Introduction to Denotational Semantics

Fireside Chat

with …

Free Food

Thursday, Sept 22
5:00 pm
Thornton E303

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Meet and ask them questions in a non-academic setting.

Learn what they wish they had known when they were students, and what their lives are like outside of the office.

Ask them anything!
All are welcome.

ACM-W

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Computer Science
University of Virginia

Gone in Sixty Seconds

- **Denotation semantics** is a formal way of assigning meanings to programs. In it, the meaning of a program is a **mathematical** object.
- Denotation semantics is **compositional**: the meaning of an expression depends on the meanings of subexpressions.
- Denotational semantics uses $\bot$ (“bottom”) to mean **non-termination**.
- DS uses **fixed points** and **domains** to handle `while`.

Recall: Induction on Derivations Summary

- If you must prove $\forall x \in A.\ P(x) \Rightarrow Q(x)$
  - A is some structure (e.g., AST), P(x) is some property
  - we pick arbitrary $x \in A$ and D :: P(x)
  - we could do induction on both facts
    - $x \in A$ leads to induction on the structure of x
    - D :: P(x) leads to induction on the structure of D
  - Generally, the induction on the structure of the derivation is more powerful and a safer bet
- Sometimes there are many choices for induction
  - choosing the right one is a trial-and-error process
  - a bit of practice can help a lot

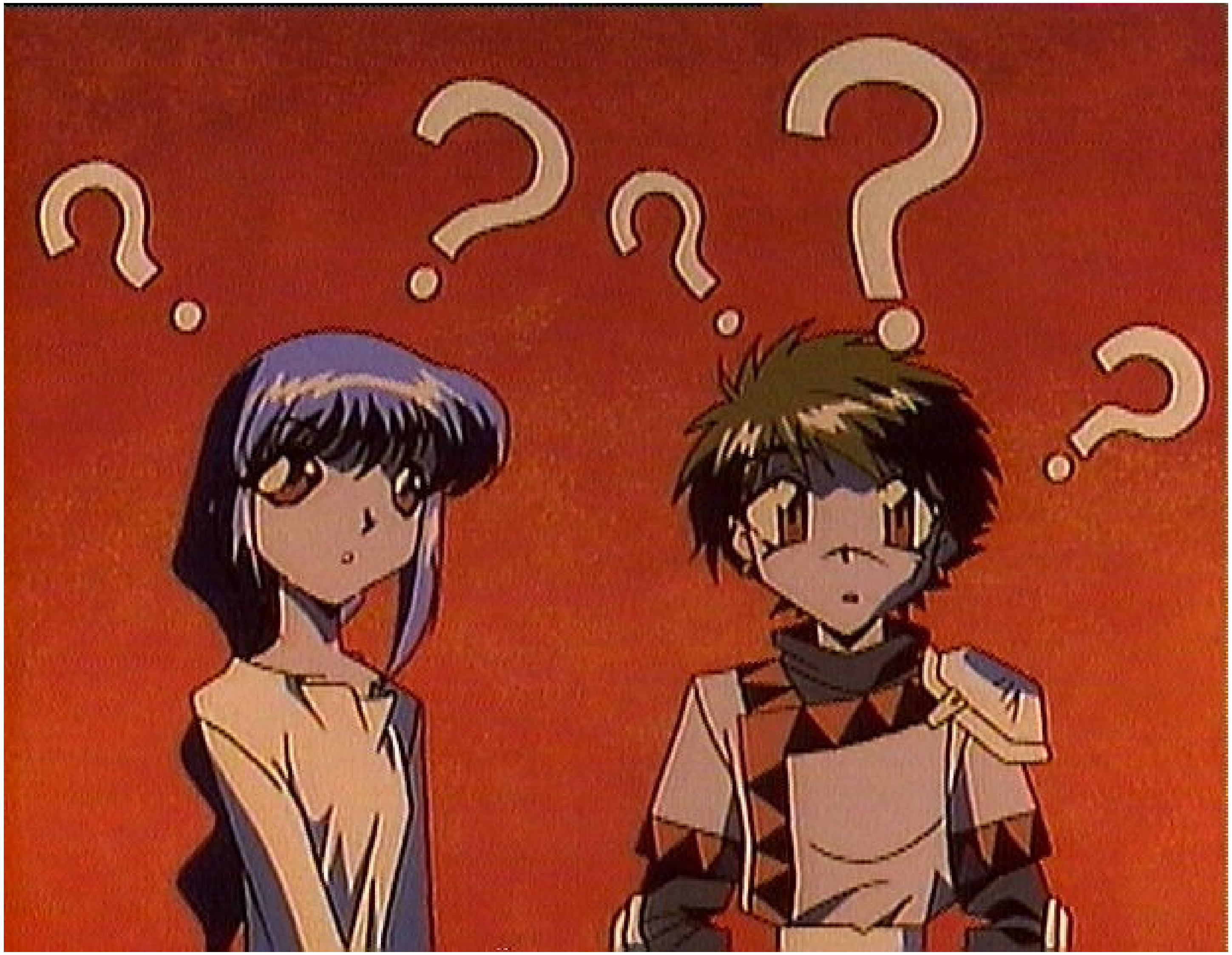
Overall Summary of Operational Semantics

