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GRAPH FACTORS

by

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This exposition is concerned with the main theorems of graph-factor theory, Hall's and Ore's Theorems in the bipartite case, and in the general case Petersen's Theorem, the 1-Factor Theorem and the f-Factor Theorem. Some published extensions of these theorems are discussed and are shown to be consequences rather than generalizations of the f-Factor Theorem. The bipartite case is dealt with in Section 2. For the proper presentation of the general case a preliminary theory of "G-triples" and "f-barriers" is needed, and this is set out in the next three Sections. The f-Factor Theorem is then proved by an argument of T. Gallai in a generalized form. Gallai's original proof derives the 1-Factor Theorem from Hall's Theorem. The generalization proceeds analogously from Ore's Theorem to the f-Factor Theorem.

1. f-Factors

The graphs of this paper are finite. They may have loops and multiple joins. We write $\operatorname{val}(G, x)$ for the *valency* of a vertex x in a graph G. It is the number of edges of G incident with x, loops being counted twice.

If S and T are disjoint sets of vertices of G we write $\lambda(S, T)$ for the number of edges of G joining S to T. If T has only one vertex t we write $\lambda(S, T)$ also as $\lambda(S, t)$. If A is an edge of G we write G'_A for the graph derived from G by deleting A. We write $|S|$ for the cardinality of a set S.

A *vertex-function* of G is a mapping f of the vertex-set $V(G)$ of G into the set of integers. Given such an f we define an associated vertex-function f' by the following rule.

(1) $$f'(x) = \operatorname{val}(G, x) - f(x)$$

for each x in $V(G)$. Thus $(f')' = f$.

Given a vertex-function f of G we define an *f-factor* of G as a spanning subgraph F of G such that

(2) $$\operatorname{val}(F, x) = f(x)$$

for each vertex x.

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Each f-factor of G has a unique associated f'-factor, defined by the remaining edges of G.

1.1. *Let f be a vertex-function of a graph G. Then G has an f'-factor if and only if it has an f-factor.* ∎

This result introduces a duality into the theory of graph-factors. When possible we shall state our general theorems in self-dual forms, symmetrical as between f and f'. Truly unsymmetrical theorems occur in dual pairs. Each member of such a pair is the result of stating the other for f' instead of f, and then reverting to f by using (1). Consider for example the statement that G has no f-factor if $f(x)<0$ for some x. The dual theorem asserts that G has no f-factor if val $(G, x)<f(x)$ for some x.

The vertex-function f of G such that $f(x)=1$ for each x is the *unit* vertex-function of G. The corresponding f-factors are the *1-factors* of G. They are of special importance in the literature. The classical theorems of Petersen and Hall are theorems about 1-factors.

2. Bipartite graphs

A *bipartition* of a graph G is an ordered pair (X, Y) of complementary subsets of $V(G)$ such that each edge of G has one end in X and one in Y. A *bipartite* graph is one with a bipartition. Thus a bipartite graph can have no loops.

The bipartite case deserves special study as being the easy part of graph-factor theory. Moreover in the present paper the general case is made to depend on it.

Let f be a vertex-function of a graph G with a bipartition (X, Y). If S and T are subsets of X and Y respectively we write

$$\gamma(S, T) = \lambda(S, T) - \sum_{s \in S} f(s) - \sum_{t \in T} f'(t), \tag{3}$$

that is

$$\gamma(S, T) = \sum_{t \in T} f(t) - \sum_{s \in S} f(s) - \lambda(X-S, T). \tag{4}$$

If also S_1 and T_1 are subsets of X and Y respectively it is clear that

$$\lambda(S \cup S_1, T \cup T_1) + \lambda(S \cap S_1, T \cap T_1) \geqq \lambda(S, T) + \lambda(S_1, T_1).$$

Hence, by (3),

$$\gamma(S \cup S_1, T \cup T_1) + \gamma(S \cap S_1, T \cap T_1) \geqq \gamma(S, T) + \gamma(S_1, T_1). \tag{5}$$

We say that f is *balanced* with respect to (X, Y) if

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(y). \tag{6}$$

The following theorem says all that need be said about the unbalanced case, since X and Y can be interchanged in its statement.

2.1. *Suppose that*

$$\sum_{x \in X} f(x) < \sum_{y \in Y} f(y).$$

Then G has no f-factor. Moreover $\gamma(X, Y) > 0$.

Proof. If F were an f-factor of G then each side of the inequality would have to be equal to the number of edges of F. The second part of the theorem follows from (4). ▮

We proceed to the proof of Ore's Theorem. The symbol $\emptyset$ denotes a null set.

2.2. *Let f be balanced with respect to (X, Y). Then G is without an f-factor if and only if there are subsets S and T of X and Y respectively such that*

$$\gamma(S, T) > 0. \tag{7}$$

Proof. Suppose (7) to hold for some S and T. If F is an f-factor of G let n be the number of its edges incident with vertices of T. Then

$$\sum_{t \in T} f(t) = n \leqq \sum_{s \in S} f(s) + \lambda(X-S, T).$$

But then $\gamma(S, T) \leqq 0$, by (4), which contradicts (7).

Conversely, consider the class of graphs G having no pair (S, T) satisfying the stated conditions. If possible choose such a G so that G has no f-factor, and so that the number α of edges of G has the least value consistent with this condition.

We observe that $f(x) \geqq 0$ for each x in X, and that $f'(y) \geqq 0$ for each y in Y. For otherwise (7) would be satisfied by a pair (S, T) of the form $(\{x\}, \emptyset)$ or $(\emptyset, \{y\})$.

We can prove also that $f'(y) > 0$ for each vertex of y that is incident with at least one edge. For suppose $f'(y) = 0$, that is $f(y) = \mathrm{val}(G, y)$ for one such y. Then we form a bipartite graph H from G by deleting y and its incident edges. We define a balanced vertex-function g of H by the rule that for each vertex v of H the number $g(v)$ is $f(v)$ diminished by the number of edges joining v to y in G.

If H has a g-factor then clearly G has an f-factor. So by the choice of G we can find $S \subseteqq X$ and $T \subseteqq Y - \{z\}$ such that $\gamma(S, T) > 0$ in H. But then we have also $\gamma(S, T \cup \{y\}) > 0$ in G, by (3). From this contradiction we infer that in fact $f'(y) > 0$.

Now suppose that $\alpha = 0$. Since f is balanced, and since $f'(y) = -f(y)$ in this case, it follows from the preceding observation that $f(v) = 0$ for each v in $V(G)$. But then G is its own f-factor, and we have a contradiction.

We deduce that G has an edge A. Any f-factor of G'_A would be an f-factor of G. So, by the choice of G, we can find $S_A \subseteq X$ and $T_A \subseteqq Y$ such that $\gamma(S_A, T_A) > 0$ in G'_A. But $\gamma(S_A, T_A) \leqq 0$ in G. Considering the definition (3) we see that the only way in which these requirements can be reconciled is for A to join a vertex z of T_A to a vertex of $X - S_A$. (The value of $f'(z)$ is one less in G'_A than in G.) Then moreover we have $\gamma(S_A, T_A) = 0$ in G.

