eecs 281 DATA STRUCTURES AND ALGORITHMS

Lecture 11: Red-Black Trees

Red-Black Tree

Designed to represent 2-3-4 tree without the additional link overhead

A Red-Black tree is a binary search tree in which each node is colored red or black

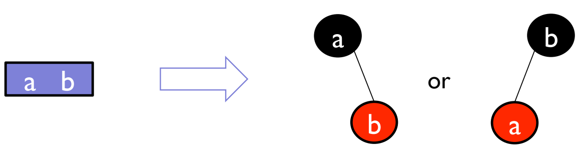
Red nodes represent the extra keys in 3-nodes and 4-nodes

2-node = black node

[Brinton,Rosenfeld,Ozbirn]

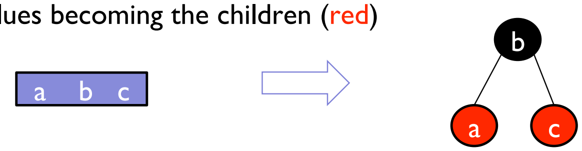
Red-Black vs. 2-3-4 Nodes

3-node = black node with one red child



4-node = black node with two red children

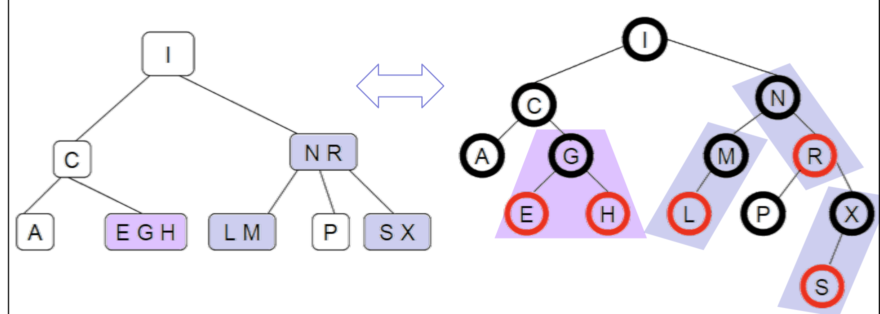
- center value becomes the parent (black) with outside values becoming the children (red)



Red-black trees are not unique, but the corresponding 2-3-4 tree is unique

[Brinton,Ozbirn]

Red-Black vs. 2-3-4 Trees

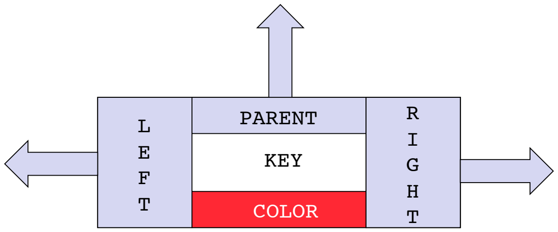


Red-Black Tree

Red-black trees are widely used:

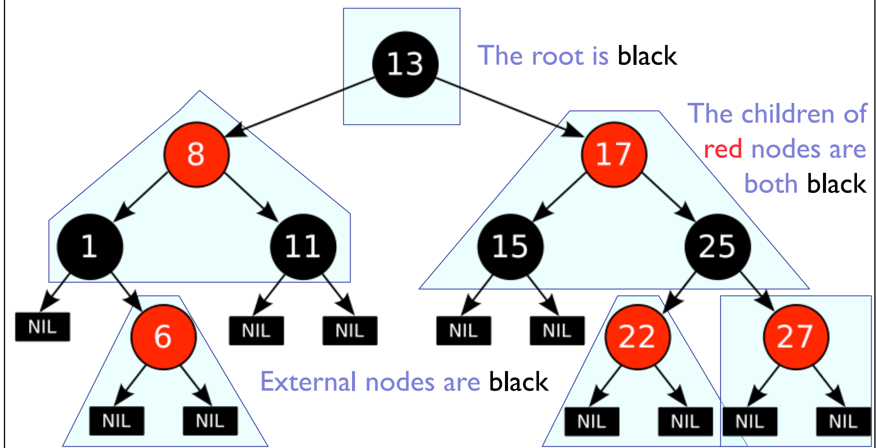
- C++ STL: `map, set, multimap, multiset`
- Java: `java.util.TreeMap, java.util.TreeSet`
- Linux kernel: `linux/rbtree.h`

RBT node representation:



[Brinton,Rosenfeld,Sedgewick,Walter]

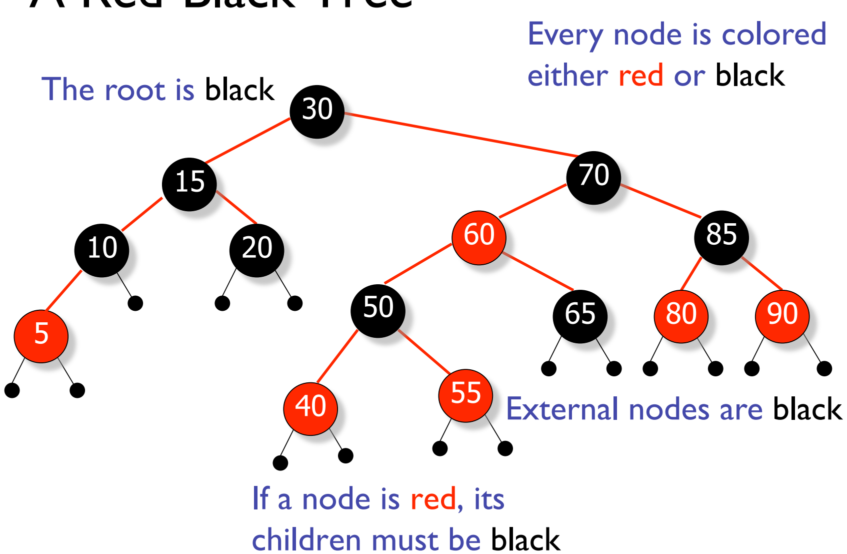
Red-Black Tree Rules and Properties



Every node is colored either red or black

[Giabbanelli]

