Asymptotic MSE Performance of ML Direction-Of-Arrival Estimation with Estimated Noise Covariance

Christ D. Richmond*

Foundations of Computational Mathematics

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*The author is a Senior Staff at MIT Lincoln Laboratory, Lexington, MA and Senior Member of the IEEE.
Outline

• Introduction
  – Adaptive sensor arrays
  – Maximum-Likelihood

• Mean-squared error prediction

• Numerical examples

• Summary
Adaptive Sensor Arrays

General Beamformer

\[ y(t) = \sum_{n} W_n \cdot x_n(t) = w^H x \]
\[ \text{cov}(x) = R \quad E\{x\} = Sv \]

- Spatial filtering
- Adapt to changing environment
- Spatial diversity increases capacity

Conventional System Design

- Array topology / # sensors
  - Resolution, sidelobes, ambiguities, etc.
- Operating frequency / bandwidth
- Power constraints, cost, etc.

Wide Application Areas

- Radar
- Comms/SIGINT
- Sonar

Algorithm Design/Analysis

- Detection, estimation / localization / classification, and tracking
- Metrics: Optimal Neyman-Pearson, Max signal-to-interference + noise ratio (SINR), Min mean squared error (MSE), etc.

\[ w \propto R^{-1} v \]

Weiner Solution

- Theoretical performance analysis
Adaptive Sensor Arrays

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TODAY

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Weiner Solution
Airborne Surveillance Radars:
Signal and Interference Environment
Airborne Surveillance Radars: Signal and Interference Environment
Airborne Surveillance Radars: Signal and Interference Environment

Transmit Power Pattern

Azimuth

Target

\( \theta \)

\( v \)
Airborne Surveillance Radars: Signal and Interference Environment

\[ s_{TX}(t) = \text{Re}\{\tilde{p}(t) \cdot e^{j2\pi f_c t}\} \]
**Transmit Power Pattern**

**Target**

\[
\mathbf{TX/RX\ Waveform}
\]

\[
s_{TX}(t) = \text{Re}\{\tilde{p}(t) \cdot e^{j2\pi f_c t}\}
\]

\[
s_{RX}(t) = \text{Re}\{\alpha \tilde{p}(t - \tau) \cdot e^{j2\pi (f_c + f_d) t}\}
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Airborne Surveillance Radars: Signal and Interference Environment

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- Azimuth \( \theta \)
- Time Delay (Range)
- Doppler (Velocity)
- Transmit Power Pattern
- Ground Clutter
- Target

\( \nu \)
Airborne Surveillance Radars: Signal and Interference Environment

Transmit Power Pattern

$s_{TX}(t) = \text{Re}\{\tilde{p}(t) \cdot e^{j2\pi f_c t}\}$

TX/RX Waveform

Ground Clutter

$s_{RX}(t) = \text{Re}\{\alpha \tilde{p}(t - \tau) \cdot e^{j2\pi (f_c + f_d) t}\}$

Target

Azimuth

$\theta$

Hostile Jamming Interferer

Transmit Power Pattern

Time Delay (Range)

$\nu$

Doppler (Velocity)
Maximum-Likelihood Estimation

• Post-detection it is desired to estimate parameters of target

• Maximum-likelihood (ML) often effective approach:

\[ \hat{\theta}_{ML} = \arg \max_{\theta} \ln p(x|\theta, R) \quad \hat{\theta}_{ML} = \arg \max_{\theta, R} \ln p(x, x(1), \ldots, x(L)|\theta, R) \]
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**R known:**

\[
\hat{\theta}_{ML} = \arg \max_{\theta} \frac{|\mathbf{v}^{H}(\theta)\mathbf{R}^{-1}x|^2}{\mathbf{v}^{H}(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}
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\]

where \( x \sim \mathcal{CN}_{N} (S\mathbf{v}(\theta_1), \mathbf{R}) \), \( \hat{\mathbf{R}} \propto \sum_{l=1}^{L} x(l)x^{H}(l) \), and \( x(l) \sim \mathcal{CN}_{N} (0, \mathbf{R}) \)
Maximum-Likelihood Estimation