- Precise specification of dynamic semantics
  - order of evaluation (or that it doesn't matter)
  - error conditions (sometimes implicitly, by rule applicability; "no applicable rule" = "get stuck")
- Simple and abstract (vs. implementations)
  - no low-level details such as stack and memory management, data layout, etc.
- Often not compositional (see while)
- Basis for many proofs about a language
  - Especially when combined with type systems!
- Basis for much reasoning about programs
- Point of reference for other semantics

Dueling Semantics

- Operational semantics is
  - simple
  - of many flavors (natural, small-step, more or less abstract)
  - not compositional
  - commonly used in the real (modern research) world
- Denotational semantics is
  - mathematical (the meaning of a syntactic expression is a mathematical object)
  - compositional
- Denotational semantics is also called: fixed-point semantics, mathematical semantics, Scott-Strachey semantics

Typical Student Reaction To Denotation Semantics



Denotational Semantics Learning Goals

- DS is compositional (!)
- When should I use DS?
- In DS, meaning is a "math object"
- DS uses $\perp$ ("bottom") to mean non-termination
- DS uses fixed points and domains to handle `while`
  - This is the tricky bit

DS In The Real World

- ADA was formally specified with it
- Handy when you want to study non-trivial models of computation
  - e.g., “actor event diagram scenarios”, process calculi
- Nice when you want to compare a program in Language 1 to a program in Language 2

Foreshadowing

- Denotational semantics assigns meanings to programs
- The meaning will be a mathematical object
  - A number $a \in \mathbb{Z}$
  - A boolean $b \in \{\text{true}, \text{false}\}$
  - A function $c : \Sigma \to (\Sigma \cup \{\text{non-terminating}\})$
- The meaning will be determined compositionally
  - Denotation of a command is based on the denotations of its immediate sub-commands (= more than merely syntax-directed)

New Notation

- ‘Cause, why not?

  $[\![\ ]\!]$ = “means” or “denotes”

- Example:

$[\![\text{foo}]\!]$	= “denotation of foo”
$[\![3 < 5]\!]$	= true
$[\![3 + 5]\!]$	= 8

- Sometimes we write $A[\![\cdot]\!]$ for arith, $B[\![\cdot]\!]$ for boolean, $C[\![\cdot]\!]$ for command

Rough Idea of Denotational Semantics

- The meaning of an arithmetic expression e in state σ is a number n
- So, we try to define $A[\![e]\!]$ as a function that maps the current state to an integer:

$$A[\![\cdot]\!] : \mathrm{Aexp} \to (\Sigma \to \mathbb{Z})$$

- The meaning of boolean expressions is defined in a similar way

$$B[\![\cdot]\!] : \mathrm{Bexp} \to (\Sigma \to \{\mathrm{true}, \mathrm{false}\})$$

- All of these denotational function are total
  - Defined for all syntactic elements
  - For other languages it might be convenient to define the semantics only for well-typed elements

Denotational Semantics of Arithmetic Expressions

- We inductively define a function

$$A[\![\cdot]\!] : \text{Aexp} \to (\Sigma \to \mathbb{Z})$$

$$
\begin{aligned}
A[\![n]\!]\,\sigma &= \text{the integer denoted by literal } n \\
A[\![x]\!]\,\sigma &= \sigma(x) \\
A[\![e_1+e_2]\!]\,\sigma &= A[\![e_1]\!]\sigma + A[\![e_2]\!]\sigma \\
A[\![e_1-e_2]\!]\,\sigma &= A[\![e_1]\!]\sigma - A[\![e_2]\!]\sigma \\
A[\![e_1*e_2]\!]\,\sigma &= A[\![e_1]\!]\sigma * A[\![e_2]\!]\sigma
\end{aligned}
$$

- This is a total function (= defined for all expressions)