A Red-Black Tree



[McCollam]

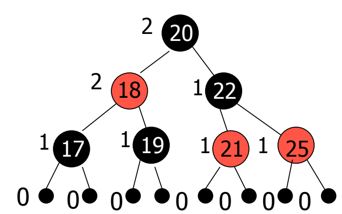
Black-height bh(x)

Black-height of node x is the number of black nodes on the path from x to an external node (including the external node but not counting x itself)

Every node has a black-height, bh(x)

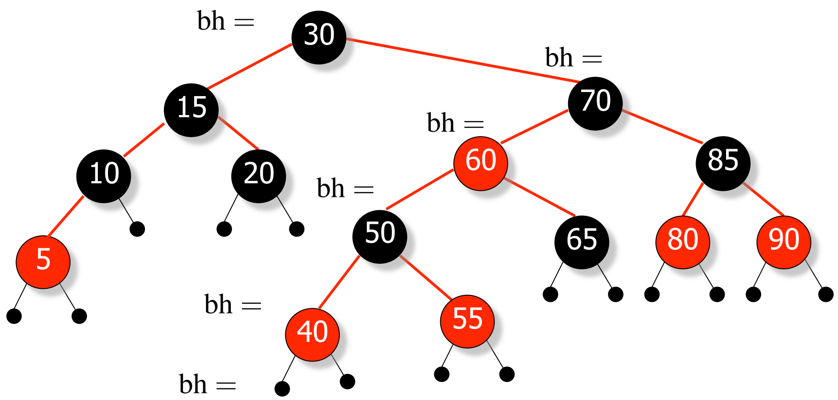
For all external nodes, bh(x) $= 0$

For root x, bh(x) $=$ bh(T)



[Walter]

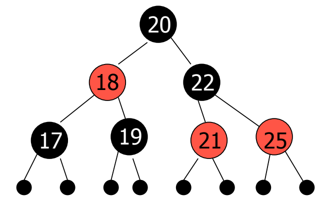
Black-height Rule



Every path from a node x to an external node must contain the same number of black nodes $=$ black-height(x)

[McCollam]

Red-Black Rules and Properties



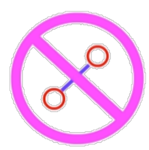
1. Every node is either red or black
2. The root is black [root rule]
3. External nodes (nulls) are black
4. If a node is red, then both its children are black [red rule]
5. Every path from a node to a null must have the same number of black nodes (black height) [black-height rule]
   a. this is equivalent to a 2-3-4 tree being a perfect tree: all the leaf nodes of the 2-3-4 tree are at the same level (black-height$=1$)
   b. a black node corresponds to a level change in the corresponding 2-3-4 tree

[Walter,Brinton]

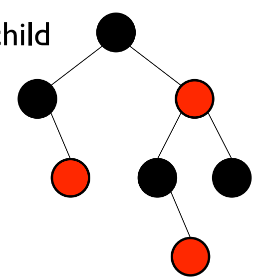
Implications of the Rules

If a red node has any children, it must have two children and they must be black (why?)

- can't have 2 consecutive reds (double red) on a path
- however, any number of black nodes may appear in a sequence



If a black node has only one child that child must be a red leaf (why?)



[Scott,Ozbirn]

Red-Black Tree Height Bound

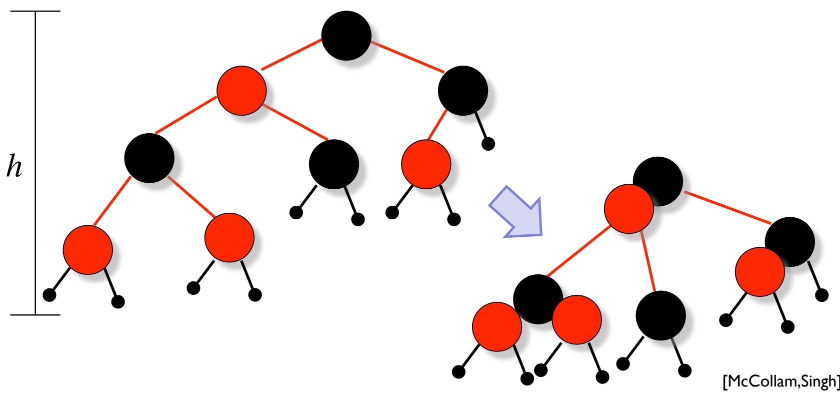
Red-black tree rules constrain the adjacency of node coloring, ensuring that no root-to-leaf path is more than twice as long as any other path, which limits how unbalanced a red-black tree may become

Theorem: The height of a red-black tree with n internal nodes is between $\log_2(n+1)$ and $2\log_2(n+1)$

[Walter,Brinton,Singh]

Red-Black Tree Height Bound

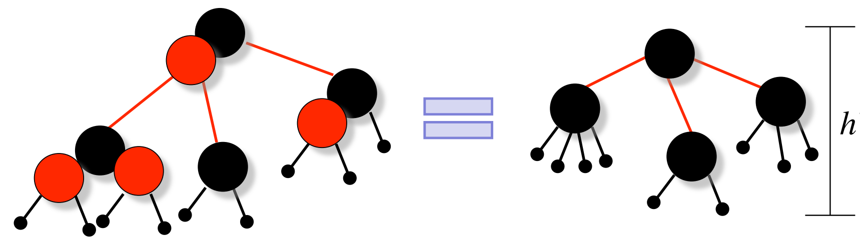
Start with a red black tree with height h
(note: *height* here includes the external nodes)
Merge all red nodes into their black parents



