New Lattice Codes for Multiple-Descriptions

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***Abstract*—A new coding scheme for the L-descriptions problem is proposed. In particular we consider continuous sources and lattice quantizers. New covering and packing bounds for using nested lattices are derived. We prove through an example that using nested lattice quantizers instead of independently generated codebooks results in gains.**

I. Introduction

Lattice quantizers have been of great interest in compression of continuous sources [1], [2]. In the point-to-point (PtP) communication settings, the interest towards such codes is mainly due to reduced complexity of encoding and decoding. In multi-terminal communications, the significance of lattice codes is augmented because they give performance gains over unstructured codes in terms of achievable rate-distortion (RD) regions. The improvement stems from the fact that due to the structure of lattice quantizers, encoders utilizing such quantizers can transmit summations of different quantizations more efficiently than those using unstructured quantizers. These gains are observed in a variety of multi-terminal settings [1], [2], [4], and they are analogous to those seen when using linear codes for quantizing discrete sources [5], [6], [7].

Direct performance analysis of lattice coding techniques in general multi-terminal source coding set-ups turns out to be difficult. The analysis is usually carried out for quantization of Gaussian sources with Gaussian test channels. Hence, characterizations of inner bounds to the optimal RD region for general sources and test channels are not available. Recently, a method for finding the achievable RD region using lattice quantizers was proposed in [4]. The method involves analyzing the performance of linear codes for the discrete version of the problem, then convergence arguments are used to show achievability for continuous sources. Using this approach, the authors in [4], show achievability of the Wyner-Ziv rate region in the PtP set-up. Also in [3], the same technique is used to derive a new achievable RD region for the distributed source coding problem with continuous sources.

The multiple-desctiptions (MD) set-up describes a communications system consisting of one encoder and several decoders. The encoder compresses the source into several descriptions and transmits them through noiseless links. Each decoder receives a subset of these descriptions. The decoder then finds a reconstruction of the source using the descriptions it has received.

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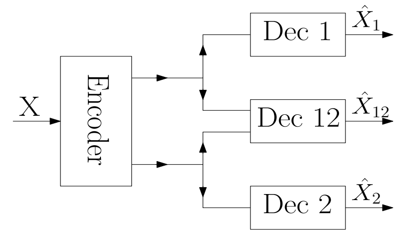

Fig. 1. The two-descriptions problem

The problem arises naturally when a transmitter wishes to send data to different receivers with varying quality of service demands. Another instance of the MD problem emerges when dealing with channel blackouts. In this situation, satisfactory source reconstruction is ensured via transmitting different descriptions of the source through multiple paths. In the latter perspective, each decoder represents the actual receiver in a specific blackout situation where a subset of the transmission links are experiencing failures; these failures are known at the decoder, but not the encoder. As an example, the two-descriptions problem is depicted in Figure 1.

The best known achievablity scheme for the discrete two descriptions problem is due to Zhang and Berger [8]. The Zhang-Berger scheme utilizes a base layer codebook and several refinement layer codebooks. The base layer is transmitted over all descriptions and is decoded at every decoder, while the refinement layer contains codebooks decoded at individual decoders. There has been several attempts to generalize the Zhang-Berger scheme for the L-descriptions problem $(L \geq 3)$ [9], [11], [10]. The CMS with binning strategy [12] unifies all of the previous schemes. The CMS strategy is based on random unstructured codes. This scheme was recently improved upon in [13]. In [13], first a new set of unstructured coding layers are added which provide strict improvement. In the second stage, this scheme is enhanced by addition of structured coding layers which provide further improvements. In [13], it was shown that if a pair of nested coset quantizers with the same inner code are used in the encoder, there would be a strict improvement in the achievable RD region for the L-descriptions problem when $L \geq 3$. In other words, structured codes provide asymptotic RD performance gains over unstructured codes. In this paper we consider continuous sources and provide a new achievable RD region for the L-descriptions problem using random lattice codes. These are new multi-terminal lattice codes. These lattice quantizers induce correlated quantization noises. They can not be decomposed as a collection of PtP lattice quantizers. We show that using a pair of nested lattice quantizers with the same inner code gives strict improvements over the CMS region in the continuous source case.