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  \[
  \hat{\theta}_{ML} = \arg \max_{\theta} \ln p(x|\theta, R) \quad \hat{\theta}_{ML} = \arg \max_{\theta, R} \ln p(x, x(1), \ldots, x(L)|\theta, R)
  \]

\[\begin{align*}
\text{R known:} & \quad \hat{\theta}_{ML} = \arg \max_{\theta} \frac{|v^H(\theta)R^{-1}x|^2}{v^H(\theta)R^{-1}v(\theta)} \\
\text{R unknown:} & \quad \hat{\theta}_{ML} = \arg \max_{\theta} \frac{|v^H(\theta)\hat{R}^{-1}x|^2}{v^H(\theta)\hat{R}^{-1}v(\theta)}
\end{align*}\]

where \( x \sim \mathcal{CN}_N(Sv(\theta_1), R), \hat{R} \propto \sum_{l=1}^{L} x(l)x^H(l), \) and \( x(l) \sim \mathcal{CN}_N(0, R) \)

- ML requires non-linear search ("beamsplitting")
  - Optimal asymptotically, but with threshold effect

**GOAL:** Predict MSE performance near threshold

Mean Squared Error (MSE):
\[E \left\{ \left( \hat{\theta}_{ML} - \theta_1 \right)^2 \right\}\]
Mean-Squared Error Performance: No Mismatch vs Mismatch

\[ \hat{\theta}_{ML} = \arg \max_{\theta} t_{ML}(\theta, \text{data}) \]
Mean-Squared Error Performance: No Mismatch vs Mismatch

No Mismatch

Array Element Positions

$\hat{\theta}_{ML} = \arg\max_\theta t_{ML}(\theta,\text{data})$

Ambiguity Function

Noise Free

Scan Angle

Mean Squared Error (dB)

Mean-Squared Error Performance:
No Mismatch vs Mismatch

Array Element Positions

$\theta_1$

No Information

Threshold

Cramér-Rao Bound

Asymptotic

SNR (dB)

$\text{SNR}_{TH}$

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Mean-Squared Error Performance: No Mismatch vs Mismatch

No Mismatch

Array Element Positions

\[ \hat{\theta}_{ML} = \arg \max_{\theta} t_{ML}(\theta, \text{data}) \]

Ambiguity Function

\[ \text{Scan Angle} \]

High SNR

Small Errors

Driven by Local Mainlobe Errors

Cramér-Rao Bound

Asymptotic

Threshold

No Information
Mean-Squared Error Performance: No Mismatch vs Mismatch

No Mismatch

Array Element Positions

θ₁

No Information

Driven by Global Ambiguity/Sidelobe Errors

Cramér-Rao Bound

Asymptotic

Threshold

SNRTH

Mean Squared Error (dB)

SNR (dB)

Low SNR

High SNR

Large Errors

Driven by Local Mainlobe Errors

Large Errors

Small Errors

θ₁

θML

θML = arg maxθ tML(θ, data)

Noise Free

Scan Angle

θ

Scan Angle

θ
Mean-Squared Error Performance: No Mismatch vs Mismatch

**No Mismatch**

Array Element Positions

$\theta_i$

**Signal Mismatch**

Array Element Positions

$\theta_i$

Assumed True

---

Mean Squared Error (dB)

SNR (dB)

Cramér-Rao Bound

Asymptotic

Threshold

No Information

$\text{SNR}_{TH}$
Mean-Squared Error Performance: No Mismatch vs Mismatch

No Mismatch

Array Element Positions

Signal Mismatch

Array Element Positions

Threshold

Asymptotic

Cramér-Rao Bound

SNR \( \text{TH} \)

Sidelobe Target

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Mean-Squared Error Performance: No Mismatch vs Mismatch

No Mismatch

Array Element Positions

Mismatch affects threshold and asymptotic region leading to atypical performance curves

