Statistics of Wave Fields in Complex Enclosures in the Frequency and Time Domains

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Motivation: Electromagnetic Coupling to Electronic Circuits in Enclosures

Schematic

- Coupling of external radiation to computer circuits is a complex process:
  - apertures
  - resonant cavities
  - transmission lines
  - circuit elements

- Intermediate frequency range involves many interacting resonances

- System size >> Wavelength

- Chaotic Ray Trajectories
- “Wave Chaos”

• What can be said about coupling without solving in detail the complicated EM problem?
• Statistical Description!
What is Wave Chaos?

- Wave propagation - linear phenomena (response is linearly proportional to excitation)
- Therefore - not chaotic
- In complex geometries field distribution are highly sensitive to: frequency and/or small perturbations
- Classical rays are chaotic - this affects the field solutions

Two incident rays with slightly different initial directions have rapidly diverging trajectories
Random Matrix Theory and Wave Mechanics

- **Nuclear Spectra**
  Motivated by the then impossibility of calculating spectra of large nuclei, Wigner (1950’s) conjectured that these spectra have statistical properties that are the same as those of suitable ensembles of random matrices.

- **Quantum Dots & Disordered Systems**

- **Acoustic Resonators**

- **Electromagnetic Compatibility**

- **Quantum Chaos**
Eigenfrequency Statistics

\[ \hat{H} \cdot \psi_n = E_n \psi_n \]

Eigenvalues \( E_n \) are distributed particular statistics

Mean Spacing:
\[ \Delta E = \left\langle E_{n+1} - E_n \right\rangle \]

Normalized Spacing:
\[ s_n = \frac{\left\langle E_{n+1} - E_n \right\rangle}{\Delta E} \]

Spacing distributions are characteristic for many systems

Chaos A = Time reversal symmetry
Chaos B = Time reversal symmetry broken

Chaos A

Chaos B

Thorium

Harmonic Oscillator

Integrable

Tomovic (1996)
Hypothesis: Random matrix theory (RMT) applies to wave problems in the semiclassical regime (short wavelength), if the ray approximation corresponding to the given wave problem yields chaotic wave trajectories [McDonald and Kaufman, PRL (1979); Bohigas, et al., PRL (1984)].

Replace eigenfunction with superposition of random plane waves (Berry Hypothesis)

\[
\phi_n = \lim_{N \to \infty} \text{Re} \left\{ \sqrt{\frac{2}{AN}} \sum_{k=1}^{N} a_k \exp[i(k_n e_k \cdot x + \theta_k)] \right\}
\]

Note: Hypothesis now assumed to apply even for simple systems.
Our Problem - Scattering

Q: What is the nature of the interaction of the wave chaotic system with the outside world via the connecting ports?

A difficulty: While the waves within the chaotic system are presumably described by random matrix theory in a ‘universal’ (system independent) manner, the answer to Q. also depends on the non-universal aspects of the specific geometry of the coupling between the ports and the chaotic system.
Universal and System Specific Aspects

1. Entry of energy to cavity determined by properties of port.
   - System specific
   - Applies also to exit

2. Wave fields inside described by RMT and Berry Hypothesis
   - Universal statistics
   - Some system specific dependence

Note, interference is still accounted for.
Black Box Representation
N - Port System

Schematic

Incoming RF Power

Device Terminal

N- Port System
Z and S-Matrices

What is $S_{ij}$?

N ports
- voltages and currents,
- incoming and outgoing waves

$Z$ matrix
\[
\begin{pmatrix}
V_1 \\
V_2 \\
\vdots \\
V_N
\end{pmatrix}
= Z
\begin{pmatrix}
I_1 \\
I_2 \\
\vdots \\
I_N
\end{pmatrix}
\]
voltage current

$S$ matrix
\[
\begin{pmatrix}
V_1^- \\
V_2^- \\
\vdots \\
V_N^-
\end{pmatrix}
= S
\begin{pmatrix}
V_1^+ \\
V_2^+ \\
\vdots \\
V_N^+
\end{pmatrix}
\]
outgoing incoming