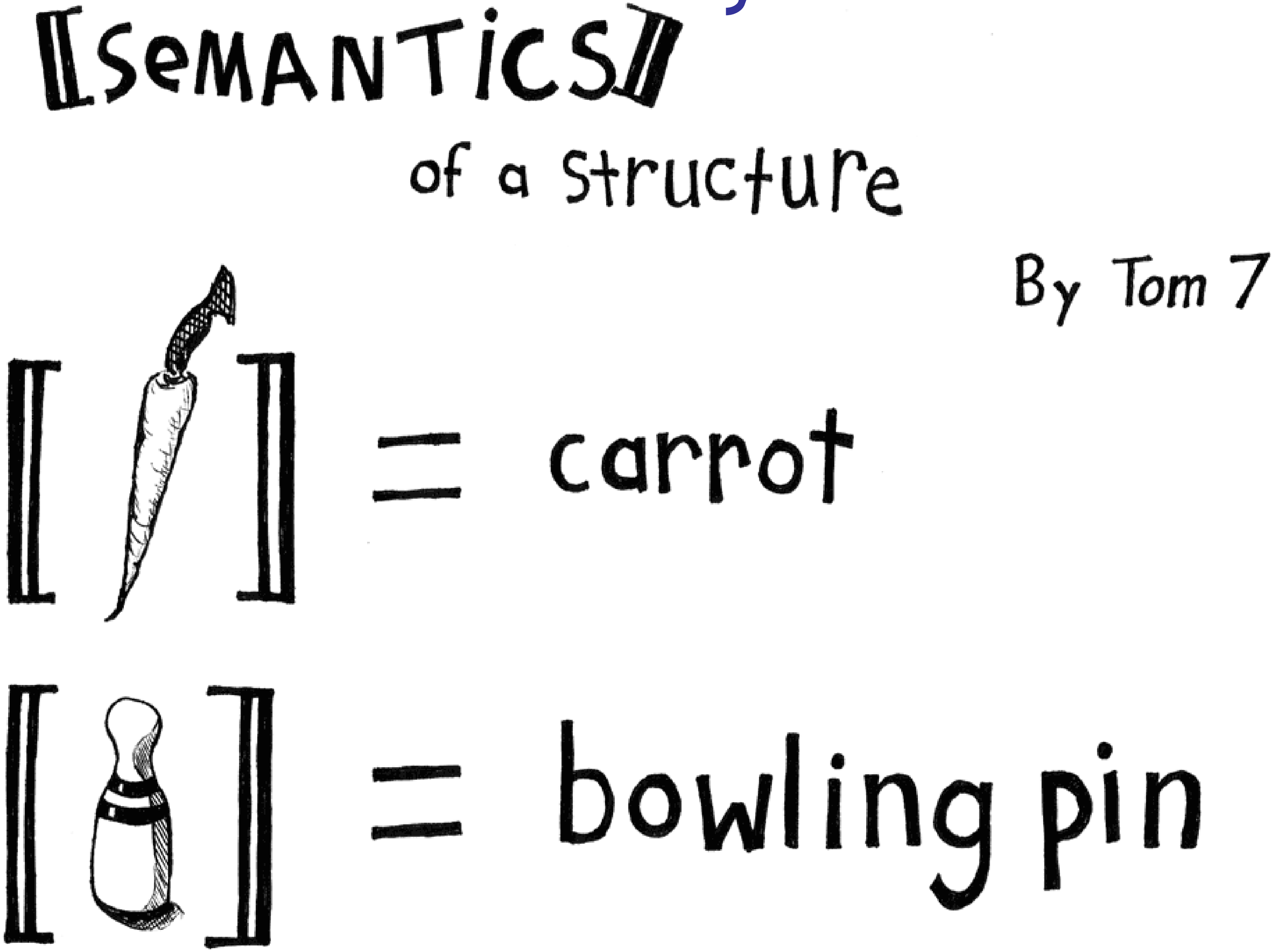
Denotational Semantics of Boolean Expressions

- We inductively define a function

$$B[\![\cdot]\!] : \text{Bexp} \rightarrow (\Sigma \rightarrow \{\textbf{true}, \textbf{false}\})$$

$$
\begin{aligned}
B[\![\text{true}]\!]\sigma &= \textbf{true} \\
B[\![\text{false}]\!]\sigma &= \textbf{false} \\
B[\![b_1 \wedge b_2]\!]\sigma &= B[\![b_1]\!]\,\sigma \wedge B[\![b_2]\!]\,\sigma \\
B[\![e_1 = e_2]\!]\sigma &= \text{if } A[\![e_1]\!]\,\sigma = A[\![e_2]\!]\,\sigma \\
&\quad\ \text{then } \textbf{true} \text{ else } \textbf{false}
\end{aligned}
$$

Seems Easy So Far



Denotational Semantics for Commands

- Running a command c starting from a state σ yields another state σ'
- So, we try to define $C[\![c]\!]$ as a function that maps σ to σ'

$$C[\![\cdot]\!] : \mathrm{Comm} \to (\Sigma \to \Sigma)$$

- Will this work? Bueller?

$\perp$ = Non-Termination

- We introduce the special element $\perp$ ("bottom") to denote a special resulting state that stands for **non-termination**
- For any set X, we write

$$X_\perp \text{ to denote } X \cup \{\perp\}$$

Convention:

whenever $f \in X \to X_\perp$ we extend f to $X_\perp \to X_\perp$ so that $f(\perp) = \perp$

- This is called strictness

Denotational Semantics of Commands

- We try:

$$C[\![\cdot]\!] : \text{Comm} \to (\Sigma \to \Sigma_\perp)$$

$$
\begin{array}{ll}
C[\![\text{skip}]\!]\ \sigma & = \sigma \\
C[\![x := e]\!]\ \sigma & = \sigma[x := A[\![e]\!]\ \sigma] \\
C[\![c_1;\ c_2]\!]\ \sigma & = C[\![c_2]\!]\ (C[\![c_1]\!]\ \sigma) \\
C[\![\text{if } b \text{ then } c_1 \text{ else } c_2]\!]\ \sigma & = \\
\qquad \text{if } B[\![b]\!]\sigma \text{ then } C[\![c_1]\!]\sigma \text{ else } C[\![c_2]\!]\sigma & \\
C[\![\text{while } b \text{ do } c]\!]\ \sigma & = \text{? (later)}
\end{array}
$$

Examples

- C⟦x:=2; x:=1⟧ σ =

  σ[x := 1]
- C⟦if true then x:=2; x:=1 else …⟧ σ =

  σ[x := 1]
- The semantics does not care about intermediate states (cf. "big-step")
- We haven't used ⊥ yet

Q: Theatre (012 / 842)

- Name the author or the 1953 play about McCarthyism that features John Proctor's famous cry of *"More weight!"* .

Q: Games (557 / 842)

- Name the company that manufactures **Barbie** (a $1.9 billion dollar a year industry in 2005 with two dolls being bought every second).