Signal Mismatch

Array Element Positions

Assumed True

Mismatch affects threshold and asymptotic region leading to atypical performance curves
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Nonlinear Estimation and Ambiguity Functions

\[ \hat{\theta}_{ML} = \arg \max_{\theta} \frac{|v^H(\theta) \hat{R}^{-1} x|^2}{v^H(\theta) \hat{R}^{-1} v(\theta)} \]

- PDF aggregates density around maxima of ambiguity function (AF)

- Divide interval into M sub-intervals
  - “No Interval Error” (NIE)
  - “Intervals of Error” (IE)

- Use intervals to approximate MSE
  - Estimation approximately a M-ary hypothesis testing problem
Nonlinear Estimation and Ambiguity Functions

\[ \hat{\theta}_{ML} = \arg \max_{\theta} \frac{|v^H(\theta)\hat{R}^{-1}x|^2}{v^H(\theta)\hat{R}^{-1}v(\theta)} \]

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\[ |S|^2 \frac{|v^H(\theta)R^{-1}v(\theta_1)|^2}{v^H(\theta)R^{-1}v(\theta)} \]

Histgram of ML Estimates

\[ \theta \]

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Nonlinear Estimation and Ambiguity Functions

\[ \hat{\theta}_{ML} = \arg \max_\theta \frac{|v^H(\theta)\hat{R}^{-1}x|^2}{v^H(\theta)\hat{R}^{-1}v(\theta)} \]

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Approximating MSE Performance: Based on Method of Interval Errors (MIE)

• MSE given by

\[
E \left\{ \left( \hat{\theta} - \theta_1 \right)^2 \right\} \equiv \int (\omega - \theta_1)^2 p_\theta(\omega) d\omega
\]
Approximating MSE Performance: Based on Method of Interval Errors (MIE)

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\[ E \left\{ \left( \hat{\theta} - \theta_1 \right)^2 \right\} \equiv \int (\omega - \theta_1)^2 p_\hat{\theta}(\omega) d\omega \]

\[ \approx \int (\omega - \theta_1)^2 \times \text{IE} \times \text{NIE} \times \text{IE} \]
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- MSE given by:
  \[ E \left\{ \left( \hat{\theta} - \theta_1 \right)^2 \right\} \equiv \int (\omega - \theta_1)^2 \ p_\theta(\omega) d\omega \]

\[ \approx \int (\omega - \theta_1)^2 \times \int d\omega \]

**“Local Errors”**

\[ E \left\{ \left( \hat{\theta}_{ML} - \theta_1 \right)^2 \left| \theta_1 \right\} \right. \equiv \left[ 1 - \sum_{m=2}^{M} p(\hat{\theta}_{ML} = \theta_m | \theta_1) \right] \cdot \sigma_{ML}^2(\theta_1) + \sum_{m=2}^{M} p(\hat{\theta}_{ML} = \theta_m | \theta_1) \cdot (\theta_m - \theta_1)^2 \]
Approximating MSE Performance: Based on Method of Interval Errors (MIE)

- MSE given by
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- “Local Errors”
  \[ E\left\{ (\hat{\theta}_{ML} - \theta_1)^2 \left| \theta_1 \right. \right\} \approx \left[ 1 - \sum_{m=2}^{M} p(\hat{\theta}_{ML} = \theta_m \left| \theta_1 \right.) \right] \cdot \sigma^2_{ML}(\theta_1) + \sum_{m=2}^{M} p(\hat{\theta}_{ML} = \theta_m \left| \theta_1 \right.) \cdot (\theta_m - \theta_1)^2 \]

- “Global Errors”

- Challenge is calculation of error probabilities and asymptotic MSE:
  \[ p(\hat{\theta}_{ML} = \theta_m \left| \theta_1 \right.) = ? \quad \sigma^2_{ML}(\theta_1) = ? \]

Both are functions of the estimated covariance
Interval Error Probability for Adaptive ML Estimation

- Union bound suggests pairwise error approximation:

\[
p(\hat{\theta}_{ML} = \theta_m | E\{x\} = Sd) \simeq \Pr\left( \frac{|v^H(\theta_m)\hat{R}^{-1}x|^2}{v^H(\theta_m)\hat{R}^{-1}v(\theta_m)} > \frac{|v^H(\theta_1)\hat{R}^{-1}x|^2}{v^H(\theta_1)\hat{R}^{-1}v(\theta_1)} \bigg| E\{x\} = Sd \right)
\]

