Some fundamental limitations of learning for dynamics and control

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American Control Conference

New Orleans, LA

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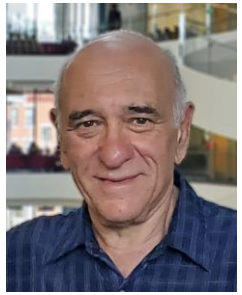
Outline

- **The role of learning in dynamics and control**

- **On some fundamental limitations of learning immersions (Koopman embeddings)**



Joint work with
Zexiang Liu (Michigan) and Eduardo Sontag (Northeastern)
arXiv:2312.17045 **(IFAC WC'23, Automatica'25)**

- **On the hardness of learning to stabilize linear systems**

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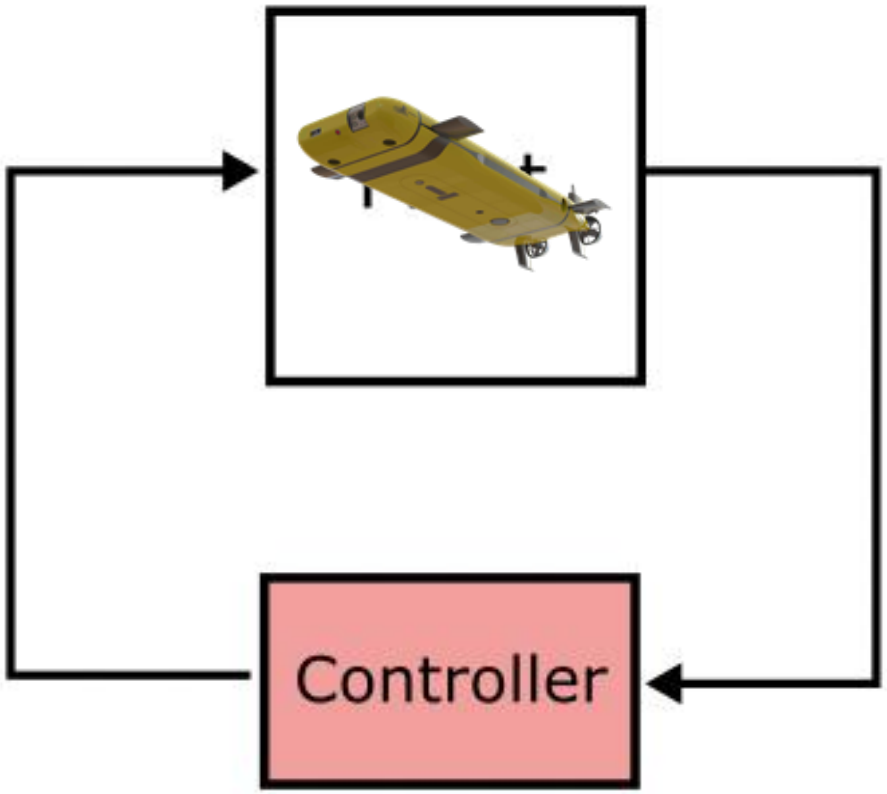
Xiong Zeng (Michigan), Zexiang Liu (Michigan), Zhe Du (Michigan), and Mario Sznaier (Northeastern)
arXiv:2311.11151 **(CDC'23)**

Plant (+ operating environment)

Task or mission specification

φ



Many components in a feedback loop can be learned:

- **Plant model:** model-based learning, system identification
- **Task specification:** inverse optimal control, inverse reinforcement learning
- **Controller:** model-free learning
- **Perception/estimation modules**

GOAL: **Ensure** system-level safety and performance when some of these pieces are learned!

Plant (+ operating environment)

Task or mission specification

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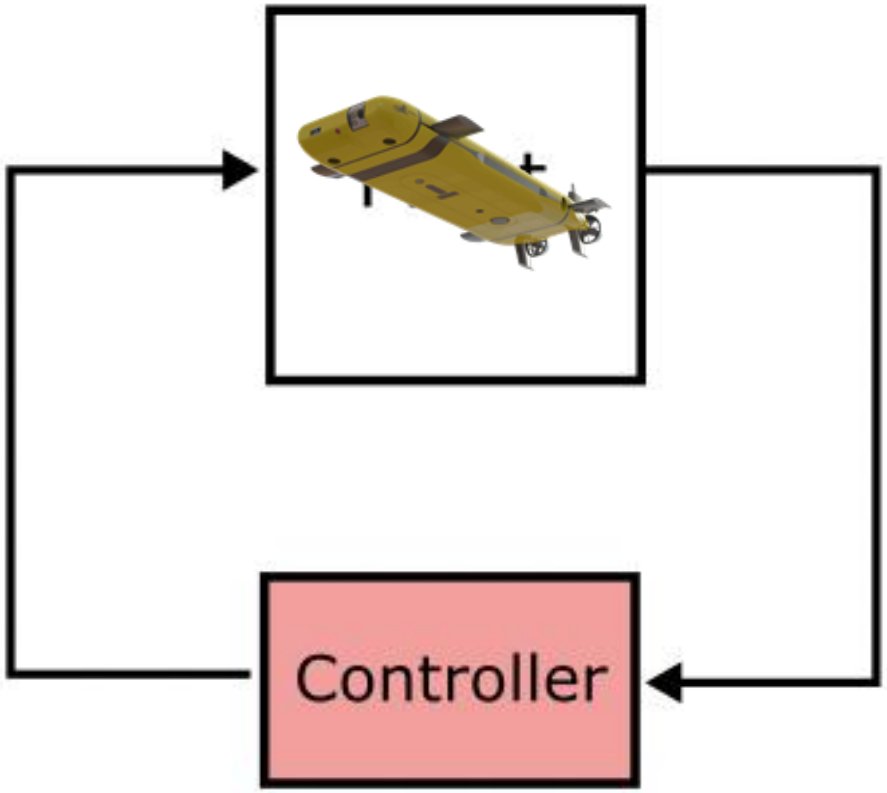

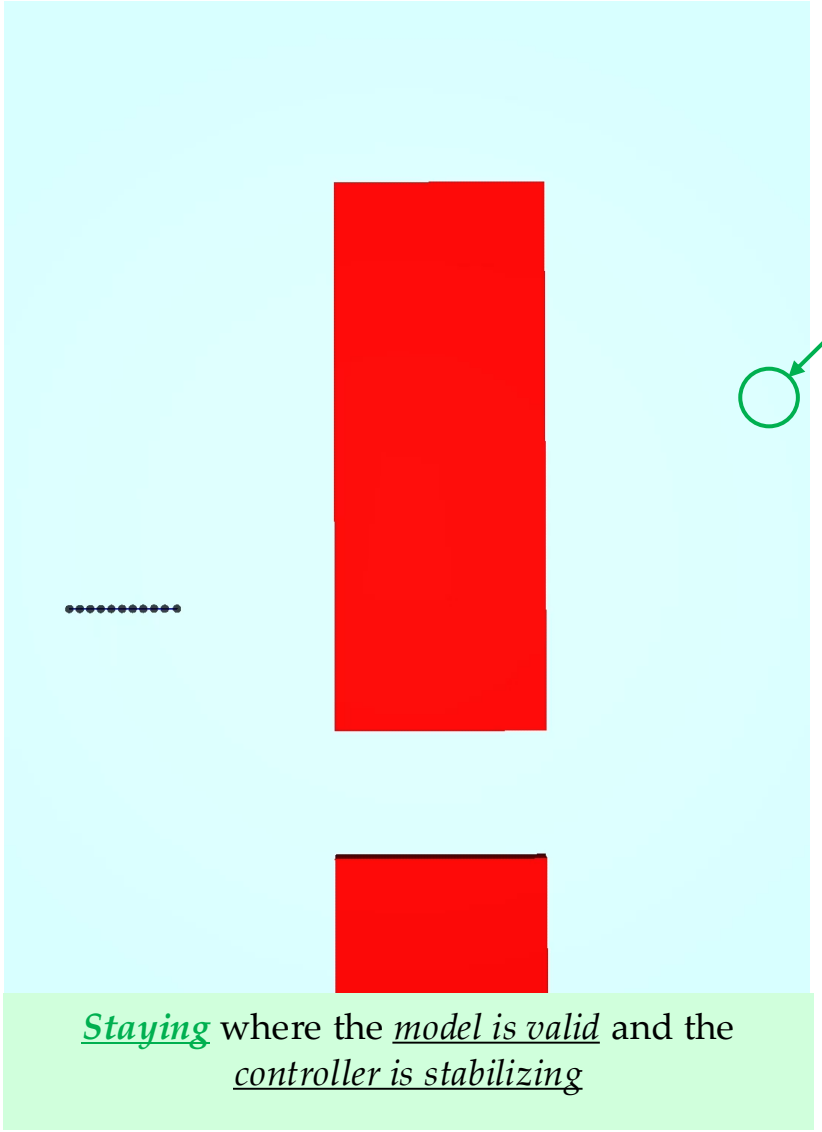

Learned plant model
(Chou, Ozay, Berenson CDC'21)