It follows from these observations that we can find $S \subseteqq X$ and $T \subseteqq Y$ so that

$$\gamma(S, T) = 0 \quad \text{in } G, \tag{8}$$

and so that some edge of G joins T to $X - S$. Choose such a pair (S, T) so that $|S \cup T|$ has the least possible value. Since $f'(z) > 0$ for at least one z in T, it follows from (3) and (8) that there is an edge B of G joining S to T.

As with A we can find $S_B \subseteq X$ and $T_B \subseteq Y$ so that

$$\gamma(S_B, T_B) = 0 \quad in\ G, \tag{9}$$

and so that B joins T_B to $X - S_B$.

By the choice of G neither $\gamma(S \cup S_B, T \cup T_B)$ nor $\gamma(S \cap S_B, T \cap T_B)$ can be positive in G. Hence, by (5), (8) and (9) we have

$$\gamma(S \cap S_B, T \cap T_B) = 0 \quad in\ G. \tag{10}$$

But the edge B joins $T \cap T_B$ to a vertex not in $S \cap S_B$. It follows, by the choice of S and T, that $S \subseteq S_B$ and $T \subseteq T_B$. But this is impossible; the edge B joins S to T, but not S_B to T_B. The theorem follows. ▮

The above form 2.2 of Ore's Theorem has the advantage of being self-dual. But the following form is often preferred.

2.3. *Let f be balanced with respect to (X, Y). Then G is without an f-factor if and only if there is a subset T of Y such that*

$$\sum_{t \in T} f(t) > \sum_{x \in X} \text{Min}\,\{f(x), \lambda(T, x)\}. \tag{11}$$

Proof. If (7) holds, then so does (11), by (4). If (4) holds we define S as the set of all vertices s of X such that $f(s) > \lambda(T, x)$. We can then deduce (7) from (4). Accordingly Propositions 2.2 and 2.3 are equivalent. ▮

If $T \subseteq Y$ let us write $D(T)$ for the set of all vertices of X that are joined to T. We can now state Hall's Theorem as follows.

2.4. *Let (X, Y) be a bipartition of G such that $|X| = |Y|$. Then G is without a 1-factor if and only if there is a subset T of Y such that*

$$|T| > |D(T)|. \tag{12}$$

Proof. The requirement $|X| = |Y|$ is merely a restriction to the balanced case. Theorem 2.3 reduces immediately to 2.4 when we take f to be the unit vertex-function. For if x is in X then Min $\{f(x), \lambda(T, x)\}$ is 1 if x is in $D(T)$, and is zero otherwise. ▮

We conclude this Section by noting one more property of the function $\gamma(S, T)$.

2.5. *Let f be balanced with respect to the bipartition (X, Y) of G. Let S and T be subsets of X and Y respectively. Then*

$$\gamma(S, T) = \gamma(Y - T, X - S), \tag{13}$$

where the left and right sides are defined in terms of the bipartitions (X, Y) and (Y, X) respectively.

Proof. Since f is balanced we can rewrite (4) as

$$\gamma(S, T) = \sum_{s \in X - S} f(s) - \sum_{t \in Y - T} f(t) - \lambda(X - S, T).$$

But the expression on the right is $\gamma(Y - T, X - S)$, by another application of (4). ▮

3. G-triples and f-barriers

Let G be a graph, and let f be a vertex-function of G.

Any subset U of $V(G)$ defines an induced subgraph Ind (G, U) of G. It is made up of the vertices of U and the edges of G having both ends in U. (A loop is said to have two coincident ends.) We refer to the components of Ind (G, U) as the *components* of U in G.

A *G-triple* is an ordered triple (S, T, U), where S, T and U are disjoint subsets of $V(G)$ whose union is $V(G)$. If $B=(S, T, U)$ is a G-triple we write B' for the G-triple (T, S, U).

If $B=(S, T, U)$ is a G-triple, and C is any component of U in G, we write

(14) $$J(B,f,C) = \sum_{c \in V(C)} \{f(c)+\lambda(T, c)\}.$$

We say that C is an *odd* or *even* component of U in G, with respect to B and f, according as $J(B,f, C)$ is odd or even. We denote the number of odd components of U in G, with respect to B and f, by $h(B,f)$.

3.1. *Let $B=(S, T, U)$ be a G-triple, and let C be a component of U in G. Then*

(15) $$J(B',f', C) \equiv J(B,f, C) \pmod 2.$$

Proof.

$$J(B',f', C)+J(B,f, C) = \sum_{c \in V(C)} \{\mathrm{val}(G, c)+\lambda(T, c)+\lambda(S, c)\},$$

by (1) and (14)

$$\equiv \sum_{c \in V(C)} \mathrm{val}(C, c) \equiv 0 \pmod 2. \blacksquare$$

Thus a component of U in G is odd with respect to B' and f' if and only if it is odd with respect to B and f. We thus have

(16) $$h(B',f') = h(B,f).$$

We define the *deficiency* $\delta(B,f)$ of the G-triple $B=(S, T, U)$, with respect to f, as follows.

(17) $$\delta(B,f) = h(B,f) - \sum_{s \in S} f(s) - \sum_{t \in T} f'(t) + \lambda(S, T).$$

By (16) we have

(18) $$\delta(B,f) = \delta(B',f').$$

We can indeed regard (17) as a self-dual definition, symmetrical with respect to the double interchange of f with f' and B with B'.

An *f-barrier* of G is a G-triple $B=(S, T, U)$ such that $\delta(B,f)>0$. If B is an f-barrier then B' is an f'-barrier, by (18).

If K is any subgraph of G we write $\Sigma(K,f)$ for the sum of the numbers $f(x)$ over all vertices x of K. We say that f is *odd-summing* or *even-summing* on K according as $\Sigma(K,f)$ is odd or even.

3.2. *Let $B=(S, T, U)$ be a G-triple. Then*

(19) $$\delta(B,f) \equiv \Sigma(G,f) \pmod 2.$$

Proof. By (14) and (17)

$$\begin{aligned}\delta(B,f) &\equiv \sum_{u\in U}\{f(u)+\lambda(T,u)\}+\sum_{s\in S}f(s)+\sum_{t\in T}\{\mathrm{val}\,(G,t)+f(t)\}+\lambda(S,T)\\ &\equiv \Sigma(G,f)+\lambda(T,U)+\lambda(S,T)+\sum_{t\in T}\mathrm{val}\,(G,t)\\ &\equiv \Sigma(G,f)\pmod 2. \quad \blacksquare\end{aligned}$$

We conclude this Section by seeking out some examples of f-barriers. We note some obvious but important ones in the following two theorems.

