Noise Sensitivity of Direct Data-Driven Linear Quadratic Regulator by Semidefinite Programming

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***Abstract*— In this paper, we study the noise sensitivity of the semidefinite program (SDP) used in the direct data-driven infinite horizon linear quadratic regulator (LQR) problem for discrete-time linear time-invariant systems. While this SDP is shown to find the true LQR controller in the noise-free setting, we show that it leads to a trivial solution when data is corrupted by noise, even when the noise is arbitrarily small. Hence, a "certainty equivalence" approach that uses the original SDP with noisy data is not appropriate.**

I. Introduction

Two of the most dominant paradigms in learning-based control are the certainty equivalence approach and the robust control approach. Roughly speaking, in certainty equivalence, we *pretend* that our data is not corrupted by noise, the estimated model is the true system model, or the estimated control policy is designed based on the true system and clean data. On the other hand, in robust control approach, we try to bound the effect of the noise in the data and aim to find a controller that achieves the desired properties for all possible noise values within this bound.

These two different paradigms can be applied both in the model-based case, where system identification is followed by control design, or in direct data-driven control, which uses data to directly synthesize a controller using the behavioral approach (see, e.g., [1], [2]). In the context of model-based LQR, Mania et al. [3] show that certainty equivalence is statistically consistent and is more sample-efficient than the robust approach given in [4]. The success of certainty equivalent control, in this case, lies in the fact that there is some inherent robustness in the solutions of the Riccati equations with respect to perturbations in system matrices.

A natural question is how the certainty equivalence approach and a robust control approach compare in terms of statistical properties for direct data-driven LQR. Several works consider a robust approach for direct data-driven control for different control objectives or noise settings (e.g., [5]–[7]). On the other hand, De Persis and Tesi [8] analyze a certainty equivalent approach to direct data-driven LQR, where they provide a sufficient condition for stabilizability and regularization techniques for improving noise robustness. Here, the certainty equivalence approach amounts to using the noisy data directly in the semidefinite program developed for the noise-free case. It is observed in [8] that even small noise can lead to a violation of their proposed sufficient condition for stabilizability and the semidefinite program may favor low gain solutions. However, a thorough understanding of the noise sensitivity of this semidefinite program for direct data-driven LQR is missing. In this paper, using a scalar system, we prove that the semidefinite program for direct data-driven LQR is very sensitive to noise and yields, with probability one, *trivial control gains* independent of the data even when there is an arbitrarily small amount of noise.

There are also recent works that propose alternative optimization formulations for direct data-driven control that aim to mimic the solution of the model-based certainty equivalent control [9], [10]. Since the controllers synthesized by these alternative formulations are equivalent to model-based certainty equivalent control, they inherit the nice statistical consistency properties of the former. Our analysis does not pertain to these alternative formulations.

The remainder of the paper is organized as follows. After we briefly introduce the notation used throughout the paper, in Section II we review some basic results from direct data-driven LQR focusing on a scalar system. Section III introduces the main results and their proofs. We provide some numerical examples in Section IV before concluding the paper in Section V.

Notation: We use lower case, lower case boldface, and upper case boldface letters to denote scalars, vectors, and matrices respectively. $\mathcal{N}$ denotes the Gaussian distribution. $\mathbb{R}$ denotes the real number domain. We denote the identity matrix of size n as $\mathbf{I}_n$.

II. Direct Data-Driven (DDD) LQR

For simplicity, we only consider the following scalar discrete-time linear time-invariant (LTI) system in this paper:

$$x_{t+1} = ax_t + bu_t + w_t, \tag{1}$$

where $x_t \in \mathbb{R}$, $u_t \in \mathbb{R}$, $w_t \in \mathbb{R}$ are the state, input, and noise at time t, respectively. We assume that the initial state and the noise are independent random variables with distributions satisfying $w_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$, $\mathbb{E}[x_0] = 0$, and $\mathbb{E}[x_0^2] = \sigma_{x_0}^2$. The system parameters a and b are unknown but, throughout the paper, we assume that $b \neq 0$, which ensures that the system is controllable. The standard LQR problem for the

