Proof Techniques for Operational Semantics

Small-Step Contextual Semantics

- In small-step contextual semantics, derivations are not tree-structured
- A contextual semantics derivation is a sequence (or list) of atomic rewrites:

$$\langle x+(7-3),\sigma\rangle \rightarrow \langle x+(4),\sigma\rangle \xrightarrow[\sigma(x)=5]{} \langle 5+4,\sigma\rangle \rightarrow \langle 9,\sigma\rangle$$

If $\langle r, \sigma\rangle \rightarrow \langle e, \sigma'\rangle$

then $\langle H[r], \sigma\rangle \rightarrow \langle H[e], \sigma'\rangle$

r = redex
H = context (has hole)

Context Decomposition

- Decomposition theorem:

  If c is not "skip" then there exist unique H and r such that c is H[r]

  - "Exist" means progress
  - "Unique" means determinism

Short-Circuit Evaluation

- What if we want to express short-circuit evaluation of $\wedge$?
  - Define the following contexts, redexes and local reduction rules

$$H ::= \ldots \mid H \wedge b_2$$

$$r ::= \ldots \mid \text{true} \wedge b \mid \text{false} \wedge b$$

$$<\text{true} \wedge b, \sigma> \rightarrow <b, \sigma>$$

$$<\text{false} \wedge b, \sigma> \rightarrow <\text{false}, \sigma>$$

  - the local reduction kicks in before b_2 is evaluated

Contextual Semantics Summary

- Can view • as representing the program counter
- Contextual semantics is inefficient to implement directly
- The major advantage of contextual semantics: it allows a mix of local and global reduction rules
  - For IMP we have only local reduction rules: only the redex is reduced
  - Sometimes it is useful to work on the context too
  - We'll do that when we study memory allocation, etc.

Cunning Plan for Proof Techniques

- Why Bother?
- Mathematical Induction
- Well-Founded Induction
- Structural Induction
  - "Induction On The Structure Of The Derivation"

One-Slide Summary

- **Mathematical Induction** is a proof technique: If you can prove P(0) and you can prove that P(n) implies P(n+1), then you can conclude that for all natural numbers n, P(n) holds.
- Induction works because the natural numbers are **well-founded**: there are no **infinite descending chains** n > n-1 > n-2 > ... > ... .
- **Structural induction** is induction on a formal structure, like an AST. The base cases use the leaves, the inductive steps use the inner nodes.
- **Induction on a derivation** is structural induction applied to a derivation D (e.g., D::$\langle \mathbf{c}, \sigma \rangle \Downarrow \sigma'$).

Why Bother?

- I am loathe to teach you anything that I think is a waste of your time.
- Thus I must convince you that inductive opsem proof techniques are useful.
  - Recall class goals: understand PL research techniques and apply them to your research
- This motivation should also highlight where you might use such techniques in your own research.

Never Underestimate

"Any counter-example posed by the Reviewers against this proof would be a useless gesture, no matter what technical data they have obtained. **Structural Induction** is now the ultimate proof technique in the universe. I suggest we use it." --- Admiral Motti, *A New Hope*

Classic Example (Schema)

- "A well-typed program cannot go wrong."
  - Robin Milner
- When you design a new type system, you must show that it is safe (= that the type system is sound with respect to the operational semantics).
- *A Syntactic Approach to Type Soundness.* Andrew K. Wright, Matthias Felleisen, 1992.
  - Type preservation: "if you have a well-typed program and apply an opsem rule, the result is well-typed."
  - Progress: "a well-typed program will never get stuck in a state with no applicable opsem rules"
- Done for real languages: SML/NJ, SPARK ADA, Java
  - PL/I, plus basically every toy PL research language ever.

Classic Examples

- **CCured Project (Berkeley)**
  - A program that is instrumented with CCured run-time checks (= "adheres to the CCured type system") will not segfault (= "the x86 opsem rules will never get stuck").
- **Vault Language (Microsoft Research)**
  - A well-typed Vault program does not leak any tracked resources and invokes tracked APIs correctly (e.g., handles IRQL correctly in asynchronous Windows device drivers, cf. Capability Calculus)
- **RC - Reference-Counted Regions For C (Intel Research)**
  - A well-typed RC program gains the speed and convenience of region-based memory management but need never worry about freeing a region too early (run-time checks).
- **Typed Assembly Language (Cornell)**
  - Reasonable C programs (e.g., device drivers) can be translated to TALx86. Well-typed TALx86 programs are type- and memory-safe.
- **Secure Information Flow (Many, e.g,. Volpano et al. '96)**
  - Lattice model of secure flow analysis is phrased as a type system, so type soundness = noninterference.