3.3. *If $f(x)<0$ for some x in $V(G)$ then $(\{x\}, \emptyset, V(G)-\{x\})$ is an f-barrier. Dually if $f'(x)<0$, that is if $f(x)>\mathrm{val}\,(G,x)$, then $(\emptyset, \{x\}, V(G)-\{x\})$ is an f-barrier.*

3.4. *If f is odd-summing on G, then $(\emptyset, \emptyset, V(G))$ is an f-barrier.*

Theorem 3.3 is an immediate consequence of (17). Theorem 3.4 can be regarded as a consequence of 3.2. For the deficiency of $(\emptyset, \emptyset, V(G))$ is non-negative by (17), and odd by 3.2. ▮

We can discover occurrences of f-barriers in the bipartite theory of Section 2. Let us return to the case in which G has a bipartition (X, Y).

Let S and T be subsets of X and Y respectively. Let B be the G-triple $(S, T, V(G)-(S\cup T))$. By comparing Equations (3) and (17) we find that

$$\delta(B,f) \geqq \gamma(S,T). \tag{20}$$

Applying this result to 2.2 we obtain the following.

3.5. *Let G have a bipartition (X, Y) and let f be balanced with respect to (X, Y). Then if G has no f-factor it has an f-barrier (S, T, U) such that $S\subseteqq X$ and $T\subseteqq Y$.* ▮

Another observation of interest can be made in the bipartite case. Let P and Q be complementary subsets of $V(G)$, and let B be the G-triple $(P, Q, \emptyset)$. Then

$$\begin{aligned}\delta(B,f) &= \lambda(P\cap X, Q\cap Y)+\lambda(Q\cap X, P\cap Y)-\sum_{p\in P}f(p)-\sum_{q\in Q}f'(q), \qquad \text{by (17)},\\ &= \gamma(P\cap X, Q\cap Y)+\gamma(P\cap Y, Q\cap X)\end{aligned}$$

by (3). Here the first γ is defined in terms of (X, Y) and the second in terms of (Y, X). Hence

$$\delta(B,f) = 2\gamma(P\cap X, Q\cap Y), \qquad \text{by (13)}. \tag{21}$$

We note that if $S\subseteqq X$ and $T\subseteqq Y$ we can arrange that $P\cap X=S$ and $Q\cap Y=T$ by writing

$$P = S\cup(Y-T) \quad \text{and} \quad Q=(X-S)\cup T.$$

3.6. *Let G be a bipartite graph. Then G has either an f-factor or an f-barrier of the form $(P, Q, \emptyset)$, but not both.*

Proof. Choose a bipartition (X, Y) of G. If f is not balanced with respect to (X, Y) the theorem follows from 2.1 and Equation (20). In the balanced case we use 2.2 and Equation (21). ▮

4. Transformations of G-triples

Let f be a vertex-function of a graph G.

Let $B=(S, T, U)$ be a G-triple. If x is a vertex of S or T we write $\mu(x)$ for the number of components C of U, odd with respect to B and f, such that x is joined to C by some edge of G.

4.1. *Let x be a vertex of S, and let B_1 be the G-triple $(S-\{x\}, T, U\cup\{x\})$. Then*

$$\delta(B_1,f)-\delta(B,f) = f(x)-\mu(x)-\lambda(T, x)+\eta(x), \tag{22}$$

where $\eta(x)$ is 0 *or* 1 *and is chosen to make the right side of* (22) *even.*

Proof. Consider Equation (17), the definition of $\delta(B,f)$. When we change from B to B_1 the term $\lambda(S, T)$ is diminished by $\lambda(T, x)$, the term $-\Sigma f'(t)$ is left unaltered, and the term $-\Sigma f(s)$ is increased by $f(x)$. The graph Ind(G, U) loses $\mu(x)$ odd components, and perhaps some even ones too. The lost components are all absorbed into the single new component D containing x. We deduce that in the transformation $h(B,f)$ is diminished by $\mu(x)-\alpha$, where α is 1 or 0 according as D is odd or even with respect to B_1 and f. We conclude that

$$\delta(B_1,f)-\delta(B,f) = f(x)-\mu(x)-\lambda(T, x)+\alpha.$$

But $\delta(B_1,f)$ and $\delta(B,f)$ have the same parity, by (3.2). Hence $\alpha=\eta(x)$. ∎

4.2. *Let x be a vertex of T, and let B_1 be the G-triple $(S, T-\{x\}, U\cup\{x\})$. Then*

$$\delta(B_1,f)-\delta(B,f) = f'(x)-\mu(x)-\lambda(S, x)+\eta'(x), \tag{23}$$

where $\eta'(x)$ is 0 *or* 1, *and is chosen to make the expression on the right even.*

Proof. This theorem is the dual of 4.1. To prove it we first state 4.1 with B_1', B' and f' replacing B_1, B and f respectively. We then use (18). We also use 3.1 to show that $\mu(x)$ is invariant under duality. ∎

We refer to the change from B to B_1 in these two theorems as a *transferrence* of x from S or T to U. When using the theorems we should bear in mind that

$$\mu(x) \leqq \lambda(U, x). \tag{24}$$

Two applications follow.

4.3. *Let $B=(S, T, U)$ be a G-triple, and let x be a vertex of S such that $f(x)=$*val (G, x) *or* val $(G, x)-1$. *Then the transferrence of x from S to U does not diminish the deficiency of B.*

Proof. By (24) the expression on the right of (22) must be non-negative. ∎

4.4. *Let $B=(S, T, U)$ be a G-triple, and let x be a vertex of T such that $f(x)=0$ or 1. Then the transferrence of x from T to U does not diminish the deficiency of B.*

This is the dual of 4.3, derivable analogously from 4.2. ∎

Our next theorem relating different G-triples concerns a G-triple $B=(S, T, U)$, a component C of U, and a C-triple $B_C=(S_C, T_C, U_C)$. It deals also with a vertex-function f_C of C.

We define the *augmentation* of B by B_C as the G-triple $B_1=(S_1, T_1, U_1)$, where $S_1=S\cup S_C$ and $T_1=T\cup T_C$.

