Online Generalized Network Design Under (Dis)Economies of Scale

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Online Multicommodity Flow

- Directed graph $G = (V, E)$
- Each request $i$ has a
  - source $s_i$
  - destination $t_i$
  - demand $d_i \geq 1$
- Energy costs are sum of cost functions $f_e(\ell_e)$, where $\ell_e$ is the total demand going through edge $e$
- Goal: minimize total energy cost to route requests
In our setting, each edge cost is

\[ f_e(\ell_e) = \begin{cases} 
0 & \text{if } \ell_e = 0 \\
\sigma_e + \ell_e^{\alpha_e} & \text{if } \ell_e > 0
\end{cases} \]

We define \( q_e = \sigma_e^{1/\alpha_e} \) and

\[ q = \max_{e \in E} q_e \]
Online Multicommodity Flow Example
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## Previous Results: Multicommodity Flow

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<th>Function</th>
<th>Offline</th>
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<td>Concave</td>
<td>[Chekuri, Hajiaghayi, Kortsarz, Salavatipour, ’10]</td>
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| Convex   | [Andrews, Anta, Zhang, Zhao ’12]  
  [Makarychev, Sviridenko ’18] | [Gupta, Krishnaswamy, Pruhs ’12] |
| (D)oS    | [Andrews, Antonakopoulos, Zhang ’16] | [Antoniadis, Im, Krishnaswamy,  
  Moseley, Nagarajan, Pruhs, Stein, ’20] |

- All the above results are for undirected networks.
- Recently [Emek, Kutten, Lavi, Shi ’20] obtained (offline) approximation algorithms even for directed networks.
Result for Multicommodity flow

**Theorem**
Deterministic online algorithm for multicommodity flow with competitive ratio $O(q + (e^\alpha)^\alpha)$ where cost functions are (Dis)Economies of scale.

- First online result for (D)oS costs in directed graphs
- Matches the *offline* bound of [Emek, Kutten, Lavi, Shi ’20]
- Tight result in online setting
Our Setting - Online Generalized Network Design

- Input: set of $E$ resources and $N$ requests
- Each request $i$ is associated with replies $P_i$ which can satisfy $i$ and weight vector $w_i$ with $w_{i,e}$ representing load on resource $e$.
- Load on each resource $e$ is $l_e = \sum_{i:e \in P_i} w_{i,e}$
- Want to minimize total cost $\sum_{e \in E} f_e(l_e)$ with (D)oS functions
Our Results

Theorem (upper bound)

There is a polynomial time $O(q\tau + (e^{\alpha}\tau)^\alpha)$-competitive deterministic online algorithm for GND assuming a $\tau$-approximation algorithm for the min-cost oracle.

- The min cost oracle finds a $\tau$-approximate min-cost reply from $\mathcal{P}_i$ for any single request $i$ efficiently.
- The previous online multicommodity flow competitive ratio of $O(q + (e^{\alpha})^\alpha)$ follows from this result and having an exact min-cost oracle for online multicommodity flow.

Theorem (lower bound)

Any deterministic online algorithm has competitive ratio $\Omega(q + (1.44^{\alpha})^\alpha)$.
Overview

- Simplify (D)oS cost functions
- Convex program and dual
- Fractional online algorithm
- Integrality gap
- Integer online algorithm
- Lower bound
We model (D)oS cost functions as

\[ f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\sigma + x^\alpha & \text{if } x > 0 
\end{cases} , \]

Can approximate by a convex power function by a loss factor of \(2q\), with \(q := \sigma^{1/\alpha}\) using the function

\[ h(x) := q^{\alpha-1} \cdot x + x^\alpha \]
Lemma

For $x \in \{0\} \cup \mathbb{R}_{\geq 1}$, we have

$$\frac{1}{2} \cdot h(x) \leq f(x) \leq \max\{q, 1\} q^{\alpha - 1} \cdot x + x^{\alpha} = \max\{q, 1\} \cdot h(x).$$

Proof.
Lemma

For \( x \in \{0\} \cup \mathbb{R}_{\geq 1} \), we have

\[
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Proof.

If \( x < q \),

\[
h(x) = q^{\alpha - 1} \cdot x + x^\alpha \leq q^\alpha + x^\alpha = \sigma + x^\alpha = f(x).
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For \( x \in \{0\} \cup \mathbb{R}_{\geq 1} \), we have

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Proof.

If \( x < q \), \( h(x) = q^{\alpha-1} \cdot x + x^\alpha \leq q^\alpha + x^\alpha = \sigma + x^\alpha = f(x) \).

If \( x \geq q \), \( h(x) = q^{\alpha-1} \cdot x + x^\alpha \leq 2x^\alpha \leq 2(x^\alpha + \sigma) = 2f(x) \).
Lemma

For $x \in \{0\} \cup \mathbb{R}_{\geq 1}$, we have

$$\frac{1}{2} \cdot h(x) \leq f(x) \leq \max\{q, 1\} q^{\alpha - 1} \cdot x + x^\alpha = \max\{q, 1\} \cdot h(x).$$

Proof.

If $x < q$, $h(x) = q^{\alpha - 1} \cdot x + x^\alpha \leq q^\alpha + x^\alpha = \sigma + x^\alpha = f(x)$.

