A Combinatorial Optimization Problem in Wireless Communications and Its Analysis

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The Problem

Let

\[ E := \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger J x \]

with \( x \in \mathbb{C}^K \) and \( J \in \mathbb{C}^{K \times K} \).
Introduction

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Example 1 (sphere):

\[ \mathcal{X} = \{ x : x^\dagger x = K \} \quad \implies \quad E = \min \lambda(J) \]
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Example 2 (cube):
\[ \mathcal{X} = \{+1, -1\}^K \quad \implies \quad ??? \]
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\[ \mathcal{X} = \{ +1, -1 \}^K \quad \Rightarrow \quad ??? \]

Example 3 (vector precoding):
\[ \mathcal{X} = (4\mathbb{Z} + 1)^K \quad \Rightarrow \quad ??? \]
The Gaussian Vector Channel

Let the received vector be given by

\[ y = Ht + n \]

where

- \( t \) is the transmitted vector
- \( n \) is uncorrelated (white) Gaussian noise
- \( H \) is a coupling matrix accounting for crosstalk

In many applications, e.g. antenna arrays, code-division multiple-access, the coupling matrix is modelled as a random matrix with independent identically distributed entries (i.i.d. model).

Crosstalk can be processed either at receiver or transmitter
A combinatorial optimization problem in wireless communications ...
Processing at Transmitter

If the transmitter is a base-station and the receiver is a hand-held device one would prefer to have the complexity at the transmitter.

E.g. let the transmitted vector be

$$t = H^\dagger (H H^\dagger)^{-1} x$$

where $x = s$ is the data to be sent.

Then,

$$y = s + n.$$  

No crosstalk anymore due to channel inversion.
Problems of Simple Channel Inversion

Channel inversion implies a significant power amplification, i.e.

\[ x^\dagger (HH^\dagger)^{-1} x > x^\dagger x. \]

In particular, let

- \( \alpha = \frac{K}{N} \leq 1; \)
- the entries of \( H \) are i.i.d. with variance \( 1/N \).

Then, for fixed aspect ratio \( \alpha \)

\[ \lim_{K \to \infty} \frac{x^\dagger (HH^\dagger)^{-1} x}{x^\dagger x} = \frac{1}{1 - \alpha} \]

with probability 1.
Lattice-Relaxation Precoding

Tomlinson '71, Harashima & Miyakawa '72
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Lattice-Relaxation Precoding

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Instead of representing the logical "0" by +1, we present it by any element of the set \{\ldots, -7, -3, +1, +5, \ldots\} = 4\mathbb{Z} + 1. Correspondingly, the logical "1" is represented by any element of the set 4\mathbb{Z} - 1.

Choose that representation that gives the smallest transmit power.
General Relaxation Precoding

Let $\mathcal{B}_0$ and $\mathcal{B}_1$ denote the sets presenting 0 and 1, resp.

Let $(s_1, s_2, s_3, \ldots, s_K) \in \{0, 1\}^K$ denote the data to be transmitted.

Then, the transmitted energy per data symbol is given by

$$E = \frac{1}{K} \min_{\mathbf{x} \in \mathcal{X}} \mathbf{x}^\dagger \mathbf{J} \mathbf{x}$$

with

$$\mathcal{X} = \mathcal{B}_{s_1} \times \mathcal{B}_{s_2} \times \cdots \times \mathcal{B}_{s_K}$$

and

$$\mathbf{J} = (\mathbf{H} \mathbf{H}^\dagger)^{-1}.$$
Zero Temperature Formulation

Quadratic programming is the problem of finding the zero temperature limit of a quadratic energy potential.

The transmitted power is written as a zero temperature limit

\[
E = - \lim_{\beta \to \infty} \frac{1}{\beta K} \log \sum_{\mathbf{x} \in \mathcal{X}} e^{-\beta \text{tr}(\mathbf{x}^\dagger \mathbf{J} \mathbf{x})}
\]

with \(\frac{1}{\beta}\) denoting temperature.
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\]

\[
\longrightarrow - \lim_{\beta \to \infty} \lim_{K \to \infty} \frac{1}{\beta K} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(J x x^\dagger)}
\]

with \(\frac{1}{\beta}\) denoting temperature.
The Harish-Chandra Integral
(also called the Itzykson-Zuber integral)

