EECS 230
ENGINEERING ELECTROMAGNETICS
Leland Pierce

Time-Varying Fields 3

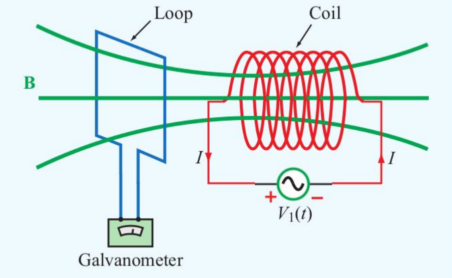
University of Michigan - Fall 2024

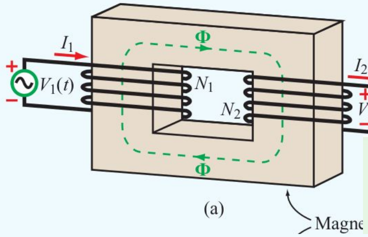
Chapter 6 Overview

Time-Varying Fields
Faraday's Law
Stationary Loop in
  time-varying field
Ideal Transformer
Moving conductor in
  static field
The Generator
Moving conductor in
  time-varying field
Displacement Current
Boundary Conditions
Charge-Current Continuity
Free-Charge dissipation
  in a conductor
Potentials

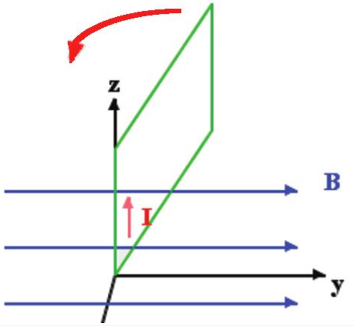
$$\nabla\cdot\mathbf{D}=\rho_v,$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},$$
$$\nabla\cdot\mathbf{B}=0,$$
$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}.$$

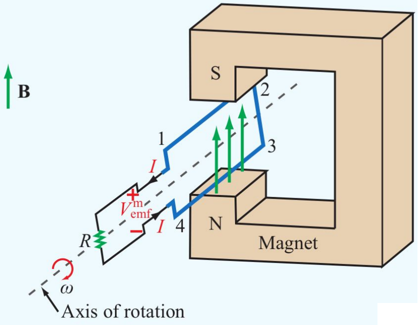
$$\oint_C\mathbf{E}\cdot d\mathbf{l}=-\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{s}$$

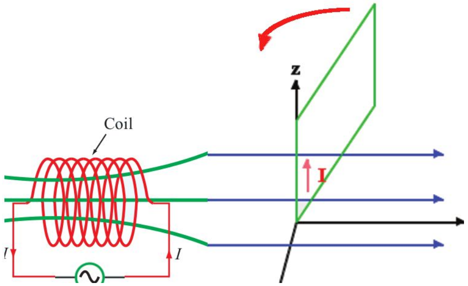




$$V_{\text{emf}}^{\text{tr}}=-N\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{s},\quad \textbf{(transformer emf)}$$



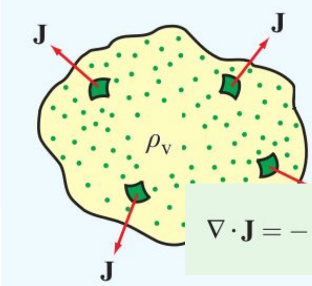




$$V_{\text{emf}}^{\text{m}}=\oint_C(\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}.\qquad \textbf{(motional emf)}$$

$$V_{\text{emf}}=-\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{s}+\oint_C(\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}.$$

$$I_{\text{d}}=\int_S\mathbf{J}_{\text{d}}\cdot d\mathbf{s}=\int_S\frac{\partial\mathbf{D}}{\partial t}\cdot d\mathbf{s},$$



$$\nabla\cdot\mathbf{J}=-\frac{\partial\rho_v}{\partial t}$$

$$\widetilde{V}(\mathbf{R})=\frac{1}{4\pi\varepsilon}\int_{\upsilon'}\frac{\tilde{\rho}_v(\mathbf{R}_i)\,e^{-jkR'}}{R'}\,d\upsilon'$$

$$\rho_v(t)=\rho_{vo}e^{-(\sigma/\varepsilon)t}$$

Lecture Coverage

Today's lecture:

Review sections 6-1 through 6-6 of the book:

6-1: Faraday's Law
6-2: Stationary Loop in time-varying **B** Field
6-3: Ideal Transformer
6-4: Moving Loop in static **B** field
6-5: The Electromagnetic Generator
6-6: Moving Loop in time-varying **B** Field

Sections 6-7 through 6-9 of the book:

6-7: Displacement Current
6-8: Electromagnetic Boundary Conditions
6-9: Charge-Current Continuity Relation

Review of Chapter 6

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = Q$$

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}$$

$$\oint_S \mathbf{B}\cdot d\mathbf{s} = 0$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{s}$$

Maxwell's Equations Empirically derived from many measurements

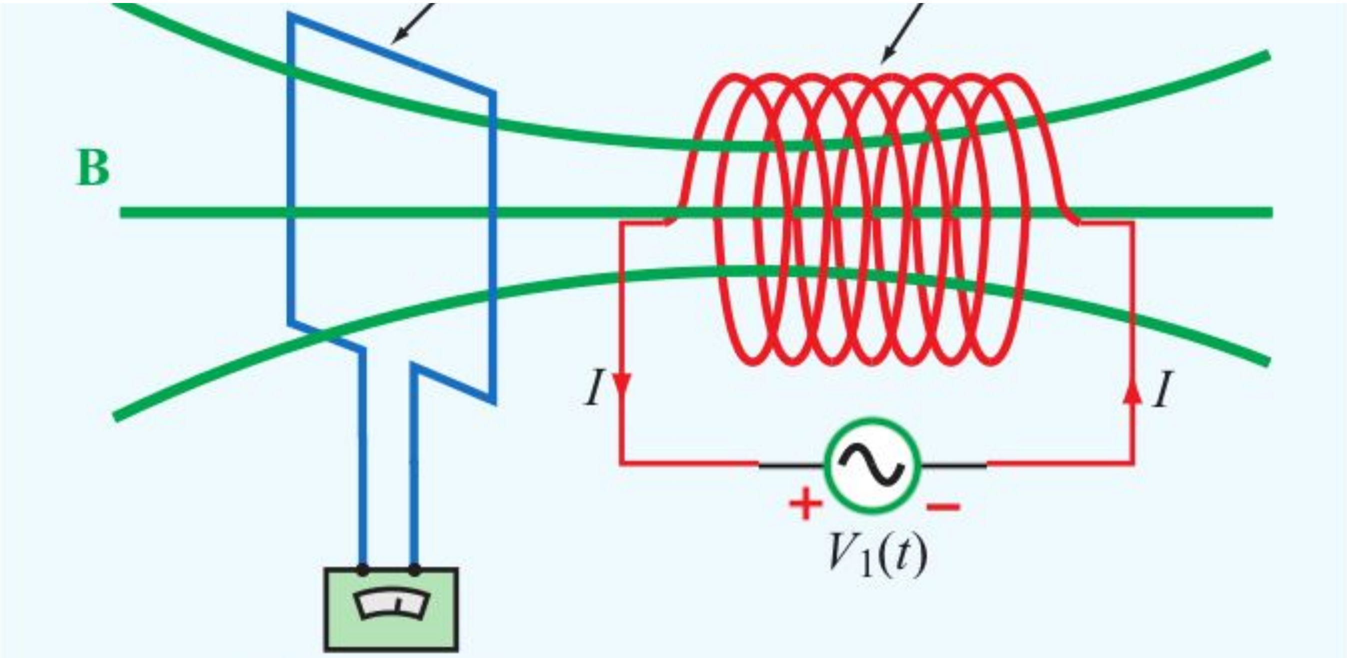
$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}.$$