Recent Examples

- "The proof proceeds by <u>rule induction</u> over the target term producing translation rules."
  - Chakravarty et al. '05
- "Type preservation can be proved by standard <u>induction on the derivation</u> of the evaluation relation."
  - Hosoya et al. '05
- "Proof: By <u>induction on the derivation</u> of $N \Downarrow W$."
  - Sumi and Pierce '05
- Method: chose four POPL 2005 papers at random, the three above mentioned structural induction.

  (emphasis mine)

Induction

- Most important technique for studying the formal semantics of prog languages
  - If you want to perform or understand PL research, you must grok this!

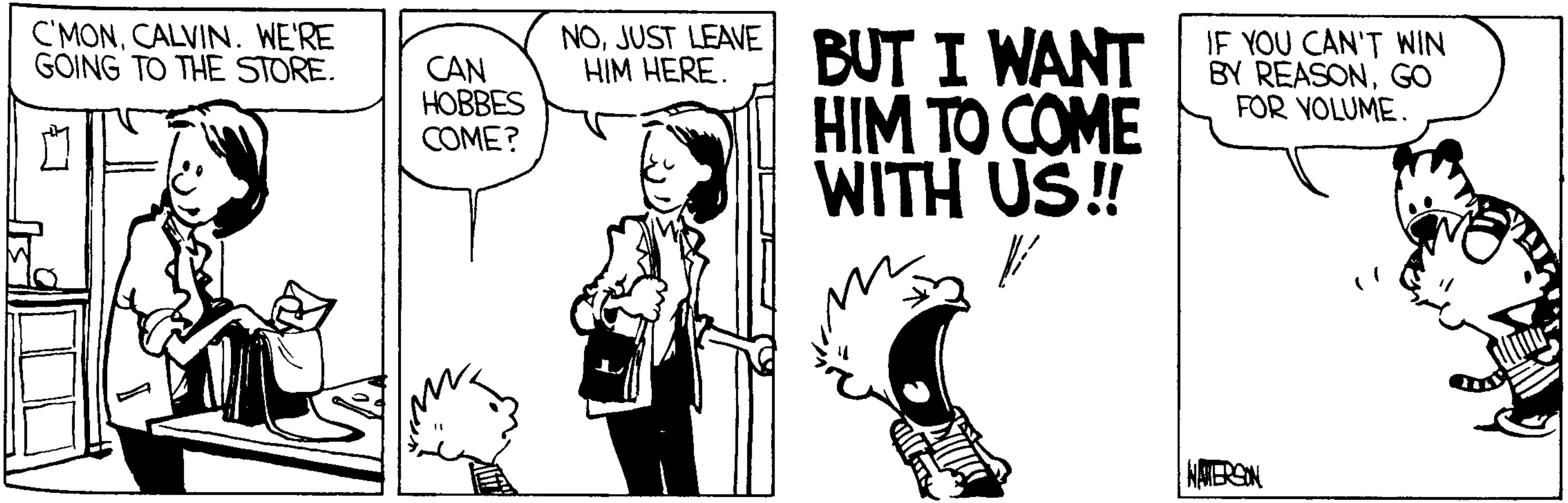
- Mathematical Induction (simple)
- Well-Founded Induction (general)
- Structural Induction (widely used in PL)

Mathematical Induction

- Goal: prove $\forall n \in \mathbb{N}.\ P(n)$
- Base Case: prove $P(0)$
- Inductive Step:
  - Prove $\forall\ n>0.\ P(n) \Rightarrow P(n+1)$
  - "Pick arbitrary n, assume $P(n)$, prove $P(n+1)$"
- Why does induction work?

Why Does It Work?