Let $\mathbf{P}$ be any positive semi-definite matrix of bounded rank $n$ and let $\mathbf{J}$ be bi-unitarily invariant. Then,

$$
\lim_{K \to \infty} \frac{1}{K} \log \mathbb{E} e^{-K \text{tr} \, \mathbf{JP}} = - \sum_{a=1}^{n} \lambda_a(\mathbf{P}) \int_{0}^{\infty} R_{\mathbf{J}}(-w) \, dw
$$

with $\lambda_a$ denoting the positive eigenvalues of $\mathbf{P}$ and $R_{\mathbf{J}}(w)$ denoting the R-transform of the spectral measure of $\mathbf{J}$ (Marinari et al. '94; Guionnet & Maïda '05).
The Replica Method

We want

$$\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \left( \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} \right)^n$$
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\]

\[
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E}_J \prod_{a=1}^n \sum_{x_a \in \mathcal{X}} e^{-\beta \text{tr}(Jx_a x_a^\dagger)}
\]
The Replica Method

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$$\lim_{K \to \infty} \frac{1}{K} \mathbf{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(\mathbf{J}x x^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbf{E} \left( \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(\mathbf{J}x x^\dagger)} \right)$$

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$$= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbf{E} \sum_{x_1 \in \mathcal{X}} \cdots \sum_{x_n \in \mathcal{X}} e^{-\text{tr} \left( \mathbf{J} \beta \sum_{a=1}^{n} x_a x_a^\dagger \right)}$$
The Replica Method

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$$= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbf{E} \sum_{x_1 \in \mathcal{X}} \cdots \sum_{x_n \in \mathcal{X}} e^{-\text{tr} \left( J \beta \sum_{a=1}^n x_a x_a^\dagger \right)}$$

$$= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbf{E} \exp \left[ -K \sum_{a=1}^n \int_0^{\beta \lambda_a(Q)} R_{J}(-w) dw \right]$$

with

$$Q_{ab} := \frac{1}{K} x_a^\dagger x_b.$$
Laplace Integration

We find

$$\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in X} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \exp \left[ -K \sum_{a=1}^{n} \beta \lambda_a(Q) \right] \left[ -K \sum_{a=1}^{n} \int_{0}^{\infty} R_J(-w)dw \right]$$
Laplace Integration

We find

\[
\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \exp \left[ -K \sum_{a=1}^{n} \beta \lambda_a(Q) \int_0^R R_J(-w)dw \right]
\]

\[
= \min_{Q: \text{Pr}(Q) > 0} \lim_{n \to 0} \frac{1}{n} \log \left[ -\sum_{a=1}^{n} \int_0^R R_J(-w)dw \right]
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Laplace Integration

We find

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\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \exp \left[ -K \sum_{a=1}^{n} \beta \lambda_a(Q) \int_{0}^{\mathcal{R}} R_J(-w)dw \right] \\
= \min_{Q: \Pr(Q) > 0} \lim_{n \to 0} \frac{1}{n} \log \left[ -\sum_{a=1}^{n} \int_{0}^{\mathcal{R}} R_J(-w)dw \right] \\
\sim \min_{Q: \Pr(Q) > 0} \lim_{n \to 0} \frac{1}{n} \text{tr} \left[ QR_J(-\beta Q) \right].
\]

How to optimize over \( Q \)?
Replica Symmetry

Replica Symmetric (RS) Ansatz

We assume a certain structure for a matrix $Q$. The easiest try is

$$Q := \begin{bmatrix}
q + \frac{\chi}{\beta} & q & \cdots & q & q \\
q & q + \frac{\chi}{\beta} & \cdots & q & q \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
q & q & \cdots & q + \frac{\chi}{\beta} & q \\
q & q & \cdots & q & q + \frac{\chi}{\beta}
\end{bmatrix}$$

with some parameters $q$ and $\chi$.

This is a critical step. Sometimes, the structure of $Q$ is more complicated.
RS Solution

Let $P(s)$ denote the limit of the empirical distribution of the information symbols $s_1, s_2, \ldots, s_K$ as $K \to \infty$. Let $q$ and $\chi$ be the simultaneous solutions to

\[
q = \int \int \arg\min_{x \in B_s} \left| z \sqrt{2q R'(-\chi) - 2x R(-\chi)} \right| Dz \, dP(s)
\]

\[
\chi = \frac{1}{\sqrt{2q R'(-\chi)}} \int \int \arg\min_{x \in B_s} \left| z \sqrt{2q R'(-\chi) - 2x R(-\chi)} \right| z^* Dz \, dP(s)
\]

where $Dz = \exp(-z^2/2)dz/\sqrt{2\pi}$, $R(\cdot)$ is the R-transform of the limiting eigenvalue spectrum of $J$, and $0 < \chi < \infty$.