Task or mission specification

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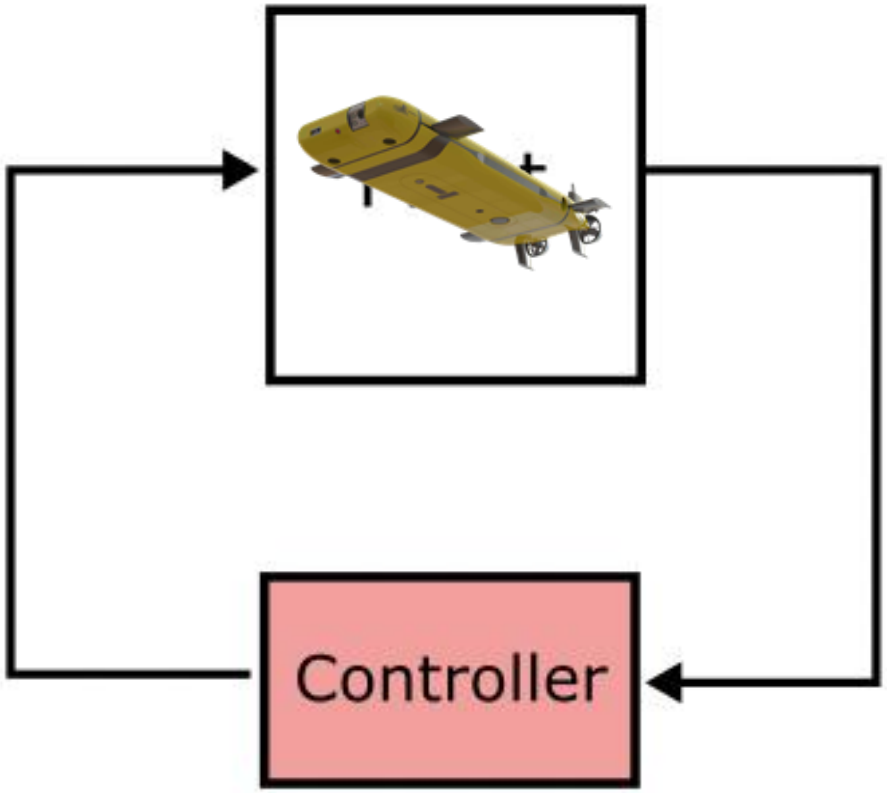
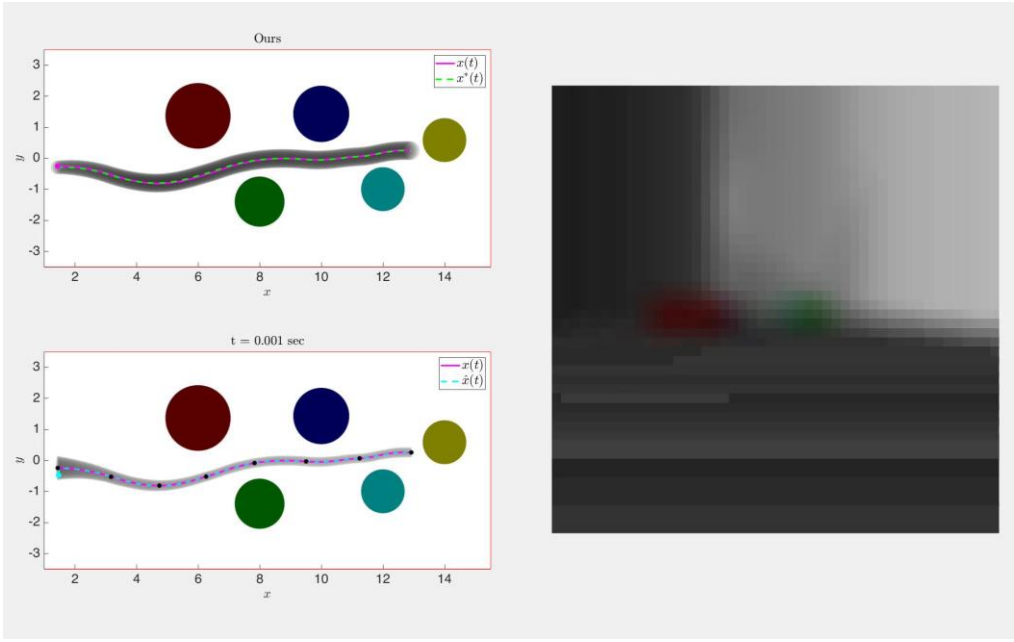
Plant (+ operating environment)





Legend

Tubes executed trajectory planned trajectory estimated trajectory

Nominal dynamics:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v\cos(\phi) \\ v\sin(\phi) \\ u_1 \\ u_2 \end{bmatrix}$$

Goal: safely reach goal using RGB-D images at runtime

Learned perception module

(Chou, Ozay, Berenson WAFR'22)

Plant (+ operating environment)

Task or mission specification

Demonstration (in VR)

Execution in new scene

Task:

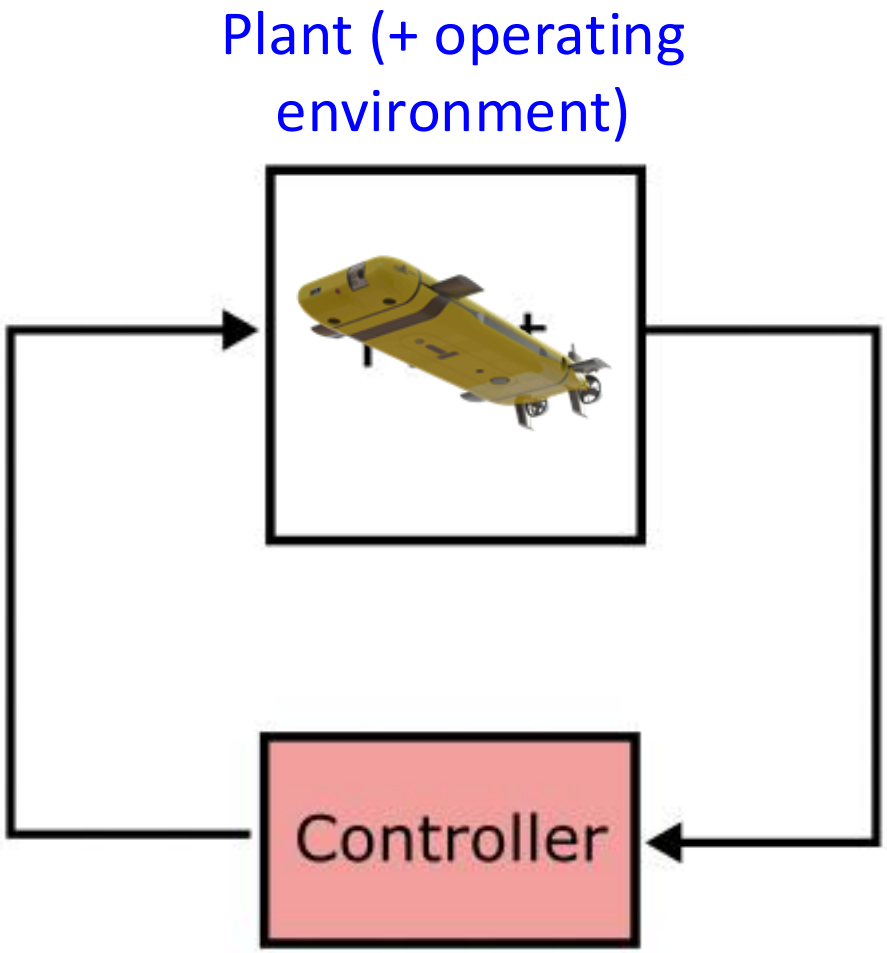
grasp soup → place in small box → deliver to blue region → grasp Cheez-Its → place in large box

Learned specification

(Chou, Ozay, Berenson RSS'20, AURO'22)

Task or mission specification

φ



Many components in a feedback loop can be learned:

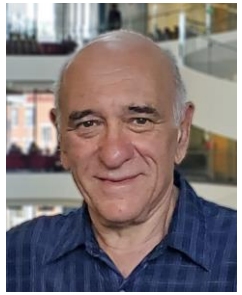
- **Plant model:** model-based learning, system identification
- **Task specification:** inverse optimal control, inverse reinforcement learning
- **Controller:** model-free learning
- **Perception/estimation modules**

What are the fundamental limits in learning-based control?

Outline

- The role of learning in dynamics and control

- **On some fundamental limitations of learning immersions (Koopman embeddings)**



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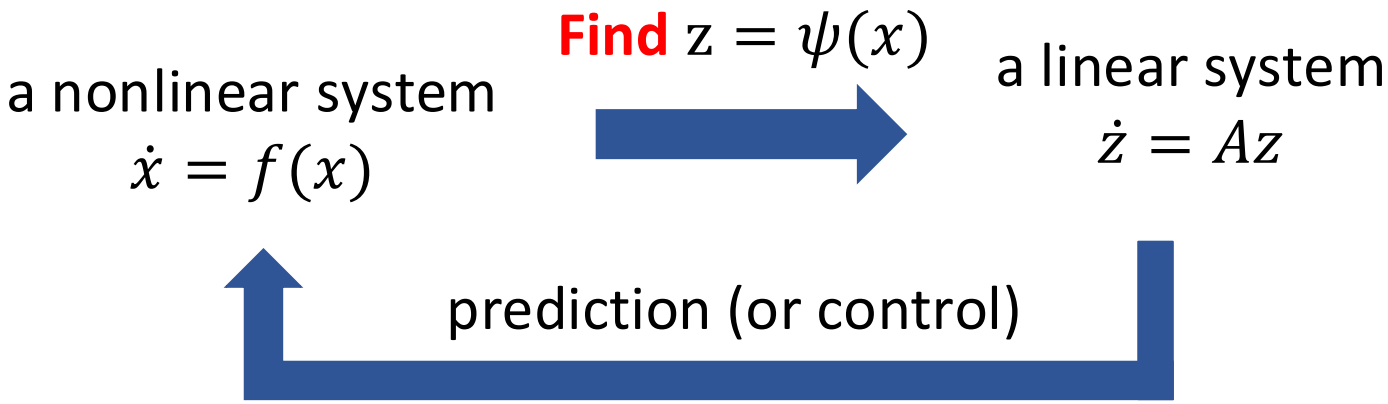
Linear immersions

The idea of Koopman-inspired methods:

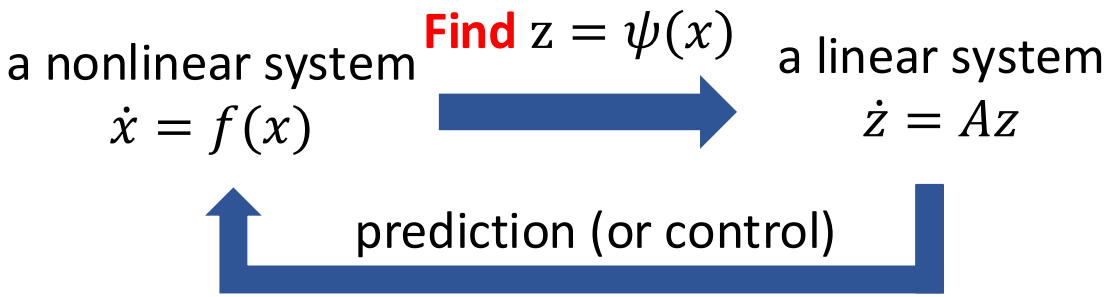


In Practice: Represent $\psi(x)$ as a neural network and learn the network weights and *A* from data

Question: Does such a $\psi(x)$ **exist or not**?