- There are no infinite descending chains of natural numbers
- For any *n*, P(*n*) can be obtained by starting from the base case and applying *n* instances of the inductive step



Well-Founded Induction

- A relation $\preceq \; \subseteq A \times A$ is well-founded if there are no infinite descending chains in A
  - Example: $<_1 = \{ (x, x+1) \mid x \in \mathbb{N} \}$
    - aka the predecessor relation
  - Example: $< \; = \{ (x, y) \mid x, y \in \mathbb{N}$ and $x < y \}$
- Well-founded induction:
  - To prove $\forall x \in A.\ P(x)$ it is enough to prove $\forall x \in A.\ [\forall y \preceq x \Rightarrow P(y)] \Rightarrow P(x)$
- If $\preceq$ is $<_1$ then we obtain mathematical induction as a special case

Structural Induction

- Recall $e ::= n \mid e_1 + e_2 \mid e_1 * e_2 \mid x$
- Define $\preceq \subseteq Aexp \times Aexp$ such that

  $e_1 \preceq e_1 + e_2 \qquad e_2 \preceq e_1 + e_2$

  $e_1 \preceq e_1 * e_2 \qquad e_2 \preceq e_1 * e_2$

  - no other elements of $Aexp \times Aexp$ are $\preceq$-related
- To prove $\forall e \in Aexp.\ P(e)$
  - $\vdash \forall n \in Z.\ P(n)$
  - $\vdash \forall x \in L.\ P(x)$
  - $\vdash \forall e_1, e_2 \in Aexp.\ P(e_1) \wedge P(e_2) \Rightarrow P(e_1 + e_2)$
  - $\vdash \forall e_1, e_2 \in Aexp.\ P(e_1) \wedge P(e_2) \Rightarrow P(e_1 * e_2)$

Notes on Structural Induction

- Called structural induction because the proof is guided by the structure of the expression
- One proof case per form of expression
  - Atomic expressions (with no subexpressions) are all base cases
  - Composite expressions are the inductive case
- This is the *most useful form of induction* in the study of PL

Example of Induction on Structure of Expressions

- Let
  - L(e) be the # of literals and variable occurrences in e
  - O(e) be the # of operators in e
- Prove that $\forall e \in$ Aexp. L(e) = O(e) + 1
- Proof: by induction on the structure of e
  - Case e = n. L(e) = 1 and O(e) = 0
  - Case e = x. L(e) = 1 and O(e) = 0
  - Case $e = e_1 + e_2$.
    - $L(e) = L(e_1) + L(e_2)$ and $O(e) = O(e_1) + O(e_2) + 1$
    - By induction hypothesis $L(e_1) = O(e_1) + 1$ and $L(e_2) = O(e_2) + 1$
    - Thus $L(e) = O(e_1) + O(e_2) + 2 = O(e) + 1$
  - Case $e = e_1 * e_2$. Same as the case for +

Other Proofs by Structural Induction on Expressions

- Most proofs for Aexp sublanguage of IMP
- Small-step and natural semantics obtain equivalent results:

$$\forall e \in \text{Exp}.\ \forall n \in \mathbb{N}.\quad e \rightarrow^* n \Leftrightarrow e \Downarrow n$$

- Structural induction on expressions works here because all of the semantics are syntax directed

Stating The Obvious (With a Sense of Discovery)

- You are given a concrete state σ.
- You have $\vdash \langle x + 1, \sigma \rangle \Downarrow 5$
- You also have $\vdash \langle x + 1, \sigma \rangle \Downarrow 88$
- Is this possible?

Why That Is Not Possible

- Prove that IMP is deterministic

  $\forall e \in \text{Aexp}.\ \forall \sigma \in \Sigma.\ \forall n, n' \in \mathbb{N}.\ \ \langle e, \sigma\rangle \Downarrow n \ \wedge\ \langle e, \sigma\rangle \Downarrow n' \ \Rightarrow\ n = n'$

  $\forall b \in \text{Bexp}.\ \forall \sigma \in \Sigma.\ \forall t, t' \in \mathbb{B}.\ \ \langle b, \sigma\rangle \Downarrow t \ \wedge\ \langle b, \sigma\rangle \Downarrow t' \ \Rightarrow\ t = t'$