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scalar LTI system (1) is

$$\min_{u_0,u_1,\cdots} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(qx_t^2+u_t^2\right)\right], \quad \text{s.t. Dynamics in (1)} \tag{2}$$

where $q>0$, $\mathbb{E}$ denotes the expectation over the randomness from the initial state x_0 and the process noise w_t. When a and b are known, this LQR problem can be solved by finding the positive solution p of the scalar discrete-time algebraic Riccati equation: $p = a^2p - \frac{a^2b^2p^2}{1+b^2p} + q$. Then, the solution of (2) is $u_t = k_{lqr}x_t$, with the optimal controller given by

$$k_{lqr} = \frac{-abp}{1+b^2p}.$$

In direct data-driven control, the parameters a and b are unknown and the goal is to directly estimate k_{lqr} from data without explicitly estimating a and b. We assume the data is collected offline by driving the system with a random input such that $u_t \overset{i.i.d.}{\sim} \mathcal{N}(0,\sigma_u^2)$ with $\sigma_u^2>0$, and this input is independent of the noise process and the initial condition. Let us define the data matrices formed from this data collected over some horizon T:

$$\begin{aligned} \mathbf{x}_0^\top(T) &:= \left[\begin{array}{cccc} x_0 & x_1 & \dots & x_{T-1}\end{array}\right], \\ \mathbf{u}_0^\top(T) &:= \left[\begin{array}{cccc} u_0 & u_1 & \dots & u_{T-1}\end{array}\right], \\ \mathbf{x}_1^\top(T) &:= \left[\begin{array}{cccc} x_1 & x_2 & \dots & x_T\end{array}\right]. \end{aligned} \tag{3}$$

Most of the time, when T is clear from context, we will drop it and use $\mathbf{x}_0^\top$, $\mathbf{u}_0^\top$, $\mathbf{x}_1^\top$ to indicate the above quantities. We similarly define $\mathbf{w}_0^\top(T) := \left[\begin{array}{cccc} w_0 & w_1 & \dots & w_{T-1}\end{array}\right]$ for the (unknown) noise sequence; and use $\mathbf{w}_0^\top$ when T is clear from context.

De Persis and Tesi proposed a semidefinite program, called the DDD LQR, to estimate the optimal LQR gain in [1, Theorem 4]. In the case of a scalar system, this SDP takes the form:

$$\begin{aligned} \min_{s\in\mathbb{R},\mathbf{y}\in\mathbb{R}^T} \quad & q\mathbf{x}_0^\top\mathbf{y} + s \\ \text{s.t.} \quad & \left[\begin{array}{cc} \mathbf{x}_0^\top\mathbf{y}-1 & \mathbf{x}_1^\top\mathbf{y} \\ \mathbf{y}^\top\mathbf{x}_1 & \mathbf{x}_0^\top\mathbf{y}\end{array}\right] \succeq 0 \\ & \left[\begin{array}{cc} s & \mathbf{u}_0^\top\mathbf{y} \\ \mathbf{u}_0^\top\mathbf{y} & \mathbf{x}_0^\top\mathbf{y}\end{array}\right] \succeq 0. \end{aligned} \tag{4}$$

If $\mathbf{y}_{ce}^*$ denotes an optimal solution to (4), then the LQR gain estimate is given by

$$k_{ce} := -\frac{\mathbf{u}_0^\top\mathbf{y}_{ce}^*}{\mathbf{x}_0^\top\mathbf{y}_{ce}^*}. \tag{5}$$

The following result from [1] shows that when the system does not have any noise (i.e., $w_t = 0$ for all t) and a persistency of excitation condition holds (which can be shown to hold with probability 1 when the input is Gaussian as assumed), we have $k_{lqr} = k_{ce}$.

Theorem 1 ([1, Theorem 4]). *Let* $\operatorname{rank}\left(\left[\begin{array}{c}\mathbf{x}_0^\top \\ \mathbf{u}_0^\top\end{array}\right]\right) = 2$. *When* $\sigma_w^2 = 0$, $k_{ce} = k_{lqr}$.

Some limitations of the estimate k_{ce}, which is also known as a certainty equivalence DDD LQR solution, are discussed in [8]; and alternative semidefinite programs with regularization are proposed to improve noise robustness. Our main result (Theorem 2) is to show that even when the noise is arbitrarily small, k_{ce} can be an arbitrarily bad estimate.

For the noisy setting, in [8], the following sufficient condition for stabilization is proposed.