[McCollam,Singh]

Red-Black Tree Height Bound

Nodes in resulting tree have degrees between 2 and 4
All external nodes are at the same level



It's the equivalent 2-3-4 tree to the red-black tree!
Height of the resulting tree is $h' \geq h/2$

[McCollam,Singh]

Red-Black Tree Height Bound

Let $h' \geq h/2$ be the height of the collapsed tree

The tree is tallest if all internal nodes have degree 2, i.e., there were no red-node in the original red-black tree, $h' = h$, and number of internal nodes is $n = 2^{h'}-1$ and $h' = 2 \log_2(n+1)$

The tree is shortest if all internal nodes have degree > 2, and $h' = h/2$; e.g., if all internal nodes have degree 4, the number of internal nodes is $n = 4^{h'}-1$ and $h' = \log_2(n+1)$

In the mixed case, $\log_2(n+1) \leq h' \leq 2 \log_2(n+1)$

[Singh]

Red-Black Tree Height Bound (Alternate Proof)

Prove: an n-internal node RB tree has height
$h \leq 2 \log(n+1)$

Claim: A subtree rooted at a node x contains
at least $2^{\mathrm{bh}(x)} - 1$ internal nodes

- proof by induction on height h
- base step: x has height 0 (i.e., external node)
  - What is $\mathrm{bh}(x)$?
  - 0
  - So…subtree contains $2^{\mathrm{bh}(x)} - 1$
    $= 2^0 - 1$
    $= 0$ internal nodes (claim is TRUE)

[Luebke]

Red-Black Tree Height Bound (Alternate Proof)

Inductive proof that subtree at node x contains at least $2^{\mathrm{bh}(x)} - 1$ internal nodes

- inductive step: x has positive height and 2 children
  - each child has black-height of $\mathrm{bh}(x)$ or $\mathrm{bh}(x)-1$ (Why?)
  - the height of a child $=$ (height of x) $- 1$
  - so the subtrees rooted at each child contain at least $2^{\mathrm{bh}(x)} - 1$ internal nodes by induction hypothesis
  - thus subtree at x contains $(2^{\mathrm{bh}(x)-1} - 1) + (2^{\mathrm{bh}(x)-1} - 1) + 1$ $= 2*2^{\mathrm{bh}(x)-1} - 1 = 2^{\mathrm{bh}(x)} - 1$ nodes

[Luebke]

Red-Black Tree Height Bound (Alternate Proof)

Thus at the root of the red-black tree:

$n \geq 2^{\mathrm{bh}(root)} - 1$

$n \geq 2^{\mathrm{bh}(root)} - 1 \geq 2^{h/2} - 1$ (Why?)

$\log(n+1) \geq h/2$

$h \leq 2\log(n+1)$

Thus $h = O(\log_2 n)$

By the black-height rule, the additional nodes in paths longer than the black height of the tree can consist only of red nodes

By the red rule, at least $1/2$ of the nodes on any path from root to an external node are black

Since the longest path of the tree is h, the black-height of the root must be at least $h/2$

[Luebke,Walter]

Time Complexity of Red-Black Trees

All non-modifying BST operations (min, max, successor, predecessor, search) run in $O(h) = O(\log n)$ time on red-black trees

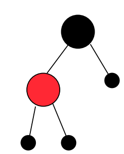
- small storage issue per node to include a color flag (no big deal)

Insertion and deletion must maintain rules of red-black trees and are therefore more complex: still $O(\log_2 n)$ time but a bit slower empirically than in ordinary BST

[Kellih,Walter]

Red-Black Insert

1. as with BST, insert new node as leaf, must be red
   - can't be black or will violate black-height rule
   - therefore the new leaf must be red

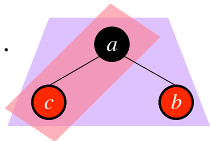


2. insert new node, if inserting into a 2-node representation (black parent), done
3. if inserting into a 3-node, could result in double red $\rightarrow$ need to rotate and recolor nodes to represent a 4-node, with a black parent
4. if inserting into a 4-node, "split" 4-node $\rightarrow$ recolor children black, parent red, and "promote" parent
5. maintain root as black node

[Brinton]

Inserting into a 3-node: Two Cases

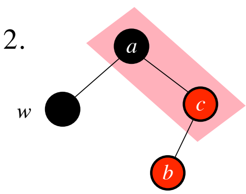
1.



Inserting node b to a black parent that is part of a 3-node, creating a 4-node, done

⇒ inserting a new node to a black parent is always simple

2.



Inserting node b to a red parent that is part of a 3-node, creating double red

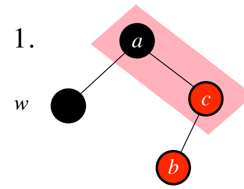
→ how to recognize that parent and grandparent are part of a 3-node?
parent is red, grandparent and uncle (w) are black

→ need to rotate to create a new 4-node

[Ozbirn]

3-Node, Red-Parent

1.