English Prose

- 43. I think its amusing fact that most people think that my earcuffs are just some kind of excentrick decoration.
- 113. They might have been the two youngest advisors, (both 17), but they were her most trust.
- 118. He sunk so lo as to go after some blonde bimbo names Tracie.
- 389. Her book was really a one of a kind item, she could not make one the same. As she passed a more freighting prospect came to her, St. John, had in his possession, her very thoughts, desires, and loves, or recorded for him in one book, John had the equivalent of her sole.

Denotational Semantics of WHILE

- Notation: $W = C[\![\text{while } b \text{ do } c]\!]$
- Idea: rely on the equivalence (see end of notes)

  while b do c ≈ if b then c; while b do c else skip
- Try

$$W(\sigma) = \text{if } B[\![b]\!]\sigma \text{ then } W(C[\![c]\!]\sigma) \text{ else } \sigma$$

- This is called the unwinding equation
- It is not a good denotation of W because:
  - It defines W in terms of itself
  - It is not evident that such a W exists
  - It does not describe W uniquely
  - It is not compositional

More on WHILE

- The unwinding equation does not specify W uniquely
- Take C⟦while true do skip⟧
- The unwinding equation reduces to W(σ) = W(σ), which is satisfied by every function!
- Take C⟦while x ≠ 0 do x := x – 2⟧
- The following solution satisfies equation (for any σ')

$$W(\sigma) = \begin{cases} \sigma[x := 0] & \text{if } \sigma(x) = 2k \wedge \sigma(x) \geq 0 \\ \sigma' & \text{otherwise} \end{cases}$$

Denotational Game Plan

- Since WHILE is recursive
  - always have something like: $W(\sigma) = F(W(\sigma))$
- Admits *many* possible values for $W(\sigma)$
- We will *order* them
  - With respect to non-termination = "least"
- And then find the least fixed point
- LFP $W(\sigma)=F(W(\sigma))$ == meaning of "while"

WHILE k-steps Semantics

- Define $W_k: \Sigma \to \Sigma_\perp$ (for $k \in \mathbb{N}$) such that

$$W_k(\sigma) = \begin{cases} \sigma' & \text{if "while b do c" in state } \sigma \text{ terminates in \underline{fewer than k} iterations in state } \sigma' \\ \perp & \text{otherwise} \end{cases}$$

- We can define the W_k functions as follows:

$$W_0(\sigma) = \perp$$

$$W_k(\sigma) = \begin{cases} W_{k-1}(C[\![c]\!]\sigma) & \text{if } B[\![b]\!]\sigma \text{ for } k \geq 1 \\ \sigma & \text{otherwise} \end{cases}$$

WHILE Semantics

- How do we get W from W_k?

$$W(\sigma) = \begin{cases} \sigma' & \text{if } \exists k. W_k(\sigma) = \sigma' \neq \bot \\ \bot & \text{otherwise} \end{cases}$$

- This is a valid compositional definition of W
  - Depends only on $C⟦c⟧$ and $B⟦b⟧$
- Try the examples again:
  - For C⟦while true do skip⟧

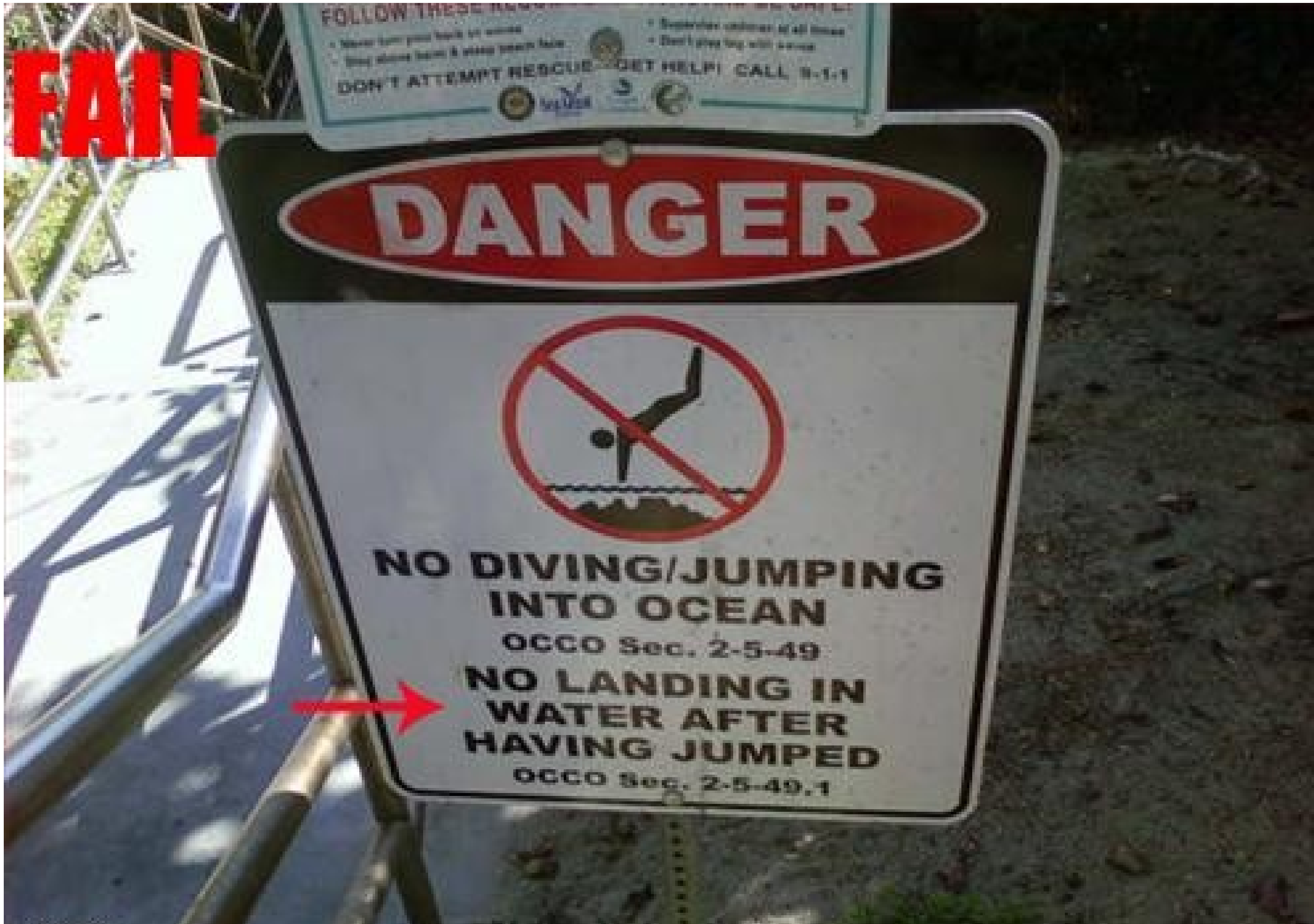
    $W_k(\sigma) = \bot$ for all k, thus $W(\sigma) = \bot$
  - For C⟦while $x \neq 0$ do x := x - 2⟧