The rest of the paper is organized as follows: Section II contains the required preliminaries and notation. Section III gives the generalized version of the improved CMS with binning scheme. In section IV, we prove achievability of the scheme for continuous sources. We derive the RD region using nested lattices with common inner codes in section V. Section VI includes two examples illustrating the gains from application of nested lattice codes. Section VI concludes the paper.

II. Preliminaries

In this section we introduce our notation and give a brief review of the tools we use for constructing and analyzing lattice quantizers.

1) **Coset codes:** Let q be a prime number. Let $\mathbb{Z}_q$ denote the unique finite field of size q. A coset code is a shifted version of a linear code and is characterized by a generator matrix $G_{k\times n}$ and a dither B^n:

$$\mathcal{C} = \{u^k G_{k\times n} + B^n | u^k \in \mathbb{Z}_q^k\}.$$

2)**Nested Coset Codes:** A pair of coset codes ($\mathcal{C}_i$, $\mathcal{C}_o$), are called nested if $\mathcal{C}_i$ lies inside $\mathcal{C}_o$. $\mathcal{C}_o$ and $\mathcal{C}_i$ are called the outer and inner codes, respectively. A nested coset code is characterized by two generator matrices $G_{k\times n}$ and $\Delta G_{l\times n}$ and a dither B^n. Here $(G_{k\times n}, B^n)$ is a characterization for $\mathcal{C}_i$ and $([G, \Delta G]^t, B^n)$ characterizes $\mathcal{C}_o$. Each of these shifted versions of $\mathcal{C}_i$ is called a bin of $\mathcal{C}_o$ and is denoted by $\mathcal{B}_m$:

$$\mathcal{B}_m = \{a\mathbf{G} + m\mathbf{\Delta G} + B^n | a \in \mathbb{Z}_q^k\}.$$

3) **Lattice Code Generation:** A lattice code is a subset of $\mathbb{R}^n$ which is closed under integer addition. To generate a lattice code, consider an arbitrary coset code $\mathcal{C}$ over $\mathbb{Z}_q$. Choose $\gamma \in \mathbb{R}^+$ called the step size of the lattice code. First the coset code is shifted so that it is centered at the origin and scaled for the step-size γ. Define this shifted and scaled version as follows:

$$\Lambda(\mathcal{C}, \gamma, q) = \{\gamma(c_i - \frac{q-1}{2})_{i=1}^n | (c_i)_{i=1}^n \in \mathcal{C}\}.$$

$\Lambda(\mathcal{C}, \gamma, q)$ is used as the building block for constructing the lattice code. We generate the lattice code by considering disjoint shifted copies of $\Lambda(\mathcal{C}, \gamma, q)$:

$$\bar{\Lambda}(\mathcal{C}, \gamma, q) = \bigcup_{v \in \gamma q \mathbb{Z}^n} \{v + \Lambda(\mathcal{C}, \gamma, q)\}.$$

Nested lattice codes are also defined in a similar fashion by constructing a pair (Λ_i, Λ_o) from an underlying pair of nested coset codes $(\mathcal{C}_i, \mathcal{C}_o)$. Similar to nested coset codes, for $m \in \mathbb{Z}_q^l$, bin m can be defined as:

$$\bar{\mathcal{B}}_m = \{\gamma(c_i - \frac{q-1}{2})_{i=1}^n | (c_i)_{i=1}^n \in \mathcal{B}_m\} \quad (1)$$

where $\mathcal{B}_m$ is a bin in the underlying nested coset code.

We proceed with some set theory definitions. Each decoder in the L-descriptions problem is denoted by the set N of descriptions the decoder is receiving.