No Mismatch
\[v(\theta_1) = d\]

Array Response Mismatch
\[v(\theta_1) \neq d\]

- Asymptotic MSE approximated:
  - Adjustment of Cramér-Rao Bound (CRB) [No Mismatch]
  - Taylor series [Array Response Mismatch]

*Exact formulae derived in Richmond, IEEE T-IT, Vol. 52, No. 5, May 2006
Asymptotic ML MSE with Signal Mismatch

- Asymptotic MSE (jitter) given by

\[ \sigma_{ML}^2(\theta_1) = E\left\{ \left( \hat{\theta}_{ML} - \theta_1 \right)^2 \right\} \]
Asymptotic ML MSE with Signal Mismatch

- Asymptotic MSE (jitter) given by

\[
\sigma_{ML}^2(\theta_1) = E \left\{ (\hat{\theta}_{ML} - \theta_1)^2 \right\} = \left[ E \left\{ (\hat{\theta}_{ML} - \theta_1) \right\} \right]^2 + E \left\{ (\hat{\theta}_{ML} - \tilde{\theta})^2 \right\}
\]

where bias can be written as

\[
E \left\{ (\hat{\theta}_{ML} - \theta_1) \right\} = E \left\{ (\hat{\theta}_{ML} - \tilde{\theta}) \right\} + (\tilde{\theta} - \theta_1)
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\]

Goal is to determine the necessary expectations
ML Asymptotic MSE:
Taylor Series Based Approach (1/3)

- Define function $f(\theta, A, B) = \frac{v^H(\theta)ABA^H v(\theta)}{v^H(\theta)A v(\theta)}$ and note that

\[ \hat{\theta}_{ML} = \arg \max_{\theta} f(\theta, \hat{R}^{-1}, \hat{R}_T) \text{ where } \hat{R} \propto \sum_{l=1}^{L} x(l)x^H(l) \text{ and } \hat{R}_T = xx^H \]
ML Asymptotic MSE: Taylor Series Based Approach (1/3)

- Define function \( f(\theta, A, B) = \frac{v^H(\theta)ABA^Hv(\theta)}{v^H(\theta)Av(\theta)} \) and note that

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- Define partials \( \dot{f}(\theta, A, B) = \frac{\partial f(\theta, A, B)}{\partial \theta} \) and \( \ddot{f}(\theta, A, B) = \frac{\partial^2 f(\theta, A, B)}{\partial^2 \theta} \)
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- Taylor’s theorem allows the first order approximation:

\[
\dot{f}(\hat{\theta}_{ML}, \hat{R}^{-1}, \hat{R}_T) \simeq \dot{f}(\hat{\theta}, R^{-1}, R_T) + \ddot{f}(\hat{\theta}, R^{-1}, R_T)(\hat{\theta}_{ML} - \hat{\theta})
\]

\[
+ \tr \left[ \left( \frac{\partial \dot{f}}{\partial B} \right)_{B=R_T}^T \left( \hat{R}_T - R_T \right) \right] + \tr \left[ \left( \frac{\partial \dot{f}}{\partial A} \right)_{A=R^{-1}}^T \left( \hat{R}^{-1} - R^{-1} \right) \right] + \text{h.o.t.}
\]

where \( \hat{\theta} \) is the argument obtaining the peak value of \( f(\theta, R^{-1}, R_T) \)
ML Asymptotic MSE:
Taylor Series Based Approach (1/3)

- Define function \( f(\theta, A, B) = \frac{\nu^H(\theta)ABA^H\nu(\theta)}{\nu^H(\theta)A\nu(\theta)} \) and note that