$$W(\sigma) = \begin{cases} \sigma[x:=0] & \text{if } \sigma(x) = 2n \wedge \sigma(x) \geq 0 \\ \bot & \text{otherwise} \end{cases}$$

More on WHILE

- The solution is not quite satisfactory because
  - It has an operational flavor (= “run the loop”)
  - It does not generalize easily to more complicated semantics (e.g., higher-order functions)
- However, precisely due to the operational flavor this solution is easy to prove sound w.r.t operational semantics

That Wasn’t Good Enough!?


FAIL
DON'T ATTEMPT RESCUE - GET HELP! CALL 9-1-1
DANGER
NO DIVING/JUMPING INTO OCEAN
OCCO Sec. 2-5-49
NO LANDING IN WATER AFTER HAVING JUMPED
OCCO Sec. 2-5-49.1

Simple Domain Theory

- Consider programs in an eager, deterministic language with one variable called "x"
  - All these restrictions are just to simplify the examples
- A state σ is just the value of x
  - Thus we can use $\mathbb{Z}$ instead of Σ
- The semantics of a command give the value of final x as a function of input x

$$C[\![c]\!] : \mathbb{Z} \rightarrow \mathbb{Z}_\perp$$

Examples - Revisited

- Take C⟦while true do skip⟧
  - Unwinding equation reduces to W(x) = W(x)
  - Any function satisfies the unwinding equation
  - Desired solution is W(x) = $\perp$
- Take C⟦while x $\neq$ 0 do x := x - 2⟧
  - Unwinding equation:

    W(x) = if x $\neq$ 0 then W(x - 2) else x
  - Solutions (for all values n, m $\in \mathbb{Z}_\perp$):

    W(x) = if x $\geq$ 0 then
        if x even then 0 else n
      else m
  - Desired solution: W(x) = if x $\geq$ 0 $\wedge$ x even then 0 else $\perp$

An Ordering of Solutions

- The desired solution is the one in which all the arbitrariness is replaced with non-termination
  - The arbitrary values in a solution are not uniquely determined by the semantics of the code
- We introduce an ordering of semantic functions
- Let f, g $\in \mathbb{Z} \to \mathbb{Z}_\perp$
- Define f $\sqsubseteq$ g as

  $$\forall x \in \mathbb{Z}.\ f(x) = \perp \text{ or } f(x) = g(x)$$

  - A "smaller" function terminates *at most as often*, and when it terminates it produces the same result

Alternative Views of Function Ordering

- A semantic function $f \in \mathbb{Z} \to \mathbb{Z}_\perp$ can be written as $S_f \subseteq \mathbb{Z} \times \mathbb{Z}$ as follows:

$$S_f = \{ (x, y) \mid x \in \mathbb{Z}, f(x) = y \neq \perp \}$$

  – set of "terminating" values for the function

- If $f \sqsubseteq g$ then
  – $S_f \subseteq S_g$ (and vice-versa)
  – We say that g <u>refines</u> f
  – We say that f <u>approximates</u> g
  – We say that g provides more information than f

The "Best" Solution

- Consider again $C[\![\text{while } x \neq 0 \text{ do } x := x - 2]\!]$
  - Unwinding equation:

    $W(x) = \text{if } x \neq 0 \text{ then } W(x - 2) \text{ else } x$
- Not all solutions are comparable:

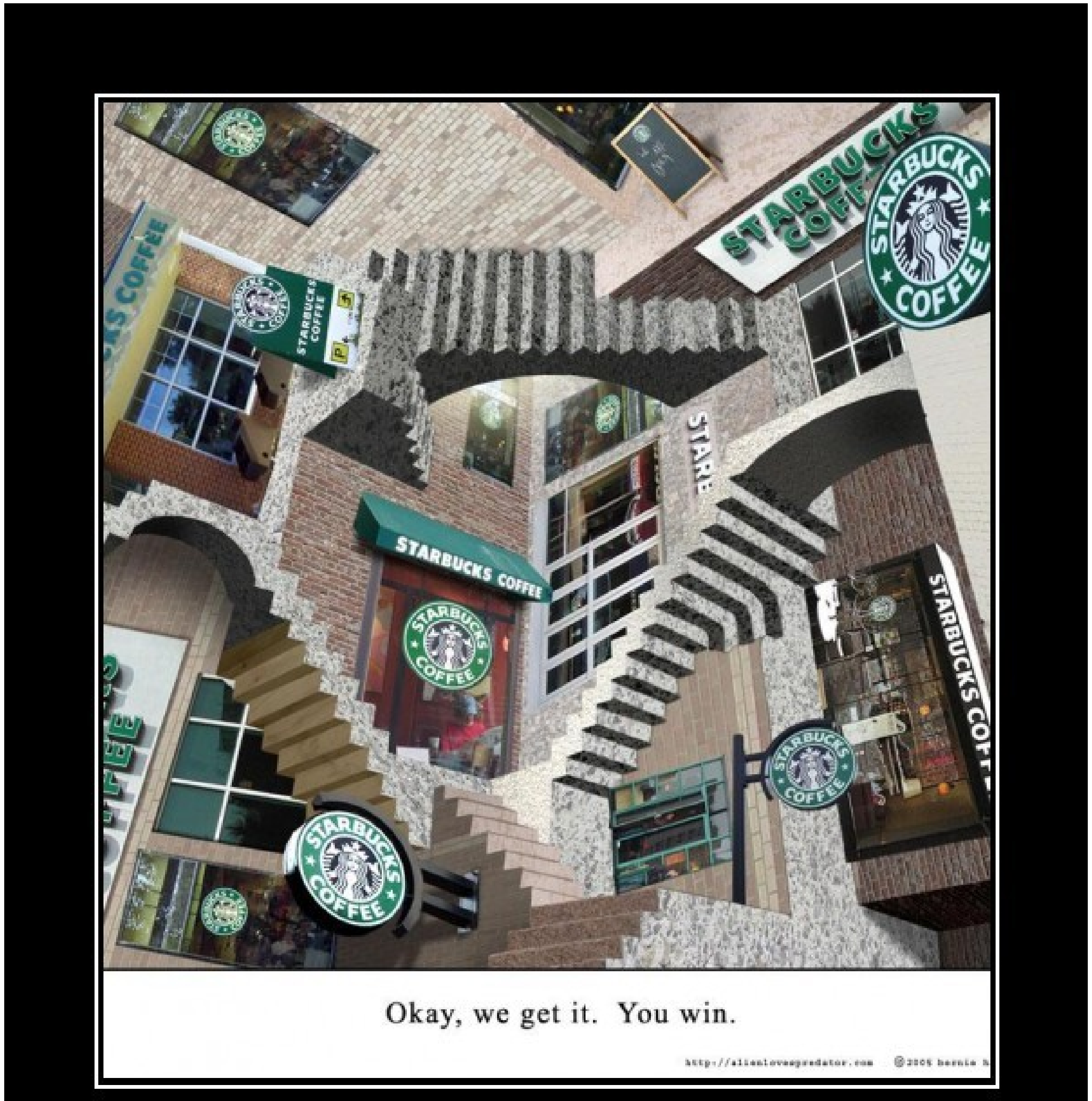
  $W(x) = \text{if } x \geq 0 \text{ then if } x \text{ even then } 0 \text{ else } 1 \text{ else } 2$

  $W(x) = \text{if } x \geq 0 \text{ then if } x \text{ even then } 0 \text{ else } \perp \text{ else } 3$

  $W(x) = \text{if } x \geq 0 \text{ then if } x \text{ even then } 0 \text{ else } \perp \text{ else } \perp$

  (last one is least and best)
- Is there always a least solution?
- How do we find it?
- *If only we had a general framework* for answering these questions …

A Recursive Labyrinth



Fixed-Point Equations

- Consider the general unwinding equation for while

  while b do c ≡ if b then c; while b do c else skip

- We define a context C (command with a hole)

  C = if b then c; • else skip

  while b do c ≡ C[while b do c]

  - The grammar for C does <u>not</u> contain "while b do c"

- We can find such a (recursive) context for any looping construct
  - Consider: fact n = if n = 0 then 1 else n * fact (n – 1)
  - C(n) = if n = 0 then 1 else n * • (n – 1)
  - fact = C [fact]

Fixed-Point Equations

- The meaning of a context is a semantic functional $F : (\mathbb{Z} \to \mathbb{Z}_\perp) \to (\mathbb{Z} \to \mathbb{Z}_\perp)$ such that

$$F\ [\![C[w]]\!] = F\ [\![w]\!]$$

- For "while": $C = \text{if } b \text{ then } c; \bullet \text{ else skip}$

$$F\ w\ x = \text{if } [\![b]\!]\ x \text{ then } w\ ([\![c]\!]\ x) \text{ else } x$$

  - F depends only on $[\![c]\!]$ and $[\![b]\!]$
- We can rewrite the unwinding equation for while
  - $W(x) = \text{if } [\![b]\!]\ x \text{ then } W([\![c]\!]\ x) \text{ else } x$
  - or, $W\ x = F\ W\ x$ for all x,
  - or, $\underline{W = F\ W}$ (by function equality)

Fixed-Point Equations

- The meaning of “while” is a solution for W = F W
- Such a W is called a fixed point of F
- We want the least fixed point
  - We need a general way to find least fixed points
- Whether such a least fixed point exists depends on the properties of function F
  - Counterexample: F w x = if w x = ⊥ then 0 else ⊥
  - Assume W is a fixed point
  - F W x = W x = if W x = ⊥ then 0 else ⊥
  - Pick an x, then if W x = ⊥ then W x = 0 else W x = ⊥
  - Contradiction. This F has no fixed point!