The power set of a set N is shown by 2^N. Note that for the L-descriptions problem $\mathcal{L} \triangleq 2^{[1:L]}$ is the set of all decoders. Subsets of $\mathcal{L}$ are called families of sets. For a family of sets $\mathcal{F}$, we define $\mathcal{F}_\sigma$ as the set of all unions of elements of $\mathcal{F}$:

$$\mathcal{F}_\sigma \triangleq \{\cup F_i, F_i \in \mathcal{F}\}.$$

A family of sets $\mathcal{F}$ is called a Sperner family if it satisfies the following property:

$$\nexists F, G \in \mathcal{F} \ni F \subset G$$

If $\mathcal{F}$ and $\mathcal{G}$ are two Sperner families then $\mathcal{F} \neq \mathcal{G}$ iff $\mathcal{F}_\sigma \neq \mathcal{G}_\sigma$. In other words, there is a bijection between the set of Sperner families and families of sets of the form $\mathcal{F}_\sigma$; the bijective map takes $\mathcal{F}$ to $\mathcal{F}_\sigma$. For an arbitrary family of sets $\mathcal{F}$ we denote the corresponding Sperner family of sets as $\mathcal{F}_s$, where the above bijective map takes $\mathcal{F}_s$ to $\mathcal{F}_\sigma$.

Next we provide a brief problem formulation for the L-descriptions set-up. Consider the continuous source X, and distortion functions $d_N(X, \hat{X}_N) : \mathbb{R}^2 \to \mathbb{R}_+, N \in \mathcal{L}$. Achievability for the rate-distortion vector $(R_i, D_N)_{i\in[1:L], N\in\mathcal{L}}$ is defined in the usual Shannon sense.

III. Enhanced CMS with Unstructured Codes

First, we provide an improvement over CMS with binning using unstructured codes for the general L-descriptions problem in the discrete source case.

Theorem 1. *Let $P_{\mathbf{U},X}$ be a joint distribution on random variables $U_\mathcal{M}, \mathcal{M} \in 2^\mathcal{L}$ and X. Assume $(\rho_{i,\mathcal{M}}, r_\mathcal{M})_{i\in[1:L],\mathcal{M}\in 2^\mathcal{L}}$ satisfy the following bounds.*

$$H(U_\mathbf{M}|X) \geq \sum_{\mathcal{M}\in\mathbf{M}} (H(U_\mathcal{M}) - r_\mathcal{M}), \forall \mathbf{M} \subset 2^\mathcal{L} \quad (2)$$

$$H(U_{\mathbf{M}_N}|U_\mathbf{L}) \leq \sum_{M\in\mathbf{M}_N\backslash\mathbf{L}} (H(U_\mathcal{M}) + \sum_{i\in[1:L]} \rho_{\mathcal{M},i} - r_\mathcal{M}), \forall \mathbf{L} \subset \mathbf{M}_N \quad (3)$$

where $\mathbf{M}_N = \{\mathcal{M} | N \in \mathcal{M}\}$. The following RD vector is achievable:

$$R_i = \sum_\mathcal{M} \rho_{\mathcal{M},i}$$
$$D_N = E\{d_N(h_N(U_{\mathbf{M}_N}, X))\}$$

Proof. **Codebook Generation:** Let $C_\mathcal{M}, \mathcal{M} \in 2^\mathcal{L}$ be the set of codebooks. $C_\mathcal{M}$ is decoded at decoder N, if $N \in \mathcal{M}$. $U_\mathcal{M}$ is the underlying random variable for codebook $C_\mathcal{M}$. Define a joint probability distribution $P_\mathbf{U}$ on random variables

$U_{\mathcal{M}}, \mathcal{M} \in 2^{\mathcal{L}}$. Each codebook $C_{\mathcal{M}}$ is generated randomly and independently based on $P_{U_{\mathcal{M}}}$ with rate $r_{\mathcal{M}}$. For the ith description, we bin the codebook randomly, uniformly and independently with binning rate $\rho_{\mathcal{M},i}$. These bin numbers are sent through the description. Decoder N, upon receiving the descriptions finds a unique vector $(u^n_{\mathcal{M}})_{N\in\mathcal{M}}$ of jointly typical sequences. If the vector does not exist or is not unique, the decoder declares error.

Covering Bounds: Since codebooks are generated randomly and independently, to find a jointly typical set of sequences $U^n_{\mathcal{M}}$ with the source vector X^n, the mutual covering bounds (2) are necessary.