E: Electric Field
H: Magnetic Field
J: Current Density
ρ_v: Charge Density

Review of Chapter 6

Faraday's Law



$$V_{\text{emf}} = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$$

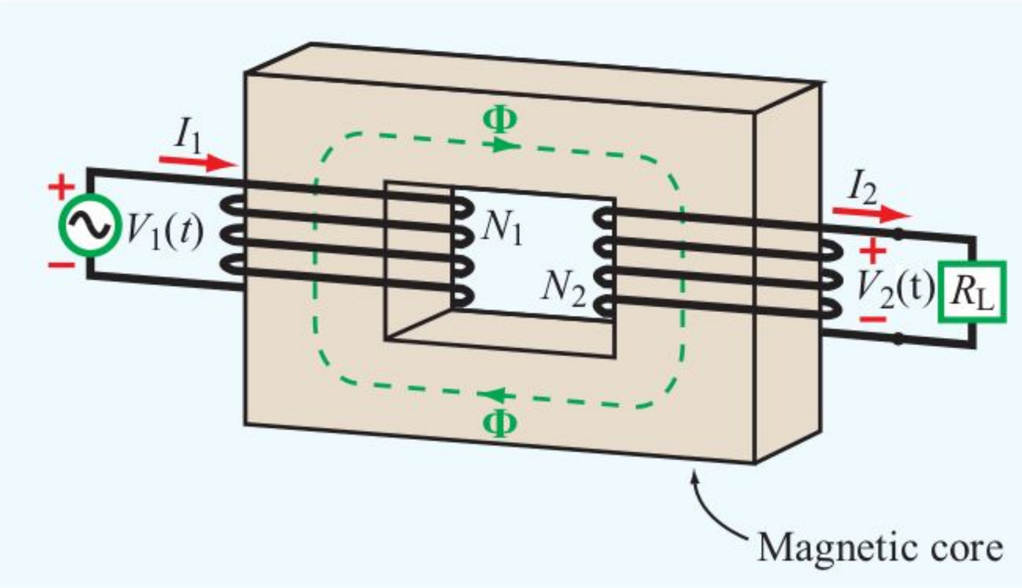
Review of Chapter 6

Time-varying **B** field, Stationary loop

$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}, \qquad \textbf{(transformer emf)}$$

Review of Chapter 6

The Ideal Transformer



For example, if $N_2=10N_1$,
Phasors:

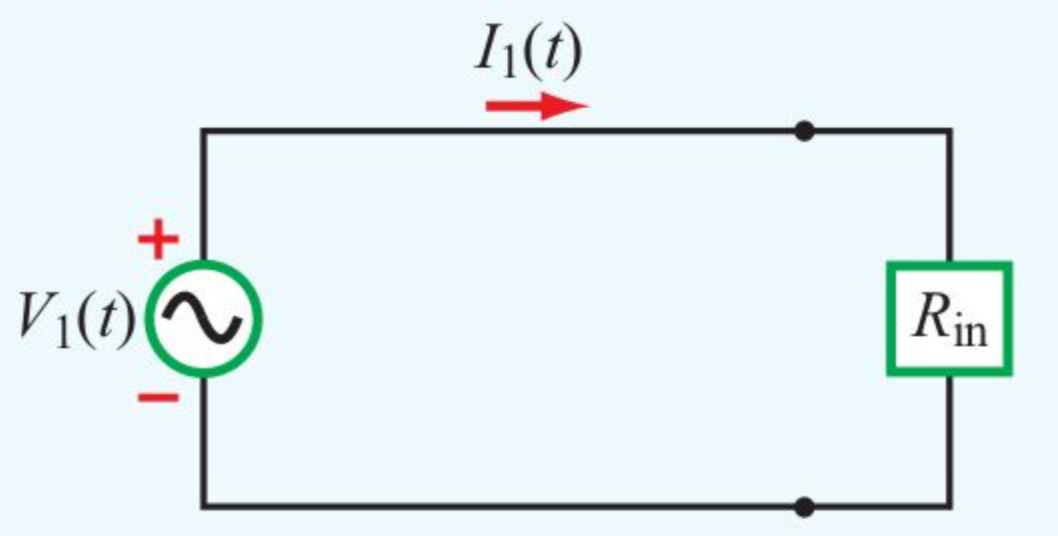
$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

$V_2 = 10\ V_1$
$I_2 = 0.1\ I_1$

Review of Chapter 6

The Ideal Transformer



Equivalent Circuit:

$$\boxed{R_{\text{in}} = \left(\frac{N_1}{N_2}\right)^2 R_L}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} \qquad \frac{V_1}{V_2} = \frac{N_1}{N_2}$$

Phasor domain:

$$\boxed{Z_{\text{in}} = \left(\frac{N_1}{N_2}\right)^2 Z_{\text{L}}}$$

Review of Chapter 6

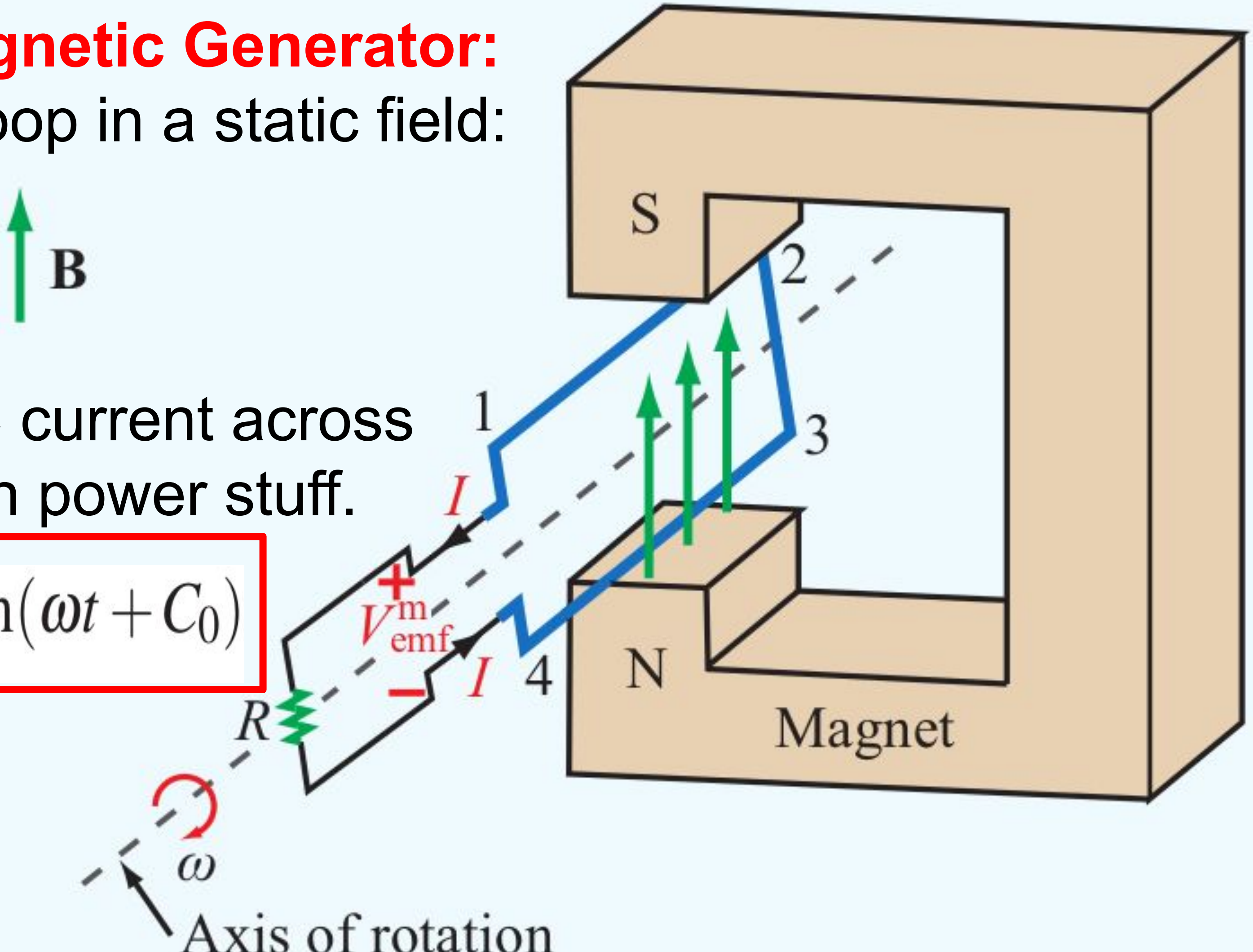
Motional EMF: moving loop, static **B**:

$$V_{\mathrm{emf}}^{\mathrm{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

Review of Chapter 6

Electromagnetic Generator:

A rotating loop in a static field:



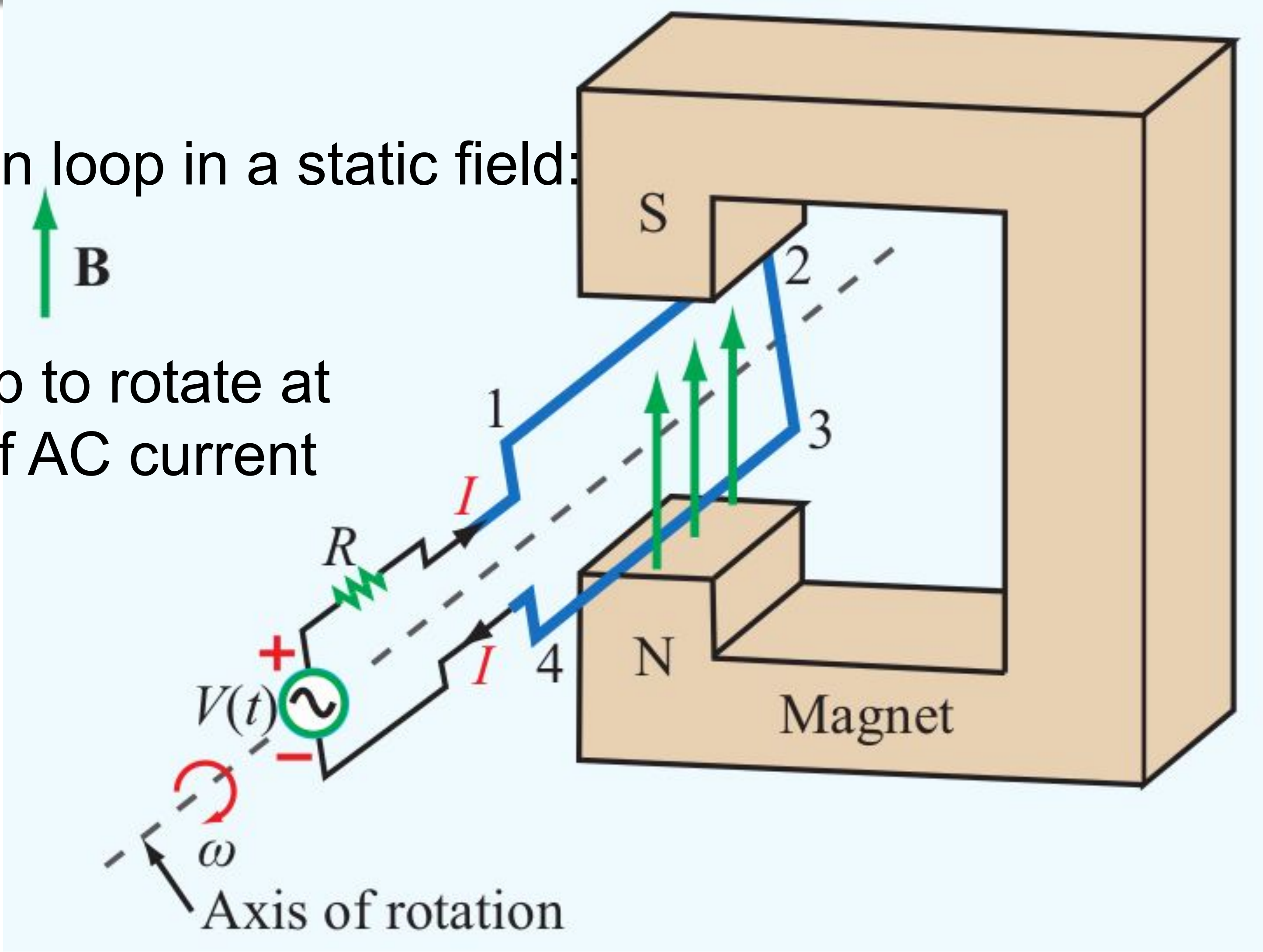
Induced AC current across the load can power stuff.

$$\boxed{V_{\mathrm{emf}} = A\omega B_0 \sin(\omega t + C_0)}$$

Review of Chapter 6

AC Motor:

AC current in loop in a static field:



Induces loop to rotate at frequency of AC current

Review of Chapter 6

When **BOTH**: loop is moving **and** field is varying:

$$\begin{aligned} V_{\text{emf}} &= V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}} \\ &= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}. \end{aligned}$$

General expression of Faraday's Law still valid:

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}$$

6-7 Displacement Current

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(Ampère's law).}$$

H is the magnetic field
J is the current density
The other term:
has the same units as current density
named: "displacement current density"

6-7 Displacement Current

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \mathbf{J}_d$$

H is the magnetic field

$\mathbf{J}_c$ is the conduction current density

$\mathbf{J}_d$ is the "displacement" current density

6-7 Displacement Current

Integral form:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}.$$

since the "conduction current" is:

$$\int_S \mathbf{J} \cdot d\mathbf{s} = I_c$$

and Stokes' Theorem says:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \oint_C \mathbf{H} \cdot d\mathbf{l},$$

get:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

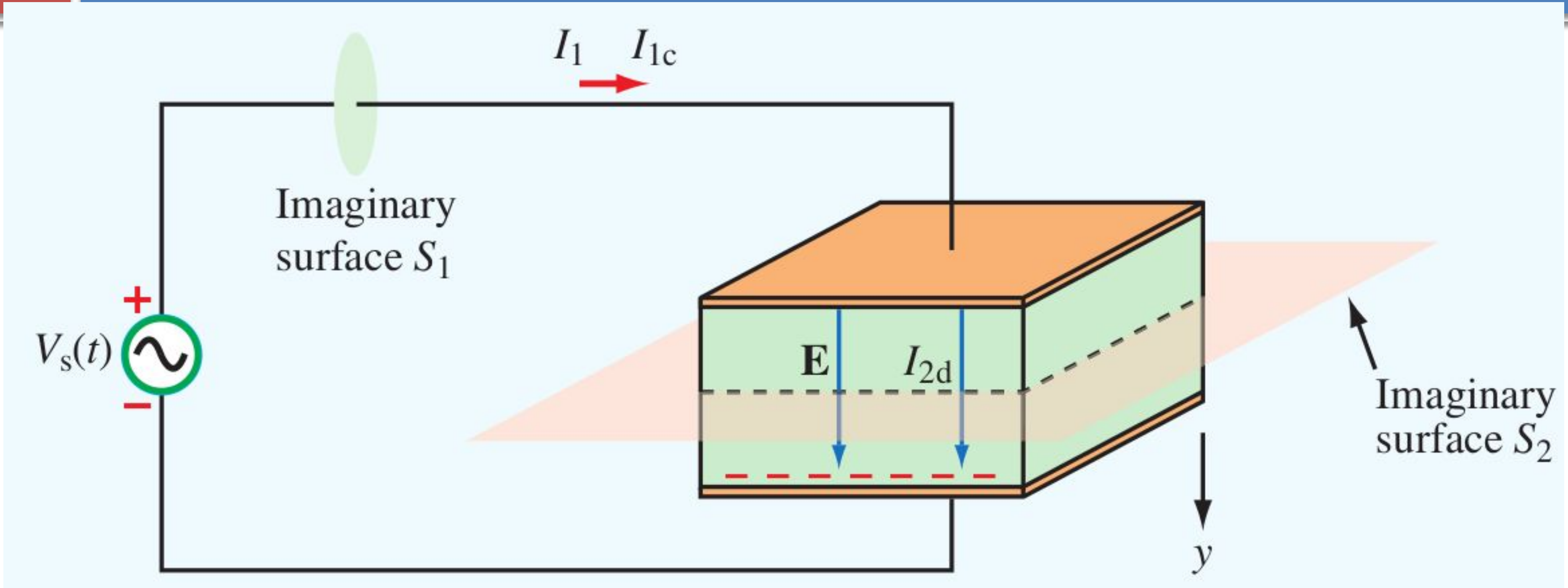
(Ampère's law)

6-7 Displacement Current

So, the total current is the sum of the conduction current and the displacement current:

$$I_{\mathrm{c}} + I_{\mathrm{d}} = I,$$

Example 1



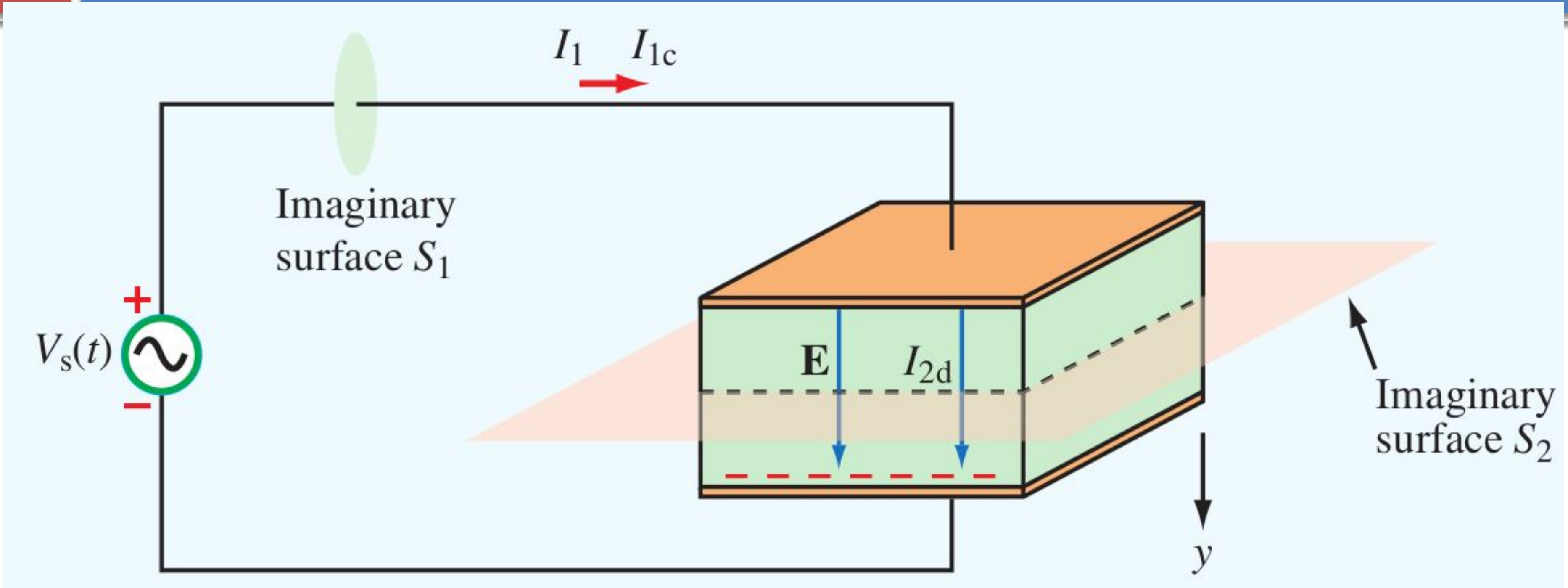
Given: voltage source: $V_s(t) = V_0 \cos \omega t$

capacitor: plate-spacing=d

y directed down

Find: I_c and I_d, in the wire and in the capacitor

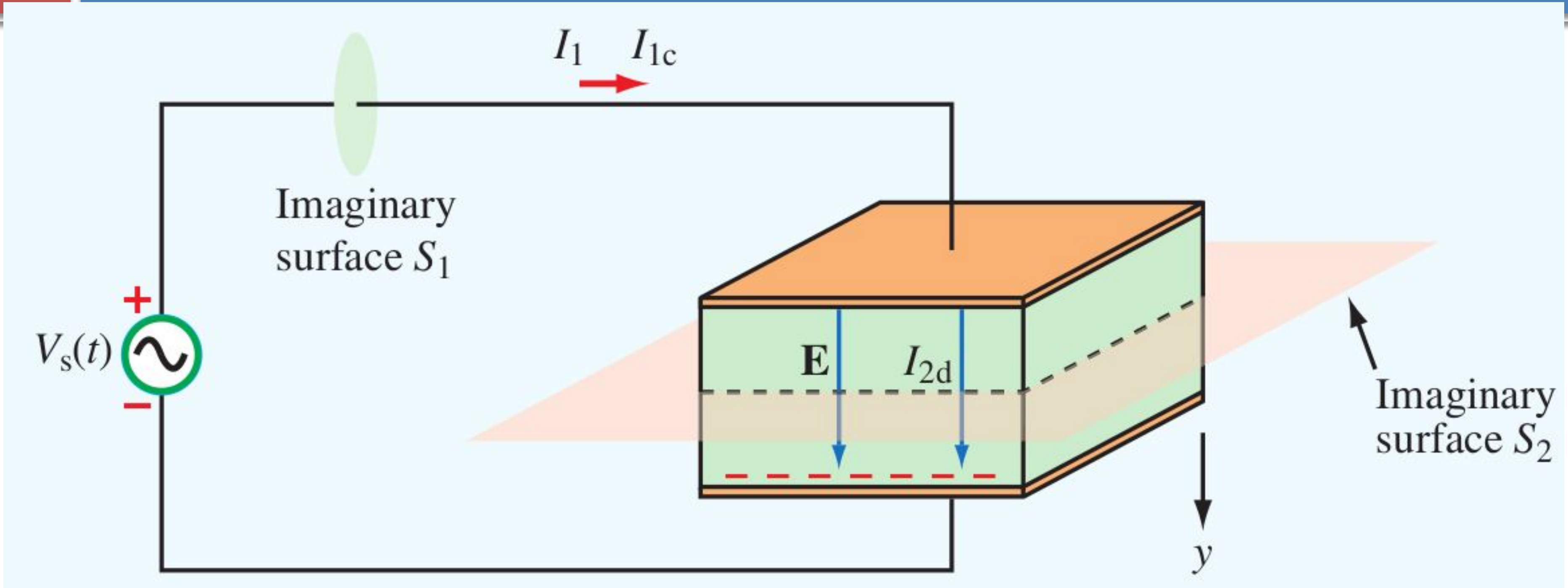
Example 1



Solution: Name the components of the current in the wire: $I_1 = I_{1c} + I_{1d}$ and in the capacitor: $I_2 = I_{2c} + I_{2d}$