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\hat{\theta}_{ML} = \arg \max_{\theta} f(\theta, \hat{R}^{-1}, \hat{R}_T) \quad \text{where} \quad \hat{R} \propto \sum_{l=1}^{L} x(l)x^H(l) \quad \text{and} \quad \hat{R}_T = xx^H
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\]

\[
+ \text{tr} \left[ \left( \frac{\partial \dot{f}}{\partial B} \right)^T_{B=R_T} \left( \hat{R}_T - R_T \right) \right] + \text{tr} \left[ \left( \frac{\partial \dot{f}}{\partial A} \right)^T_{A=R^{-1}} \left( \hat{R}^{-1} - R^{-1} \right) \right] + \text{h.o.t.}
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where \( \tilde{\theta} \) is the argument obtaining the peak value of \( f(\theta, R^{-1}, R_T) \)
ML Asymptotic MSE:
Taylor Series Based Approach (2/3)

- Since \( \hat{f}(\hat{\theta}_{ML}, \hat{R}^{-1}, \hat{R}_T) = 0 \) and \( \hat{f}(\tilde{\theta}, R^{-1}, R_T) = 0 \) the MSE can be expressed as

\[
\hat{\theta}_{ML} - \tilde{\theta} \approx \frac{1}{\hat{f}(\tilde{\theta}, R^{-1}, R_T)} \cdot \left( \text{tr} \left[ \left( \frac{\partial \hat{f}}{\partial B} \right)^T_{B=R_T} \Delta R_T \right] + \text{tr} \left[ \left( \frac{\partial \hat{f}}{\partial A} \right)^T_{A=R^{-1}} \Delta R^{-1} \right] \right)
\]

where \( \Delta R_T = \hat{R}_T - R_T \) and \( \Delta R^{-1} = \hat{R}^{-1} - R^{-1} \)
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where \( \Delta R_T = \hat{R}_T - R_T \) and \( \Delta R^{-1} = \hat{R}^{-1} - R^{-1} \)

- Note that \( E\left\{ (\hat{\theta}_{ML} - \tilde{\theta}) \right\} = 0 \) since \( E\{\Delta R_T\} = 0 \) and normalization of \( \hat{R}^{-1} \) can be chosen such that \( E\{\Delta R^{-1}\} = 0 \)
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where \( \Delta R_T = \hat{R}_T - R_T \) and \( \Delta R^{-1} = \hat{R}^{-1} - R^{-1} \)

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• Mismatch analysis addressed via choice of

\[
R_T = E\{xx^H\} = R + |S|^2dd^H \text{ such that } v(\theta_1) \neq d
\]
ML Asymptotic MSE: Taylor Series Based Approach (3/3)

- Squaring quadratic, and taking expectation one obtains the desired asymptotic MSE:

\[
E \left\{ (\hat{\theta}_{ML} - \tilde{\theta})^2 \right\} \simeq \frac{E \text{tr}^2 \left[ \left( \frac{\partial \hat{f}}{\partial B} \right)_{B=R_T}^T \Delta R_T \right]}{[\hat{f}(\tilde{\theta}, R^{-1}, R_T)]^2} + \frac{E \text{tr}^2 \left[ \left( \frac{\partial \hat{f}}{\partial A} \right)_{A=R^{-1}}^T \Delta R^{-1} \right]}{[\hat{f}(\tilde{\theta}, R^{-1}, R_T)]^2}
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*Richmond, Proceedings of IEEE SAM Workshop, 2006*
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E \left\{ \left( \hat{\theta}_{ML} - \tilde{\theta} \right)^2 \right\} \approx \frac{E \text{ tr}^2 \left[ \left( \frac{\partial \hat{f}}{\partial B} \right)^T_{B=R_T} \Delta R_T \right]}{[\hat{f}(\tilde{\theta}, R^{-1}, R_T)]^2} + \frac{E \text{ tr}^2 \left[ \left( \frac{\partial \hat{f}}{\partial A} \right)^T_{A=R^{-1}} \Delta R^{-1} \right]}{[\hat{f}(\tilde{\theta}, R^{-1}, R_T)]^2}
\]

\[
\approx \text{MSE R known*} + \text{Accuracy Loss due to Unknown R}
\]