Packing Bounds: For decoder N, description i is received if $i \in N$. Since binning is done independently and uniformly, to find a unique set of jointly typical sequences $(u^n_{\mathcal{M}})_{N\in\mathcal{M}}$, the mutual packing bounds (3) are required. □

Remark 1. It can be shown that the codebook $C_{\mathcal{M}}$ is non-redundant in this scheme if and only if 1) $\mathcal{M} = \mathcal{M}_\sigma$ and 2) it is not any of $\{\}, \{\{\}\}$ or $\mathcal{L}$. This observation decreases the number of necessary codebooks significantly. For example in the three descriptions problem $|2^{2^{[1:3]}}| = 256$, whereas there are only 20 Sperner families (i.e. 17 necessary codebooks).

IV. L-Descriptions Coding for Continuous Sources

Theorem 2. *Consider a continuous source X. Fix a joint pdf $f_{\mathbf{U},X}$. The following RD vectors are achievable for continuous sources:*

$$h_d(U_{\mathbf{M}}|X) \geq \sum_{\mathcal{M}\in\mathbf{M}} (h_d(U_{\mathcal{M}}) - r_{\mathcal{M}}), \forall \mathbf{M} \subset 2^{\mathcal{L}}$$

$$h_d(U_{\mathbf{M}_N}|U_{\mathbf{L}}) \leq \sum_{M\in\mathbf{M}_N\backslash\mathbf{L}} (h_d(U_{\mathcal{M}}) + \sum_{i\in[1:L]} \rho_{\mathcal{M},i} - r_{\mathcal{M}}), \forall \mathbf{L} \subset \mathbf{M}_N$$

$$R_i = \sum_{\mathcal{M}} \rho_{\mathcal{M},i}$$
$$D_N = E\{d_N(h_N(U_{\mathbf{M}_N}, X))\}$$

Proof. The proof involves two steps, first we approximate the n-length vector of X with an n-length vector of the random variable $X_{q,\gamma}$. $X_{q,\gamma}$ is a discrete random variable defined on $\mathcal{X}_{q,\gamma} = [-\gamma\frac{q-1}{2}, \gamma\frac{q-1}{2}] \cap \gamma\mathbb{Z}$, by the same arguments as in the previous section, the improved CMS with binning is achievable for a discrete source with the distribution of $X_{q,\gamma}$. In the next step, we take $\gamma \to 0$ while taking $\gamma q \to \infty$. Note that making γ smaller is intuitively equivalent to observing X with a finer grid, while $\gamma q \to \infty$ means we are increasing the range of values of X which we are observing. So, intuitively by taking the above two limits we get $X_{\gamma,q} \to X$.

Now we proceed with a more rigorous description of the proof. Fix n, q and γ. Fix a pdf $f_{\mathbf{U_M},X}$, where $\mathbf{U_M}$ is the set of random variables from the last section. The definition of $X_{q,\gamma}$ is as follows:

$$X^n_{q,\gamma} = argmin\{d_2(X^n, x^n_{q,\gamma}) | x^n_{q,\gamma} \in \mathcal{X}^n_{q,\gamma}\}$$

Where $d_2(a^n, b^n) = \sum_{i=1}^n (a_i - b_i)^2$. Note that if $\gamma \to 0$ and $\gamma q \to \infty$ then $d_2(X^n, X^n_{q,\gamma}) \to 0$.

Theorem III.6 in [4] shows that for continuous $\mathbf{U_M}$ we can take $\mathbf{U}_{\mathbf{M},q,\gamma}$ such that as $\gamma \to 0$ and $\gamma q \to \infty$, mutual information terms and distortions containing $\mathbf{U}_{\mathbf{M},q,\gamma}$, converge to the terms containing $\mathbf{U_M}$. Lemma 1 (below) shows that the bounds in the previous section can be written in terms of mutual informations, so they converge to the bounds in the theorem as $\gamma \to 0$ and $\gamma q \to \infty$. □

Lemma 1. *The bounds in the improved CMS region can be written in terms of mutual informations.*

Proof. The proof is included in [14]. □

In the next proposition we show that the improved CMS with binning region can also be achieved using lattice quantizers.