Assume perfectly conducting wire: **E**=0, **D**=0, so $I_{1d} = 0$
Assume σ=0 in capacitor: $I_{2c} = 0$

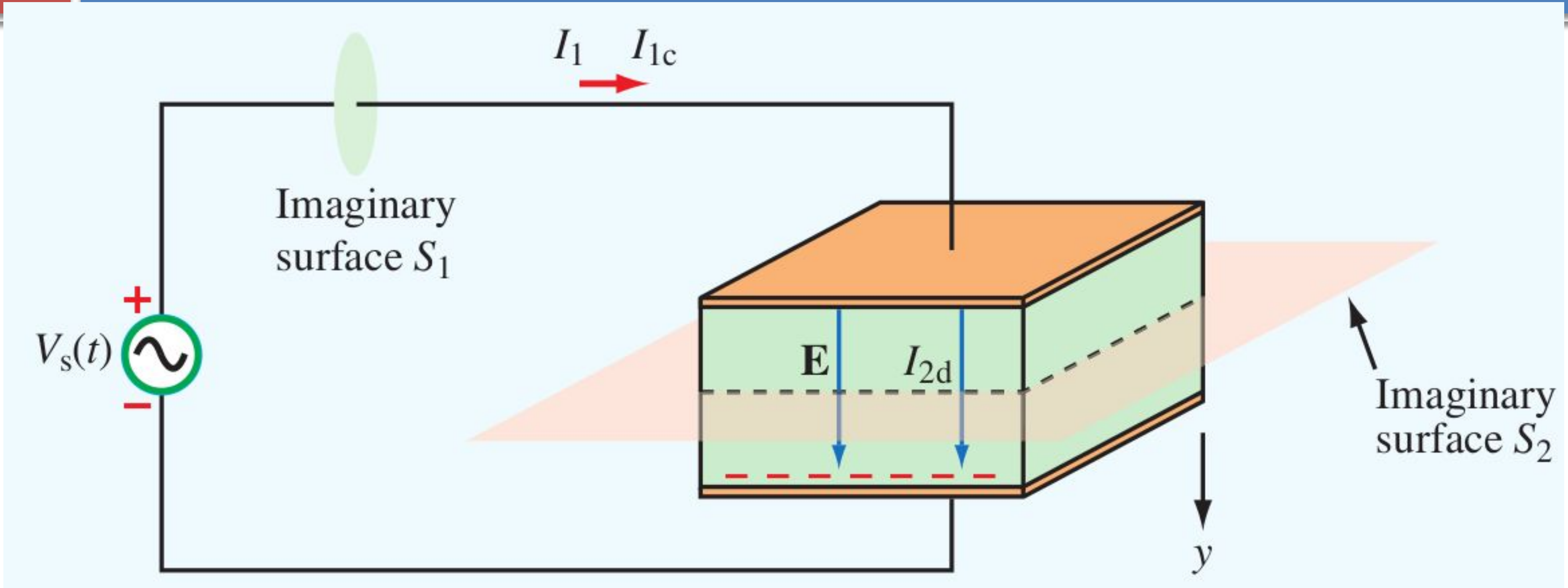
Example 1



Solve for I_{1c} using circuit theory:

$$I_{1c} = C\frac{dV_C}{dt} = C\frac{d}{dt}\left(V_0\cos\omega t\right) = -CV_0\omega\sin\omega t,$$

Example 1



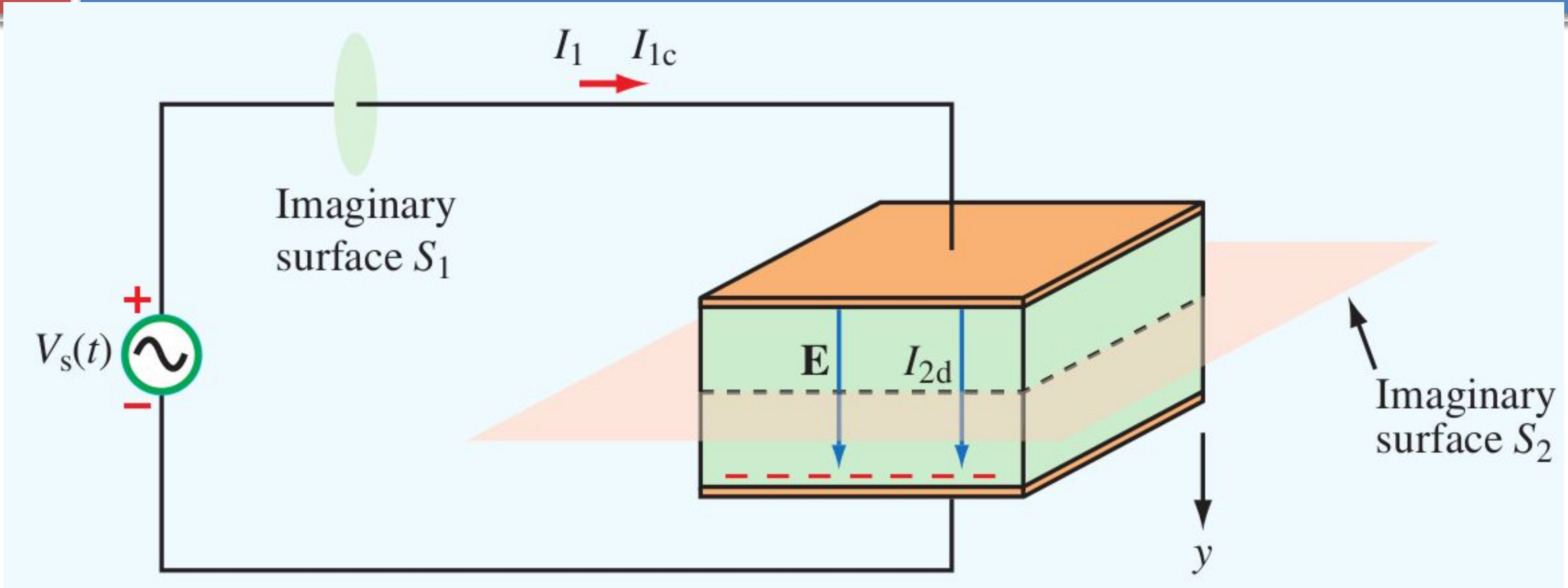
Solve for I_{2d} using Ampère's Law:

$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

D=ε **E** and:

$$\mathbf{E} = \hat{\mathbf{y}} \frac{V_c}{d} = \hat{\mathbf{y}} \frac{V_0}{d} \cos \omega t$$