Existence of linear immersion?

The idea of Koopman-inspired methods:



Question: Does such a $\psi(x)$ **exist or not**?

Answer: Sure! Take $\psi(x) = c \ \ \forall x$, and $A = 0$
(constant function)

Question: Does such a one-to-one $\psi(x)$ **exist or not**?

Existence of linear immersion?

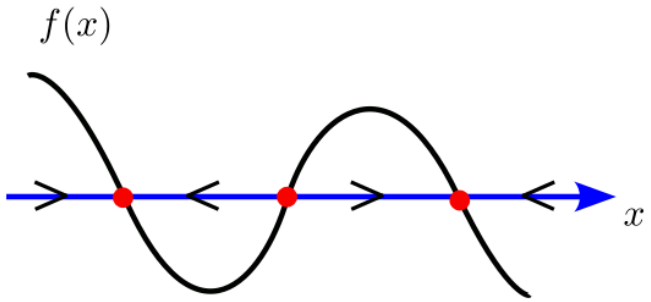
The idea of Koopman-inspired methods:

a nonlinear system $\dot{x} = f(x)$ → **Find** $z = \psi(x)$ → a linear system $\dot{z} = Az$

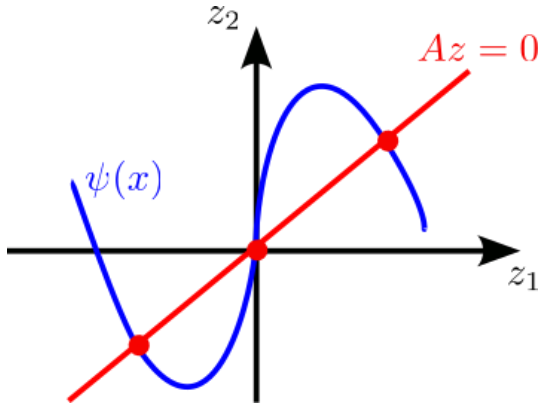
prediction (or control)

Question: Does a one-to-one $\psi(x)$ **exist or not**?

1D example:



$\psi(x)$?



The only possibility is that A is rank-deficient!

The power of discontinuity

1D example:

Consider the 1-d system with **three isolated equilibria**

$$\dot{x} = \frac{x(1-x^2)}{1+x^2}.$$

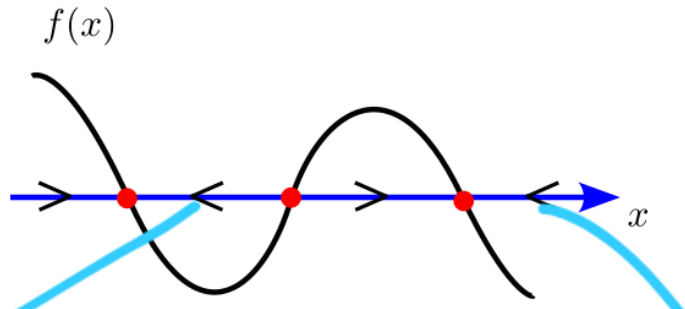
Equilibrium points are $\{-1, 0, 1\}$.

Then,

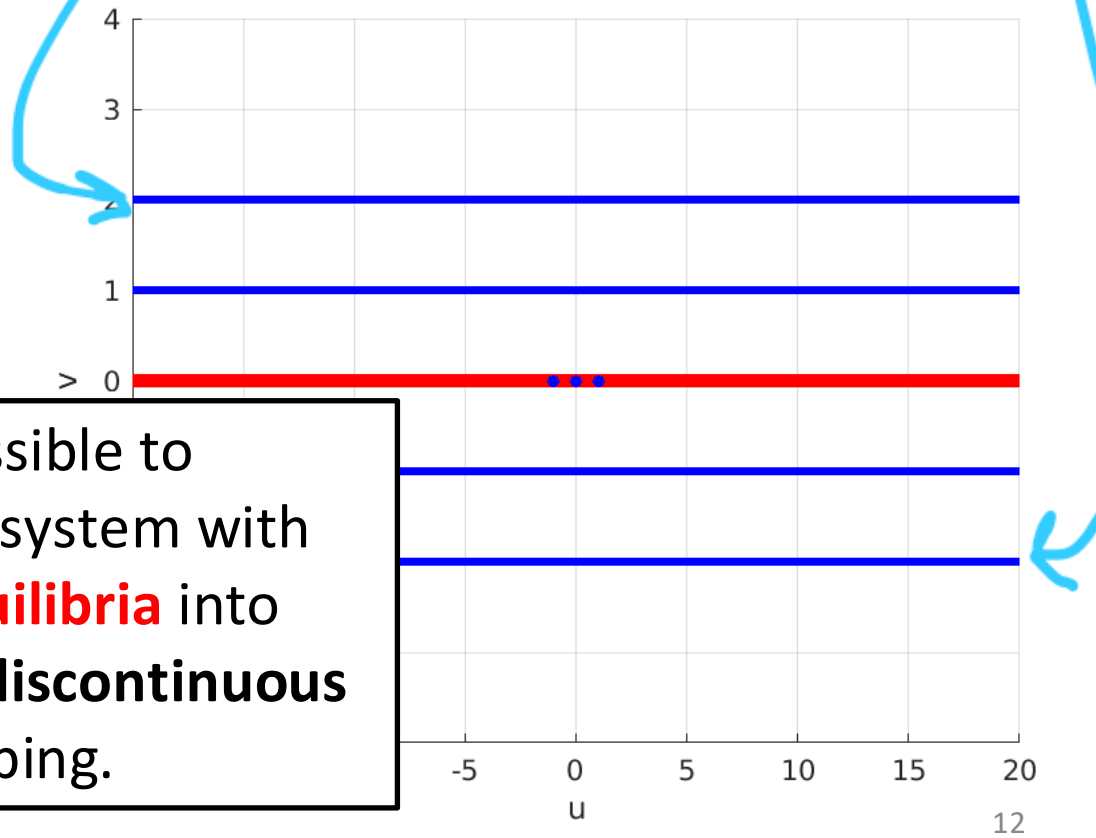
$$\dot{x} = \frac{x(1-x^2)}{1+x^2} \xrightarrow{\psi(x)} \begin{cases} \dot{u} = v \\ \dot{v} = 0 \end{cases}$$

where

$$\psi(x) = \begin{cases} (x, 0) & f(x) = 0 \\ \left(\ln\left|\frac{3x}{-2+2x^2}\right|, 1\right) & \\ \left(2\ln\left|\frac{3x}{-2+2x^2}\right|, 2\right) & \\ \left(-\ln\left|\frac{3x}{-2+2x^2}\right|, -1\right) & \\ \left(-2\ln\left|\frac{3x}{-2+2x^2}\right|, -2\right) & x > 1. \end{cases}$$



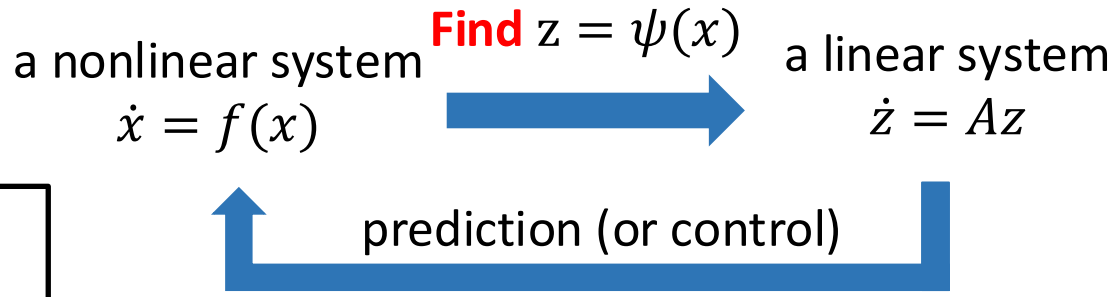
Question: Does a one-to-one $\psi(x)$ **exist or not**?



Observation: It is possible to immerse a nonlinear system with **multiple isolated equilibria** into **linear systems** via a **discontinuous** and **one-to-one** mapping.