Proposition 1. *Fix $f_{\mathbf{U},X}$. The following RD region is achievable using lattice quantizers.*

$$h_d(U_{\mathbf{M}}|X) \geq \sum_{\mathcal{M}\in\mathbf{M}} (\log q - r_{o,\mathcal{M}}), \forall \mathbf{M} \subset 2^{\mathcal{L}} \tag{4}$$

$$h_d(U_{\mathbf{M}_N}|U_{\mathbf{L}}) \leq \sum_{\mathcal{M}\in\mathbf{M}_N\backslash\mathbf{L}} (\log q + \sum_{j\in[1:L]} \rho_{\mathcal{M},j} - r_{o,\mathcal{M}}), \forall \mathbf{L} \subset \mathbf{M}_N \tag{5}$$

$$R_i = \sum_{\mathcal{M}} \rho_{\mathcal{M},i}$$
$$D_N = E\{d_N(h_N(U_{\mathbf{M}_N}, X))\}$$

Proof. The proof for achievability follows from the same arguments given for the random coding scheme by replacing the covering and packing bounds for random coding with those of linear coding. □

Lemma 2. *After the Fourier-Motzkin elimination, the region reduces to the improved CMS region.*

Proof. The proof is included in [14].

V. Achievable Region for Nested Lattices

Theorem 3. *Fix $\mathcal{M}_1, \mathcal{M}_2 \in 2^{\mathcal{L}}$, and set $N_1 \in \mathcal{L}$ such that $N_1 \notin \mathcal{M}_1 \cup \mathcal{M}_2$. The following RD vectors are achievable in the discrete L-descriptions setting.*

$$H(U_{\mathbf{M}}|X) \geq \sum_{\mathcal{M} \in \mathbf{M}} (\log q - r_{o,\mathcal{M}}), \forall \mathbf{M} \subset 2^{\mathcal{L}}$$

$$r_{\mathcal{M}_1} + r_{\mathcal{M}_2} - r_{\mathcal{M}_1,\mathcal{M}_2} \geq \log q - H(\alpha U_{\mathcal{M}_1} + \beta U_{\mathcal{M}_2}|X), \quad \forall \alpha, \beta \in \mathbb{Z}_q \tag{6}$$

$$\min(r_{\mathcal{M}_1}, r_{\mathcal{M}_2}) > r_{\mathcal{M}_1 \mathcal{M}_2}$$

$$H(U_{\mathbf{M}_N}|U_{\mathbf{L}}) \leq \sum_{\mathcal{M} \in \mathbf{M}_N \backslash \mathbf{L}} (\log q + \sum_{j \in [1:L]} \rho_{\mathcal{M},j} - r_{o,\mathcal{M}}), \forall \mathbf{L} \subset \mathbf{M}_N$$

$$H(U_{\mathbf{M}'_{N_1}}|U_{\mathbf{L}}) \leq \sum_{\mathcal{M} \in \mathbf{M}'_{N_1} \backslash \mathbf{L}} (\log q + \sum_{j \in [1:L]} \rho_{\mathcal{M},j} - r_{o,\mathcal{M}}), \forall \mathbf{L} \subset \mathbf{M}'_{N_1}$$

$$R_i = \sum_{\mathcal{M}} \rho_{\mathcal{M},i}$$

$$D_N = E\{d_N(h_N(U_{\mathbf{M}_N}, X))\}, N \neq N_1$$

$$D_{N_1} = E\{d_{N_1}(h_{N_1}(U_{\mathbf{M}'_N}, X))\}$$

Where $\mathbf{M}'_{N_1} = \mathcal{M}_{N_1} \cup \{\mathcal{M}_1 \oplus_q \mathcal{M}_2\}$.

Proof. In the the previous sections we used the fact that codebooks are generated independently and found the covering bounds necessary to ensure existence of jointly typical quantization vectors, however, in the new scheme codes with the same inner codes are used. Hence the mutual covering bounds are not enough to ensure the existence of jointly typical sequences at the encoder and the uniqueness of the decoded sequences at the decoders. Let $(\mathcal{C}, \mathcal{C}')$ be a pair of nested coset codes over $\mathbb{Z}_q$ such that they have the same inner codes. Since the inner code is the same we need new covering bounds to insure the existence of jointly typical sequences. The following lemma gives the required covering bounds.

