Which Sets of Strings are Pseudospherical?

R. Bruce Richter and Xinyu L. Wang

Department of Combinatorics and Optimization,
University of Waterloo, Waterloo, Canada
{brichter, x654wang}@uwaterloo.ca

The idea of pseudospherical drawings comes from a natural generalization of pseudolinear drawings, where each edge extends to a pseudoline and every pair of pseudolines crosses exactly once. The geometry of these drawings can be applied to techniques for crossing numbers, for example, to prove special cases for the Harary–Hill conjecture about the crossing number of K_n. Pseudospherical arrangements can be thought of as drawings where each edge extends to a simple closed curve, every pair of which intersect exactly twice, and no edge crosses any of the simple closed curves more than once.

A **string** is a simple bounded arc. We consider collections of strings embedded in S^2. Such a collection is denoted by Σ, which will be extended at each string from their endpoints to **pseudocircles**, simple closed curves. A **pseudospherical arrangement** is an extension of Σ to pseudocircles where every string e is extended to a pseudocircle γ_e so that:

PS1: For each string e, no vertex except an endpoint of e is contained in γ_e.

PS2: For distinct e, f, $|\gamma_e \cap \gamma_f| = 2$, and all intersections are crossings.

PS3: For any edge e, if its endpoints u and v are contained in the closure Δ of one of the components of $S^2 \setminus \gamma_e$, then $e \subset \Delta$.

For any cycle, vertices (where strings intersect) on the cycle are categorized as either rainbow or reflecting: v is **reflecting** in C if two edges incident to v from inside C or on C belong to the same string, and v is **rainbow** in C if all edges incident to v from inside or on C belong to distinct strings [1].

When drawing K_n as a pseudospherical arrangement, each side of a great circle induces a pseudolinear drawing of a smaller complete graph [2]. There is a known classification of obstruction cycles where a set of strings fails to be pseudolinear by Arroyo et al. if and only if the obstruction cycle has at most two rainbow vertices [1]. These cycles turn out to be related to pseudocircular obstructions except for one type of cycle, which we will detail below.

Every collection of strings with a pseudolinear extension also has a pseudospherical extension which can be obtained from the former by contracting the ends of the pseudolines into a point and perturbing the edges.

There exist four types of pseudolinear obstruction cycles. We will refer to cycles with no rainbow vertices as **clouds**, to those with one rainbow vertex as **fish**, to those with two adjacent rainbow vertices as **shrubs**, and to those with two non-adjacent rainbow vertices as **croissants**. The last type turns out to be different than the previous ones in the context of pseudocircles.

Theorem 1. *Every cloud, fish, and shrub type cycle has some pseudocircle extended from one of its strings contained entirely in the cycle's closure in any* *pseudospherical extension. However, croissant type cycles can have all their strings extended into a pseudospherical arrangement so that each pseudocircle has some portion outside the cycle.*

We will label cloud, fish, and shrub type cycles as **(pseudospherical) obstruction cycles**. From the above result we deduce that if we have two disjoint obstruction cycles in some set of strings Σ, then Σ has no pseudospherical extension.

This condition is not necessary for collections of strings without pseudospherical extensions. If every string on an obstruction cycle is intersected by some string with both ends outside the cycle, then we cannot find a pseudospherical extension for the set of strings, since some pseudocircle is in the cycle and contains some part a the string with both ends outside the cycle, which violates **PS3**.

Theorem 2. *Every cloud type cycle has at least 4 pseudocircles contained entirely in its interior in any pseudospherical extension.*

Theorem 3. *Every fish or shrub type cycle has at least 2 pseudocircles contained entirely in its interior in any pseudospherical extension.*

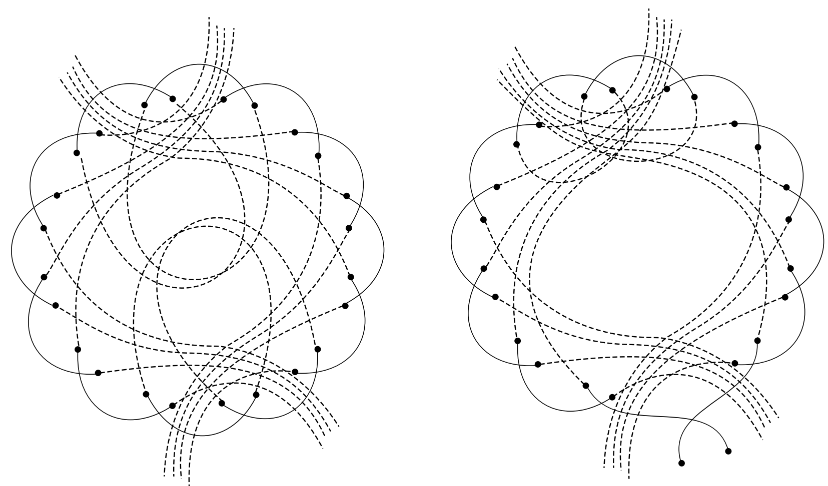

Fig. 1. Examples of extending cloud(left) and fish (right) cycles pseudospherically containing a minimum number of pseudocircles inside the cycle.

Question: Is there an excluded minor characterization of pseudospherical drawings along the lines of the pseudolinear characterization?

References

1. Arroyo, A., Richter, R.B., Sunohara, M.: Extending drawings of complete graphs into arrangements of pseudocircles. (Submitted)
2. Arroyo, A., Bensmail, J., Richter, R.B.: Extending Drawings of Graphs to Arrangements of Pseudolines. (Submitted)