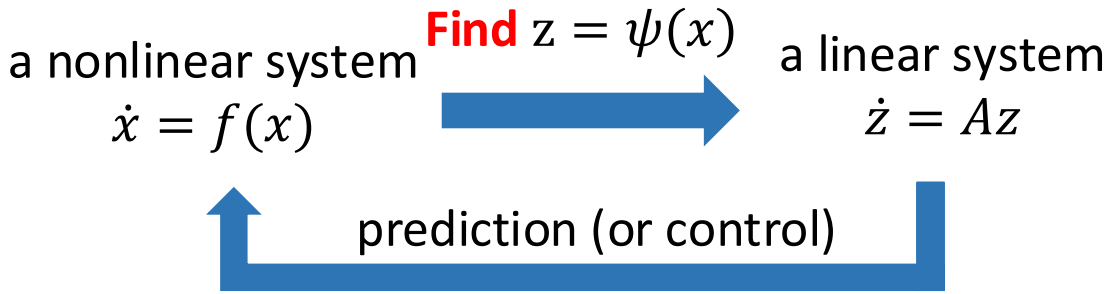
Discontinuity?

When ψ is learned via a neural network NN (with continuous activation functions), ψ is a **continuous** function.



Question: Does a one-to-one $\psi(x)$ **exist or not**?

Some definitions



Given an initial state $\xi \in \mathcal{X}$, the **ω-limit set** $\omega^+(\xi)$ is the set of all $x \in \mathcal{X}$ where there exists a sequence $t_n \to +\infty$ such that $\varphi(t_n, \xi) \to x$.

Given an initial state $\xi \in \mathcal{X}$, the trajectory $\varphi(t, \xi)$ is **precompact** *in* $\mathcal{X}$ if the closure of the set $\{\varphi(t, \xi) | t \geq 0\}$ is compact with respect to the subspace topology on $\mathcal{X}$.

For each ω-limit set Ω, its **domain of attraction** is defined by $D^+(\Omega) := \{\xi \in \mathcal{X} \mid \omega^+(\xi) = \Omega\}$.

Question: A **linear system** does not possess **more than one isolated ω-limit sets**. Can nonlinear systems with **multiple isolated ω-limit sets** be immersed into **linear systems** via **continuous** and **one-to-one** mappings?

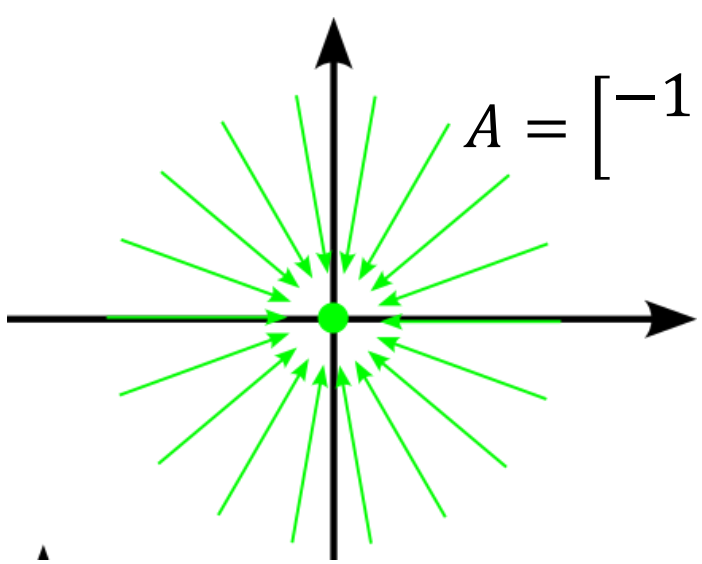
Systems with closed basins

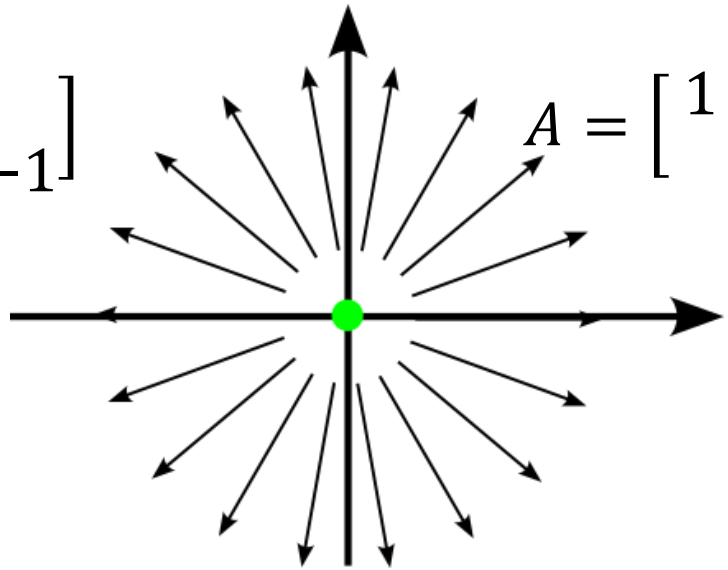
a nonlinear system $\dot{x} = f(x)$ — **Find** $z = \psi(x)$ → a linear system $\dot{z} = Az$

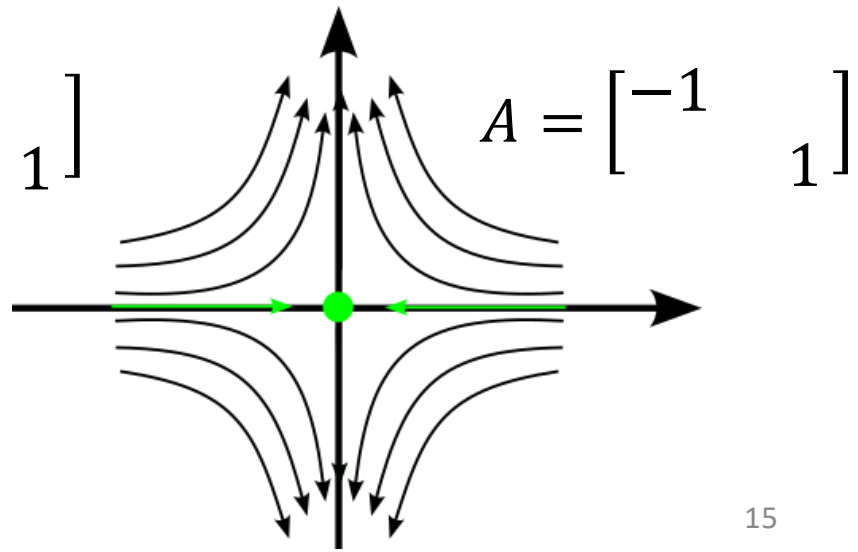
prediction (or control)

Defn. A system $\dot{x} = f(x)$ on $\mathcal{X}$ has ***closed basins*** if the domain of attraction $D^+(\Omega)$ is closed for all ω-limit set Ω in $\mathcal{X}$.

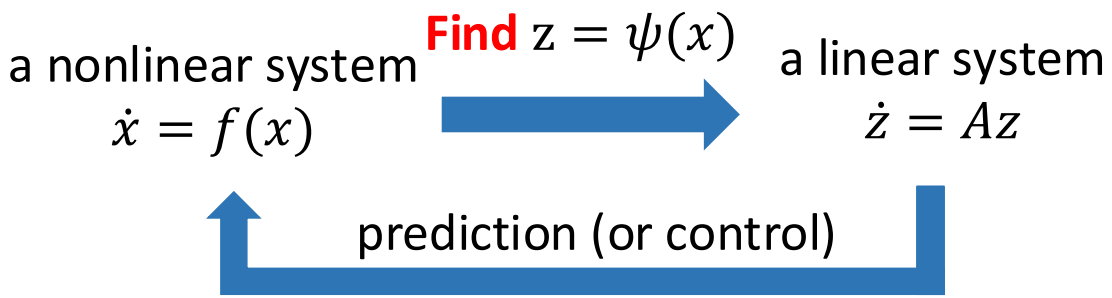
Lemma: Every linear system with $\mathcal{X} = \mathbb{R}^n$ has closed basins.







On the non-existence of linear immersion



Main result:

Theorem: Under the assumptions on $\mathcal{X}$ (connected, forward invariant), suppose that:

(T1) $\dot{x} = f(x)$ on $\mathcal{X}$ can be immersed in a system with closed basins by a one-to-one continuous mapping ψ;

(T2) trajectories of $\dot{x} = f(x)$ on $\mathcal{X}$ are precompact in $\mathcal{X}$;

(T3) the set $\mathcal{W}$ of all the ω-limit sets in $\mathcal{X}$ is finite or countable.

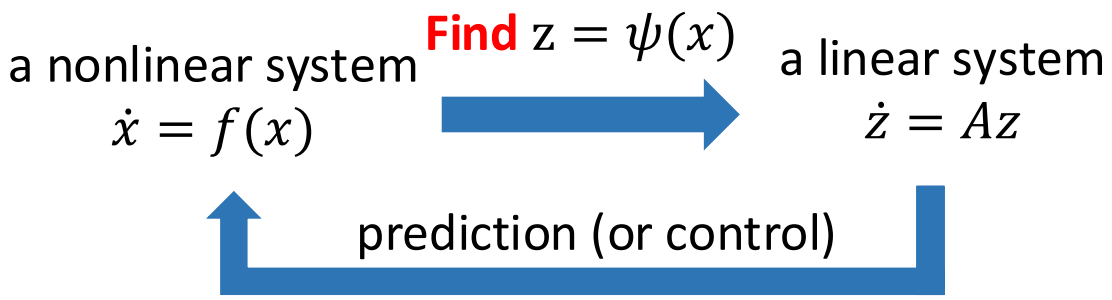
Then $\mathcal{W}$ has exactly one element.

In other words,

- if ψ is continuous and one-to-one, and
- if trajectories of $\dot{x} = f(x)$ on $\mathcal{X}$ are precompact in $\mathcal{X}$
- if f has countable but more than one ω-limit sets in $\mathcal{X}$

Then, f cannot be immersed in a linear system!