  $\forall c \in \text{Comm}.\ \forall \sigma, \sigma', \sigma'' \in \Sigma.\ \ \langle c, \sigma\rangle \Downarrow \sigma' \ \wedge\ \langle c, \sigma\rangle \Downarrow \sigma'' \ \Rightarrow\ \sigma' = \sigma''$

- No immediate way to use *mathematical* induction
- For commands we cannot use *induction on the structure of the command*
  - `while`'s evaluation does ***not*** depend only on the evaluation of its strict subexpressions

$$\frac{\langle b, \sigma\rangle \Downarrow \text{true} \qquad \langle c, \sigma\rangle \Downarrow \sigma' \qquad \langle \text{while } b \text{ do } c, \sigma'\rangle \Downarrow \sigma''}{\langle \text{while } b \text{ do } c, \sigma\rangle \Downarrow \sigma''}$$

Q: Movies (292 / 842)

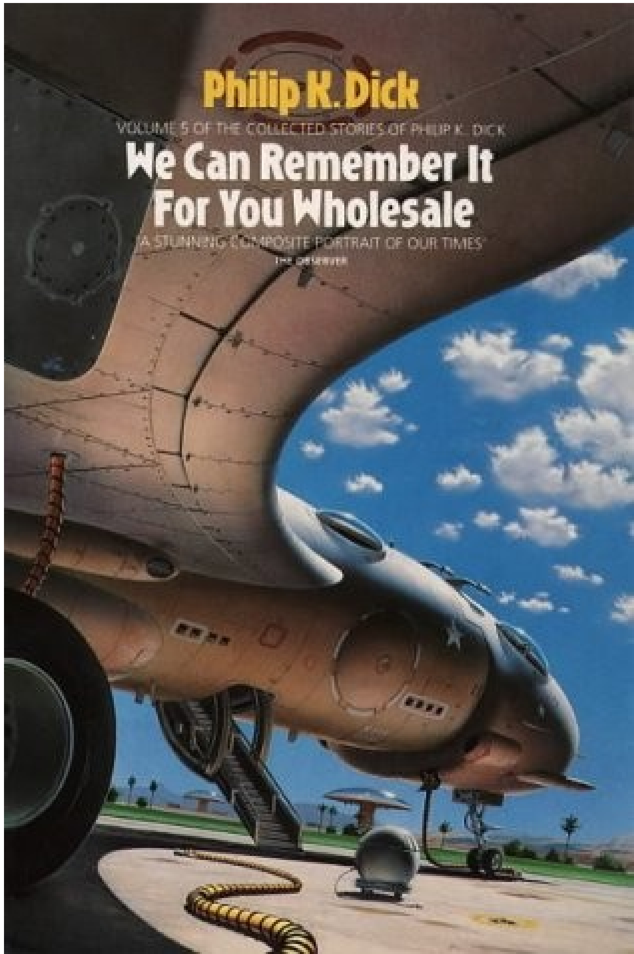
- From the 1981 movie **Raiders of the Lost Ark**, give either the protagonist's phobia xor the composer of the musical score.

Computer Science

- This Dutch Turing-award winner is famous for the semaphore, “THE” operating system, the Banker's algorithm, and a shortest path algorithm. He favored structured programming, as laid out in the 1968 article *Go To Statement Considered Harmful*. He was a strong proponent of formal verification and correctness by construction. He also penned *On The Cruelty of Really Teaching Computer Science*, which argues that CS is a branch of math and relates provability to correctness.

Recall Opsem

- Operational semantics assigns meanings to programs by listing rules of inference that allow you to prove judgments by making derivations.
- A derivation is a tree-structured object made up of valid instances of inference rules.



We Need Something New

- Some more powerful form of induction …
- With all the bells and whistles!