• MSE is that obtained assuming \( R^{-1} \) is known plus an additional term accounting for accuracy loss due to estimation of \( R^{-1} \)
ML Asymptotic MSE: Taylor Series Based Approach (3/3)

• Squaring quadratic, and taking expectation one obtains the desired asymptotic MSE:

\[
E \left\{ \left( \hat{\theta}_{ML} - \tilde{\theta} \right)^2 \right\} \approx E \operatorname{tr}^2 \left[ \left( \frac{\partial \hat{f}}{\partial B} \right)^T_{B=R_T} \Delta R_T \right] + E \operatorname{tr}^2 \left[ \left( \frac{\partial \hat{f}}{\partial A} \right)^T_{A=R^{-1}} \Delta R^{-1} \right]
\]

\[
e \left( \hat{\theta}_{ML} - \tilde{\theta} \right)^2 \approx \text{MSE} \quad \text{R known}^* + \text{Accuracy Loss due to Unknown R}
\]

• MSE is that obtained assuming \( R^{-1} \) is known plus an additional term accounting for accuracy loss due to estimation of \( R^{-1} \)

• Moments of complex Gaussian process, and moments of inverted Wishart complete analysis

*Richmond, Proceedings of IEEE SAM Workshop, 2006
Outline

• Introduction
• Mean-squared error prediction
  ✔ • Numerical examples
• Summary
Broadside Planewave Signal in White Noise: No Mismatch, ULA

- $N=18$ element uniform linear array (ULA), $(\lambda/2.25)$ element spacing
  - $3\text{dB}$ Beamwidth $\approx 7.2$ degs, search space $[60\ 120]$ degs
  - $0\text{dB}$ white noise, True Signal @ 90 degs (broadside)
- Asymptotic ML MSE agrees with CRB above threshold SNR
- MIE MSE predictions very accurate above and below threshold
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Broadside Planewave Signal in White Noise: No Mismatch, ULA, $L=1.5N$

- $N=18$ element uniform linear array (ULA), $(\lambda/2.25)$ element spacing
  - 3dB Beamwidth $\approx 7.2$ degs, search space $[60 \ 120]$ degs
  - 0dB white noise, True Signal @ 90 degs (broadside)
- Asymptotic ML MSE agrees with CRB above threshold SNR
- MIE MSE predictions very accurate above and below threshold
**Broadside Planewave Signal in White Noise: No Mismatch, ULA, $L=2N$**

- $N=18$ element uniform linear array (ULA), $(\lambda/2.25)$ element spacing
  - 3dB Beamwidth $\approx 7.2$ degs, search space $[60, 120]$ degs
  - 0dB white noise, True Signal @ 90 degs (broadside)

- Asymptotic ML MSE agrees with CRB above threshold SNR
- MIE MSE predictions very accurate above and below threshold
Off Broadside Planewave Signal in White Noise: Perturbed ULA, $L=2N$

- **$N=18$ element ULA positions perturbed by 3-D Gaussian noise**
  - **Zero mean with stand. dev.** $0.04\lambda$
  - **Array perturbation from single realization of Gaussian noise process**
- CRB no longer useful, while MIE accurate above and below threshold
- Mismatch can increase threshold SNR and introduce asymptotic bias

**Equation:**

$$z_n = N_3 \left( 0, I_3 \sigma_{RMS}^2 \right)$$

$$\sigma_{RMS} = 0.04\lambda$$
Broadside Planewave Signal in White Noise: Perturbed ULA, $L=2N$

- $N=18$ element ULA positions perturbed by 3-D Gaussian noise
  - Zero mean with stand. dev. $0.1\lambda$
  - Array perturbation from single realization of Gaussian noise process
- CRB no longer useful, while MIE accurate above and below threshold
- Mismatch can increase threshold SNR and introduce asymptotic bias
• $N=18$ element uniform linear array (ULA), $(\lambda/2.25)$ element spacing
  – *Sidelobe target 2dB above noise @ 75degs*
• SINR loss and threshold region results from competing targets
• Asymptotic MSE $\propto 1/$SNR due to signal living on ML manifold
Signal in White Noise: Perturbed ULA + 2dB Sidelobe Target (SLT), $L=2N$