On the non-existence of linear immersion



Main result "less formal":

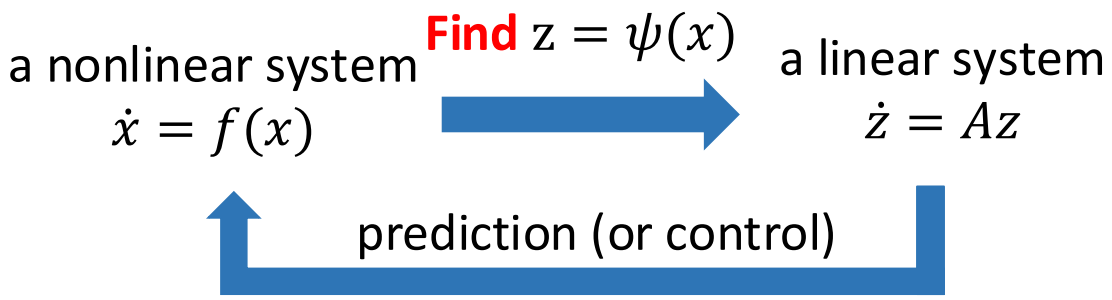
- if ψ is continuous and one-to-one, and
- if trajectories of $\dot{x} = f(x)$ on $\mathcal{X}$ are precompact (or backward precompact) in $\mathcal{X}$
- if f has countable but more than one ω-limit sets in $\mathcal{X}$

Then, f cannot be immersed in a linear system with ψ!

These conditions are not only sufficient for non-existence but also necessary in the sense that

- if ψ is discontinuous but other conditions hold, we saw an example of a 1-1 linear immersion
- if there are trajectories that are neither precompact nor backward precompact, there exists a system with countable ω-limit sets that admits a continuous 1-1 linear immersion (example due to Kvalheim and Arathoon)
- if f has uncountably many ω-limit sets, there exist examples with continuous 1-1 linear immersions

On the non-existence of linear immersion



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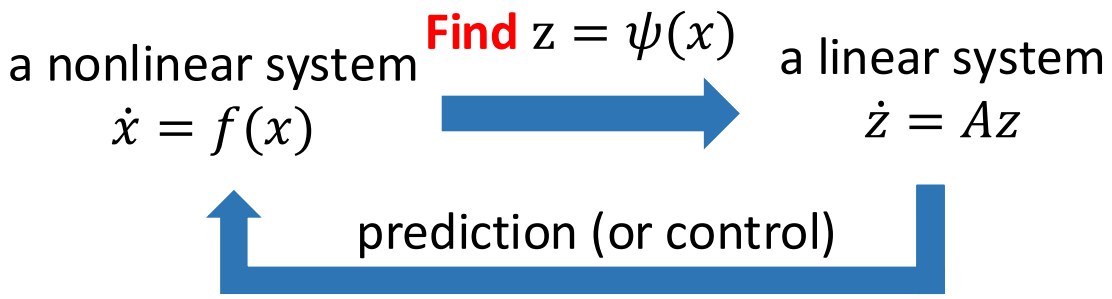
- if f has countable but more than one ω-limit sets in $\mathcal{X}$

Then, f cannot be immersed in a linear system with ψ!

Implications for finding/learning $\boldsymbol{\psi(x)}$: If using a NN with continuous activation functions:

- All omega limit sets will collapse to one (cannot be inverted to predict x)
- If we embed x in ψ (i.e., $\psi(x) = [x; \phi(x)]$), need discontinuities (can lead to high Lipschitz constants for the NN, degrading robustness)

More examples

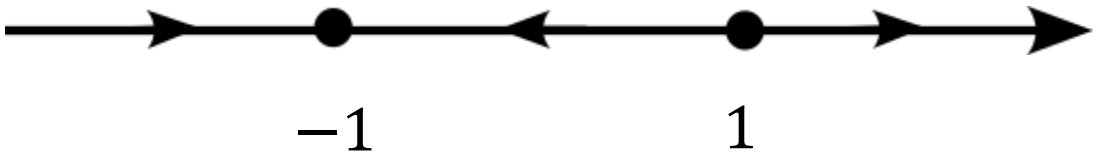


Consider the 1-d system

$$\dot{x} = x^2 - 1.$$

The ω-limit sets are $x = 1$ and $x = -1$.

Let $\mathcal{X} = (-\infty, 1)$. Then,



$$\dot{x} = x^2 - 1 \quad \longrightarrow \quad \dot{z} = -2z$$

$$\psi(x) = \frac{x+1}{x-1}$$

If we extend $\mathcal{X}$ by a point to $\mathcal{X}' = (-\infty, 1]$, ψ is not an immersion anymore as $\psi(1)$ is undefined.

More examples

a nonlinear system $\dot{x} = f(x)$ **Find** $z = \psi(x)$ a linear system $\dot{z} = Az$

prediction (or control)

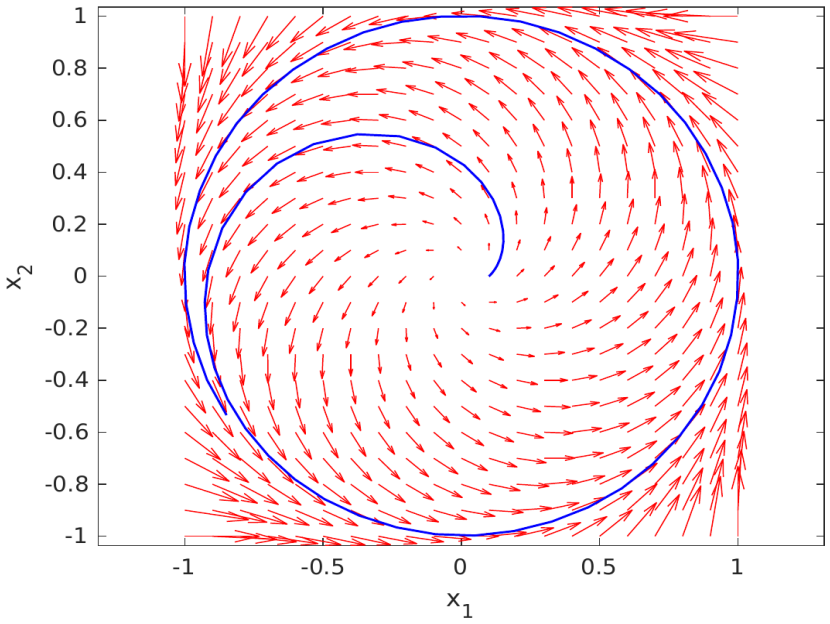
Consider the 2-d system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{cases}$$

The ω-limit sets are:

- the origin $\{0\}$ and the unit circle $\{x \mid \|x\|_2 = 1\}$.

Let $\mathcal{X} = \mathbb{R}^2/\{0\}$. Then,



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$$\psi(x) = \left(\frac{x_1}{\|x\|_2}, \frac{x_2}{\|x\|_2}, \|x\|_2^{-2} - 1 \right)$$

If we extend $\mathcal{X}$ by a point to $\mathbb{R}^2$,ψ is not an immersion anymore as $\psi(0)$ is undefined.

Implications on learning

Formally, given data $\{x_l, x_l^+\}_{l=1}^N$

$$(\psi^*, A^*) \in \underset{\substack{\text{contin.}\psi \\ A\in\mathbb{R}^{m\times m}}}{\operatorname{argmin}} \sum_{l=1}^{N} \|\psi(x_l^+) - e^{A\tau}\psi(x_l)\|$$

$$\text{s.t. } x_l^+ = \varphi(\tau, x_l)\ \forall l \in [1, N]$$

τ: sampling time.

N: number of sampled states

φ: flow of the nonlinear system

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The set of optima:

$$\mathcal{F}^*(\tau, N) = \{the\ set\ of\ cont.interpolating\ func.for\ the\ data\ set\ \{x_l, x_l^+\}_{l=1}^N\}$$

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$\mathcal{F}^*(\tau, N)$ is non-empty because there is a trivial solution: ψ with $\psi(x) = 0,\ \forall x.$
This ψ is always contained by $\mathcal{F}^*(\tau, N)$ for any τ and N

If you want to eliminate trivial solutions, you can add regularization or additional constraints to make $\mathcal{F}^*$ smaller.

Implications on learning

Implication on learning formalized:

$$\mathcal{F}^*(\tau, N) = \{the\ set\ of\ cont.interpolating\ func.for\ the\ data\ set\ \{x_l, x_l^+\}_{l=1}^N\}$$

Theorem: Let $\{x_l\}_{l=1}^{\infty}$ be a dense subset of $\mathcal{X}$. Let ψ be a continuous function such that $\psi(\Omega_1) \neq \psi(\Omega_2)$ for some ω-limit sets Ω_1 and Ω_2 of the system. Then, there exists a small enough $t^* > 0$ such that for all sampling times $\tau \in (0, t^*]$, ψ is not in $\mathcal{F}^*(\tau, N)$ *for some* $N > 0$.

Intuitively, any continuous function that ***can*** distinguish different ω-limit sets cannot be an optimizer of the learning problem as more data is collected.

Conclusions

Key contribution: Under mild conditions, for nonlinear systems **with multiple isolated ω-limit sets**, there are **NO** linear immersions that are **continuous** and **one-to-one** (even if A is low rank).

Implications for finding/learning $\psi(x)$: When **multiple ω-limit sets** are present, a valid one-to-one $\psi(x)$ needs to be **discontinuous** at certain points.

$\Rightarrow$ Continuous parameterization will not work. **Hybrid** structure is required!