Can We Solve This?

- Good news: the functions F that *correspond to contexts in our language* have least fixed points!
- The only way F w x uses w is by invoking it
- If any such invocation diverges, then F w x diverges!
- It turns out: F is monotonic, continuous
  - Not shown here!

We need more power!

New Notation: λ

- **$\lambda x.\ e$**
  - an anonymous function with body **e** and argument **x**
- Example: double(x) = x+x

  double = λx. x+x
- Example: allFalse(x) = false

  allFalse = λx. false
- Example: multiply(x,y) = x*y

  multiply = λx. λy. x*y

The Fixed-Point Theorem

- If F is a semantic function corresponding to a context in our language
  - F is monotonic and continuous (we assert)
  - For any fixed-point G of F and $k \in \mathbb{N}$

    $$F^k(\lambda x.\bot) \sqsubseteq G$$

  - The least of all fixed points is

    $$\sqcup_k F^k(\lambda x.\bot)$$

- Proof (not detailed in the lecture):
  1. By mathematical induction on k.

     Base: $F^0(\lambda x.\bot) = \lambda x.\bot \sqsubseteq G$

     Inductive: $F^{k+1}(\lambda x.\bot) = F(F^k(\lambda x.\bot)) \sqsubseteq F(G) = G$
  - Suffices to show that $\sqcup_k F^k(\lambda x.\bot)$ is a fixed-point

    $$F(\sqcup_k F^k(\lambda x.\bot)) = \sqcup_k F^{k+1}(\lambda x.\bot) = \sqcup_k F^k(\lambda x.\bot)$$

WHILE Semantics

- We can use the fixed-point theorem to write the denotational semantics of while:

  $[\![\text{while } b \text{ do } c]\!] = \sqcup_k F^k\ (\lambda x.\bot)$

  where $F\ f\ x = \text{if } [\![b]\!]\ x \text{ then } f\ ([\![c]\!]\ x) \text{ else } x$

- Example: $[\![\text{while true do skip}]\!] = \lambda x.\bot$
- Example: $[\![\text{while } x \neq 0 \text{ then } x := x - 1]\!]$
  - $F\ (\lambda x.\bot)\ x = \text{if } x = 0 \text{ then } x \text{ else } \bot$
  - $F^2\ (\lambda x.\bot)\ x = \text{if } x = 0 \text{ then } x \text{ else if } x\text{-}1 = 0 \text{ then } x\text{-}1 \text{ else } \bot$

    $= \text{if } 1 \geq x \geq 0 \text{ then } 0 \text{ else } \bot$
  - $F^3\ (\lambda x.\bot)\ x = \text{if } 2 \geq x \geq 0 \text{ then } 0 \text{ else } \bot$
  - $LFP_F = \text{if } x \geq 0 \text{ then } 0 \text{ else } \bot$
- Not easy to find the closed form for general LFPs!

Discussion

- We can write the denotational semantics but we cannot always compute it.
  - Otherwise, we could decide the halting problem
  - H is halting for input 0 iff $[\![H]\!]\ 0 \neq \perp$
- We have derived this for programs with one variable
  - Generalize to multiple variables, even to variables ranging over richer data types, even higher-order functions: domain theory

Can You Remember?

You just survived the hardest lecture.
It's all downhill from here.

Recall: Learning Goals

- DS is compositional
- When should I use DS?
- In DS, meaning is a "math object"
- DS uses $\perp$ ("bottom") to mean non-termination
- DS uses fixed points and domains to handle `while`
  - This is the tricky bit

Homework

- Homework 2
- Start Homework 3
  - Not as long as it looks – separated out every exercise sub-part for clarity.
  - Your denotational answers must be compositional (e.g., $W_k(\sigma)$ or LFP)
- Read!

Equivalence

- Two expressions (commands) are equivalent if they yield the same result from all states

  $e_1 \approx e_2$ iff

  $\forall \sigma \in \Sigma.\ \forall n \in \mathbb{N}.$

  $<e_1, \sigma> \Downarrow n$ iff $<e_2, \sigma> \Downarrow n$

  and for commands

  $c_1 \approx c_2$ iff

  $\forall \sigma, \sigma' \in \Sigma.$

  $<c_1, \sigma> \Downarrow \sigma'$ iff $<c_2, \sigma> \Downarrow \sigma'$

Notes on Equivalence

- Equivalence is like logical validity
  - It must hold in all states (= all valuations)
  - $2 \approx 1 + 1$ is like "$2 = 1 + 1$ is valid"
  - $2 \approx 1 + x$ might or might not hold.
    - So, 2 is not equivalent to $1 + x$
- Equivalence (for IMP) is undecidable
  - If it were decidable we could solve the halting problem for IMP. *How?*
- Equivalence justifies code transformations
  - compiler optimizations
  - code instrumentation
  - abstract modeling
- Semantics is the basis for proving equivalence