For each vertex c of C we define the *divergence* $\operatorname{div}(f_C, c)$ *of* f_C *at* c as follows.

$$\operatorname{div}(f_C, c) = |f(c)-\lambda(T, c)-f_C(c)|. \tag{25}$$

We define the *total divergence* $\operatorname{div}(f_C)$ *of* f_C as follows.

$$\operatorname{div}(f_C) = \sum_{c\in V(C)} \operatorname{div}(f_C, c). \tag{26}$$

4.5. *Let $B=(S, T, U)$ be a G-triple, C a component of U, and $B_C=(S_C, T_C, U_C)$ a C-triple. Let $B_1=(S_1, T_1, U_1)$ be the augmentation of B by B_C, and let f_C be a vertex-function of C. Then*

$$\delta(B_1,f) \geqq \delta(B,f)+\delta(B_C,f_C)-\operatorname{div}(f_C)-1. \tag{27}$$

Proof. We begin with some deductions from the definition of an odd component. We observe first that any odd component of U with respect to B and f, other than C, is also an odd component of U_1 with respect to B_1 and f. Moreover if D_C is an odd component of U_C with respect to B_C and f_C, and if the divergence of f_C is zero at every vertex of D_C, then D_C is an odd component of U_1 with respect to B_1 and f. Hence

$$h(B_1,f) \geqq h(B,f)-1+h(B_C,f_C)-\sum_{u\in U_C} \operatorname{div}(f_C, u). \tag{28}$$

But, by (17),

$$\begin{aligned}
\delta(B_1,f)-\delta(B,f) &= h(B_1,f)-h(B,f)-\sum_{s\in S_C} f(s) \\
&\quad -\sum_{t\in T_C} \{\operatorname{val}(G, t)-f(t)\}+\lambda(S_C, T)+\lambda(S, T_C)+\lambda(S_C, T_C) \\
&\geqq h(B_C,f_C)-1-\sum_{u\in U_C} \operatorname{div}(f_C, u) \\
&\quad -\sum_{s\in S_C} \{f(s)-\lambda(T, s)\}-\sum_{t\in T_C} \{\operatorname{val}(C, t)-f(t)+\lambda(T, t)\} \\
&\quad +\lambda(S_C, T_C), \qquad \text{by (28)}, \\
&\geqq \delta(B_C, f_C)-1-\operatorname{div}(f_C),
\end{aligned}$$

by (17) and (26). ▮

Let A be an edge of a graph G. A G-triple $B=(S, T, U)$ can also be regarded as a G'_A-triple. Let its deficiency be $\delta(B,f)$ in G and $\delta_A(B,f)$ in G'_A.

Let us first consider the change in $\operatorname{Ind}(G'_A, U)$ when A is restored. If A has both ends in a component C of $\operatorname{Ind}(G'_A, U)$ then A is added to that component. The other components persist unchanged in $\operatorname{Ind}(G, U)$, and no component changes its parity with respect to B and f. If A joins two distinct components of $\operatorname{Ind}(G'_A, U)$ these are united with A to form a single new component of $\operatorname{Ind}(G, U)$. The new component is even if and only if the two original components were both even or both odd. The other components of $\operatorname{Ind}(G'_A, U)$ persist with their parities unchanged.

In all the remaining cases $\mathrm{Ind}(G'_A, U)$ is the same graph as $\mathrm{Ind}(G, U)$. A component of this graph has different parities in G and G'_A if A joins it to a vertex of T, but not otherwise.

Having made these observations we can use (17) to establish the following rule.

4.6. $$\delta_A(B, f) = \delta(B, f)+2,$$

if A joins two odd components of $\mathrm{Ind}(G'_A, U)$, *if it joins one odd component to T, or if it has both ends in T. In all other cases*

$$\delta_A(B,f) = \delta(B,f). \quad \blacksquare$$

5. f-Prefactors

Let f be a vertex-function of a graph G.

Let $B=(S, T, U)$ be a G-triple. An *f-prefactor of G based on B* is a spanning subgraph H of G satisfying the following three conditions.

(i) *Each edge of H is incident with a vertex of S or T.*

(ii) *If x is in S or T then* $\mathrm{val}(H, x)=f(x)$.

(iii) *If C is any component of U in G, then*

$$\sum_{c\in V(C)} \mathrm{val}(H, c) \equiv \sum_{c\in V(C)} f(c) \pmod 2. \tag{30}$$

The term "f-prefactor" is justified by the following theorem.

5.1. *Let F be an f-factor of G and let $B=(S, T, U)$ be a G-triple. Let H be the spanning subgraph of G defined by those edges of F that have an end in S or T. Then H is an f-prefactor of G based on B.*

This theorem is an easy consequence of the fact that F satisfies (2). It may suggest to us that in seeking for an f-factor of G we should first try to construct an f-prefactor H based on some $B=(S, T, U)$. Having succeeded in this we might hope to turn H into an f-factor by adding edges in the components of U. That is the basic idea of the generalized Gallai method. The next theorem shows that we must accept certain restrictions on H and B.

5.2. *Let H be an f-prefactor of G based on a G-triple $B=(S, T, U)$ such that $\delta(B,f)\geqq 0$. Then $\delta(B,f)$ must be zero.*

Moreover if C is any component of U in G then the edges of H having an end in C are the edges of G joining C to T, except that just one edge $e(C)$ must be added to or omitted from these when C is odd with respect to B and f.

Furthermore each edge of G having both ends in T belongs to H, and no edge of G with both ends in S belongs to H.

Proof. Consider any odd component C of U. By Equation (14) and Condition (iii) the number of edges of H joining C to $S\cup T$ has the parity of

$$J(B, f, C)+\sum_{c\in V(C)} \lambda(T, c).$$

But $J(B,f, C)$ is odd. Hence we can find an edge $e(C)$ of G such that either $e(C)$ is in H and joins C to S, or $e(C)$ is not in H and joins C to T.

We choose one such edge $e(C)$ for each odd component C of U. Of the resulting $h(B,f)$ edges $e(C)$ let there be p in H and q not in H.

Let k be the number of edges of H joining S to T. Then, by (2) the following inequalities must hold.

$$k \leqq \sum_{s \in S} f(s) - p, \tag{31}$$

$$k \geqq \sum_{t \in T} \{f(t) - (\text{val}\,(G, t) - \lambda(S, t))\} + q. \tag{32}$$

Combining these we find that

$$0 \geqq p + q - \sum_{s \in S} f(s) - \sum_{t \in T} f'(t) + \lambda(S, T) = \delta(B, f).$$

Since $\delta(B,f)$ is non-negative by hypothesis we deduce that it is in fact zero.

But for this deficiency to be zero equality must hold in both (31) and (32). Equality in (31) means that any edge of H with an end in S must either join S to T or be one of the p such edges of the form $e(C)$. Equality in (32) means that if an edge of G has an end in T but no end in S, then either it belongs to H or it is one of the q such edges of the form $e(C)$. The theorem follows. ▮

From 5.1 and 5.2 we can deduce the following theorem 5.3, partially revealing the relationship between f-factors and f-barriers.

5.3. *If G has an f-barrier it has no f-factor.* ▮

Theorem 5.2 indicates that G-triples of zero deficiency are likely to be important in our theory. We therefore insert an existence theorem about them.

5.4. *Suppose G to have at least one edge. Suppose further that G has no f-barrier, but G'_A has an f-barrier for each edge A of G. Then there exists a G-triple $B=(S, T, U)$ such that $\delta(B,f)=0$ in G and such that S and T are not both null.*

Proof. By 3.4 f is even-summing on G, and therefore on G'_A for each edge A.