If $x \geq q$, $h(x) = q^{\alpha - 1} \cdot x + x^\alpha \leq 2x^\alpha \leq 2(x^\alpha + \sigma) = 2f(x)$

For the second inequality, we have

$$\max\{q, 1\} q^{\alpha - 1} \cdot x + x^\alpha \geq q^\alpha \cdot x + x^\alpha = \sigma x + x^\alpha \geq \sigma + x^\alpha = f(x).$$
We simplified (D)oS into sum of power functions (convex)
Algorithm for Convex Sums of Powers

We simplified (D)oS into sum of power functions (convex)
Consider cost functions $g_e(\ell_e) = c_e \cdot \ell_e^{\alpha_e}$
Assume that $\alpha_e \geq 1$
Algorithm for Convex Sums of Powers

- We simplified (D)oS into sum of power functions (convex)
- Consider cost functions $g_e(\ell_e) = c_e \cdot \ell_e^{\alpha_e}$
  Assume that $\alpha_e \geq 1$
- Obtain an $\alpha^\alpha$ competitive algorithm for a fractional relaxation.
We simplified (D)oS into sum of power functions (convex)

Consider cost functions $g_e(\ell_e) = c_e \cdot \ell_e^{\alpha_e}$

Assume that $\alpha_e \geq 1$

Obtain an $\alpha^\alpha$ competitive algorithm for a fractional relaxation.

Extend to get $(e\alpha)^\alpha$ competitive algorithm for the integral problem
Convex program relaxation for GND:

\[
\begin{align*}
\min \quad & \sum_{e \in E} c_e \cdot \left( \sum_{i=1}^{N} w_{i,e} \sum_{p \in \mathcal{P}_i : e \in p} x_{i,p} \right)^{\alpha_e} \\
\text{s.t.} \quad & \sum_{p \in \mathcal{P}_i} x_{i,p} \geq 1, \quad \forall i \in [N] \\
\quad & x \geq 0.
\end{align*}
\]
Fractional Online Algorithm

For the analysis, will use dual convex program for GND:

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{N} y_i - \sum_{e \in E \setminus E_1} \frac{c_e \alpha_e}{\beta_e} \cdot z_e^\beta_e \\
\text{s.t.} & \quad \sum_{e \in p} w_{i,e} c_e \alpha_e \cdot z_e \geq y_i, \quad \forall p \in P_i, \forall i \in [N] \\
& \quad z_e \leq 1, \quad \forall e \in E_1 \\
& \quad y, z \geq 0.
\end{align*}
\]

Lemma

For any primal \( x \in (P) \) and dual \((y, z) \in (D)\) solutions,

\[
\sum_{e \in E} c_e \cdot \left( \sum_{i=1}^{N} w_{i,e} \sum_{p \in P_i : e \in p} x_{i,p} \right)^{\alpha_e} \geq \sum_{i=1}^{N} y_i - \sum_{e \in E \setminus E_1} \frac{c_e \alpha_e}{\beta_e} \cdot z_e^\beta_e.
\]
Upon arrival of request $i$, do the following:
For each continuous time $t \in [0, 1]$:

- Choose reply $p^* \in \mathcal{P}_i$ using the min-cost oracle under costs $d_e = \alpha_e c_e \cdot \ell_e^{\alpha_e-1} \cdot w_{i,e}$ for each $e \in E$, where $\ell_e$ is the current fractional load on $e$.
- Raise primal variable $x_{i,p^*}$ at rate one, i.e. $\frac{\partial}{\partial t} x_{i,p^*} = 1$

The fractional online algorithm has competitive ratio at most $\alpha^\alpha$, where $\alpha = \max_{e \in E} \alpha_e$. 
Example where fractional relaxation has a large integrality gap:

Let all costs be $f(x) = x^2$ with a single request, and $n$ edges. If we divide the request up uniformly, total cost is $1/n$. In the integer setting, the total cost is 1.
Convex program relaxation for GND:

\[
\begin{align*}
\min & \sum_{e \in E} c_e \cdot \left( \sum_{i=1}^{N} w_{i,e} \sum_{p \in \mathcal{P}_i : e \in p} x_{i,p} \right)^{\alpha_e} + \sum_{e \in E} \frac{c_e \alpha_e}{e^{\alpha}} \cdot \sum_{i=1}^{N} w_{i,e}^{\alpha_e} \sum_{p \in \mathcal{P}_i : e \in p} x_{i,p} \\
\text{s.t.} & \sum_{p \in \mathcal{P}_i} x_{i,p} \geq 1, \quad \forall i \in [N] \\
& x \geq 0.
\end{align*}
\]

- Optimal convex program value at most 
  \((1 + \alpha e^{-\alpha}) \cdot OPT \leq 2 \cdot OPT.\)
Upon the arrival of request $i$, we do the following:

- Choose reply $p_i \in \mathcal{P}_i$ using the min-cost oracle under the costs

$$
\psi_e = \alpha_e c_e \cdot \ell_e^{\alpha_e-1} \cdot w_{i,e} + \frac{\rho}{e^\alpha} \cdot c_e \alpha_e w_{i,e}, \text{ for each } e \in E,
$$

where $\ell_e := \sum_{j<i; e \in p_j} w_{j,e}$ is the current load on $e$. 
Lower Bound

Machine Scheduling

- $\Omega((1.44\alpha)^\alpha)$ lower bound from restricted assignment scheduling with $\ell_p$-norm load balancing (Caragiannis, 2008)

Online Directed Steiner Tree

- Similar idea as $\Omega(q)$ lower bound (Faloutsos, Pankaj, Sevcik, 2002)
Conclusion
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Convex program and dual
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- Fractional and Integer algorithms
- Future work: primal-dual algorithms with convex programs for more general problems?
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- GND and multicommodity flow
- Convex program and dual
- Fractional and Integer algorithms
- Future work: primal-dual algorithms with convex programs for more general problems?
- Thanks for watching!