S matrix
\[
S = (Z + Z_0)^{-1}(Z - Z_0)
\]
$Z(\omega), S(\omega)$

- Complicated function of frequency
- Details depend sensitively on unknown parameters
Frequency Dependence of Reactance for a Single Realization

Lossless:

\[ V = Z_{\text{cav}} I \]
\[ Z_{\text{cav}} = jX_{\text{cav}} \]

\[ S = \frac{Z_{\text{cav}} - Z_0}{Z_{\text{cav}} + Z_0} = e^{j\phi} \]

Mean spacing \( \delta f \approx 0.016 \) GHz
Frequency Dependence of Reactance for a Single Realization

Lossless:

\[ V = Z_{cav} I \]
\[ Z_{cav} = jX_{cav} \]

Mean spacing \( \delta f \approx .016 \text{ GHz} \)

Reflection coefficient:

\[ S = \frac{Z_{cav} - Z_0}{Z_{cav} + Z_0} = e^{j\phi} \]
Statistical Model of Z Matrix

Statistical Model Impedance

\[ Z_{ij}(\omega) = - \frac{j}{\pi} \sum_n R_{Ri}^{1/2}(\omega_n) R_{Rj}^{1/2}(\omega_n) \frac{\Delta\omega_n^2 w_{in} w_{in}}{\omega^2(1 + jQ^{-1}) - \omega_n^2} \]

System parameters
- Radiation Resistance \( R_{Ri}(\omega) \)
- \( \Delta\omega_n^2 \) - mean spectral spacing
- \( Q \) - quality factor

Statistical parameters
- \( \omega_n \) - random spectrum
- \( w_{in} \) - Guassian Random variables

Free-space radiation resistance
- \( R_R(\omega) \)
- \( Z_R(\omega) = R_R(\omega) + jX_R(\omega) \)
Compact Expression for Impedance Matrix

Statistical Model for Cavity Impedance

\[ Z_{cav} = i \text{Im}(Z_{rad}) + \left[ R_{rad} \right]^{1/2} \cdot \xi \cdot \left[ R_{rad} \right]^{1/2} \]

Radiation Impedance - Ports

\[ Z_{rad} = i \text{Im}(Z_{rad}) + R_{rad} \]

Statistics generated by RMT

\[ \xi = \frac{i}{\pi} \sum_n \frac{\Delta k^2 w_n \tilde{w}_n}{k_n^2(1 + i/Q) - k_n^2} \]

Lorentzian Random Matrix (\( \alpha = k_o^2/(Q\Delta k^2) \))
Two Approaches

• $Z(\omega)$ construction - Microscopic approach
  Uses pieces of problem: $R_R$, $\Delta \omega^2$, $Q$ to construct $Z$ and $S$.

• Poisson Kernel - Global Approach to $S$
  (P. A. Mello, P. Pereyra, and A. Frankel, Ann of Phys. 161, 251 (1985))
  Statistics of $S$ - Matrix determined by its average $E\{S\}$
    - $P(S| E\{S\})$ includes details of coupling


Some Relevant References

Expect: \[ Z = j(X_R + R_R \xi) \]
\[ \xi - \text{unit Lorentzian} \]

Mello, “Mesoscopic Quantum Physics” 1995

Moveable conducting disk - 0.6 cm diameter
“Proverbial soda can”

Cavity impedance calculated for
100 locations of disk
4000 frequencies
6.75 GHz to 8.75 GHz
Frequency Dependence of Reactance for a Single Realization

Mean spacing $\delta f \approx 0.016$ GHz

$Z_{cav} = jX_{cav}$
Distribution of Fluctuating Cavity Impedance

\[ \xi = \frac{(X - X_R(\omega))}{R_R(\omega)} \]

\[ Z = j(X_R + R_R\xi) \]

\[ R_R \approx 35 \, \Omega \]

Graph showing PDF of \( \xi \) with different frequency bands:
- 6.75-7.25 GHz
- 7.25-7.75 GHz
- 7.75-8.25 GHz
- 8.25-8.75 GHz
Frequency Dependence of Median Cavity Reactance

Median Impedance for 100 locations of disc

Effect of strong Reflections?