Make the new node (b) along with parent (c) and grandparent (a) a 4-node

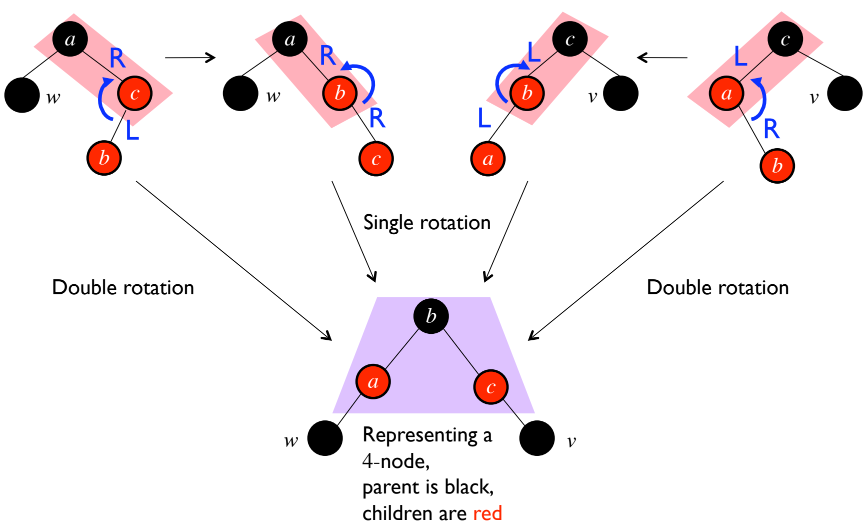
Rotate to make parent (c) the middle value of the 4-node

There are four possible combinations of a, b, and c corresponding to LL, RR, LR, RL rotations (see next slide)

As the middle value of a 4-node, parent (c) will be black, and the two outer nodes (a) and (b) will be red

[Ozbirn]

3-Node, Red-Parent Rotations



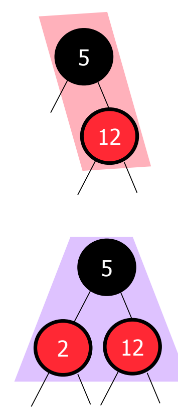
[Ozbirn]

Inserting into a 3-Node

Insert 2

3-node

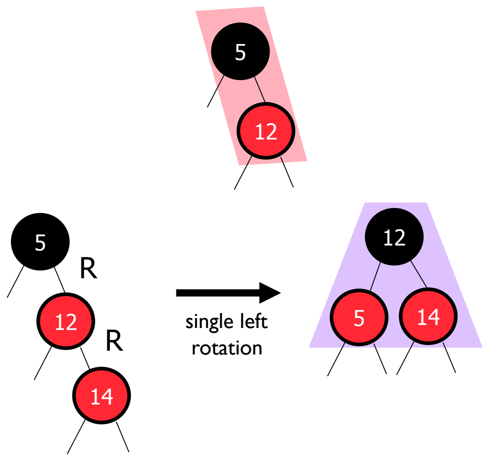
4-node



[Brinton]

Inserting into a 3-Node

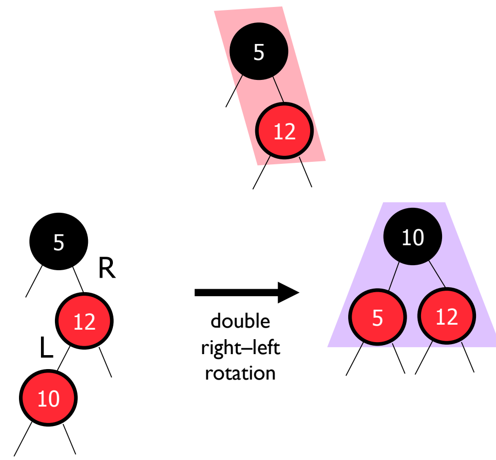
Insert 14



[Brinton]

Inserting into a 3-Node

Insert 10



[Brinton]

Red-Black Insert

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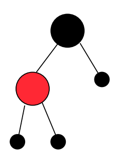

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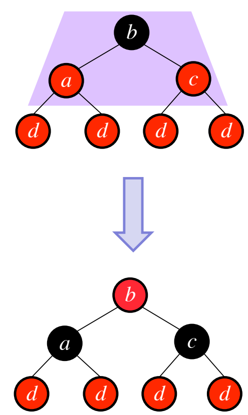
4. if inserting into a 4-node, "split" 4-node → recolor children black, parent red, and "promote" parent
5. maintain root as black node

[Brinton]

Inserting into a 4-node



Inserting node d causes double red, and d' s parent has red sibling w

→ parent, aunt, and grandparent are part of a 4-node

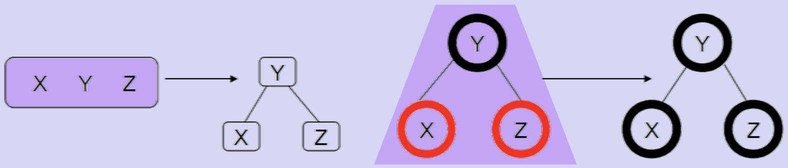
→ need to recolor, to split the 4-node and "promote" grandparent
 parent and aunt become black
 grandparent becomes red

If grandparent is root, change it back to black
Otherwise, insert grandparent to great-grandparent, applying the same insertion rules as before depending on whether great-grandparent is a 2-node, 3-node, or 4-node

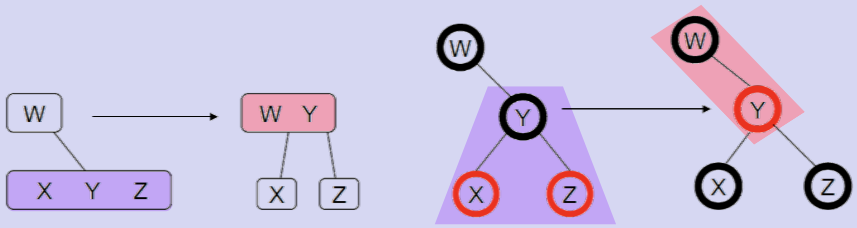
[Ozbirn]