Example 1

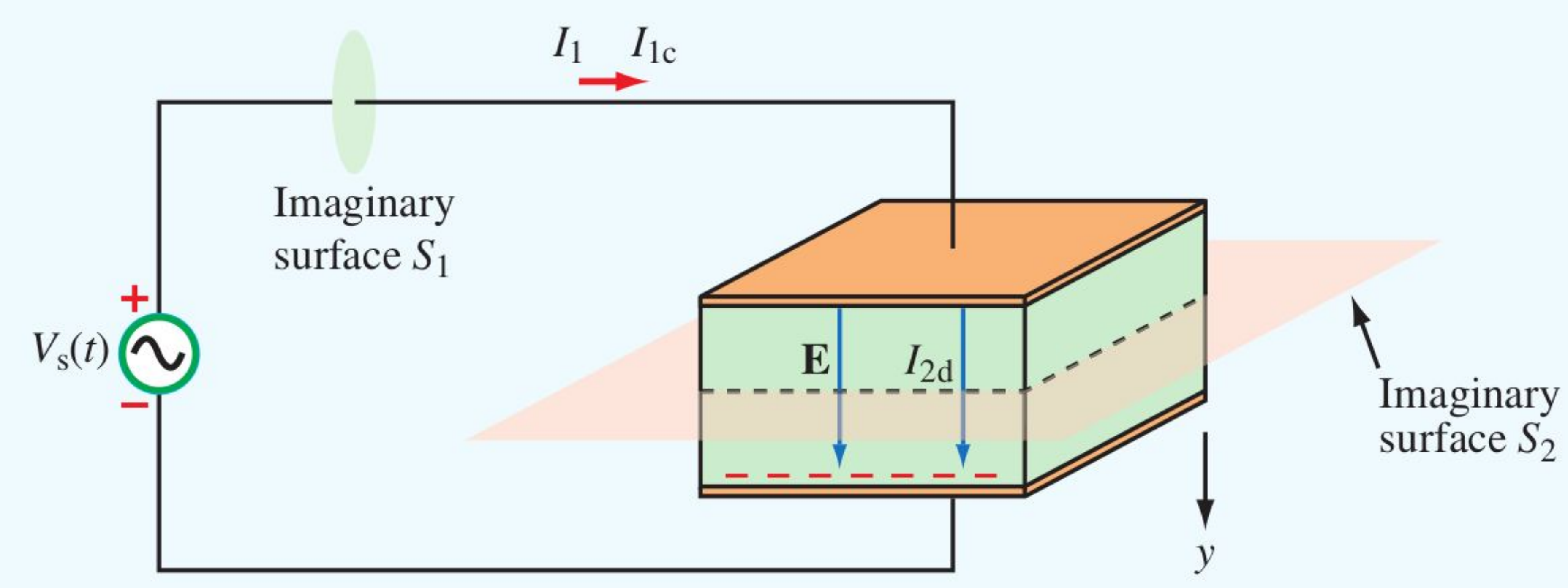


$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad = \int_A \left[\frac{\partial}{\partial t} \left(\hat{\mathbf{y}} \, \frac{\varepsilon V_0}{d} \cos \omega t \right) \right] \cdot (\hat{\mathbf{y}} \, ds)$$

$$= -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t \quad = -C V_0 \omega \sin \omega t,$$

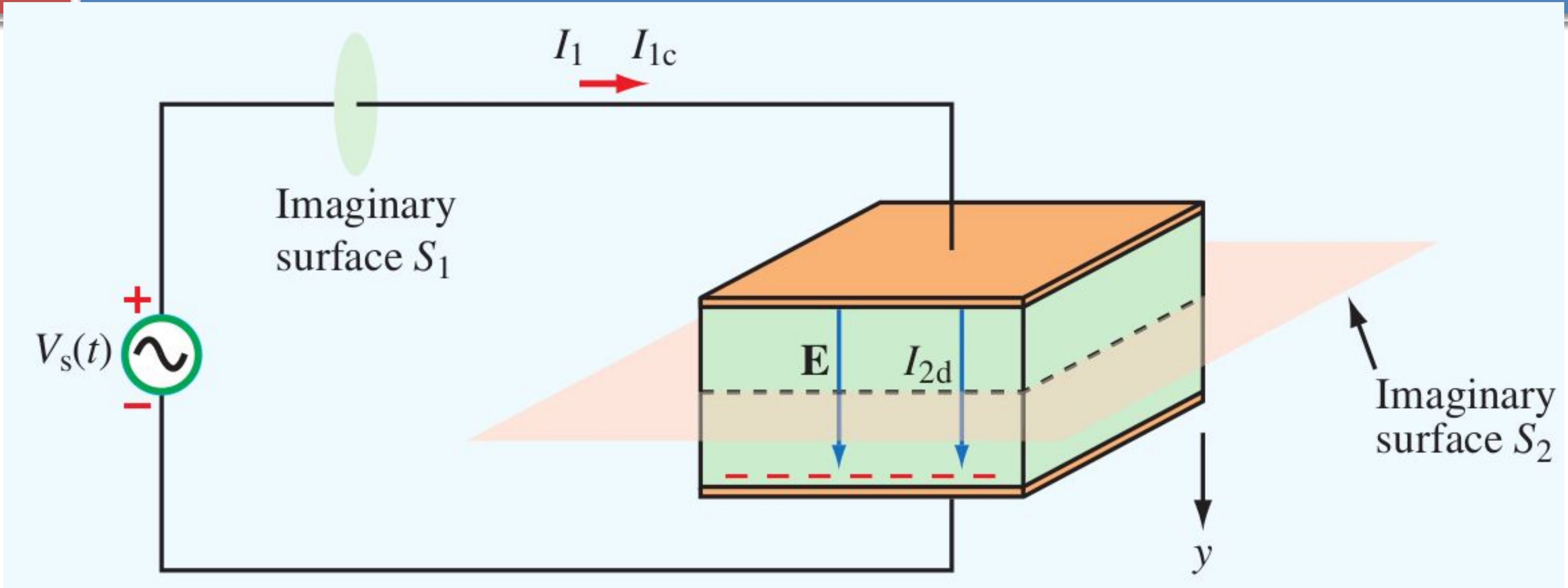
(we know: $C = \varepsilon A/d$)

Example 1



So, we get: $I_{1c} = I_{2d}$, as expected.

Example 1



The **conduction current is moving free electrons** back-and-forth with the sinusoidal force of the applied voltage

The **displacement current is moving bound electrons** in atoms/molecules back-and-forth

Example 2

Given: Wire with finite conductivity: $\sigma < \infty$, ε

Conduction current: $I_c = I_0 \sin \omega t$ Amps,

Find: the Displacement Current, I_d

Solution: Use Ampère's Law:

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s},$$

with A=wire cross-sectional area:

$$I_c = JA = \sigma EA, \qquad \text{so: } E = I_c / (\sigma A)$$

$$\text{so: } D = \varepsilon E \quad = \varepsilon\, I_c / (\sigma A) \quad = [I_0 \varepsilon / (\sigma A)] \sin \omega t$$

ds is over the cross-section of the wire

D and d**s**: both oriented along the length of the wire

Example 2

Using Ampère's Law:

$$I_{\mathrm{d}} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

$$I_{\mathrm{d}} = \int_S \frac{\partial}{\partial t} \left(\hat{\mathbf{n}} \frac{I_0 \varepsilon}{\sigma A} \sin \omega t \right) \cdot \hat{\mathbf{n}} ds$$

$$I_{\mathrm{d}} = \frac{I_0 \varepsilon}{\sigma A} A\omega \, \cos \omega t$$

$$\boxed{I_{\mathrm{d}} = (I_0 \varepsilon \omega / \sigma) \, \cos \omega t}$$

Example 2

$$I_\mathrm{d} = (I_0 \varepsilon \omega / \sigma) \cos \omega t$$

Example 6-7: I_0 = 2mA, σ= 2X10^7 S/m, ε= 8.85X10^{-12} F/m,
ω=10^9 rad/sec

Get:

I_d = 0.885X10^{-12} cos ωt Amps

which is 9 orders of magnitude smaller than I_c
and with a 90^o phase offset as well.

Displacement Current usually ignored in good conductors.

