A Plane-wave Expansion Method for Analyzing Propagation in 3D Periodic Ceramic Structures

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Introduction

The plane-wave expansion method has been the basis for much of the understanding of photonic and electrical behavior in crystals. As it was originally presented, Floquet's theorem was a solution to Mathieu's equation and later extended to Hill's equation for periodic media [1]. Among the results of Floquet's theorem is quantum band theory, which is used to describe electron behavior in periodic potentials. Bloch's theorem, which can be regarded as the 3d generalization of Floquet's theorem, states that the eigenfunctions of the Schrödinger equation for crystals are composed of the products of a plane wave and a function with the same periodicity at the crystalline lattice. Though it's less used, Bloch's theorem applies equally to electromagnetic propagation in periodic structures.

Two-dimensional periodic structures are commonly used in a number of Electromagnetic Bandgap (EBG) applications. There has been somewhat less concern, however, with the electromagnetic propagation and bandgap behavior of 3d dielectric structures. Much of this is to due the fact that most 3d periodic dielectric structures cannot be fabricated efficiently, but with developments of Ceramic Stereolithography, this is beginning to change [2]. The principle concern of this study is to understand electromagnetic propagation behavior in a ceramic cubical lattice (Fig. 1). The cubical lattice has several advantages for the design of millimeter-wave components; particularly it has an isotropic symmetry and a wide effective dielectric contrast over the achievable linewidth-to-period ratios [3].

Plane-wave Expansion

For a given propagation vector k, Bloch's theorem states that the fields in a periodic lattice are given by:

$$\overline{H}(x,y,z) = \overline{H}^{p}(x,y,z)e^{-j\bar{k}\bullet\bar{r}} \quad (1)$$

where H^p represents the 3D periodic function that has the same period as the lattice. The fields in the structure obey the source free wave equation.

$$-\frac{1}{\varepsilon^r}\nabla^2\overline{H} + \nabla\frac{1}{\varepsilon^r}\times\left(\nabla\times\overline{H}\right) = k_0^2\overline{H} \quad (2)$$

Owing to their periodic properties, the Fourier basis is the natural choice to describe the dielectric lattice and the periodic component of the magnetic field. By defining the periodic magnetic field and the reciprocal of the dielectric lattice function by its Fourier coefficients—H^p_{mnp}, and $\varepsilon^{-r}_{m'n'p'}$ respectively—quantities may be expressed by equivalent simple Fourier expansions.

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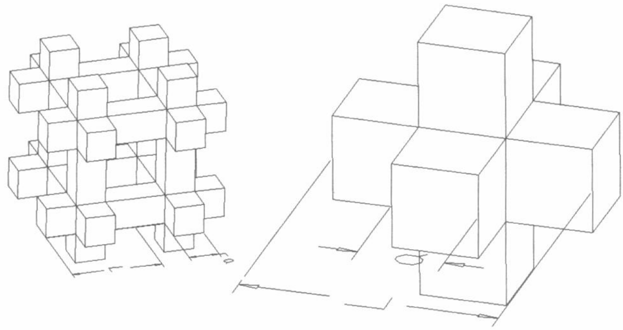

a b

Fig 1: (a) 8 periods of the cubical substructure, (b) one period of the cubical substructure with equivalent circuit

$$\frac{1}{\varepsilon^{r}(\bar{r})} = \sum_{m'n'p'} \varepsilon^{-r}_{m'n'p'} e^{-j\bar{\xi}_{m'n'p'}\bullet\bar{r}} \tag{3a}$$

$$\bar{H}(\bar{r}) = \bar{H}^{p}(\bar{r}) e^{-j\bar{k}\bullet\bar{r}} = \sum_{mnp} \bar{H}^{p}_{mnp} e^{-j\bar{\xi}_{mnp}\bullet\bar{r}} e^{-j\bar{k}\bullet\bar{r}} = \sum_{mnp} \bar{H}^{p}_{mnp} e^{-j\bar{\kappa}_{mnp}\bullet\bar{r}} \tag{3b}$$