For each A the graph G'_A has an f-barrier $B_A=(S_A, T_A, U_A)$. By 4.6 the restoration of A can decrease the deficiency of B_A by at most 2. We infer from 3.2 that $\delta(B_A,f)=2$ in G'_A and $\delta(B,f)=0$ in G.

We may now assume that $B_A=(\emptyset, \emptyset, V(G))$ for each A, since otherwise the theorem holds. By 4.6 this means that each A is an isthmus of G, whose deletion splits the corresponding component of G into two connected pieces, the end-graphs of A in G, and moreover f is odd-summing on each of these end-graphs.

We see that G is a forest. It therefore has a monovalent vertex x with the single incident edge X. Write $B=(\{x\}, \emptyset, V(G)-\{x\})$. Then $V(G)-\{x\}$ has one odd component with respect to B and f, namely the end-graph of X not including x. But $f(x)\leqq 1$, by 3.3. Accordingly $\delta(B,f)=0$ by our hypothesis. This completes the proof. ▮

We conclude the Section with an existence theorem for f-prefactors. It represents the application of bipartite theory in the generalized Gallai method.

5.5. *Let $B=(S, T, U)$ be a G-triple with $\delta(B,f)=0$. Then either G has an f-prefactor based on B or it has an f-barrier $Z=(P, Q, R)$ such that $P\subseteqq S$ and $Q\subseteqq T$.*

Proof. We construct from G a graph G_1, as follows. The edges of G_1 are the edges of G with at least one end in S or T, together with one new edge $a(C)$ for each component C of U in G. The vertices of G_1 are the members of S and T, together with two new vertices $x(C)$ and $y(C)$ for each component C of U. The ends in G_1 of $a(C)$ are $x(C)$ and $y(C)$. If an edge of G joins a component C of U to S, then it is incident with $y(C)$ in G_1. If instead it joins C to T then it is incident with $x(C)$ in G_1. The other incidences of G_1 are as in G. The construction is illustrated in Figure 1, for a case in which U has just two components C_1 and C_2.

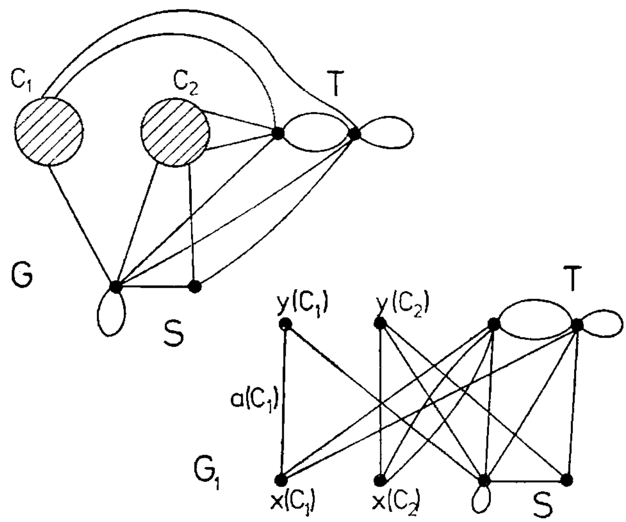

Fig. 1

We define a vertex-function f_1 of G_1 by the following rules. $f_1(x)=f(x)$ if x is in S or T. If C is a component of U in G then $f_1(x(C))=\lambda(x(C), T)$ and $f_1(y(C))$ is 0 or 1 according as C is even or odd in G with respect to B and f. As a consequence of (14),

$$f_1(x(C))+f_1(y(C)) = \sum_{c\in V(C)} f(c) \tag{33}$$

for each C.

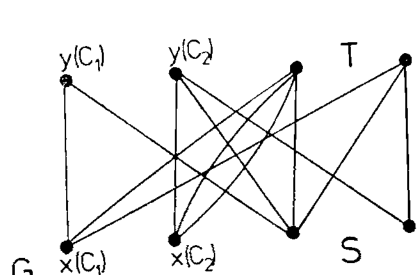

Fig. 2

We now make a further change by deleting from G_1 every edge having both ends in S, or both in T. Let the new graph be G_2. For it we define a vertex-function f_2 as follows. First we define $m(t)$, where t is in T, as the number of edges of G incident with t and having both ends in T, loops on t being counted twice. For each vertex x of G_2 not in T we write $f_2(x)=f_1(x)$. But for t in T we put $f_2(t)=f_1(t)-m(t)$. The graph G_2 is shown in Figure 2.

Let U' be the set of vertices $x(C)$, and U'' the set of vertices $y(C)$. We note that G_2 has a bipartition (X, Y), where $X=S\cup U'$ and $Y=T\cup U''$. We can show that f_2 is balanced with respect to (X, Y). For, in notation appropriate to G and f,

$$\begin{aligned}\sum_{y\in Y} f_2(y) &= h(B,f)+\sum_{t\in T}\{f(t)-m(t)\}\\ &= h(B,f)-\sum_{t\in T} f'(t)+\lambda(S,T)+\lambda(U,T)\\ &= \delta(B,f)+\sum_{s\in S} f(s)+\sum_{u\in U}\lambda(T,u), \qquad \text{by (17)},\\ &= \sum_{x\in X} f_2(x).\end{aligned}$$

Suppose G_2 to have an f_2-factor F_2. Let H_1 be the spanning subgraph of G_1 defined by the edges of F_2 with at least one end in S or T, together with the edges of G_1 having both ends in T. Then H_1 is an f_1-prefactor of G_1 based on the G_1-triple $(S, T, V(G_1)-(S\cup T))$, as we may see by applying 5.1 to G_2. The components of $V(G_1)-(S\cup T)$ in G_1, as in G_2, are the single-edge connected subgraphs defined by the edges of the form $a(C)$. Using (33) we deduce that the edges of H_1 define in G an f-prefactor H based on B.

In the remaining case we deduce from 2.2 that there exist $S_1\subseteq X$ and $T_1\subseteq Y$ such that, in terms of G_2,

$$\lambda(S_1, T_1)-\sum_{s\in S_1} f_2(s)+\sum_{t\in T_1}\{\mathrm{val}(G_2,t)-f_2(t)\}>0. \tag{34}$$

Putting this in terms of G_1 we have

$$\lambda(S_1, T_1)-\sum_{s\in S_1} f_1(s)+\sum_{t\in T_1}\{\mathrm{val}(G_1,t)-f_1(t)\}>0. \tag{35}$$

By (35) the G_1-triple (S_1, T_1, U_1), where $U_1=V(G_1)-(S_1\cup T_1)$, is an f_1-barrier of G_1. But by 4.3 any vertex of U' occurring in S_1 can be transferred to U_1 without diminishing the deficiency of the G_1-triple. By 4.4, any vertex of U'' occurring in T_1 can likewise be transferred to U_1. We can therefore assert that G_1 has an f_1-barrier $Z_1=(P, Q, R_1)$ such that $P\subseteq S$ and $Q\subseteq T$. Let $Z=(P, Q, R)$ be the G-triple with the same first and second members as Z_1.