Inserting into a 4-node

Grandparent is root: recolor the two children black

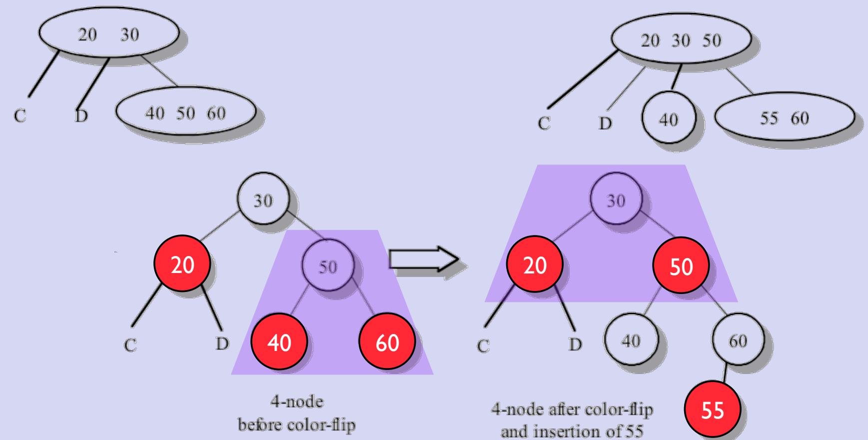


Insert red grandparent into a 2-node great-grandparent:



Inserting into a 4-node

After inserting 55, promote red grandparent to a 3-node, black great-grandparent:



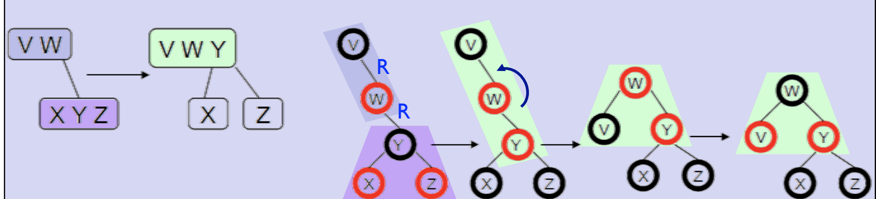
[Brinton]

Inserting into a 4-node

Promoting red grandparent to a 3-node, red-great grandparent:

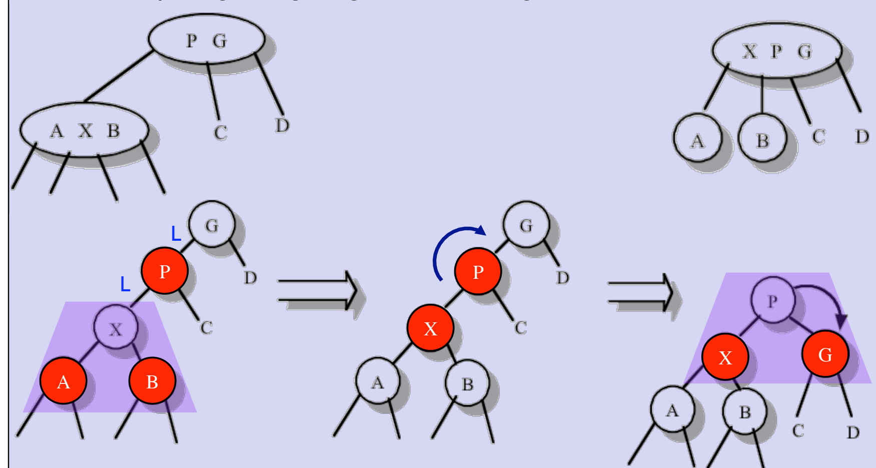
Four cases:

- RR: requiring a single left rotation, e.g.,



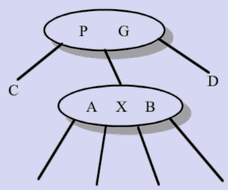
3-Node, Red-Great Grandparent

- LL: requiring a single right rotation, e.g.:

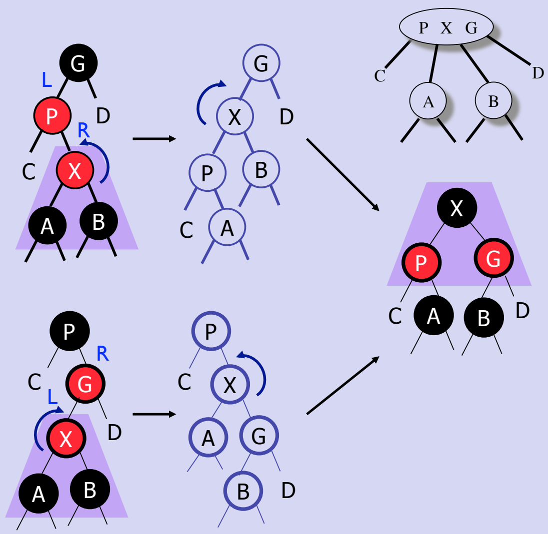


3-Node, Red-Great Grandparent

- LR: requiring a double left-right rotation, e.g.:

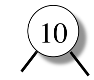


- RL: requiring a double right-left rotation, e.g.,



RBT Insertion Examples

Equivalent 2-3-4 tree:

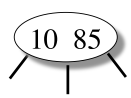


Insert 10 – root, must be black

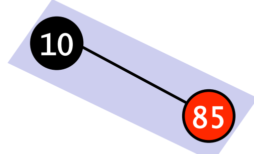
[Rosenfeld]

RBT Insertion Examples

Equivalent 2-3-4 tree:



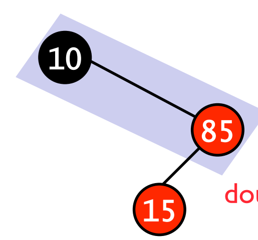
Insert 85 (root is now a 3-node)



[Rosenfeld]

RBT Insertion Examples

Insert 15

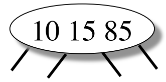


double red!