Radiation Reactance
HFSS with perfectly absorbing Boundary conditions

Δf = .3 GHz, L = 100 cm
Comparison of HFSS Results and Model for Pdf’s of Normalized Impedance

Normalized Reactance

Normalized Resistance

\[ Z_{cav} = jX_R + (\rho + j\xi) R_R \]
EXPERIMENTAL Test

S. Hemmady et al., PRL 94, 014102 (2005)

- 2 Dimensional Quarter Bow Tie cavity
- 1-port S and Z measurements in the 6 – 12 GHz range.
- Ensemble average through 100 locations of the perturbation
Importance of Normalization

\[ Z = j(X_R + R_R \xi) \]

\[ \xi = \frac{(X - X_R)}{R_R} \]
Testing the Effects of Varying Loss

\[ Z = jX_R + R_R(\rho + j\xi) \]

\[ \alpha = \frac{\omega}{2Q\Delta \omega} \]

\[ P_\rho(\rho, \alpha) \]
\[ \alpha = 7.6 \]
\[ \alpha = 4.2 \]
\[ \alpha = 0.8 \]

\[ P_\xi(\xi, \alpha) \]
\[ \alpha = 7.6 \]
\[ \alpha = 4.2 \]
\[ \alpha = 0.8 \]
Experimental Setup [Cavity Case]

- Frequency Range: 2GHz to 20 GHz (\(\lambda \ll L\))
- Ensemble Averaging over \(~20\) positions of the mode-stirrer.

Port Radiation Measurement Setup

Microwave absorber

Paddle-Wheel Mode-Stirrer

Port 1

Port 2
PDF of voltages on port 2 for different power-spectral densities radiated at Port 1

- Flat PSD radiated from port 1
- Gaussian-shaped PSD radiated from port 1
Unexpected Frequency Dependence of Median Cavity Reactance

**Calculated:** Median Reactance for 100 locations of disc

**Expected:** Radiation Reactance

HFSS with perfectly absorbing Boundary conditions

\[ \Delta f_{\text{res}} = 0.016 \text{ GHz} \]

\[ \Delta f = 0.3 \text{ GHz, } L = 100 \text{ cm} \]
Effect of Direct Ray Paths

Original Random Coupling Model (RCM)
- RF energy is randomized on entering cavity
- Only radiation impedance of ports, cavity volume and average Q are important

In some geometries, or in narrow frequency bands specifics of internal geometry are important

Modified Random Coupling Model
- J. Hart et al., PHYSICAL REVIEW E 80, 041109 (2009)
- Allows for systematic improvement by inclusion of geometric details if known
- Can be used in conjunction with measured data

Using the Random Coupling Model, extended to include the effect of direct paths, we can now derive these two fading distributions.
Effect of Short Ray Paths
Hart et al. PRE 80, 041109 (2009).

\[ Z = j(X_{R,N}+R_{R,N}\xi) \]

Where:
\[ Z_{R,N} = R_{R,N} + jX_{R,N} \]

Radiation impedance including the effect of N-bounces off the wall calculated in the semi-classical approximation.

Semi-classical Green’s function in 2D

\[
G(r,r') = \frac{1}{4} \left[ H_0^{(1)}(k|r-r'|) + \sum_{\text{paths}-j} i \left( \frac{2}{\pi} \left| \frac{\partial^2 L_j(r,r')}{\partial r \partial r'} \right| \right)^{1/2} \exp\left[ ikL_j(r,r') + i\pi(n - 1/4) \right] \right]
\]
Modified RCM

J. Hart et al., PHYSICAL REVIEW E 80, 041109 (2009)

Original RCM:  \[ Z_{\text{cav}} = j \text{Im}(Z^{\text{rad}}) + \left[ R^{\text{rad}} \right]^{1/2} \cdot \xi \cdot \left[ R^{\text{rad}} \right]^{1/2} \]