There is a 1—1 correspondence Φ relating the components K of R in G to the components $\Phi(K)$ of R_1 in G_1. The rule is that K is obtained from $\Phi(K)$ by replacing the edges of the form $a(C)$ in it, each deleted with its two ends, by the corresponding subgraphs C of G. We note that the sum of the numbers $f_1(x)$ in $\Phi(K)$ has the same parity as the sum of the numbers $f(x)$ in K, by (33). It follows that Φ preserves parity, taken with respect to Z and f in G, and with respect to Z_1 and f_1 in G_1. Accordingly $h(Z,f)=h(Z_1,f_1)$ and therefore $\delta(Z,f)=\delta(Z_1,f_1)>0$. So Z is an f-barrier of G with the required properties.

This completes the proof. ∎

6. The general case

We can now state and prove the f-Factor Theorem, as follows.

6.1. *Let f be a vertex-function of a graph G. Then G has an f-factor or an f-barrier, but not both.*

Proof. In view of 5.3 it suffices to prove that G has an f-factor or an f-barrier.

If possible choose G and f so that G has no f-factor and no f-barrier, so that the number of edges of G has the least value α consistent with this condition, and so that the number β of vertices has the least value consistent with these conditions.

Suppose first that $\alpha=0$. Then $f(x)=0$ at each vertex x, by 3.3. But then G is its own f-factor, and we have a contradiction.

If A is any edge of G the graph G'_A has no f-factor. For any such f-factor would be an f-factor of G. So, by the choice of G, G'_A has an f-barrier. We deduce from 5.4 that there is a G-triple $B=(S, T, U)$ such that $\delta(B,f)=0$, and such that S and T are not both null. Choose such a B so that $|U|$ has the least possible value. By 5.5 G has an f-prefactor H based on B.

For each component C of U in G we define a vertex-function f_C as follows.

(36) $$f_C(c) = f(c) - \mathrm{val}\,(H, c),$$

for each c in $V(C)$.

Referring to 5.2 and the definition of even and odd components we find that f_C is even-summing on C. By 5.2 and (25) $\mathrm{div}\,(f_C, c)$ is zero at every vertex c of C if C is even, and at every vertex but one if C is odd. The exceptional vertex in the odd case is the end of $e(C)$ in C. There $\mathrm{div}\,(f_C, c)=1$. We note that, in all cases,

(37) $$\mathrm{div}\,(f_C) \leqq 1.$$

Assume that C has an f_C-barrier $B_C=(S_C, T_C, U_C)$, for a particular C. Let $B_1=(S_1, T_1, U_1)$ be the augmentation of B by B_C. By 3.2 we have

(38) $$\delta(B_C, f_C) \geqq 2.$$

It follows from (17) and the connection of C that S_C and T_C are not both null. Accordingly

(39) $$|U_1| < |U|.$$

Applying (37) and (38) to (27), and remembering that $\delta(B,f)=0$, we find that $\delta(B_1,f)\geqq 0$. Hence, by our hypothesis, $\delta(B_1,f)=0$. But because of (39) this contradicts the choice of B.

We deduce that no C has an f_C-barrier. But each C has fewer vertices than G. Hence, by the choice of G and f, there exists an f_C-factor F_C for each component C of U. But then the edges of H and the graphs F_C, taken together, define an f-factor of G, by (36). Thus our initial assumption that G and f can be chosen so that G has no f-factor and no f-barrier is false. The theorem follows. ∎

Let us now consider some specializations. A necessary and sufficient condition for the existence of a 1-factor appears in [8]. In order to state it we need the following definition. If $S \subseteq V(G)$ then we write $h(S)$ for the number of components of $V(G)-S$ in G with an odd number of vertices. The "1-Factor Theorem" is as follows.

6.2. *G is without a* 1*-factor if and only if there is a subset S of V*(*G*) *such that*

$$|S| < h(S). \tag{40}$$

Proof. We apply 6.1 to the case in which f is the unit vertex-function. Using 4.4 we find that G is without a 1-factor if and only if it has an f-barrier of the form $B=(S, \emptyset, V(G)-S)$, that is if and only if there is a subset S of $V(G)$ satisfying (40). ▮

The most famous of graph-factor theorems, Petersen's Theorem, runs as follows.

6.3. *Let G be a cubic graph, either without isthmuses or having all its isthmuses in a single arc. Then G has a* 1*-factor.*

Proof. Suppose G to have no 1-factor. Then by 6.2 there exists $S \subseteq V(G)$ satisfying (40). Since the number of vertices of a cubic graph is even $|S|$ and $h(S)$ have the same parity. Hence

$$|S| \leqq h(S)-2. \tag{41}$$

Of the $h(S)$ components of $V(G)-S$ with an odd number of vertices let there be p joined to S by a single edge, and q joined to S by 3 or more. (The number of joining edges of such a component must be odd.) Then

$$3|S| \geqq \lambda(S, V(G)-S) \geqq p+3q. \tag{42}$$

Since $h(S)=p+q$ it follows from (41) and (42) that p is at least 3. Accordingly G has three isthmuses not lying on a single arc. ▮

There is an interesting application of the f-Factor Theorem in a proof of the Erdős—Gallai Theorem on graphic sequence [1].

Let $P=(f_1, f_2, ..., f_p)$ be a non-increasing sequence of p non-negative integers f_i, such that the f_i sum to an even number $2q$. We say that P is *strictly graphic* if there is a graph G, without loops or multiple joins, whose p vertices can be enumerated as $v_1, v_2, ..., v_p$ so that val $(G, v_i)=f_i$ for each suffix i. The Erdős—Gallai Theorem gives a necessary and sufficient condition for P to be strictly graphic.

Let K_p be a complete p-graph, with vertices $v_1, v_2, ..., v_p$. Let f be the vertex-function on K_p such that $f(v_i)=f_i$ for each i. Evidently P is strictly graphic if and only if K_p has an f-factor. So, by 6.1 P fails to be strictly graphic if and only if K_p has an f-barrier $B=(S, T, U)$.

The theory simplifies in this case since U has at most one component in G. Hence $h(B,f)=0$ or 1. Moreover f is even-summing on K_p and therefore $\delta(B,f)$ is even, by 3.2. We can therefore assert that P fails to be strictly graphic if and only if there is a K_p-triple $B=(S, T, U)$ such that

$$\lambda(S, T)-\sum_{s\in S} f(s)-\sum_{t\in T} f'(t) > 0. \tag{43}$$

Some improvements on (43) are possible. For example we can arrange that if t is in T then $f(t)\geqq|T|+|U|$, for otherwise we could transfer t to U without diminishing the left side of (43). Considering possible transfers from U to T we see that we can then arrange that if u is in U then $f(u)<|T|+|U|$. By transfers between

S and U we can further arrange that $f(s) \leqq |T|$ if s is in S, and that $f(u) > |T|$ if u is in U.