Induction on the Structure of Derivations

- Key idea: The hypothesis does not just assume a $c \in$ Comm but the **existence of a derivation of $\langle c, \sigma \rangle \Downarrow \sigma'$**
- Derivation trees are also defined inductively, just like expression trees
- A derivation is built of <u>subderivations</u>:

$$\dfrac{\dfrac{\langle x, \sigma_{i+1} \rangle \Downarrow 5 - i \qquad 5 - i \leq 5}{\langle x \leq 5, \sigma_{i+1} \rangle \Downarrow \text{true}} \qquad \dfrac{\dfrac{\langle x + 1, \sigma_{i+1} \rangle \Downarrow 6 - i}{\langle x{:=}x{+}1, \sigma_{i+1} \rangle \Downarrow \sigma_i} \qquad \langle W, \sigma_i \rangle \Downarrow \sigma_0}{\langle x{:=}x{+}1;\ W, \sigma_{i+1} \rangle \Downarrow \sigma_0}}{\langle \text{while } x \leq 5 \text{ do } x := x + 1, \sigma_{i+1} \rangle \Downarrow \sigma_0}$$

- Adapt the structural induction principle to work on the structure of derivations

Induction on Derivations

- To prove that for all derivations D of a judgment, property P holds
- For each derivation rule of the form

$$\frac{H_1 \ \dots \ H_n}{C}$$

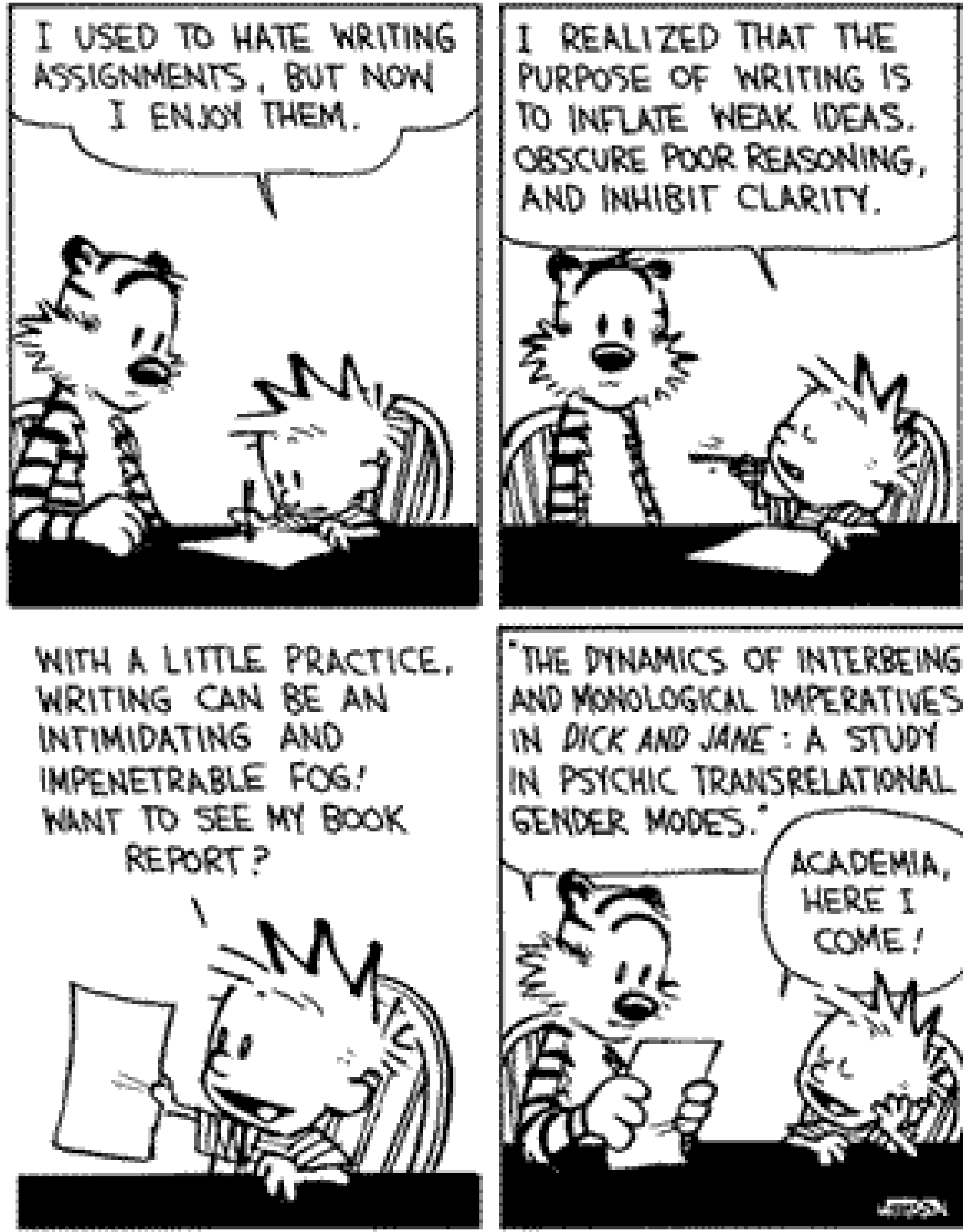
- Assume P holds for derivations of H_i ($i = 1..n$)
- Prove the the property holds for the derivation obtained from the derivations of H_i using the given rule