Definition 1. Consider 3 finite sets $\mathcal{X}, \mathcal{U}, \mathcal{V}$. Fix a PMF $P_{X,U,V}$ on $\mathcal{X} \times \mathcal{U} \times \mathcal{V}$. A sequence of code pairs (C_1^n, C_2^n) is called P_{XUV}-covering if:

$$\forall \epsilon > 0, P(\{x^n | \exists (u^n, v^n) \in A_\epsilon^n(U, V|x^n) \cap C_1 \times C_2\}) \to 1$$
$$n \to \infty$$

Lemma 3. *For any P_{XUV} on $\mathcal{X} \times \mathcal{U} \times \mathcal{V}$ and rates r_o, r'_o and r_i satisfying 7-10, there exists a sequence of pairs of nested coset codes $\mathcal{C}_1^n = (\mathcal{C}_o^n, \mathcal{C}_i^n)$ and $\mathcal{C}_2^n = (\mathcal{C}_o'^n, \mathcal{C}_i^n)$ which are P_{XUV}-covering.*

$$r_o \geq \log q - H(U|X) \tag{7}$$
$$r'_o \geq \log q - H(V|X) \tag{8}$$
$$r_o + r'_o \geq 2 \log q - H(U, V|X) \tag{9}$$
$$r_o + r'_o - r_i \geq \log q - H(\alpha U + \beta V|X), \forall \alpha, \beta \in \mathbb{Z}_q, \tag{10}$$

where r_o, r'_o are the rates of the outer codes $\mathcal{C}_o^n$ and $\mathcal{C}_o'^n$, respectively and r_i denotes the rate of the inner code $\mathcal{C}_i^n$.

Proof. The proof is included in [14]. □

Remark 2. Note in the above lemma if we take $r_i = 0$, then the two codes are generated independently, so the covering bounds reduce to mutual covering bounds in the original scheme. However transmitting the linear combination of U and V would require a larger rate since they are coming from two independent codebooks (i.e. the rate of the codebook for $U + iV$ would be equal to $r_U + r_V$.) On the other hand if we take $r_o = r_i$, then the covering bounds become tighter, since we are using the exact same linear code.

Remark 3. Note that the new covering bounds are different from the old bounds only in the addition of (10). The intuitive explanation for this additional bound is that if $(u^n, v^n, x^n) \in A^n(U, V, X)$, where $A^n(U, V, X)$ is the set of jointly typical sequences of length n, then $(\alpha u^n + \beta v^n, x^n) \in A^n(\alpha U + \beta V, X)$. There are $q^{n(r_o + r'_o - r_i)}$ vectors in $\alpha C_o + \beta C'_o$. So $q^{n(r_o + r'_o - r_i)}$ should be large enough to insure the existence of $\alpha u^n + \beta v^n$ typical with x^n which results in (10).

It follows by the same arguments as in the previous section that the region is achievable using lattice quantizers. To see this, note that there is only one additional covering bound in the new system of inequalities, but the difference between this bound and the previous bounds can be written in terms of mutual informations.

Lemma 4. *The new bounds for the achievable RD region using nested coset codes can be written in terms of mutual informations.*

Proof. The proof is included in [14]. □

VI. Improvements for Using Nested Lattice Quantizers

We proceed to show through an example that using nested lattice codes gives gains in terms of achievable rate-distortion.

A. Example 1

The set-up is shown in figure 2. Here $(X, Z) \sim N(0, 1)$. The distortion function for the individual decoders is mean squared error. Decoder 1 and 2 want to reconstruct their respective source with MSE less than or equal P, and decoder 3 wants to reconstruct $Y = X + Z$ with distortion $2P$. The distortion constraint in the joint decoders is as follows:

$$E(([X, Z] - [\hat{X}_{12}, \hat{Z}_{12}])^t([X, Z] - [\hat{X}_{12}, \hat{Z}_{12}])) \leq \begin{bmatrix} P & 1 \\ 1 & P \end{bmatrix}$$
$$E(([X, Y] - [\hat{X}_{13}, \hat{Y}_{13}])^t([X, Y] - [\hat{X}_{13}, \hat{Y}_{13}])) \leq \begin{bmatrix} P & P \\ P & 2P \end{bmatrix}$$
$$E(([X, Y] - [\hat{X}_{23}, \hat{Y}_{23}])^t([X, Y] - [\hat{X}_{23}, \hat{Y}_{23}])) \leq \begin{bmatrix} P & P \\ P & 2P \end{bmatrix}$$

Where $\hat{A}_{ij}, A \in \{X, Z, Y\}, i, j \in \{1, 2, 3\}$ is the reconstruction of A at decoder ij.