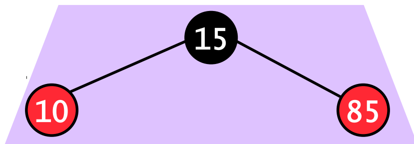
[Rosenfeld]

RBT Insertion Examples

Equivalent 2-3-4 tree:



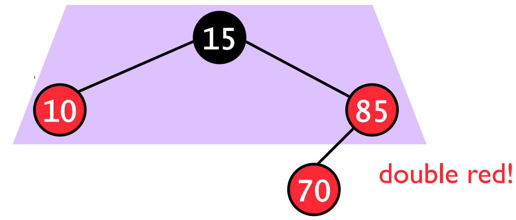
Rotate – Recolor (root becomes a 4-node)



[Rosenfeld]

RBT Insertion Examples

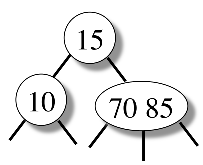
Insert 70 (split the 4-node)



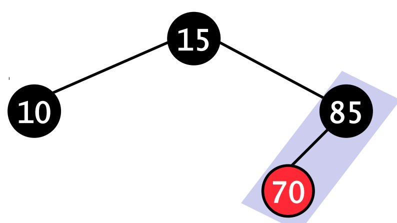
[Rosenfeld]

RBT Insertion Examples

Equivalent 2-3-4 tree:



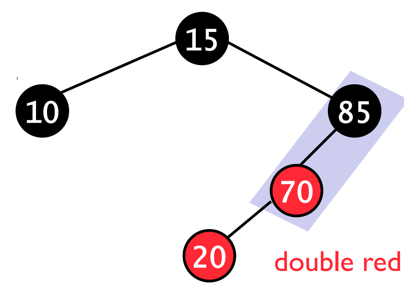
Recolor (root must be black)



[Rosenfeld]

RBT Insertion Examples

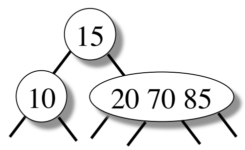
Insert 20 (sibling of parent is black, a 3-node)



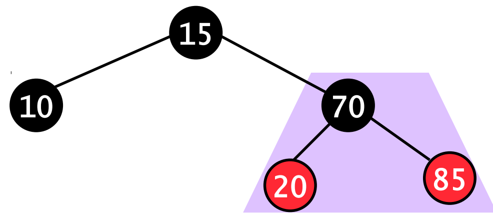
[Rosenfeld]

RBT Insertion Examples

Equivalent 2-3-4 tree:



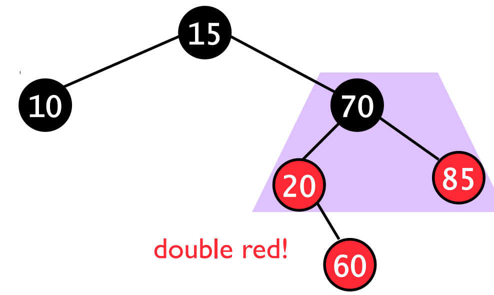
Rotate (becomes a 4-node)



[Rosenfeld]

RBT Insertion Examples

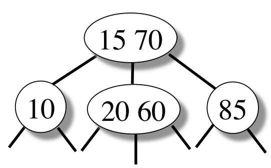
Insert 60 (sibling of parent is red, a 4-node, need to split)



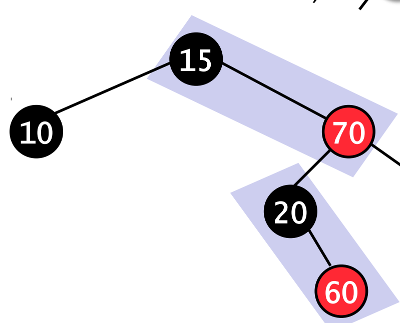
[Rosenfeld]

RBT Insertion Examples

Equivalent 2-3-4 tree:



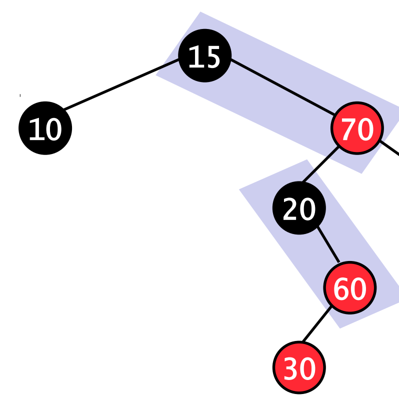
Recolor (promote middle value, 70)



[Rosenfeld]

RBT Insertion Examples

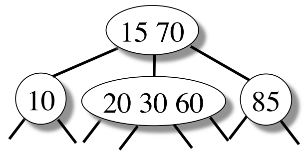
Insert 30 (sibling of parent is black, a 3-node)



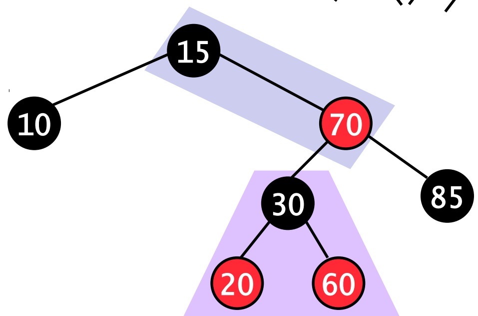
[Rosenfeld]