New Notation

- Write D :: Judgment to mean “D is the derivation that proves Judgment”

- Example:

$D :: \langle x+1, \sigma\rangle \Downarrow 2$



Induction on Derivations (2)

- Prove that evaluation of commands is deterministic:

  $<c, \sigma> \Downarrow \sigma' \Rightarrow \forall \sigma'' \in \Sigma.\ <c, \sigma> \Downarrow \sigma'' \Rightarrow \sigma' = \sigma''$
- Pick arbitrary c, σ, σ' and D :: $<c, \sigma> \Downarrow \sigma'$
- To prove: $\forall \sigma'' \in \Sigma.\ <c, \sigma> \Downarrow \sigma'' \Rightarrow \sigma' = \sigma''$
  - Proof: by induction on the structure of the derivation D
- Case: last rule used in D was the one for skip

$$D :: \quad \frac{}{<\text{skip}, \sigma> \Downarrow \sigma}$$

  - This means that c = skip, and $\sigma' = \sigma$
  - By inversion $<c, \sigma> \Downarrow \sigma''$ uses the rule for skip
  - Thus $\sigma'' = \sigma$
  - This is a base case in the induction

Induction on Derivations (3)

- Case: the last rule used in D was the one for sequencing

$$D :: \frac{D_1 :: \langle c_1, \sigma\rangle \Downarrow \sigma_1 \qquad D_2 :: \langle c_2, \sigma_1\rangle \Downarrow \sigma'}{\langle c_1; c_2, \sigma\rangle \Downarrow \sigma'}$$

- Pick arbitrary σ'' such that $D'' :: \langle c_1; c_2, \sigma\rangle \Downarrow \sigma''$.
  - by inversion D'' uses the rule for sequencing
  - and has subderivations $D''_1 :: \langle c_1, \sigma\rangle \Downarrow \sigma''_1$ and $D''_2 :: \langle c_2, \sigma''_1\rangle \Downarrow \sigma''$
- By induction hypothesis on D_1 (with D''_1): $\sigma_1 = \sigma''_1$
  - Now $D''_2 :: \langle c_2, \sigma_1\rangle \Downarrow \sigma''$
- By induction hypothesis on D_2 (with D''_2): $\sigma'' = \sigma'$
- This is a simple inductive case

Induction on Derivations (4)

- Case: the last rule used in D was `while true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow \text{true} \qquad D_2 :: \langle c, \sigma\rangle \Downarrow \sigma_1 \qquad D_3 :: \langle \text{while } b \text{ do } c, \sigma_1\rangle \Downarrow \sigma'}{\langle \text{while } b \text{ do } c, \sigma\rangle \Downarrow \sigma'}$$

- Pick arbitrary σ'' such that $D''::\langle \text{while } b \text{ do } c, \sigma\rangle \Downarrow \sigma''$
  - by inversion and determinism of boolean expressions, D'' also uses the rule for `while true`
  - and has subderivations $D''_2 :: \langle c, \sigma\rangle \Downarrow \sigma''_1$ and $D''_3 :: \langle W, \sigma''_1\rangle \Downarrow \sigma''$
- By induction hypothesis on D_2 (with D''_2): $\sigma_1 = \sigma''_1$
  - Now $D''_3 :: \langle \text{while } b \text{ do } c, \sigma_1\rangle \Downarrow \sigma''$
- By induction hypothesis on D_3 (with D''_3): $\sigma'' = \sigma'$

What Do You, The Viewers At Home, Think?