- $N=18$ element ULA positions perturbed by 3-D Gaussian noise
  - Zero mean with stand. dev. $0.1\lambda$
  - Array perturbation from single realization of Gaussian noise process
  - Sidelobe target 2dB above noise @ 75degs
- CRB no longer useful, while MIE accurate above and below threshold
- Mismatch affects both threshold region and asymptotic performance
Outline

• Introduction
• Mean-squared error prediction
• Numerical examples

✓ • Summary
Summary

• Developed theory to predict impact of signal mismatch on ML estimation accuracy
  – arbitrary deterministic signal mismatch allowed
  – above and below threshold SNR performance considered
  – finite sample effects due to noise covariance estimation included

• Generalizes previous work on ML threshold region performance and provides tools useful for system design and analysis

• Circumvents need for time consuming Monte Carlo simulation

• Examples show estimation loss introduced by uncertainties in $R$ and $\nu$ can be significant and should be considered in design
Backups
Abstract

Asymptotic Mean Squared Error Performance of Maximum-Likelihood DOA Estimation with Estimated Noise Covariance

Christ D. Richmond*
Senior Member, IEEE, christ@ll.mit.edu

The mean squared error (MSE) performance prediction of Maximum-Likelihood (ML) Direction-Of-Arrival (DOA) angle estimation has been studied extensively by several authors (see refs in [1]). Most recently ML DOA performance prediction of both the threshold and asymptotic regions in the presence of a general form of deterministic array response mismatch was considered assuming the noise-plus-interference covariance matrix (NICM) was known [1,2]. The error probabilities required for predicting the threshold region MSE of the general case of deterministic mismatch are derived in [3] including the unknown NICM case. Approximations of the asymptotic MSE performance were explored in [2] based on a stochastic representation of the ML filter weight vector, yielding only moderate success due to the approximate representation of the functional dependence of some parameters. Herein an exact general expression for the asymptotic DOA MSE performance for the unknown NICM case is derived based on a Taylor Series expansion that accounts for the exact functional dependence of all parameters. It is shown herein that the MSE expression derived in [1] is augmented by an additive term that accounts for the loss due exclusively to NICM estimation.


ML Asymptotic MSE:
Approximate Taylor Series Based Approach*

- Following Taylor Series approach it can be shown that

\[ \hat{\theta}_{ML} - \theta_G \approx -\frac{2/K}{f(\theta_G,R_T,0,1/K)} \left\{ \text{Re} \left[ \hat{w}^H(\theta_G) \Delta R_T \dot{w}(\theta_G) \right] + \right. \]

Known \( R \)

\[ \text{Re} \left[ \hat{w}^H(\theta_G) R_T t \right] + \text{Re} \left[ \hat{w}^H(\theta_G) R_T \dot{w}(\theta_G) \right] (\chi - 1/K) \}

Unknown \( R \)

MSE obtained from Mean and Variance of \( \hat{\theta}_{ML} - \theta_G \)

- Quadratic function defined

\[ f(\theta, R_T, t, \chi) = \chi \left[ w(\theta) + t \right]^H \hat{R}_T \left[ w(\theta) + t \right] \]

where

\[ w(\theta) = \frac{R^{-1} v(\theta)}{\sqrt{v^H(\theta) R^{-1} v(\theta)}}, \quad \hat{w}(\theta) = \frac{\hat{R}^{-1} v(\theta)}{\sqrt{v^H(\theta) \hat{R}^{-1} v(\theta)}} = \chi \left[ w(\theta) + t \right] \]

\[ \hat{R}_T = \bar{x} \bar{x}^H, \quad E\left\{ \bar{x} \bar{x}^H \right\} = R + \left| S \right|^2 dd^H, \quad \hat{R} = \frac{1}{L} \bar{x} \bar{x}^H \]

\( \chi \) and \( t \) are random quantities* modeling errors in \( w(\theta) \) due to unknown \( R \)

*Richmond, Proceedings of ASAP 2006