FOR ALL OF THE OPPORTUNITIES YOU BOTCHED?
IF I WERE, YOU CAN BET I'D BE RE-EVALUATING MY STRATEGY.
MOM, DAD KEEPS INSULTING ME.

Encoding Recursion

- Given a predicate P encode the function “find” such that “find P n” is the smallest natural number which is larger than n and satisfies P
- find satisfies the equation

  find p n = if p n then n else find p (succ n)

- Define

$$F = \lambda f.\lambda p.\lambda n.(p\ n)\ n\ (f\ p\ (\text{succ}\ n))$$

- We need a fixed point of F

  find = F find

  or

  find p n = F find p n

The Fixed-Point Combinator Y

- Let $Y = \lambda F.\ (\lambda y.F(y\ y))\ (\lambda x.\ F(x\ x))$
  - This is called the fixed-point combinator
  - Verify that $Y\ F$ is a fixed point of F

    $$Y\ F \rightarrow_{\beta} (\lambda y.F\ (y\ y))\ (\lambda x.\ F\ (x\ x)) \rightarrow_{\beta} F\ (Y\ F)$$

  - Thus $Y\ F =_{\beta} F\ (Y\ F)$
- Given any function in λ-calculus we can compute its fixed-point (w00t! why do we *not* win here?)
- Thus we can define "find" as the fixed-point of the function F from the previous slide
- Essence of recursion is the self-application "$y\ y$"

Expressiveness of Lambda Calculus

- Encodings are fun
  - Yes! Yes they are!
- But programming in pure λ-calculus is painful
- So we will add constants (0, 1, 2, …, true, false, if-then-else, etc.)
- Next we will add *types*

Still Going!

- One minute break
- Stretch!

Calvin and Hobbes
by WATTERSON
NO EARTHLING HAS EVER BEFORE SEEN THE CRATERED, SCARRED SURFACE OF DISTANT PLANET ZOG!
... ALTHOUGH IT'S NOT UNLIKE SOME OF THOSE ZIT CREAM COMMERCIALS...
WE JOIN THE FEARLESS SPACEMAN SPIFF, INTERPLANETARY EXPLORER EXTRAORDINAIRE, OUT AT THE FARTHEST REACHES OF THE GALAXY...
WITH NERVES OF STEEL, OUR HERO SETS FORTH ON HIS DANGEROUS MISSION!
HE FIRES HIS HYPER-JETS AND...
... BLASTS INTO THE FIFTH DIMENSION!
INTO A WORLD BEYOND HUMAN COMPREHENSION!
INTO A WORLD WHERE TIME HAS NO MEANING!
MAN, THIS CLASS LASTS FOREVER!
SO WE CARRY THE THREE INTO THE TENS COLUMN ....
WATTERSON

Q: Books (711 / 842)

- In this 1943 Antoine de Saint-Exupery novel the title character lives on an asteroid with a rose but eventually travels to Earth.

Springfield, Illinois

- Animaniacs
- Two player game
- Players take turns naming State Capitals until one player says “Springfield, Illinois”
- The player to say “Springfield, Illinois” receives points equal to the number of previously-named Capitals
- Why is this game interesting?

Q: Computer Science（姚期智）

- This Shanghai-born Turing Award winner is known for contributions to the theory of computation. He formulated the Millionaire's Problem and stated this minimax principle: “the expected cost of any randomized algorithm for solving a given problem, on the worst case input for that algorithm, can be no better than the expected cost, for a worst-case random probability distribution on the inputs, of the deterministic algorithm that performs best against that distribution.”

Types

- A program variable can assume a *range of values* during the execution of a program

- An upper bound of such a range is called a type of the variable
  - A variable of type "bool" is supposed to assume only boolean values
  - If x has type "bool" then the boolean expression "not(x)" has a sensible meaning during every run of the program

Typed and Untyped Languages

- Untyped languages
  - Do *not* restrict the range of values for a given variable
  - Operations might be applied to inappropriate arguments. The behavior in such cases might be unspecified
  - The pure λ-calculus is an extreme case of an untyped language (however, its behavior is completely specified)
- (Statically) Typed languages
  - Variables are assigned (non-trivial) types
  - A type system keeps track of types
  - Types might or might not appear in the program itself
  - Languages can be explicitly typed or implicitly typed

The Purpose Of Types

- The foremost purpose of types is *to prevent certain types of run-time execution errors*
- Traditional trapped execution errors
  - Cause the computation to stop immediately
  - And are thus well-specified behavior
  - Usually enforced by hardware
  - e.g., Division by zero, floating point op with a NaN
  - e.g., Dereferencing the address 0 (on most systems)
- Untrapped execution errors
  - Behavior is unspecified (depends on the state of the machine = this is very bad!)
  - e.g., accessing past the end of an array
  - e.g., jumping to an address in the data segment

Execution Errors

- A program is deemed safe if it does *not* cause untrapped errors
  - Languages in which all programs are safe are safe languages
- For a given language we can designate a set of forbidden errors
  - A superset of the untrapped errors, usually including some trapped errors as well
    - e.g., null pointer dereference
- Modern Type System Powers:
  - prevent race conditions (e.g., Flanagan TLDI ‘05)
  - prevent insecure information flow (e.g., Li POPL ’05)
  - prevent resource leaks (e.g., Vault, Weimer)
  - help with generic programming, probabilistic languages, …
  - … are often combined with dynamic analyses (e.g., CCured)

Preventing Forbidden Errors - Static Checking

- Forbidden errors can be caught by a combination of static and run-time checking
- Static checking
  - Detects errors early, *before testing*
  - Types provide the necessary static information for static checking
  - e.g., ML, Modula-3, Java
  - Detecting certain errors statically is undecidable in most languages

Preventing Forbidden Errors - Dynamic Checking

- Required when static checking is undecidable
  - e.g., array-bounds checking
- Run-time encodings of types are still used (e.g. Lisp)
- Should be limited since it delays the manifestation of errors
- Can be done in hardware (e.g. null-pointer)

Safe Languages

- There are typed languages that are not safe ("weakly typed languages")
- *All safe languages use types* (static or dynamic)

	Typed		Untyped
	Static	Dynamic	
Safe	ML, Java, Ada, C#, Haskell, ...	Lisp, Scheme, Ruby, Perl, Smalltalk, PHP, Python, …	λ-calculus
Unsafe	C, C++, Pascal, ...	?	Assembly

- We focus on statically typed languages

Why Typed Languages?

- Development
  - *Type checking catches early many mistakes*
  - Reduced debugging time
  - Typed signatures are a powerful basis for design
  - Typed signatures enable separate compilation
- Maintenance
  - Types act as checked specifications
  - Types can enforce abstraction
- Execution
  - Static checking reduces the need for dynamic checking
  - Safe languages are easier to analyze statically
    - the compiler can generate better code

Homework

- Read Cardelli article
- Read Wright & Matthias article
- Homework 5 Due Soon