- Let's do `if true` together!
- Case: the last rule in D was `if true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow true \qquad D_2 :: \langle c1, \sigma\rangle \Downarrow \sigma_1}{\langle \text{if } b \text{ do } c1 \text{ else } c2, \sigma\rangle \Downarrow \sigma_1}$$

- Try to do this on a piece of paper. In a few minutes I'll have some lucky winners come on down.

Induction on Derivations (5)

- Case: the last rule in D was `if true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow \text{true} \qquad D_2 :: \langle c1, \sigma\rangle \Downarrow \sigma'}{\langle \text{if } b \text{ do } c1 \text{ else } c2, \sigma\rangle \Downarrow \sigma'}$$

- Pick arbitrary σ'' such that
  $D'' :: \langle \text{if } b \text{ do } c1 \text{ else } c2, \sigma\rangle \Downarrow \sigma''$
  - By inversion and determinism, D'' also uses `if true`
  - And has subderivations $D''_1 :: \langle b, \sigma\rangle \Downarrow$ true and $D''_2 :: \langle c1, \sigma\rangle \Downarrow \sigma''$
- By induction hypothesis on D_2 (with D''_2): $\sigma' = \sigma''$

Induction on Derivations Summary

- If you must prove $\forall x \in A.\ P(x) \Rightarrow Q(x)$
  - with A inductively defined and P(x) rule-defined
  - we pick arbitrary $x \in A$ and $D :: P(x)$
  - we could do induction on both facts
    - $x \in A$ leads to induction on the structure of x
    - $D :: P(x)$ leads to induction on the structure of D
  - Generally, the induction on the structure of the derivation is more powerful and a safer bet
- Sometimes there are many choices for induction
  - choosing the right one is a trial-and-error process
  - a bit of practice can help a lot

Equivalence

Optional Material Begins

- Two expressions (commands) are **equivalent** if they yield the same result from all states

$e_1 \approx e_2$ iff

$\forall \sigma \in \Sigma. \forall n \in \mathbb{N}.$

$<e_1, \sigma> \Downarrow n$ iff $<e_2, \sigma> \Downarrow n$

and for commands

$c_1 \approx c_2$ iff

$\forall \sigma, \sigma' \in \Sigma.$

$<c_1, \sigma> \Downarrow \sigma'$ iff $<c_2, \sigma> \Downarrow \sigma'$

Notes on Equivalence

- Equivalence is like logical validity
  - It must hold in all states (= all valuations)
  - $2 \approx 1 + 1$ is like "$2 = 1 + 1$ is valid"
  - $2 \approx 1 + x$ might or might not hold.
    - So, 2 is not equivalent to $1 + x$
- Equivalence (for IMP) is **undecidable**
  - If it were decidable we could solve the halting problem for IMP. *How?*
- Equivalence justifies code transformations
  - compiler optimizations
  - code instrumentation
  - abstract modeling
- Semantics is the basis for proving equivalence

Equivalence Examples

- skip; c ≈ c
- while b do c ≈
  if b then c; while b do c else skip
- If $e_1 \approx e_2$ then $x := e_1 \approx x := e_2$
- while true do skip ≈ while true do x := x + 1
- Let c be