Then, replica symmetry (RS) implies

\[
\frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger J x \to q \frac{\partial}{\partial \chi} \chi R(-\chi)
\]

as $K \to \infty$. 
Some R-Transforms

\[ \mathbf{I} : \quad R(w) = 1 \]

\[ \mathbf{H H}^\dagger : \quad R(w) = \frac{1}{1 - \alpha w} \quad \text{Marchenko-Pastur (MP) law} \]

\[ (\mathbf{H H}^\dagger)^{-1} : \quad R(w) = \frac{1 - \alpha - \sqrt{(1 - \alpha)^2 - 4\alpha w}}{2\alpha w} \quad \text{inv. MP} \]
Odd Integer Quadrature Lattice
Complex Lattice Precoding

\[ E [\text{dB}] = \frac{10}{\pi} (1 - \alpha)^{\frac{1}{\sqrt{\pi}} - 1} \]

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A combinatorial optimization problem in wireless communications ...
1-Step Replica Symmetry Breaking

\[ Q := \begin{bmatrix}
q + p + \frac{\chi}{\beta} & q + p & q & q & \cdots & q & q \\
q + p & q + p + \frac{\chi}{\beta} & q & q & \cdots & q & q \\
q & q & q + p + \frac{\chi}{\beta} & q + p & \cdots & q & q \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
q & q & q & \cdots & q & q + p + \frac{\chi}{\beta} & q + p \\
q & q & q & \cdots & q & q + p & q + p + \frac{\chi}{\beta}
\end{bmatrix} \]

with the macroscopic parameters \( q, p \) and \( \chi \) and the blocksize \( \frac{\mu}{\beta} \).
1-Step Replica Symmetry Breaking

\[ E = \left( q + p + \frac{\chi}{\mu} \right) R(-\chi - \mu p) - \frac{\chi}{\mu} R(-\chi) - q(\mu p + \chi) R'(-\chi - \mu p) \]

The macroscopic parameters \( q, p, \chi \) and \( \mu \) are given by 4 coupled non-linear equations.

Solving those equations numerically is a tedious and tricky task.
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Complex Convex Relaxation

... allows for convex programming (and is replica symmetric).
A combinatorial optimization problem in wireless communications ...
What happens if the MP-law has a mass point at zero ($K > N$)?

Can we precode without interference?
**Inverting Singular Channels**

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Can we precode without interference?

The precoder produces

\[
\lim_{\epsilon \to 0} \arg\min_{x \in \mathcal{X}} \frac{x^\dagger (HH^\dagger + \epsilon I)^{-1} x}{K}
\]

The received signal becomes

\[
y = \lim_{\epsilon \to 0} HH^\dagger(HH^\dagger + \epsilon I)^{-1} x + n.
\]
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\mathbf{y} = \lim_{\epsilon \to 0} \mathbf{H} \mathbf{H}^\dagger (\mathbf{H} \mathbf{H}^\dagger + \epsilon \mathbf{I})^{-1} \mathbf{x} + \mathbf{n}.
\]

If the energy is finite, there is no interference.
Overloaded Convex Precoding

The probability that a random $N$ dimensional subspace in $K$ real dimensions intersects the 1. $K$-tant is

$$P(K, N) = 2^{1-K} \sum_{\ell=0}^{N-1} \left( \begin{array}{c} K - 1 \\ \ell \end{array} \right)$$
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As $K, N$ to infinity, we get

$$P(K, N) = \begin{cases} 
1 & K < 2N \\
1/2 & K = 2N \\
0 & K > 2N 
\end{cases}$$
Overloaded Convex Precoding

The probability that a random $N$ dimensional subspace in $K$ complex dimensions intersects the 1. $K$-tant is

$$P(K, N) = 2^{1-2K} \sum_{\ell=0}^{2N-1} \binom{2K-1}{\ell}$$

As $K, N$ to infinity, we get

$$P(K, N) = \begin{cases} 1 & K < 2N \\ 1/2 & K = 2N \\ 0 & K > 2N \end{cases}$$
Overloaded Convex Precoding

A combinatorial optimization problem in wireless communications ...
Wanted

\[
\lim_{K \to \infty} \frac{1}{K} \log \mathbb{E}_{A,B} e^{-K \text{tr} A P B P} = f \{ R_A(\cdot), R_B(\cdot), \ldots \}
\]

\ldots \text{ or other more complicated exponents.}
Negative Entropy

\[ S = \chi R(-\chi) - \int_{0}^{\chi} R(-w)dw \]

The closer the entropy is to zero, the better the RSB approximation.