Equivalence Examples

- skip; c ≈ c
- while b do c ≈
      if b then c; while b do c else skip
- If $e_1 \approx e_2$ then x := e_1 ≈ x := e_2
- while true do skip ≈ while true do x := x + 1
- If c is

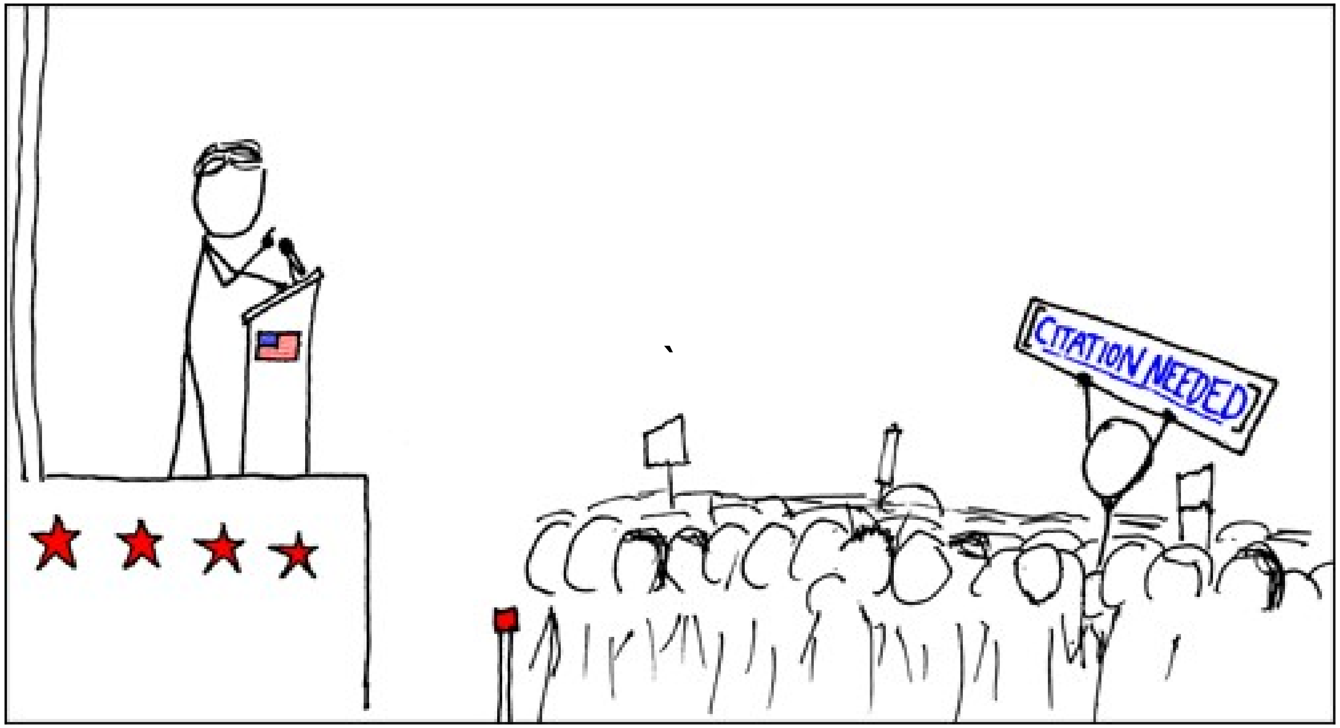
```
while x ≠ y do
      if x ≥ y then x := x - y else y := y - x
```

  then

(x := 221; y := 527; c) ≈ (x := 17; y := 17)

Potential Equivalence

- $(x := e_1; x := e_2) \approx x := e_2$
- Is this a valid equivalence?



Not An Equivalence

- $(x := e_1; x := e_2) \not\approx x := e_2$
- lie. Chigau yo. Dame desu!
- Not a valid equivalence for all e_1, e_2.
- Consider:
  - $(x := x+1; x := x+2) \not\approx x := x+2$
- But for n_1, n_2 it’s fine:
  - $(x := n_1; x := n_2) \approx x := n_2$

Proving An Equivalence

- Prove that "skip; c $\approx$ c" for all c
- Assume that D :: <skip; c, σ> ⇓ σ'
- By inversion (twice) we have that

$$D :: \dfrac{\dfrac{}{\langle \text{skip}, \sigma\rangle \Downarrow \sigma} \quad D_1 :: \langle c, \sigma\rangle \Downarrow \sigma'}{\langle \text{skip}; c, \sigma\rangle \Downarrow \sigma'}$$

- Thus, we have D_1 :: <c,σ> ⇓ σ'
- The other direction is similar

Proving An Inequivalence

- Prove that $x := y \not\approx x := z$ when $y \neq z$
- It suffices to exhibit a σ in which the two commands yield different results

- Let $\sigma(y) = 0$ and $\sigma(z) = 1$
- Then

$$\langle x := y, \sigma \rangle \Downarrow \sigma[x := 0]$$

$$\langle x := z, \sigma \rangle \Downarrow \sigma[x := 1]$$