  while x ≠ y do

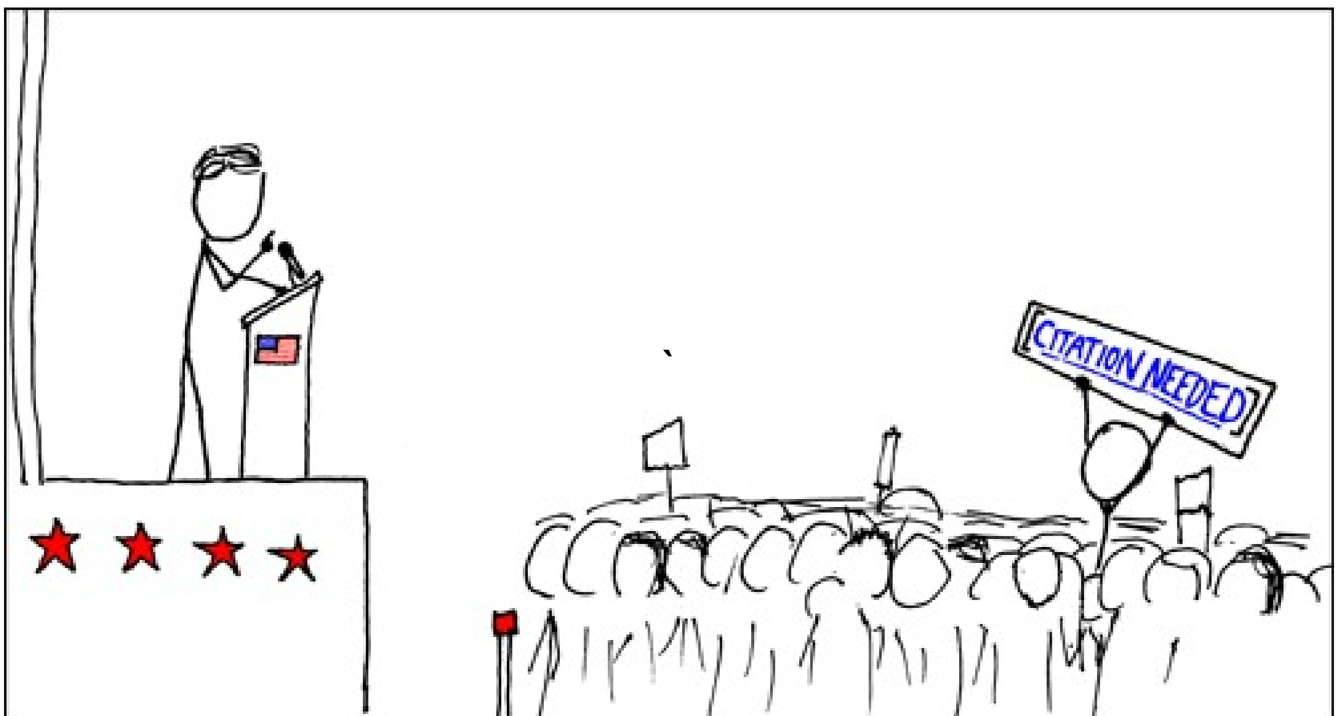
  if x ≥ y then x := x - y else y := y - x

  then

  (x := 221; y := 527; c) ≈ (x := 17; y := 17)

Potential Equivalence

- $(x := e_1;\ x := e_2) \approx x := e_2$
- Is this a valid equivalence?



Not An Equivalence

- $(x := e_1; x := e_2) \not\approx x := e_2$
- Iie. Chigau yo. Dame desu!
- Not a valid equivalence for all e_1, e_2.
- Consider:
  - $(x := x+1; x := x+2) \not\approx x := x+2$
- But for n_1, n_2 it's fine:
  - $(x := n_1; x := n_2) \approx x := n_2$

Proving An Equivalence

- Prove that "skip; c $\approx$ c" for all c
- Assume that D :: <skip; c, σ> $\Downarrow$ σ'
- By inversion (twice) we have that

$$D :: \dfrac{\dfrac{}{\langle \text{skip}, \sigma\rangle \Downarrow \sigma} \quad D_1 :: \langle c, \sigma\rangle \Downarrow \sigma'}{\langle \text{skip}; c, \sigma\rangle \Downarrow \sigma'}$$

- Thus, we have D_1 :: <c,σ> $\Downarrow$ σ'
- The other direction is similar

Proving An Inequivalence

- Prove that $x := y \not\sim x := z$ when $y \neq z$
- It suffices to exhibit a σ in which the two commands yield different results

- Let $\sigma(y) = 0$ and $\sigma(z) = 1$
- Then

$$\langle x := y, \sigma \rangle \Downarrow \sigma[x := 0]$$

$$\langle x := z, \sigma \rangle \Downarrow \sigma[x := 1]$$

Optional Material Ends

Summary of Operational Semantics

- Precise specification of dynamic semantics
  - order of evaluation (or that it doesn’t matter)
  - error conditions (sometimes implicitly, by rule applicability; “no applicable rule” = “get stuck”)
- Simple and abstract (vs. implementations)
  - no low-level details such as stack and memory management, data layout, etc.
- Often not compositional (see while)
- Basis for many proofs about a language
  - Especially when combined with type systems!
- Basis for much reasoning about programs
- Point of reference for other semantics

Homework

- Don't Neglect Your Homework
- Read Winskel Chapter 5
  - Pay careful attention.