Lemma 1 ([8, Lemma 4]). *Consider an optimal solution* $\mathbf{y}_{ce}^*$ *of* (4). *Let*

$$\mathbf{M} := \mathbf{y}_{ce}^*(\mathbf{x}_0^\top\mathbf{y}_{ce}^*)^{-1}\left(\mathbf{y}_{ce}^*\right)^\top,$$

and

$$\Psi := \mathbf{w}_0^\top\mathbf{M}\mathbf{w}_0 - \mathbf{x}_1^\top\mathbf{M}\mathbf{w}_0 - \mathbf{w}_0^\top\mathbf{M}\mathbf{x}_1.$$

Consider k_{ce} *defined in* (5). *If there exists* $\eta \ge 1$ *such that*

$$\Psi \le 1 - \frac{1}{\eta},$$

then k_{ce} *is a stabilizing state feedback gain for* (1).

We will also provide an interpretation of this sufficient condition in light of our main theorem.

III. Closed-form Solution to DDD LQR Semidefinite Program

Our first main result shows that in the presence of noise, with probability 1, the solution of the certainty equivalence DDD LQR problem in (4) is independent of the data.

Theorem 2. *Assume* $\sigma_w^2 > 0$, $T\ge 3$, *and the offline data is persistently exciting in the sense that*

$$\operatorname{rank}\left(\left[\begin{array}{c}\mathbf{x}_0^\top \\ \mathbf{u}_0^\top\end{array}\right]\right) = 2.$$

Then, for all optimal solutions $\mathbf{y}_{ce}^*$ *of* (4), *the controller gain* k_{ce} *given in* (5) *is unique and we have that*

$$\mathbb{P}_T(k_{ce}=0) = 1,$$

where the probability is with respect to the randomness of x_0, $\mathbf{u}_0(T)$, *and* $\mathbf{w}_0(T)$.

Based on Theorem 2, we have that when $T\ge 3$, the data-driven feedback gain based on certainty equivalence DDD LQR in (4) is generically equal to 0. Hence, k_{ce} will not converge to k_{lqr} no matter how large T is, i.e., k_{ce} is an inconsistent estimator of k_{lqr}.

The following corollary enables an alternative interpretation of Lemma 1.

Corollary 1. *Consider* Ψ *defined in Lemma 1. Let all the assumptions in Theorem 2 hold. Then, with probability one, we have that* $\Psi = a^2$.

Corollary 1 shows that when the open-loop system in (1) is unstable, the inequality condition in Lemma 1 will not hold with probability one. This is because when $|a|>1$, there does not exist $\eta\ge 1$ such that $\Psi = a^2 \le 1-\frac{1}{\eta}$. Therefore, Lemma 1 essentially says that, when the open-loop system is stable, control gains being zero is stabilizing, as expected.

A. Proof of the main results

Our proof builds on a few lemmas.

Lemma 2. *Assume* $\sigma_w^2 > 0$, $T \geq 3$. *Consider the events* $E_1 = \left\{ (x_0, \mathbf{u}_0, \mathbf{w}_0) : \operatorname{rank}\left(\begin{bmatrix} \mathbf{x}_0^\top \\ \mathbf{u}_0^\top \end{bmatrix} \right) = 2 \right\}$ *and* $E_2 = \left\{ (x_0, \mathbf{u}_0, \mathbf{w}_0) : \operatorname{rank}\left(\begin{bmatrix} \mathbf{x}_0^\top \\ \mathbf{u}_0^\top \\ \mathbf{x}_1^\top \end{bmatrix} \right) = 3 \right\}$. *Then,*

$$\mathbb{P}\left(E_2 \mid E_1\right) = 1. \tag{6}$$

Proof. Based on the dynamics defined in (1), we have that

$$\mathbf{x}_1 = a\mathbf{x}_0 + b\mathbf{u}_0 + \mathbf{w}_0. \tag{7}$$

Now consider the complement E_2^c of the event E_2. We will show that $\mathbb{P}\left(E_2^c \mid E_1\right) = 1 - \mathbb{P}\left(E_2 \mid E_1\right) = 0$. Note that given E_1, the only way E_2 fails is if we have $\operatorname{rank}\left(\begin{bmatrix} \mathbf{x}_0^\top \\ \mathbf{u}_0^\top \\ \mathbf{x}_1^\top \end{bmatrix} \right) = 2$. However, by (7), this requires $\mathbf{w}_0 \in \operatorname{span}(\mathbf{x}_0, \mathbf{u}_0)$. Note that $\mathbf{w}_0$ is a Gaussian random vector taking values in $\mathbb{R}^T$ and this event requires it to lie in a Lebesgue measure zero subset of $\mathbb{R}^T$. It follows from [11, Prop 1.24] that if X is a random variable absolutely continuous with respect to the Lebesgue measure (like Gaussian random variables), then $\mathbb{P}(X \in B) = 0$ for any $B \subset \mathbb{R}^n$ which has Lebesgue measure zero. Therefore, $\mathbb{P}\left(E_2^c \mid E_1\right) = 0$. □