To simply computation, the vectors are defined as:

$$\bar{\xi}_{mnp} = \left(\frac{2\pi m}{L_x}\right)\hat{x} + \left(\frac{2\pi n}{L_y}\right)\hat{y} + \left(\frac{2\pi p}{L_z}\right)\hat{z} \tag{4a}$$

$$\bar{\kappa}_{mnp} = \bar{k} + \bar{\xi}_{mnp} \tag{4b}$$

The periodic plane-wave expansion of the wave equation can be expressed as a convolution by substituting (3) into (2) and taking inner product with an orthogonal basis.

$$\begin{aligned} &\sum_{m'n'p'} \varepsilon^{-r}_{m-m',n-n',p-p'} \Big\{ \bar{\kappa}_{m'n'p'} \bullet \bar{\kappa}_{m'n'p'} \bar{H}^{p}_{m'n'p'} + \left(\bar{\xi}_{m-m',n-n',p-p'} \bullet \bar{\kappa}_{m'n'p'} \right) \bar{H}^{p}_{m'n'p'} \\ &- \left(\bar{\xi}_{m-m',n-n',p-p'} \bullet \bar{H}^{p}_{m'n'p'} \right) \bar{\kappa}_{m'n'p'} \Big\} = k_0^2 \bar{H}^{p}_{mnp} \end{aligned} \tag{5}$$

Unlike similar 1D and 2D equations for electromagnetic propagation in periodic structures, the *H*-field components in (5) cannot generally be separated in the orthogonal TE and TM modes. Rather every field component is coupled to every other field component through the $\xi \bullet H$ term. This is intuitively obvious in the 3D structure, and only as (5) generalizes to the 1- and 2-dimensional cases do the fields decouple. This does, however, present some degree of computational difficulty. Equation (5) may be rewritten:

$$\sum_{m'n'p'} \begin{pmatrix} A^{xx}_{mm'nn'pp'} & A^{xy}_{mm'nn'pp'} & A^{xz}_{mm'nn'pp'} \\ A^{yx}_{mm'nn'pp'} & A^{yy}_{mm'nn'pp'} & A^{yz}_{mm'nn'pp'} \\ A^{zx}_{mm'nn'pp'} & A^{zy}_{mm'nn'pp'} & A^{zz}_{mm'nn'pp'} \end{pmatrix} \bar{H}^{p}_{m'n'p'} = k_0^2 \bar{H}^{p}_{mnp} \tag{6}$$

Where the matrix elements are defined as

$$A^{\alpha\beta}_{mm'nn'pp'} = \varepsilon^{-r}_{m-m',n-n',p-p'} \left\{ -\xi^{\beta}_{m-m',n-n',p-p'} \kappa^{\alpha}_{m'n'p'} + \left(\left\| \bar{\kappa}_{m'n'p'} \right\|^2 + \bar{\xi}_{m-m',n-n',p-p'} \bullet \bar{\kappa}_{m'n'p'} \right) \delta_{\alpha\beta} \right\} \tag{7}$$

By mapping *mnp* to a single variable s and $m'n'p'$ to s', it becomes possible to express (6) as a matrix multiplication of a matrix K, with constituent matrices $A_{ss'}$, and the vector v, with constituent vectors H_s.

One way to state (6) in words is to say that for a given dielectric structure and a given frequency, there will be some set of vectors k which satisfy the relation. The converse interpretation is that for a given structure and a given k, only waves with certain frequencies may propagate. Since the frequency term presents itself only in k_0, the frequencies which propagate for a given periodic delay, kL, may be determined by the eigenvalues of K.

$$K\bar{v} = \lambda k_0^2 \bar{v} \tag{8}$$

The eigenvalues of the system represent the squares of the propagating frequencies, and the eigenvectors represent the corresponding field profiles. Each eigenvalue does not, however, correspond to a propagating mode; those eigenvalues which correspond to eigenmodes orthogonal to propagation do not propagate by the traditional sense. This phenomenon is a lattice equivalent to transverse and normal fields. In a symmetric lattice, the propagating and non-propagating modes can be distinguished from the multiplicity of the eigenvalues. If every mode were calculated, there would be a countable number of frequencies which would propagate with a given phase delay, but since the number of modes used in the calculation must be truncated, so must the number of solutions. In this case, the solutions of greatest interest are the smallest eigenvalues—representing the lowest order of propagation. In the lowest order of propagation, the effective dielectric constant of the medium is easily determined by the change in the propagation over frequency.