Current work (for control systems):

- Learn **local** and approximate Koopman liftings
  - Applications in safety constrained reachability (Balim, Aspeel, Liu, Ozay, L-CSS'23)
- Given multiple approximate Koopman liftings, which one is better?
  - A preorder relation between approximate Koopman liftings (Aspeel, Ozay, ADHS'24)
- Extension to discrete-time systems (Ristich, Sontag, Ozay, IFAC WC'26)

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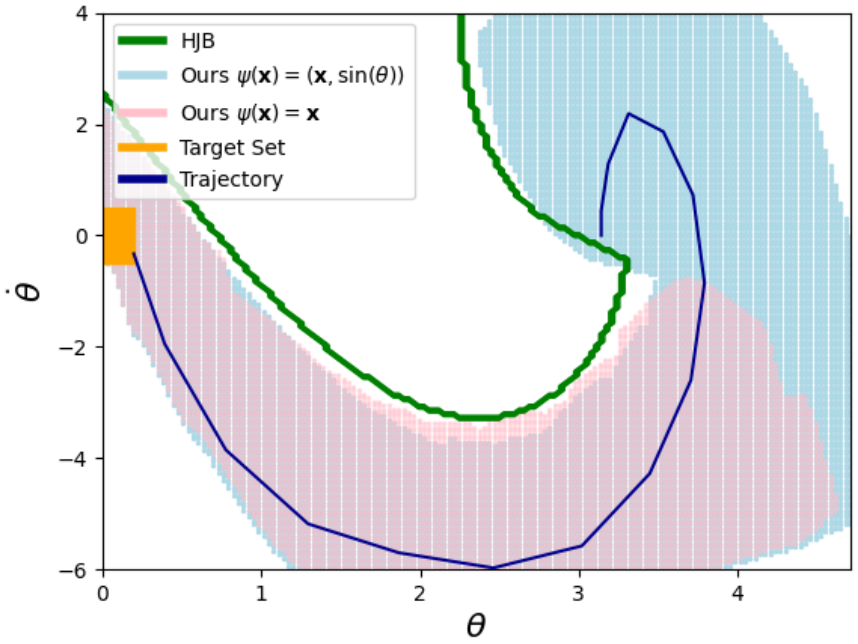
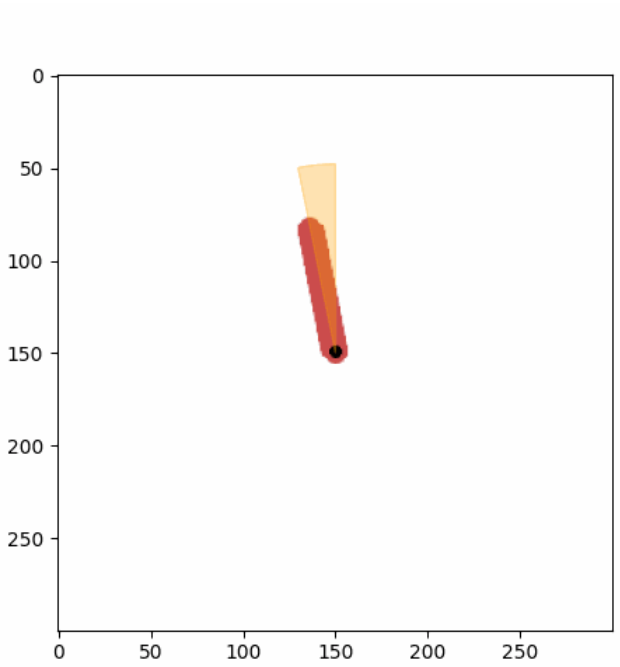
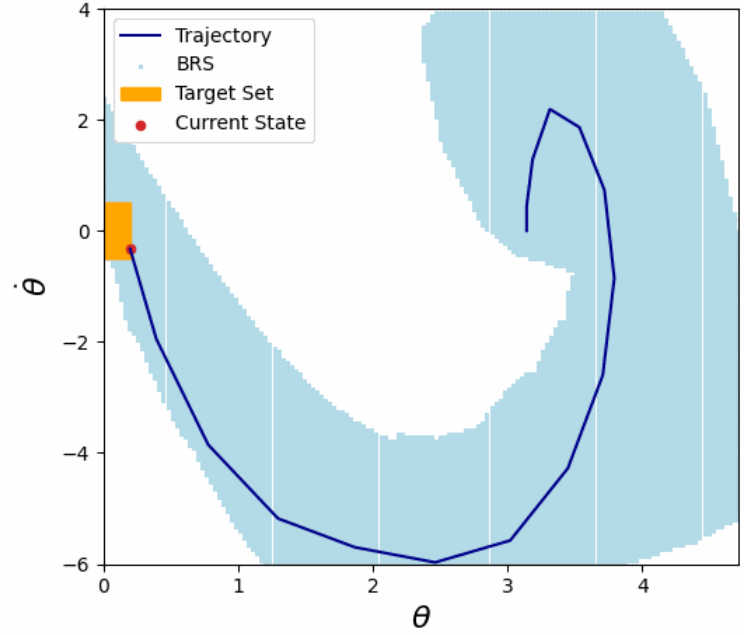
  - A preorder relation between approximate Koopman liftings (Aspeel, Ozay, ADHS'24)

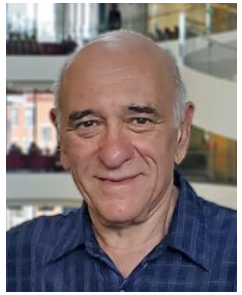






Outline

- The role of learning in dynamics and control
- On some fundamental limitations of learning immersions (Koopman embeddings)



Joint work with
Zexiang Liu (Michigan) and Eduardo Sontag (Northeastern)
arXiv:2312.17045 **(IFAC WC'23, Automatica'25)**

- **On the hardness of learning to stabilize linear systems**

Joint work with
Xiong Zeng (Michigan), Zexiang Liu (Michigan), Zhe Du (Michigan), and Mario Sznaier (Northeastern)
arXiv:2311.11151 **(CDC'23)**

How to measure hardness?

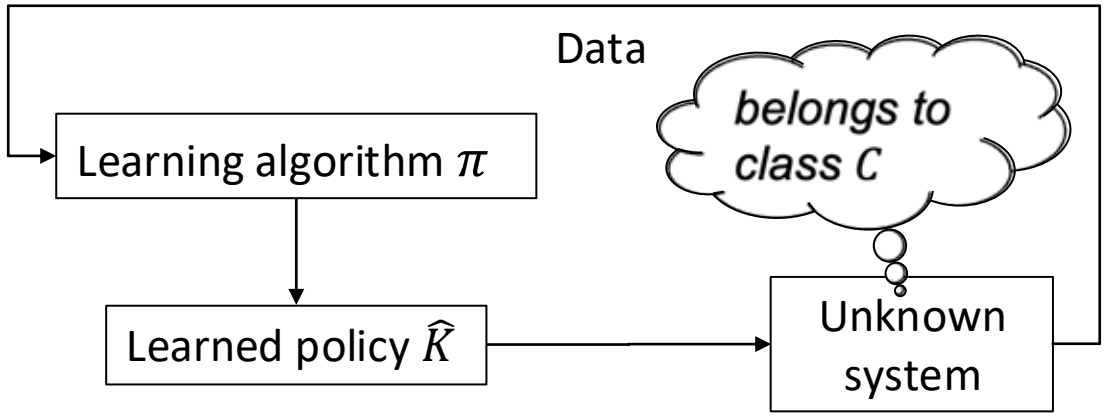
- Computational complexity: how the **computational cost** of an algorithm depends on the **input size**?

  Comp. Cost = $\mathcal{O}$(**input size**)

- Sample complexity: how a **performance metric** of a *learning* algorithm depends on the **sample size** (# of data points)?

  Performance = $\mathcal{O}$(**sample size**)

  Examples of performance metric:
  approximation error,
  cumulative cost or regret,
  probability of stability (this work)



complexity of a problem ≈ complexity of the best algorithm

Setup

Given

- A discrete-time linear time-invariant (LTI) system S from **a class $\mathcal{C}_n$**:

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^n$ and $w_t \sim \mathcal{N}(0, \sigma_w^2 \boldsymbol{I})$

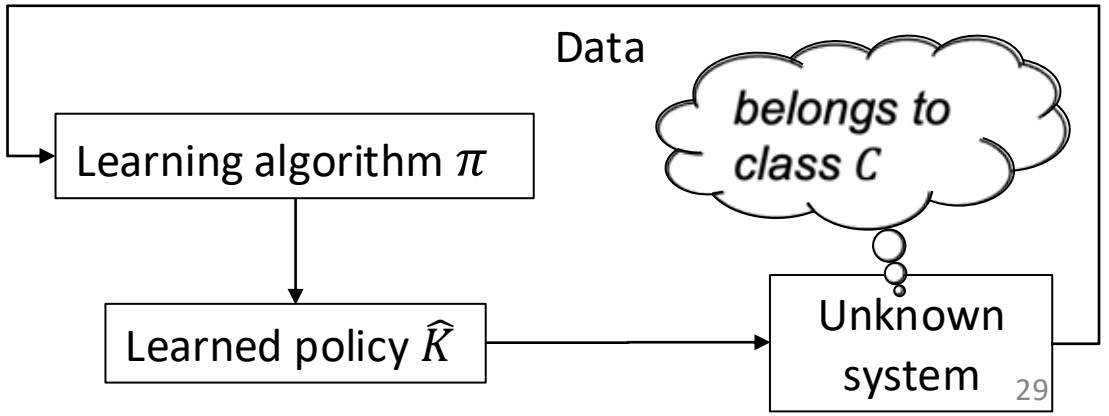
- A time budget N and an input budget $\mathbb{E}[\|u_t\|^2] \leq \sigma_u^2$

Consider a learning algorithm π that

(i) interacts with the system for N units of time, and

(ii) outputs a linear static state-feedback controller $\widehat{K}$ at time N

We want $\widehat{K}$ to stabilize the (unknown) system S.