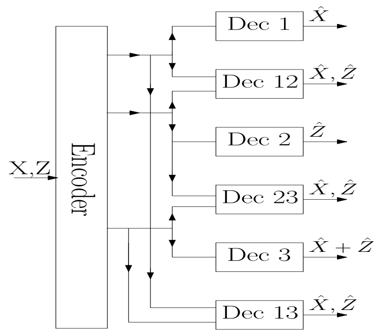

Fig. 2. Example for Vector Gaussian Source

Theorem 4. *The following rate triple is achievable for the distortions mentioned above using the nested lattice structure.*

$$(R_1, R_2, R_3) = (\frac{1}{2}log(\frac{1}{p}), \frac{1}{2}log(\frac{1}{p}), \frac{1}{2}log(\frac{2}{p}))$$

Theorem 5. *The above rates are not achievable using the enhanced CMS scheme.*

The proofs of theorems 4 and 5 can be found in [14]. Note that in the above example, as a result of the independence between the two source components X and Z, the extra covering bound in (10) is redundant. But this is not always the case, to illustrate this point we investigate another example.

B. Example 2

Assume the source X is scalar Gaussian with distribution $N(0, 1)$. Consider the random variables U and V which are jointly Gaussian with X and have the following covariance matrix ($\frac{1}{2} < P < \frac{2}{3}$):

$$Cov([X, U, V]) = \begin{pmatrix} 1 & 1-P & 1-P \\ 1-P & 1-P & 0 \\ 1-P & 0 & 1-P \end{pmatrix}$$

We intend to transmit U on the first description, V on the second description and $U + V$ on the third description. In this case the new covering bound is not redundant. To see this note that the covering bound is non-redundant if $I(U{+}V; V|X) - I(U; V|X) < 0$. Simplifying the inequality, the bound is non-redundant if $Var(V|XU) < Var(V|X, U + V)$. From the formula for conditional variance we have:

$$Var(V|X, U) = \frac{p(1-p)}{2}, Var(V|X, U + V) = \frac{1-p}{2}$$
$$Var(V|XU) < Var(V|X, U + V) \iff p < \frac{2}{3}$$

Which shows the bound is non-redundant in this setting. We calculate the achievable rates using nested lattices. FME gives:

$$R_1 = R_2 = \max\big\{ \frac{I(UV; X)}{2}, I(U; X) - I(\alpha U + \beta V; V|X) + I(U; V|X)\big\}$$
$$R_3 = R_1 - H(U) + H(U + V)$$

We have:

$$I(UV; X) = \frac{1}{2}\log(\frac{1}{2p-1}), \quad I(U; X) = \frac{1}{2}\log\frac{1}{p}$$
$$- I(\alpha U + \beta V; V|X) + I(U; V|X) =$$
$$\frac{1}{2}\log(\frac{\alpha^2 p}{\alpha^2 + \beta^2 - (\alpha+\beta)^2(1-p)})$$

Hence $R_1 = R_2 = \max(\frac{1}{2}\log(\frac{p^2}{p+(1-p)^2}), \frac{1}{4}\log(\frac{1}{2p-1})), R_3 = R_1 + \frac{1}{2}$. And the distortions are $D_1 = D_2 = p, D_3 = 2p, D_{12} = D_{13} = D_{23} = 2p - 1$.

VII. Conclusion

A new coding scheme for the general MD problem with continuous sources was proposed. The strategy allowed the use of arbitrary (non-Gaussian) test channels. Nested lattices were utilized for quantization. The achievable RD region was calculated for both the case when all lattices are generated independently and when two of the nested lattices have the same inner code. It was shown that for set-ups with three or more descriptions, it is beneficial to use the same inner code for a pair of nested lattices.

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