Modified RCM:  \[ Z_{\text{cav}} = j \text{Im}(Z^{\text{ave}}) + \left[ R^{\text{ave}} \right]^{1/2} \cdot \xi \cdot \left[ R^{\text{ave}} \right]^{1/2} \]

Here \( Z^{\text{ave}} \) is the cavity impedance averaged over a prescribed frequency band
- can be measured
- can be calculated if enough geometry is known

Calculated in geometric optics limit:  \[ Z^{\text{ave}} = j \text{Im}(Z^{\text{rad}}) + \left[ R^{\text{rad}} \right]^{1/2} \cdot \xi \cdot \left[ R^{\text{rad}} \right]^{1/2} \]

\[ \left( \xi \right)_{mn} = \sum_{\text{ray paths } \ell b} p_{b,mn} C_{b,mn} \exp[-jS_{b,mn}(\omega) - j \pi / 4] \]

\( p_{b,mn} \) - attenuation,  \( C_{b,mn} \) - ray defocusing,  \( S_{b,mn} \) ray phase change=\( kL_{b,mn} \)
Numerical Tests (HFSS) on Bow Tie Cavity

95 realizations generated by moving a circular perturber

Average (over realizations) impedance seen at port #1

![Graph showing frequency vs. median reactance with data points and error bars.](image)
Multiple Bounces in the Bow Tie
Prediction of Average Resistance

Theory includes contributions from $N = 2, 4, 6$ bounces.
\[ z_{\text{correction}} = \frac{Z - Z_{\text{Rad}}}{R_{\text{Rad}}} \]
Measurement of Z11

Single Realization

- **Measured Data**
- **Theory, 584 trajectories**
- **Radiation Impedance**
Frequency Smooth Data
Window = 240 MHz

Single Realization
Configuration Averaging
100 Realizations

Contributions to theory from trajectories blocked by perturber are dropped.
Question: What are the characteristics of the time decaying pulse emerging from the cavity?

- Incident Pulse
- Complex cavity
- Reflected Pulse
- “Coda”
Output signal is a superposition of modes

\[ V(t) = \sum_n V_n \exp\left[j(\omega_n + \gamma_n)t\right] \]

The following quantities have random distributions, determined by Random Matrix Theory (RMT)

\[ V_n, \quad \omega_n, \quad \gamma_n \]
1. Ensemble average decays with power law (already known)

2. Time average of an individual realization follows ensemble average for some time, and then decays exponentially

3. Individual realizations oscillate about their time average.

4. Transition from algebraic to exponential decay obeys universal distribution (parameters: pulse bandwidth, cavity mode spacing, port coupling strength)
Universal Transition for Gaussian Pulses
J. Hart et al. PRE (2009)

Ensemble Average
\[ \bar{P}(t) \sim t^{-5/2} \]

\( C(\alpha, \tau) \) = fraction of realizations with
\[ P(t) < \alpha \bar{P}(t) \]

Normalized time
\[ \tau = \frac{t \Delta \omega^3}{\omega_B^2 P_0} \]

\( \Delta \omega \) - mode spacing
\( \omega_B \) - pulse bandwidth

\( P_0 \) - depends on port coupling
Numerical verification via simulation

\[ C(\alpha, \tau) = \text{fraction of realizations with} \]

\[ P(t) < \alpha \bar{P}(t) \]

\[ \tau = \frac{t \Delta \omega^3}{\omega_B^2 P_0} \]

50 realizations of time domain equations
Sensing small changes in a wave chaotic scattering system

Recent Developments

1. First Principles derivation of fading distributions

2. Coupling of cavities

3. Statistics of fields in chains of coupled cavities

4. Statistics of fields in mixed systems

5. Coupling by apertures
Conclusions

**Frequency Domain:**

Universal (RMT) and system specific contributions to statistics of wave scattering. Effects of losses, antennas, direct ray paths can be included.

**Time Domain:**

Decay of wave energy exhibits universal behavior. Time reversal symmetry can be used sensitively determine changes in an environment.