Remark 1. *The results in Lemma 2 (hence, the remaining results as well) are not restricted to Gaussian noise and will hold for any sufficiently regular noise distribution whose support contains an open set around the origin.*

Lemma 3. *Assume* $\operatorname{rank}\left(\begin{bmatrix} \mathbf{x}_0^\top \\ \mathbf{u}_0^\top \\ \mathbf{x}_1^\top \end{bmatrix} \right) = 3$. *Consider the following underdetermined system of equations:*

$$\begin{bmatrix} \mathbf{x}_0^\top \\ \mathbf{u}_0^\top \\ \mathbf{x}_1^\top \end{bmatrix} \mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

Then, $\mathbf{y}^*$ *is an optimal solution of* (4) *if and only if* $\mathbf{y}^*$ *satisfies* (8).

Proof. We note that since $\operatorname{rank}\left(\begin{bmatrix} \mathbf{x}_0^\top \\ \mathbf{u}_0^\top \\ \mathbf{x}_1^\top \end{bmatrix} \right) = 3$, Equation (8) always has a solution. First, we prove that any solution of (8) is an optimal solution of (4). Since $\mathbf{x}_0^\top \mathbf{y} \geq 1 > 0$, by Schur complement, we can rewrite the first constraint in (4) as: $\mathbf{x}_0^\top \mathbf{y} - 1 - \frac{(\mathbf{x}_1^\top \mathbf{y})^2}{\mathbf{x}_0^\top \mathbf{y}} \geq 0$ and $\mathbf{x}_0^\top \mathbf{y} - 1 \geq 0$, and the second constraint as $s \geq 0$ and $s - \frac{\left(\mathbf{u}_0^\top \mathbf{y}\right)^2}{\mathbf{x}_0^\top \mathbf{y}} \geq 0$. Since s is being minimized, its optimal value will be $s = \frac{\left(\mathbf{u}_0^\top \mathbf{y}\right)^2}{\mathbf{x}_0^\top \mathbf{y}}$. Hence, we can remove s from (4) to obtain the following equivalent problem:

$$\begin{aligned} \underset{\mathbf{y} \in \mathbb{R}^T}{\text{minimize}} \quad & O_{ce}(\mathbf{y}) := q\mathbf{x}_0^\top \mathbf{y} + \frac{\left(\mathbf{u}_0^\top \mathbf{y}\right)^2}{\mathbf{x}_0^\top \mathbf{y}} \\ \text{subject to} \quad & \mathbf{x}_0^\top \mathbf{y} - 1 \geq 0 \\ & \left(\mathbf{x}_0^\top \mathbf{y} - 1\right) \mathbf{x}_0^\top \mathbf{y} - \left(\mathbf{x}_1^\top \mathbf{y}\right)^2 \geq 0 \end{aligned} . \tag{9}$$

For the optimization problem (9), because $\mathbf{x}_0^\top \mathbf{y} - 1 \geq 0$ and $q \geq 0$, we have $O_{ce}(\mathbf{y}) \geq q$ for any feasible solution $\mathbf{y}$ of (9). Therefore, if there exists $\mathbf{y}_m$ such that $O_{ce}(\mathbf{y}_m) = q$, then $\mathbf{y}_m$ is an optimal solution of (9). Take a solution $\mathbf{y}^*$ of (8), then two inequality constraints of (9) are satisfied with equality and we also have $O_{ce}(\mathbf{y}^*) = q$. Therefore, any solution of (8) is an optimal solution of (9) and the optimal objective value of (9) is q.