RBT Insertion Examples

Equivalent
2-3-4 tree:



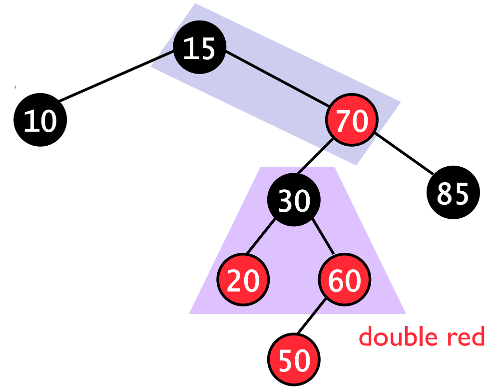
Rotate (made a 4-node)



[Rosenfeld]

RBT Insertion Examples

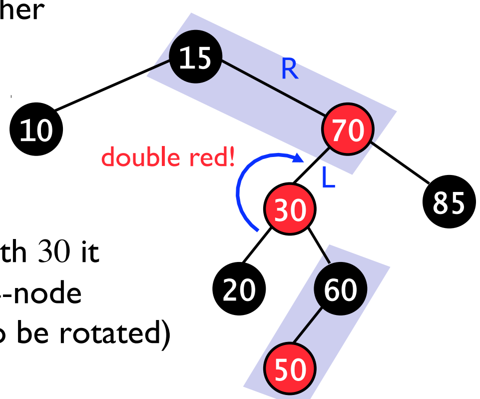
Insert 50 (sibling of parent?)



[Rosenfeld]

RBT Insertion Examples

Insert 50 (promote middle value, 30,
causing another
double red;
sibling of
30's parent,
70, is black,
→ 70 is in
a 3-node; with 30 it
becomes a 4-node
and needs to be rotated)



[Rosenfeld]

RBT Insertion Examples

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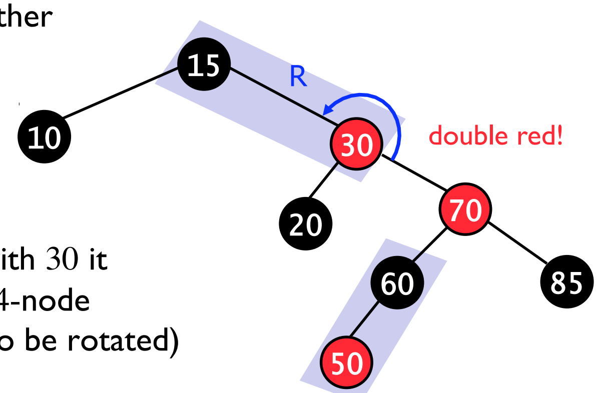
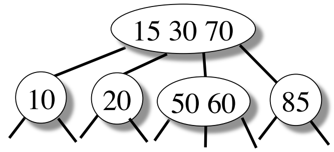


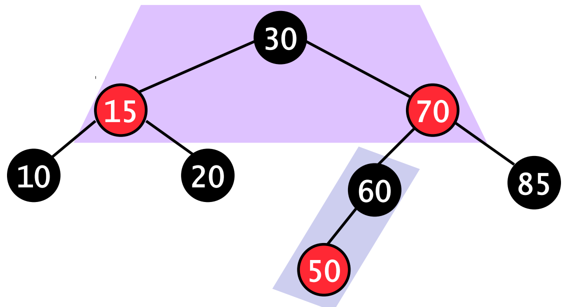
[Rosenfeld]

RBT Insertion Examples

Equivalent 2-3-4 tree:



Double Rotate – Recolor



Demo: http://gauss.ececs.uc.edu/RedBlack/redblack.html

[Rosenfeld]

RBT Removal

If we delete a node, what was the color of the node removed?

- Red? easy, since
  - we won't have changed any black heights,
  - nor will we have created 2 red nodes in a row;
  - also, it could not have been the root
- Black?
  - could violate any of root rule, red rule, or black-height rule

[Walter]

Red-Black Tree Removal

Observations:

- if we delete a red node, tree is still a red-black tree
- a red node is either a leaf node or must have two children

Rules:

1. if node to be deleted is a red leaf, remove leaf, done
2. if it is a single-child parent, it must be black (why?); replace with its child (must be red) and recolor child black
3. if it has two internal node children, swap node to be deleted with its in-order successor
   - if in-order successor is red (must be a leaf, why?), remove leaf, done
   - if in-order successor is a single child parent, apply second rule

In both cases the resulting tree is a legit red-black tree (we haven't changed the number of black nodes in paths)

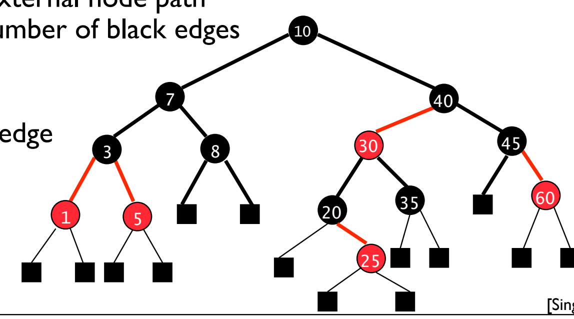
4. if in-order successor is a black leaf, or if the node to be deleted is itself a black leaf, things get complicated . . .