These arrangements having been made we observe that T now consists of the vertices with suffixes from 1 to r, where $r = |T|$. We have moreover $\lambda(S, T) = r(p-r-|U|)$ and $f'(t) = p-1-f(t)$. We can now assert that P fails to be strictly graphic if and only if there is a non-negative integer r, not exceeding p, such that

(44) $$\sum_{i=1}^{r} f_i > r(r-1) + \sum_{i=r+1}^{p} \min(r, f_i).$$

This is the condition given by Erdős and Gallai.

7. Variations

Let G be a graph. Let f and g be vertex-functions of G such that

(45) $$\mathrm{val}(G, x) \geqq f(x) \geqq g(x) \geqq 0,$$

for each vertex x of G. We define a *(g, f)-subgraph* of G as a spanning subgraph H such that

(46) $$f(x) \geqq \mathrm{val}(H, x) \geqq g(x)$$

for each x in $V(G)$. Thus an f-factor of G is an (f, f)-subgraph.

Graph-factor duality carries over into the theory of (g, f)-subgraphs. For if H is any (g, f)-subgraph of G then the remaining edges of G define an (f', g')-subgraph. However we shall not emphasize this aspect of the theory in the present Section.

Let H be a (g, f)-subgraph of G. We define its *shortcoming* $M(H, f)$ with respect to f as follows.

(47) $$M(H, f) = \sum_{x \in V(G)} \{f(x) - \mathrm{val}(H, x)\}.$$

This number measures the amount by which H fails to be an f-factor. We note that

(48) $$M(H, f) \equiv \Sigma(G, f) \pmod 2.$$

Most published extensions and generalizations of the f-Factor Theorem are theorems about (g, f)-subgraphs. Often they derive fairly easily from the following theorem 7.1. Yet this theorem is not a true generalization of the f-Factor Theorem but a simple deduction from it.

We need one more definition. Let $B = (S, T, U)$ be any G-triple. Then we define $h_0(B, f, g)$ as the number of components C of U in G such that C is odd with respect to B and f, and such that $f(c) = g(c)$ for each vertex c of C.

7.1. *Let f and g be vertex-functions of G satisfying* (45). *Let M and q be non-negative integers, M having the parity of $\Sigma(G, f)$. Then either G has a (g, f)-subgraph H such that*

(49) $$M - 2q \leqq M(H, f) \leqq M$$

or it has a G-triple $B=(S, T, U)$ satisfying one of the following conditions (i), (ii) *and* (iii). *But it cannot have both.*

(i) $\delta(B,f) > M$,

(ii) $\delta(B, g) > \sum_{x \in V(G)} \{f(x)-g(x)\}-M+2q$,

(iii) $h_0(B, f, g)-\sum_{s\in S} f(s)-\sum_{t\in T} g'(t)+\lambda(S, T) > 0.$

Proof. We construct a graph K from G as follows. We introduce one new vertex k. If x is in $V(G)$ we join k to x by exactly $f(x)-g(x)$ new edges. Finally we attach q new loops to k. Next we define a vertex-function f_1 of K. We put $f_1(k)=M$ and write $f_1(x)=f(x)$ for each x in $V(G)$. We note that f_1 is even-summing on K, by (48). It is easily verified that G has a (g,f)-subgraph H satisfying (49) if and only if K has an f_1-factor.

By 6.1 either G has such an H or there is a K-triple $Z=(P, Q, R)$ such that

(50) $$\delta(Z,f_1) > 0 \quad \text{(but not both).}$$

With such a Z there are three possibilities: k is in P, Q or R. Conditions (i), (ii) and (iii) are obtained by stating (50) in terms of G for each of these.

To deal with the first possibility we consider any G-triple $B=(S, T, U)$ and the K-triple $Z=(P, Q, R)$, with k in P, such that $S=P-\{k\}$, $T=Q$ and $U=R$. We note that Ind (G, U) and Ind (K, R) are identical, and that $J(B,f, C)=J(Z,f_1, C)$ for each component C of Ind (G, U) by (14). Hence, by (17),

$$\begin{aligned}\delta(Z,f_1) &= h(B,f)-\sum_{s\in S} f(s)-M\\ &\quad -\sum_{t\in T}\{f'(t)+(f(t)-g(t))\}+\lambda(S, T)\\ &\quad -\sum_{t\in T}(f(t)-g(t))\\ &= \delta(B,f)-M.\end{aligned}$$

We deduce that there is a Z with k in P and satisfying (50) if and only if there is a B satisfying Condition (i).

Next we consider any G-triple $B=(S, T, U)$ and the K-triple $Z=(P, Q, R)$, with k in Q, such that $S=P$, $T=Q-\{k\}$ and $U=R$. Again the graphs Ind (G, U) and Ind (K, R) are identical, but now $J(Z,f_1, C)=J(B,f, C)+\Sigma(f(c)-g(c))$ for each component C, the sum being over the vertices c of C. Hence, by (17)

$$\begin{aligned}\delta(Z,f_1) &= h(B, g)-\sum_{s\in S} f(s)\\ &\quad -\sum_{t\in T}\{f'(t)+(f(t)-g(t))\}-\Big\{\sum_{x\in V(G)}(f(x)-g(x))-M+2q\Big\}\\ &\quad +\lambda(S, T)+\sum_{s\in S}(f(s)-g(s))\\ &= h(B, g)-\sum_{s\in S} g(s)-\sum_{t\in T} g'(t)+\lambda(S, T)\\ &\quad -\sum_{x\in V(G)}(f(x)-g(x))-2q+M.\end{aligned}$$

Accordingly there is a Z with k in Q and satisfying (50) if and only if there is a B satisfying Condition (ii).

Finally we consider any G-triple $B=(S, T, U)$ and the K-triple $Z=(P, Q, R)$, with k in R, such that $S=P$, $T=Q$ and $U=R-\{k\}$. The components of Ind (G, U) for which f and g agree at each vertex persist as components of Ind (K, R), and $J(Z, f_1, C)$ is equal to $J(B, f, C)$ for each of them. But the other components of Ind (G, U) all unite with k to form a single component D of Ind (K, R). Put $d=0$ or 1 according as D is even or odd with respect to Z and f_1. By (17) we have

$$\delta(Z, f_1) = d+h_0(B, f, g)-\sum_{s\in S} f(s)$$
$$-\sum_{t\in T}\{f'(t)+(f(t)-g(t))\}+\lambda(S, T).$$

Now $\delta(Z, f_1)$ is necessarily even, by 3.2. It therefore follows from the above result that there is a Z with k in R and satisfying (50) if and only if there is a B satisfying Condition (iii). This completes the proof of 7.1. ∎

We go on to consider some specializations of 7.1. One possibility is to take M and q so large, with $2q \geqq M$, as to impose no restrictions on $M(H, f)$ other than

$$0 \leqq M(H, f) \leqq \sum_{x\in V(G)} (f(x)-g(x)), \tag{51}$$

which is required by (46) and (47). With sufficiently large M and q Conditions (i) and (ii) of 7.1 become impossible. Thus we deduce the following result, called Lovász' Theorem, from 7.1. [5].