Setup

Given

- A discrete-time linear time-invariant

$$x_{t+1} =$$

where $x_t \in \mathbb{R}^n$ and $w_t \sim \mathcal{N}(0, \sigma_w^2 \boldsymbol{I})$

- A time budget N and an input budge

Use your favorite algorithm π:

- Excite the system for N steps and do (direct) data-driven control
- Excite the system for N steps and do system id and certainty equivalent control
- Use any other online/active learning method

Consider a learning algorithm π that

(i) interacts with the system for N units of time, and

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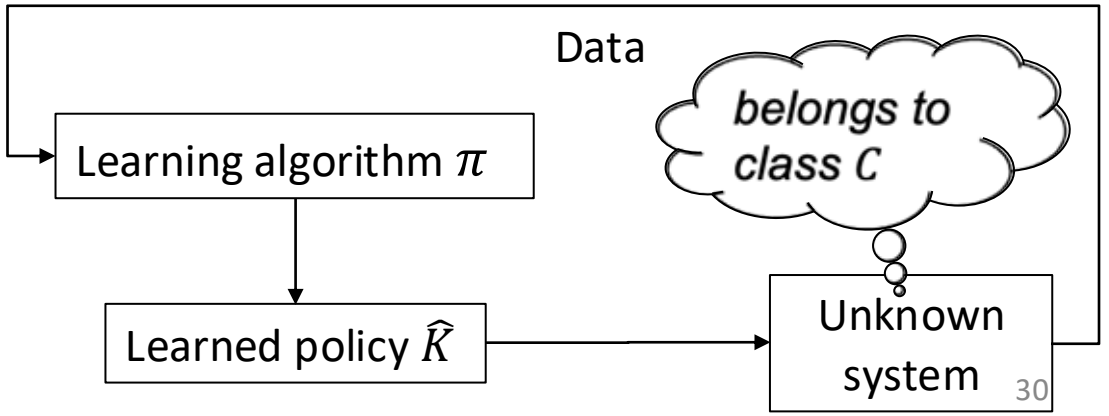
We want $\widehat{K}$ to stabilize the (unknown) system S.



Problem statement

Denote the probability that $\widehat{K}$ stabilizes an unknown system $S = (A, B) \in \mathcal{C}_n$ by

$$P_{S,\pi}^{N}\left(\left(A + B\,\widehat{K}\right)\text{is stable}\right)$$

A learning algorithm π is evaluated by the **worst-case** stabilization probability over $\mathcal{C}_n$:

$$\inf_{S \in \mathcal{C}_n} P_{S,\pi}^{N}((A + B\,\widehat{K}) \text{ is stable})$$

Sample complexity of π: Given a $\delta \in [0, 1]$, the **minimum time/data budget N** such that

$$\inf_{S \in \mathcal{C}_n} P_{S,\pi}^{N}((A + B\,\widehat{K}) \text{ is stable}) \geq 1 - \delta.$$

Question: Given a class $\mathcal{C}$ of linear systems, how does the **sample complexity** for learning stabilizing static state-feedback controllers depend on the **system dimension n**?

Roadmap to establish hardness

Find a class $\mathcal{C}_n$ of systems such that for any algorithm π and for all $\delta \in [0,0.5]$, we have

$$\inf_{S \in \mathcal{C}_n} P_{S,\pi}^{N}((A + B\,\widehat{K}) \text{ is stable}) \geq 1 - \delta.$$

only if

$$N \geq O(exp(n)).$$

Key insight: if the class $\mathcal{C}_n$ contains a pair of systems (A_1, B_1) and (A_2, B_2) that are

- hard to distinguish, and
- not co-stabilizable

then learning to stabilize is hard.

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  when driven by the same algorithm π, the resulting state trajectories are similar

- not co-stabilizable

  there does not exist a single controller that simultaneously stabilizes the two systems

then learning to stabilize is hard. $\nexists\, K$ s.t. both $A_1 + B_1 K$ *and* $A_2 + B_2 K$ are stable

A pair of systems that are hard to learn to stabilize

$$\mathbf{A} = \begin{bmatrix} r & v & 0 & \cdots & 0 \\ 0 & 0 & v & \cdots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & v \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \ \mathbf{B} = \begin{bmatrix} b^{(1)} \\ 0 \\ \vdots \\ 0 \\ v \end{bmatrix} \in \mathbb{R}^n$$

$x_{t+1} = Ax_t + Bu_t + w_t, \quad r > 1, 0 < v < \frac{r-1}{2}$, and $b^{(1)} \geq 0$.

First order system with actuation delay of length n-1

Take $\mathrm{B}_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v \end{bmatrix}, \mathrm{B}_2 = \begin{bmatrix} m \\ 0 \\ \vdots \\ v \end{bmatrix}$

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Proposition: There exists a state-feedback controller K that stabilizes both (A, B_1) and (A, B_2) only if

$$m \leq \mathcal{O}\left(\frac{1}{\exp(n)}\right)$$

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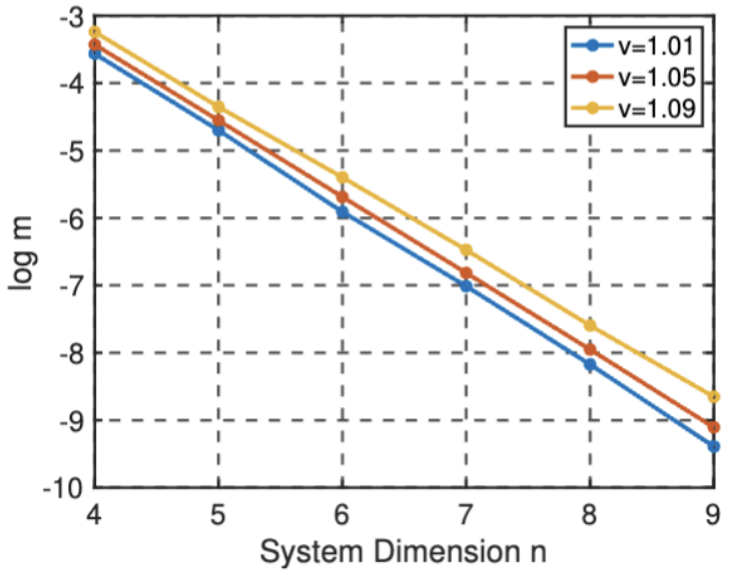
We can also do a sanity check numerically using LMI-based sufficient condition for co-stabilizability

$$\begin{aligned} find \quad & K, P \\ s.t. \quad & (A + B_1 K)^T P (A + B_1 K) \prec P \\ & (A + B_2(m) K)^T P (A + B_2(m) K) \prec P \\ & P \succ 0 \end{aligned}$$



Main result

$$\mathbf{A} = \begin{bmatrix} r & v & 0 & \cdots & 0 \\ 0 & 0 & v & \cdots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & v \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \ \mathbf{B} = \begin{bmatrix} b^{(1)} \\ 0 \\ \vdots \\ 0 \\ v \end{bmatrix} \in \mathbb{R}^{n}$$

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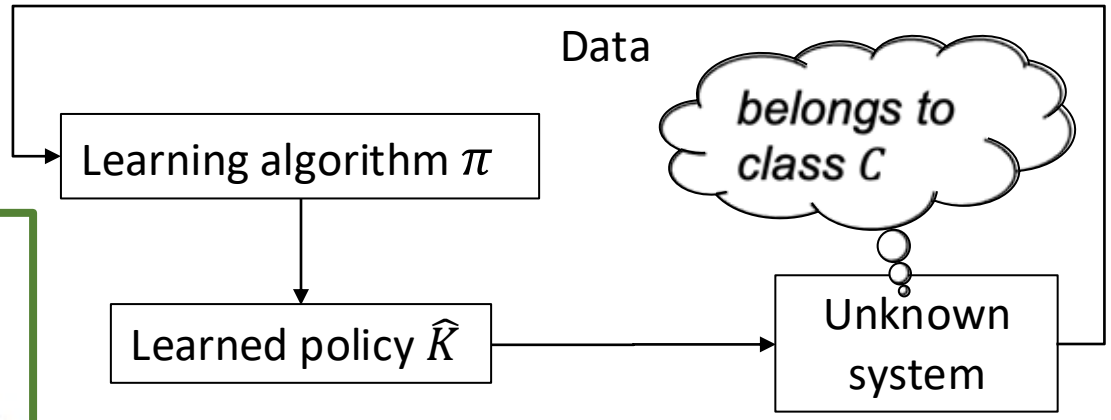

Take $\mathrm{B}_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v \end{bmatrix}, \mathrm{B}_2 = \begin{bmatrix} m_n \\ 0 \\ \vdots \\ v \end{bmatrix}$

Theorem: For any class C of systems containing such a pair with $m_n = 2\left(\frac{2v}{r-1}\right)^n$ and any learning algorithm, the least amount of data for learning to stabilize arbitrary systems in C grows exponentially with the state dimension!

$$\inf_{S\in C_n} P^N_{S,\pi}((A + B\,\widehat{K}) \text{ is stable}) \geq 1 - \delta \quad \text{only if} \quad N \geq \frac{\sigma_w^2}{2\sigma_u^2}\left(\frac{r-1}{2v}\right)^{2n} \log\left(\frac{1}{3\delta}\right)$$