Next, we will prove that any optimal solution $\mathbf{y}_{ce}^*$ of (9) must satisfy (8). First, according to the first constraint in (9), $\mathbf{x}_0^\top \mathbf{y}_{ce}^* \neq 1$ implies that $\mathbf{x}_0^\top \mathbf{y}_{ce}^* > 1$. Then, we have $O_{ce}(\mathbf{y}_{ce}^*) > q$, which contradicts the fact that $\mathbf{y}_{ce}^*$ is an optimal solution of (9). Therefore, we have that $\mathbf{x}_0^\top \mathbf{y}_{ce}^* = 1$ must hold. Based on the second constraint in (9), $\mathbf{x}_0^\top \mathbf{y}_{ce}^* = 1$ implies that $\mathbf{x}_1^\top \mathbf{y}_{ce}^* = 0$ must hold. According to the objective function definition in (9) and the fact that the optimal objective value of (9) is q, we necessarily have $\mathbf{u}_0^\top \mathbf{y}_{ce}^* = 0$ must hold. In conclusion, any optimal solution of (9) satisfies (8). □

Remark 2. *We note that in the noiseless setting, given* E_1, *we have*

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{x}_0^\top \\ \mathbf{u}_0^\top \\ \mathbf{x}_1^\top \end{bmatrix} \right) = 2,$$

with $\mathbf{x}_1 = a\mathbf{x}_0 + b\mathbf{u}_0$. *In this case, for Equation* (8) *to have a solution, we need* $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$ *to be in the column space of the above matrix, which would only happen when* $a = 0$.[1] *Hence, the constructed trivial solution is not a valid solution in the noiseless case and our result does not constitute a counterexample to the semidefinite program* (4) *resulting in the true* k_{lqr} *in noiseless settings.*

Next, we give the proof of Theorem 2.

Proof. (of Theorem 2) By Lemma 3 and by the definition of k_{ce}, any optimal solution of (4) yields the unique k_{ce} value of 0 when $\begin{bmatrix} \mathbf{x}_0^\top \\ \mathbf{u}_0^\top \\ \mathbf{x}_1^\top \end{bmatrix}$ is full row rank, which holds true with probability one due to Lemma 2. This completes the proof. □

We can prove Corollary 1 similarly.

Proof. (of Corollary 1) Under the assumptions of Theorem 2, by Lemma 2 and Lemma 3, we get that $\mathbf{x}_0^\top \mathbf{y}_{ce}^* = 1, \mathbf{u}_0^\top \mathbf{y}_{ce}^* = 0$, and $\mathbf{x}_1^\top \mathbf{y}_{ce}^* = 0$, with probability 1. Then based on (7), it implies that

$$\mathbf{w}_0^\top \mathbf{y}_{ce}^* = \left(\mathbf{x}_1 - a\mathbf{x}_0 - b\mathbf{u}_0\right)^\top \mathbf{y}_{ce}^* = -a,$$

[1] Note that when $a = 0$, the true LQR gain $k_{lqr} = 0$, hence in this case as well the semidefinite program (4) results in the true k_{lqr}.

with probability 1. Then by the definition of Ψ in Lemma 1, we conclude that $\mathbb{P}\left(\Psi = a^2\right) = 1$. □

IV. Numerical Examples

In this section, we present two numerical experiments—one for a scalar system and another for a multivariate system—to validate our main theoretical results and provide insights for future research.

A. Scalar System

We first perform the experiments to estimate the LQR gain for a scalar system, both with and without noise, to verify Theorem 1 and Theorem 2. Consider the scalar dynamics in (1) with parameters $a = 2$ and $b = 1$ and the scalar LQR problem in (2) with $q = 1$. Let the initial state be $x_0 = 1$ and set the data matrices horizon to $T = 4$. For this setup, the true LQR gain is $k_{lqr} = 1.6180$. First, consider the following noiseless data matrices:

$$\begin{aligned} \mathbf{u}_0^\top &= \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}, \\ \mathbf{w}_0^\top &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{x}_0^\top &= \begin{bmatrix} 1 & 2.1 & 4.3 & 8.7 \end{bmatrix}, \\ \mathbf{x}_1^\top &= \begin{bmatrix} 2.1 & 4.3 & 8.7 & 17.5 \end{bmatrix}. \end{aligned} \tag{10}$$

The LQR gain estimate, computed using (4), is $k_{ce} = 1.6180$, which exactly matches the true LQR gain k_{lqr}, thereby verifying Theorem 1. Next, we consider data matrices with non-zero noise:

$$\begin{aligned} \mathbf{u}_0^\top &= \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}, \\ \mathbf{w}_0^\top &= \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.2 \end{bmatrix}, \\ \mathbf{x}_0^\top &= \begin{bmatrix} 1 & 2.2 & 4.7 & 9.7 \end{bmatrix}, \\ \mathbf{x}_1^\top &= \begin{bmatrix} 2.2 & 4.7 & 9.7 & 19.7 \end{bmatrix}. \end{aligned} \tag{11}$$

In this case, the LQR gain estimate computed by (4) is $k_{ce} = 0$, verifying Theorem 2.