RB-Trees: Alternative Definition

Colored edges definition

1. child pointers are colored red or black
2. the root has black edges
3. pointer to an external node is black
4. no root-to-external-node path has two consecutive red edges
5. every root to external node path has the same number of black edges

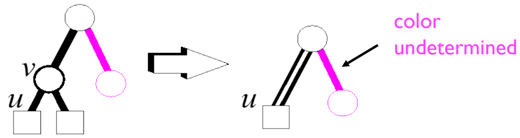
color of node == color of incoming edge



[Singh]

Black-Leaf Removal

We want to remove v, which is a black leaf
Replace v with external node u, color u **double black**



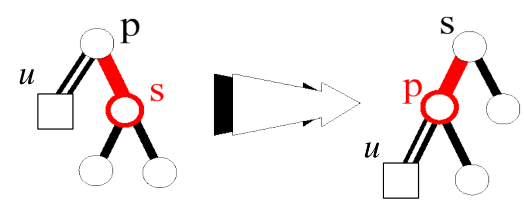
To eliminate **double black** edges, idea:

- find a red edge nearby, and change the pair (red, **double black**) into (black, black)
- as with insertion, we recolor and/or rotate
- rotation resolves the problem locally, whereas recoloring may propagate it two levels up
- slightly more complicated than insertion

[Šaltenis]

Red Sibling

If sibling is red, rotate such that a black node becomes the new sibling, then treat it as a black-sibling case (next slides)

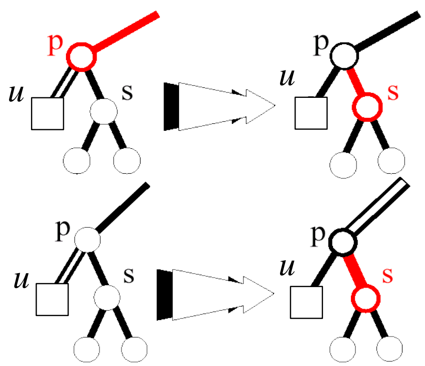


[Šaltenis]

Black Sibling and Nephew/Niece

If sibling and its children are black, recolor sibling and parent

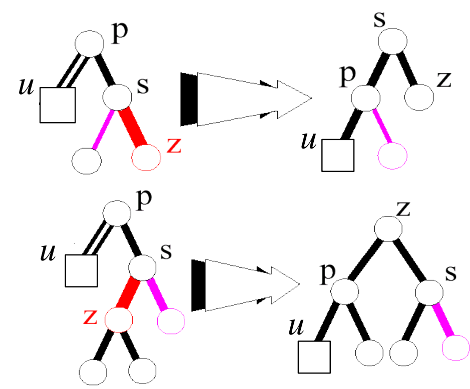
If parent becomes double black, percolate up



[Šaltenis]

Black Sibling but Red Nephew

If sibling is black and one of its children is red, rotate and recolor red nephew involved in rotation

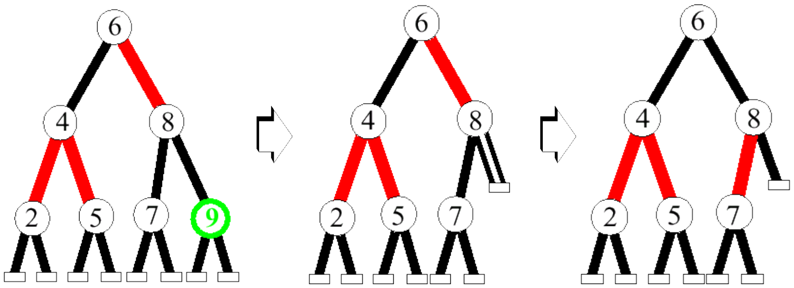


[Šaltenis]

Red-Black Tree Removal Example

Remove 9

sibling and its children are black, recolor sibling and parent

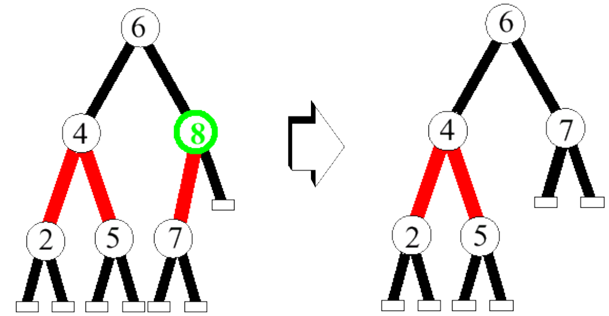


[Šaltenis]

Red-Black Tree Removal Example

Remove 8:

not a black leaf, no double black

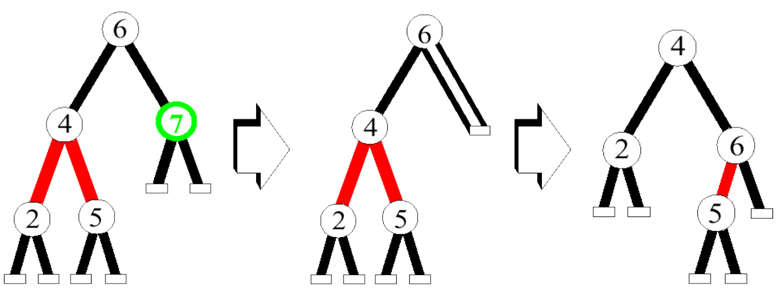


[Šaltenis]

Red-Black Tree Removal Example

Remove 7:

sibling is black and one of its children is red, rotate and recolor red nephew involved in rotation



[Šaltenis]

Efficiency of Red Black Trees

Insertions and removals require additional time due to requirements to recolor and rotate

Most insertions require on average a single rotation: still $O(\log_2 n)$ time but a bit slower empirically than in ordinary BST

[Kellih]