$$\varepsilon^{eff} = \left(\frac{c}{2\pi} \frac{\partial k}{\partial f} \right)^2 \tag{9}$$

Propagation and Bandgap Behavior of a Cubical Ceramic Lattice

The Fourier coefficients of the cubical alumina (Al_2O_3) lattice are defined as:

$$\begin{aligned} \varepsilon^{-r}_{mnp} = \frac{a^3}{L^3} \left(\frac{1}{\varepsilon^r_{Al_2O_3}} - 1 \right) \Bigg\{ & \frac{L}{a} \delta_{0m} \operatorname{sinc}\left(\frac{an\pi}{L} \right) \operatorname{sinc}\left(\frac{ap\pi}{L} \right) + \frac{L}{a} \delta_{0n} \operatorname{sinc}\left(\frac{am\pi}{L} \right) \operatorname{sinc}\left(\frac{ap\pi}{L} \right) \\ & + \frac{L}{a} \delta_{0p} \operatorname{sinc}\left(\frac{am\pi}{L} \right) \operatorname{sinc}\left(\frac{an\pi}{L} \right) - 2 \operatorname{sinc}\left(\frac{am\pi}{L} \right) \operatorname{sinc}\left(\frac{an\pi}{L} \right) \operatorname{sinc}\left(\frac{ap\pi}{L} \right) \Bigg\} + \delta_{0m} \delta_{0n} \delta_{0p} \end{aligned} \tag{10}$$

The low-frequency effective dielectric constant determined using the plane-wave expansion (Fig 2) shows the effective dielectric constant is slightly less than that predicted with the quasi-static approximation [3]. Due to the negative concavity of the lowest-order propagation (Fig 3), the effective dielectric will increase slightly as the frequency becomes higher. The linearity of the propagation curve away from the vertex suggests that the cubical lattice can be reasonably regarded as non-dispersive when the periodicity is less than 35% of the guided wavelength. Physical measurements of the effective dielectric constant of alumina cubical lattice structures have confirmed this to be true. In order to verify the bandgap nature of the analysis, simulations using Ansoft HFSS show that transmission characteristics through 7 periods of the cubical lattice are consistent with bandgap behavior predicted by (8).

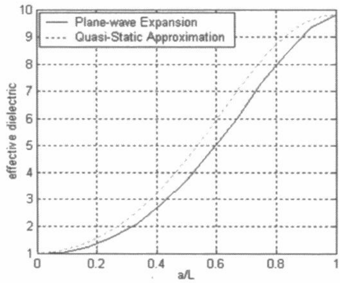

Fig 2: Low-frequency effective dielectric for plane-wave expansion and quasi-static approximation

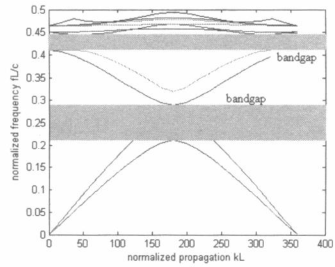

Fig 3: Bandgap behavior of a cubical lattice for a = L/2, (Al_2O_3)

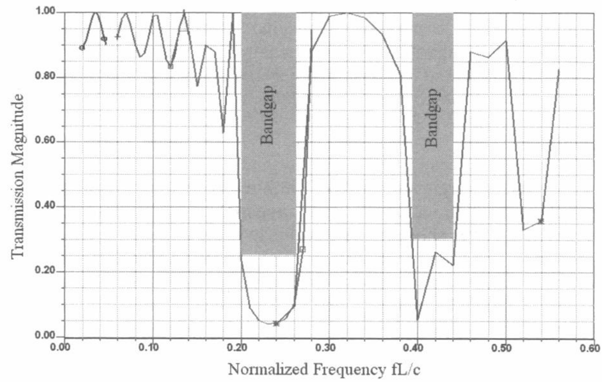

Fig 4: Ansoft HFSS simulation of transmission through

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