7.2. *Let f and g be vertex-functions of G satisfying* (45). *Then G has either a (g, f)-subgraph or a G-triple $B=(S, T, U)$ satisfying Condition* (iii) *of* 7.1, *but not both.* ∎

M. Las Vergnas [4] has made an interesting observation on the special case of Lovász' Theorem in which $g(x)$ is 0 or 1 for each x. Essentially he finds that if G has no (g, f)-subgraph then B can be chosen so as to satisfy the following condition: each vertex t of T satisfies $g(t)=1$ and is incident in G only with edges joining it to S.

We can prove this by a careful study of the circumstances in which a vertex can be transferred from T to U or from U to S without diminishing the expression on the left of the inequality of Condition (iii).

Given $B=(S, T, U)$ satisfying (iii) we write Q for the set of all components C of U in G such that C is odd with respect to B and f, and such that f and g agree at every vertex of C.

Take first a vertex t of T. Let $\alpha(t)$ be the number of members of Q joined to t. If t is transferred to U the expression on the left of the inequality of (iii) increases by not less than

$$\text{val}(G, t)-g(t)-\lambda(S, t)-\alpha(t).$$

By making transfers from T to U we arrange that each remaining vertex t of T corresponds to a negative value of the above expression. This means that $g(t)=1$, that each edge incident with t joins it either to S or to a member C of Q, and that no two edges of G join t to the same member of Q.

Let us now suppose that some edge E joins t to a vertex u of some C in Q. If $f(u)=g(u)=0$ it is permissible to transfer u to S. For the possible decrease of 1 in $h_0(B,f,g)$ is offset by the increase in $\lambda(S,T)$. If $f(u)=g(u)=1$ the transfer is still permissible, since it does not now diminish $h_0(B,f,g)$. This is because the sum of the numbers

$$f(x)+\lambda(T,x)$$

over all the remaining vertices x of C is still odd.

When no more transfers from T to U or from U to S can be made without diminishing the expression of the left of the inequality of Condition (iii) it follows from the preceding observations that B satisfies Las Vergnas' condition.

We now specialize 7.1 to the case in which $q=0$ and $g(x)=0$ for each x. We take M to be an integer satisfying the following condition (52) and having the parity of $\Sigma(G,f)$.

(52) $$0 \leqq M \leqq \Sigma(G,f).$$

If Condition (ii) of 7.1 holds we can transfer each vertex of T to U by 4.4. The transfer is also possible when Condition (iii) holds, as we have seen in our discussion of Las Vergnas' Theorem. So when $B=(S,T,U)$ satisfies (ii) or (iii) we can assume that T is null. But then $h(B,g)$, $h_0(B,f,g)$ and $\lambda(S,T)$ are all zero and the stated inequalities are impossible. Accordingly we have the following theorem.

7.3. *Let f and g be vertex-functions of G satisfying* (45), *$g(x)$ being zero for each x. Let M be an integer satisfying* (52) *and having the parity of $\Sigma(G,f)$. Then G has either a (g,f)-subgraph H such that $M(H,f)=M$ or a G-triple B satisfying Condition* (i) *of* 7.1, *but not both.* ∎

The special case of 7.3 in which f is the unit vertex-function is of special interest. We can then suppose B of the form $(S, \emptyset, V(G)-S)$, by 4.4. Condition (i) is then equivalent to the assertion that

(53) $$M+|S| < h(S).$$

Here we have Berge's generalization of the 1-Factor Theorem.

We now take note of a variation on graph-factor theory using edge-capacities. We suppose given a vertex-function f of G. To each edge E of G we assign a non-negative integer $c(E)$ called its *capacity*. A *solution of f in G*, with respect to the capacities, is a function p assigning to each edge E a non-negative integer $p(E)$ not exceeding $c(E)$, and having the following property: for each vertex x of G the sum of the numbers $p(E)$ assigned to the incident edges, loops being counted twice, is $f(x)$. J. Edmonds has pointed out to the author that such solutions can be found by algorithmic methods. Here we are concerned only to relate their theory to that of f-factors.

Let us construct from G a graph K as follows. We replace each edge E by $c(E)$ distinct edges with the same two ends. Evidently f has a solution in G, with respect to the given capacities, if and only if K has an f-factor. The theory of f-factors is thus easily modified so as to deal with solutions of f. Essentially we have only to interpret the symbol $\lambda(S,T)$ as the sum of the capacities of the edges joining S to T rather than as the number of such edges.

References

An explicit reference to the result I have called Ore's Theorem eludes me. However it, like most of the theory of Section 2, derives from Ore's theory of directed graphs set out in [6]. A graph with a bipartition (X, Y) has a natural orientation, each edge being directed from X to Y. The reference for Hall's Theorem is [3], for Petersen's Theorem [7] and for the 1-Factor Theorem [8]. The f-Factor Theorem, for loopless graphs, is proved in [9] from the theory of alternating paths, and in [10] from the 1-Factor Theorem. Some improvements in the theory are given in [11] and [12].

[1] P. Erdős and T. Gallai, Graphs with prescribed valencies, *Mat. Lapok* **11** (1960) 264—274. (Hungarian).
[2] T. Gallai, Neuer Beweis eines Tutte'schen Satzes, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), 135—139.
[3] P. Hall, On representatives of subsets, *J. London Math. Soc.* **10** (1935), 26—30.
[4] M. Las Vergnas, An extension of Tutte's 1-factor theorem, *Discrete Math.*, **23** (1978), 241—255.
[5] L. Lovász, Subgraphs with prescribed valencies, *J. Combinatorial Theory* **8** (1970), 391—416.
[6] O. Ore, Studies on directed graphs I, II, III. *Ann. Math.* **63** (1956), 383—406; **64** (1956), 142—153; **68** (1958), 526—549.
[7] J. Petersen, Die Theorie der regulären Graphs, *Acta Math.* **15** (1891), 193—220.
[8] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* **22** (1947), 107—111.
[9] W. T. Tutte, The factors of graphs, *Can. J. Math.*, **4** (1952), 314—328.
[10] W. T. Tutte, A short proof of the factor theorem for finite graphs, *Can. J. Math.* **6** (1954), 347—352.
[11] W. T. Tutte, Spanning subgraphs with specified valencies, *Discrete Math.* **9** (1974), 97—108.
[12] W. T. Tutte, The subgraph problem, *Annals of Discrete Math.* 3 (1978), 289—295.