B. Multivariate System

We next consider a multivariate LTI system, described by the following dynamics:

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t, \tag{12}$$

where $\mathbf{x}_t \in \mathbb{R}^n$, $\mathbf{u}_t \in \mathbb{R}^m$, $\mathbf{w}_t \in \mathbb{R}^n$ represent the state, input, and noise at time t, respectively. The initial state and the noise are independent random variables, with $\mathbf{w}_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2 \mathbf{I}_n)$, and the initial state $\mathbf{x}_0 = 0$. We consider the following specific system matrices $(\mathbf{A}, \mathbf{B})$:

$$\mathbf{A} = \begin{bmatrix} 0.8878 & 0.2232 \\ 0.3491 & 0.3726 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -0.6808 \\ 0.3726 \end{bmatrix},$$

where the spectral radius of $\mathbf{A}$ is 1.01, indicating that this LTI system is open-loop unstable. For the parameter matrices $(\mathbf{Q}, \mathbf{R})$ in LQR problem, we set $\mathbf{Q} = \mathbf{I}_2$ and $\mathbf{R} = 1$, yielding a true LQR gain of $\begin{bmatrix} -0.7112 & -0.2046 \end{bmatrix}$. The implementation of DDD LQR for multivariate systems follows the method outlined in Theorem 4 in [1]. The data matrices are collected offline by exciting the system with random inputs such that $\mathbf{u}_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_u^2)$, where $\sigma_u^2 = 1$. We set the trajectory length to $T = 50$. When $\sigma_w^2 = 0$, the LQR gain estimate computed by DDD LQR is $\begin{bmatrix} -0.7112 & -0.2046 \end{bmatrix}$, which matches the true LQR gain, demonstrating the accuracy of the method in Theorem 4 of [1] for noiseless multivariate systems. However, when $\sigma_w^2 = 0.00001$, the LQR gain estimate from DDD LQR becomes: $\begin{bmatrix} 0 & 0 \end{bmatrix}$. This LQR gain estimate would result in an unstable closed-loop system. This result suggests that the algorithm in Theorem 4 of [1] is highly sensitive to noise, even when the noise level is very small.

We also implement the semidefinite program in Theorem 4 in [1] for 100 random controllable multivariate LTI systems with the state dimension $n = 5$, the input dimension $m = 2$. We generate $\mathbf{B}$ with independent scalar Gaussian $\mathcal{N}(0,1)$ entries, $\hat{\mathbf{A}}$ with independent uniformly distributed entries between [0,1], and let $\mathbf{A} = r\hat{\mathbf{A}}/\rho(\hat{\mathbf{A}})$ with $r = 0.5$ and $\rho(\hat{\mathbf{A}})$ referring to the spectral radius of $\hat{\mathbf{A}}$. We run the experiments 20 times for every randomly generated system with the length of data trajectory $T = 20$, the random inputs such that $\mathbf{u}_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_u^2 \mathbf{I}_2)$, where $\sigma_u^2 = 1$, and the random noise such that $\mathbf{w}_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2 \mathbf{I}_5)$, where $\sigma_w^2 = 0.00001$. For all experiments, the LQR gain estimate from DDD LQR are always equal to a zero matrix.

The theoretical analysis of multivariate LTI systems is left for future work.

V. Conclusion and Future Work

This paper shows that the semidefinite program proposed for the direct data-driven infinite-horizon LQR problem leads to, with probability one, a trivial solution for noisy data. Therefore, a "certainty equivalent" approach where the noisy data is directly used in the semidefinite program for the noiseless case is in general not a good idea and it is better to resort to robust approaches or alternative formulations. This is in contrast to the model-based certainty equivalent control, which was shown to be superior to a robust approach in terms of statistical properties [3].

Our current work focuses on generalizing these results to multivariate systems and on the analysis of regularized versions of the direct data-driven control for noisy data.

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