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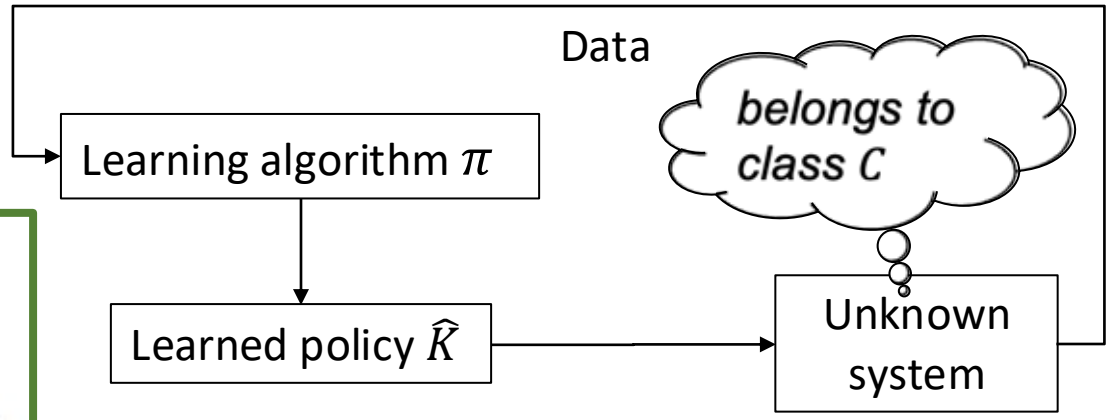

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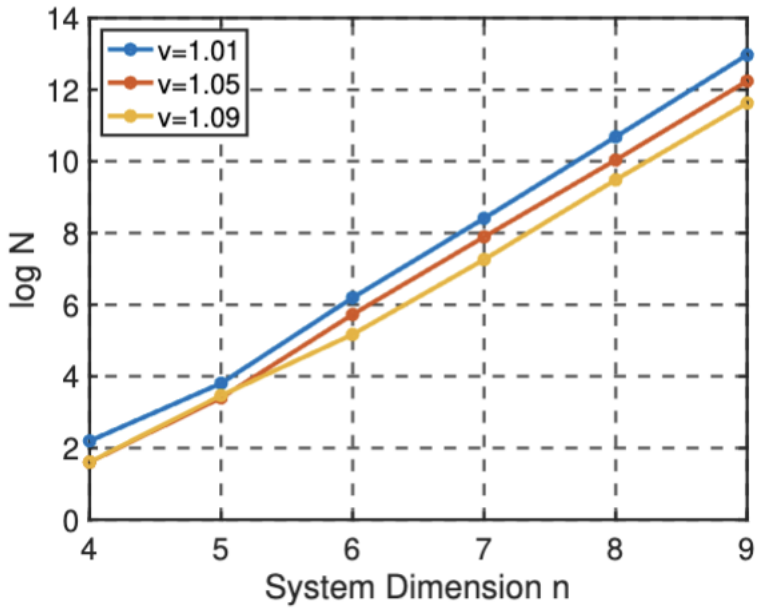
exp in dimension n

Numerical illustration

Number of samples required for 90% probability of finding a stabilizing controller using certainty equivalent LQR*:



y-axis: observed least sample size N for CE LQR being stabilizing with probability 0.9

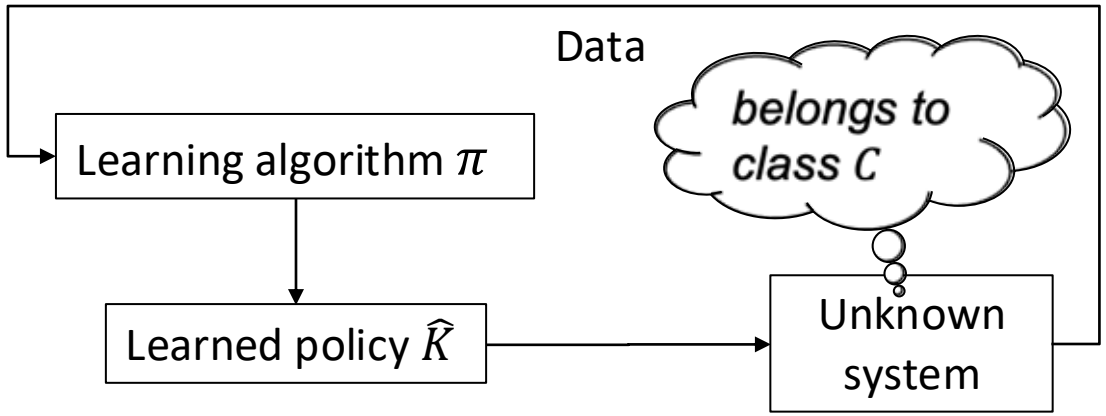
Certainty Equivalent LQR

- Excite the system with iid Gaussian input for N steps
- Learn system matrices
- Do LQR with learned model

$$\min_{u_0,u_{1,\dots}} \lim_{T\to\infty} E[\frac{1}{T}\sum_{t=0}^{T}(x_t^T Q x_t + u_t^T R u_t)]$$
$$s.t.\, x_{t+1} = Ax_t + Bu_t + w_t$$

* By our result, same trend holds for any algorithm

Conclusions



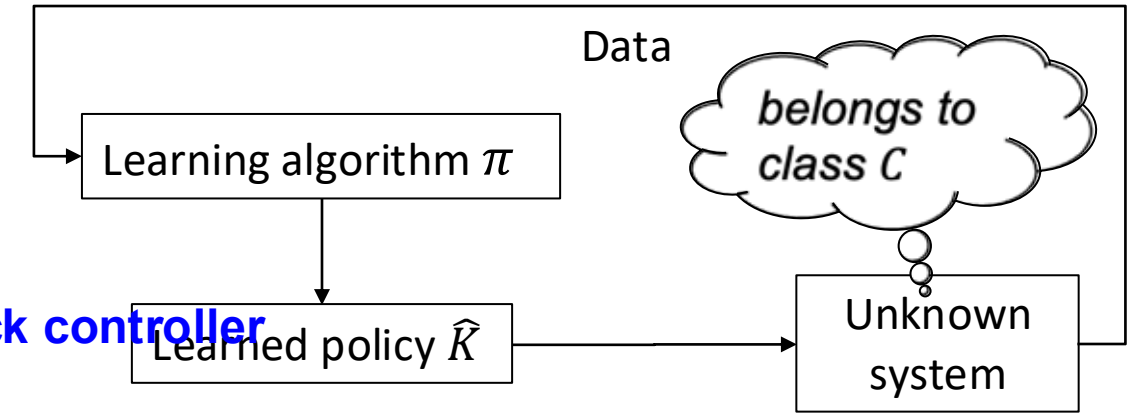
Key contributions:

- Learning to stabilize a linear system is hard if we have systems that are

  hard to distinguish AND hard to co-stabilize

- Compared with earlier work by Tsiamis et al., our analysis
  - reveals a new obstruction for the hardness of learning to stabilize;
  - covers a larger class of systems (including open-loop unstable ones) – encompasses many examples from literature where some hardness was observed

Conclusions



Key contributions: with a static linear state-feedback controller

- Learning to stabilize a linear system is hard if we have systems that are

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- Compared with earlier work by Tsiamis et al., our analysis
  - reveals a new obstruction for the hardness of learning to stabilize;
  - covers a larger class of systems (including open-loop unstable ones) – encompasses many examples from literature where some hardness was observed

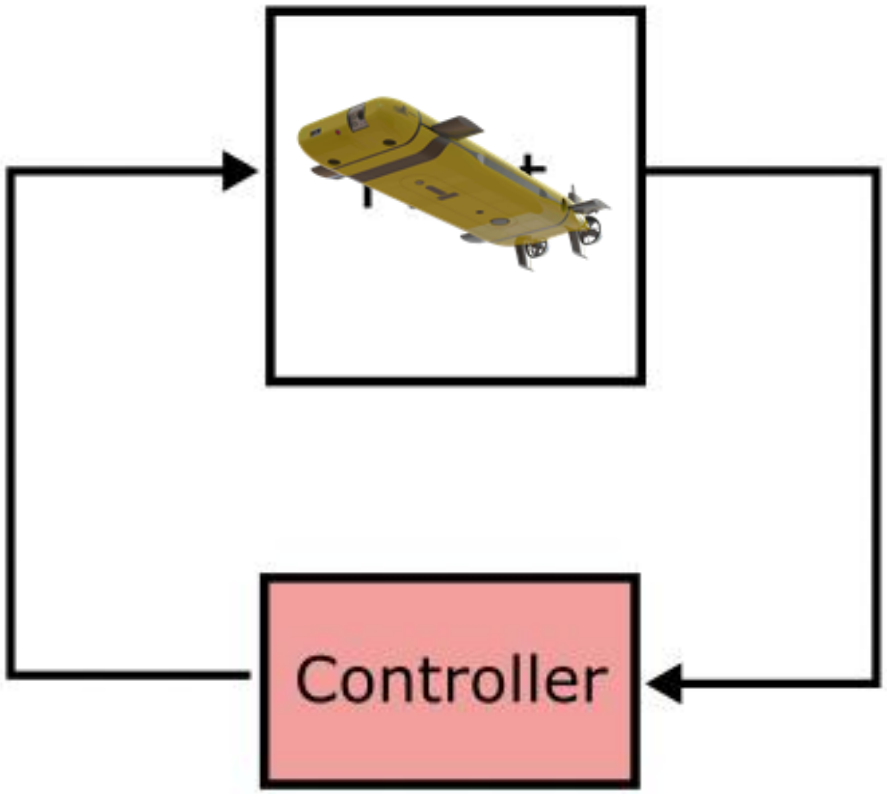
Current work:

- Observation: Static linear state feedback controllers are not a rich enough class for co-stabilizing linear systems (memory helps!)
- Can we reduce sample complexity by using dynamic time-varying controllers?

Plant (+ operating environment)

Task or mission specification

φ



Many components in a feedback loop can be learned:

- **Plant model:** model-based learning, system identification
- **Task specification:** inverse optimal control, inverse reinforcement learning
- **Controller:** model-free learning
- **Perception/estimation modules**

Learn but be aware of the fundamental limits (e.g., hardness and impossibility results) ➔ Learn better

More info and preprints @ https://web.eecs.umich.edu/~necmiye/