Exercise 6-5

Given: poor conductor

σ = 100 S/m

ε_r = 4

Find: ω such that magnitudes are equal: $|J_c| = |J_d|$

Solution: Assume: $E = E_0 \cos \omega t$

$$|J_c| = |J_d|$$

$$\sigma |E| = \left| \frac{\partial D}{\partial t} \right|$$

Exercise 6-5

$$\sigma|E| = \left|\frac{\partial D}{\partial t}\right|$$

$$\sigma E_0|\cos\omega t| = \epsilon E_0|-\omega\sin\omega t|$$

Ignore time-dependence, as well as phase:

$$\sigma E_0 = \epsilon\omega E_0$$

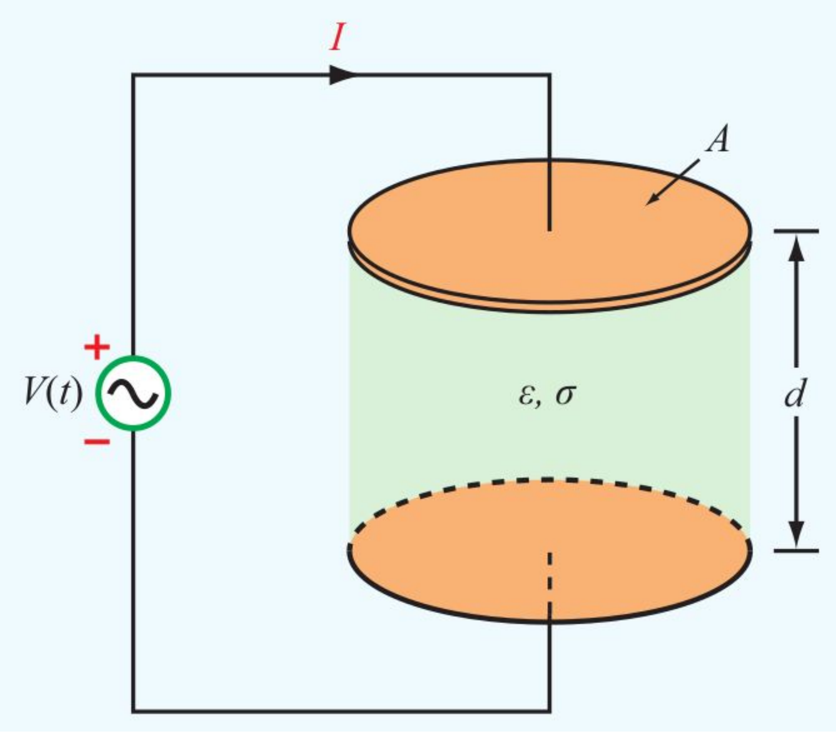
$$\omega = \frac{\sigma}{\varepsilon}$$

$$\omega = \frac{100}{(4)(8.85\times 10^{-12})}$$

$$\boxed{\omega = 2.82\times 10^{12}\ \mathrm{rad/sec}}$$

Example 4

Given: parallel-plate capacitor
filled with: ε_r, $\sigma > 0$
separation d, area A
$V(t) = V_0 \cos(\omega t)$



Find: (a) Conduction current, I_c,
(b) Displacement current, I_d,
both flowing between the two plates

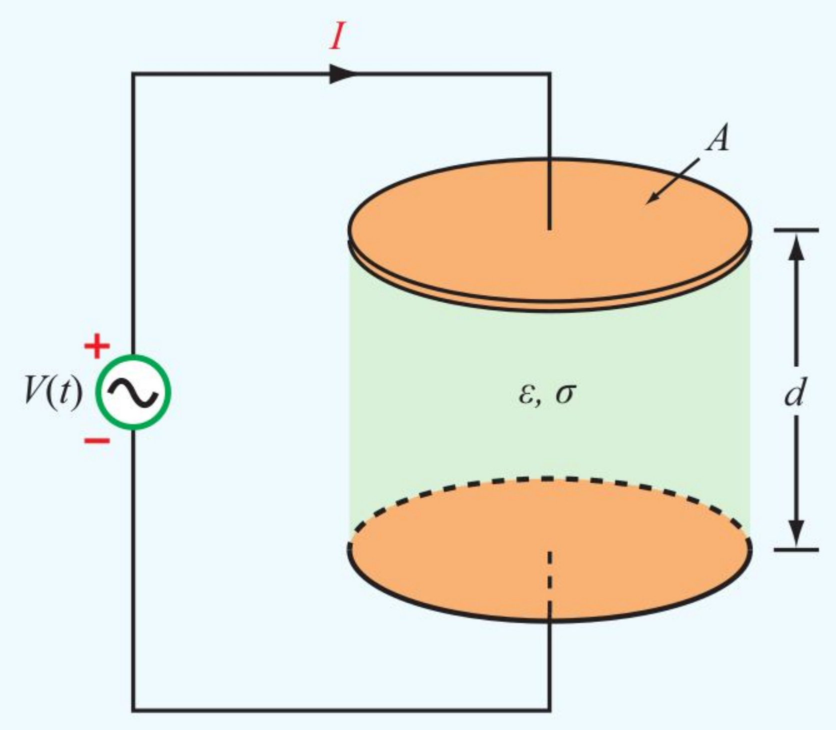
Solution: **(a) Conduction current:**

treat the material as a resistor:

$$R = \frac{d}{\sigma A}$$

Example 4

Solution: **(a) Conduction current:** treat the material as a resistor:



$$R = \frac{d}{\sigma A}$$

Ohms Law:

$$I_c = \frac{V}{R}$$

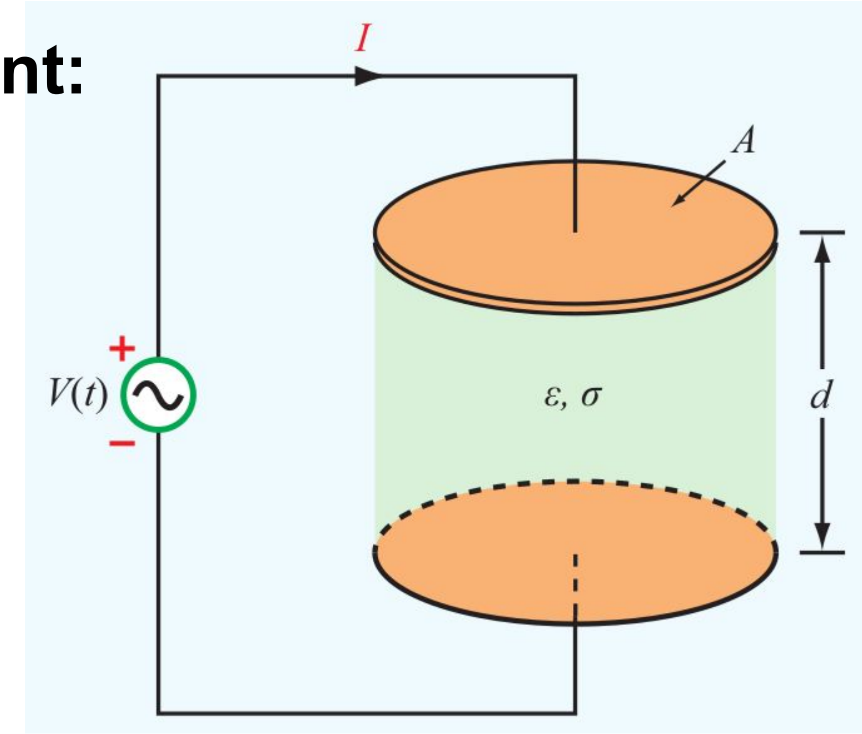
(*V* is positive on top plate)

$$I_c = \frac{V \sigma A}{d}$$

so: $$\boxed{I_c = \frac{\sigma A}{d} V_0 \cos \omega t}$$

Example 4

Solution: **(b) Displacement current:**



$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s},$$

assume field is uniform:

$$I_d = \frac{\partial D}{\partial t} \cdot A$$

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

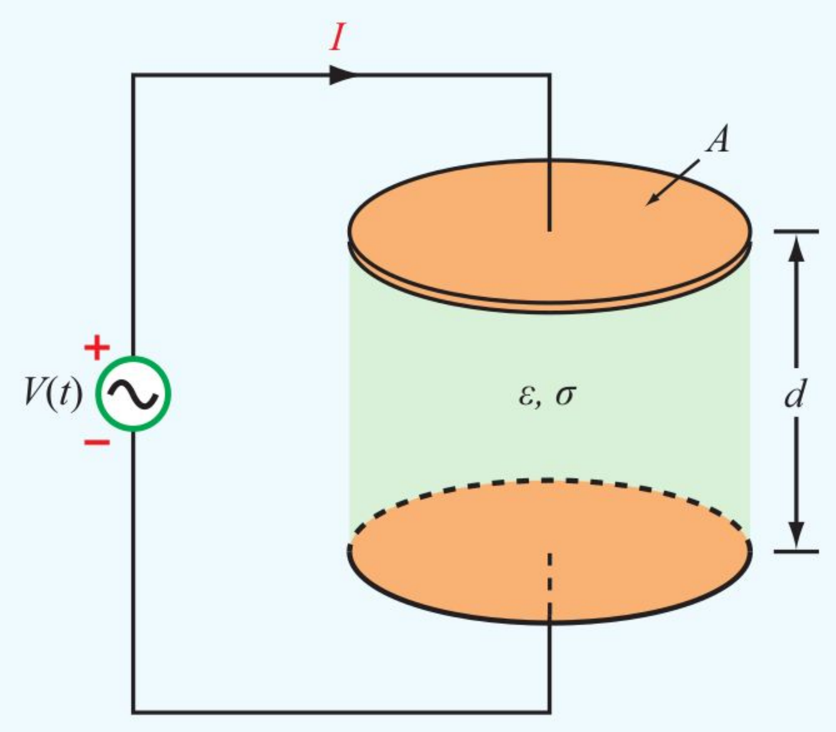
$$I_d = \frac{\varepsilon A}{d} \frac{\partial V(t)}{\partial t}$$

Example 4

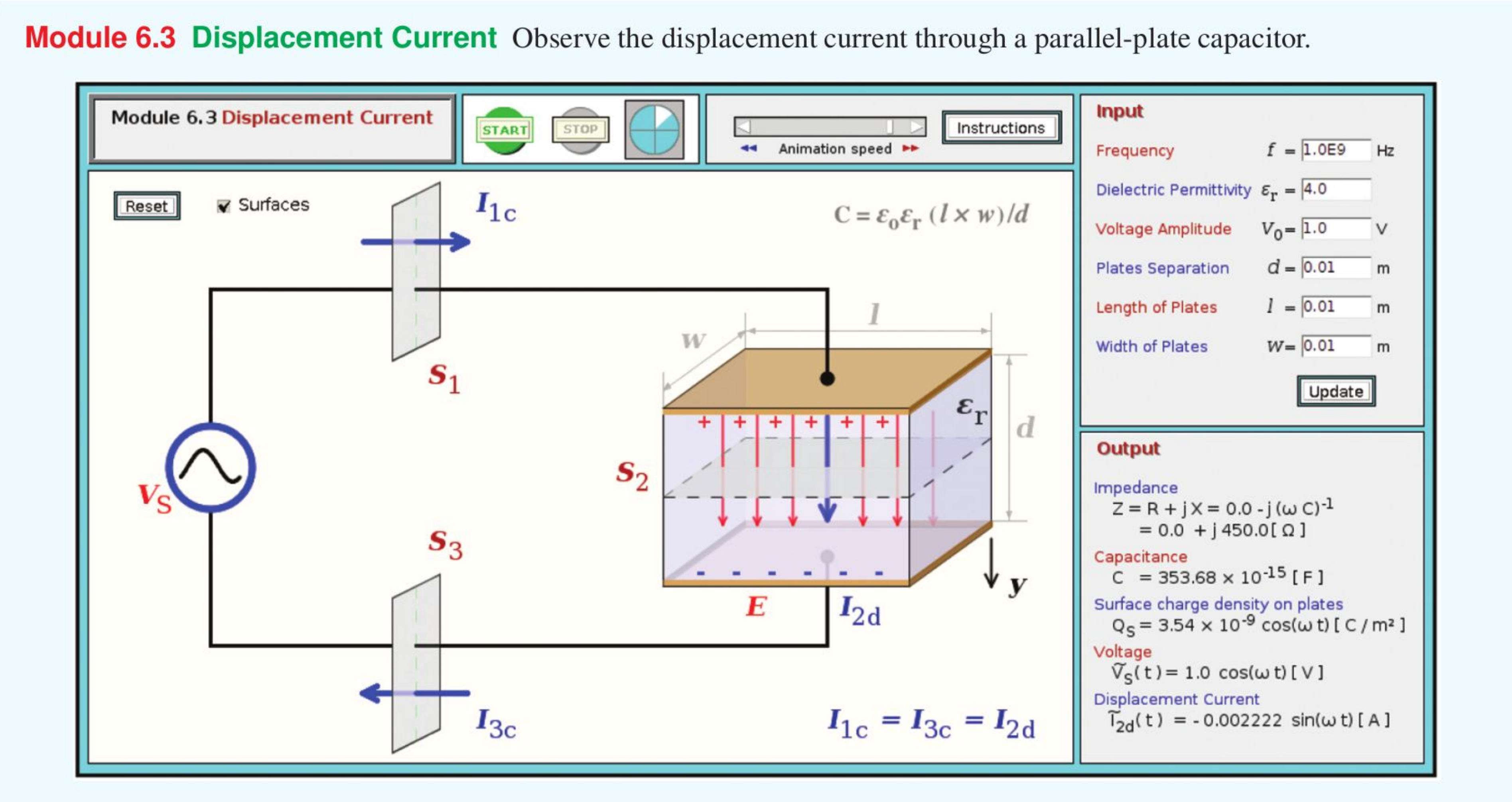
$$I_d = \frac{\varepsilon A}{d}\frac{\partial V(t)}{\partial t}$$

$$I_d = \frac{\varepsilon A}{d}\frac{\partial V_0 \cos \omega t}{\partial t}$$

$$\boxed{I_d = -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t}$$



Displacement Current



Module 6.3 **Displacement Current** Observe the displacement current through a parallel-plate capacitor.

Input

Frequency f = 1.0E9 Hz

Dielectric Permittivity ε_r = 4.0

Voltage Amplitude V_0 = 1.0 V

Plates Separation d = 0.01 m

Length of Plates l = 0.01 m

Width of Plates w = 0.01 m

Output

Impedance

$Z = R + jX = 0.0 - j(\omega C)^{-1}$

$= 0.0 + j\,450.0\,[\Omega]$

Capacitance

$C = 353.68 \times 10^{-15}\,[F]$

Surface charge density on plates

$Q_S = 3.54 \times 10^{-9} \cos(\omega t)\,[C/m^2]$

Voltage

$\tilde{V}_S(t) = 1.0 \cos(\omega t)\,[V]$

Displacement Current

$\tilde{I}_{2d}(t) = -0.002222 \sin(\omega t)\,[A]$

6-8 Electromagnetic Boundary Conditions

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}},$$

$$\nabla \times \mathbf{E} = 0.$$

In the dynamic case we must use:

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

So the condition on the normal component does not change. (1st eqn)
Let's look at the condition on the tangential component.

6-8 Electromagnetic Boundary Conditions

In the dynamic case we must use:

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

Previously we used KVL:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

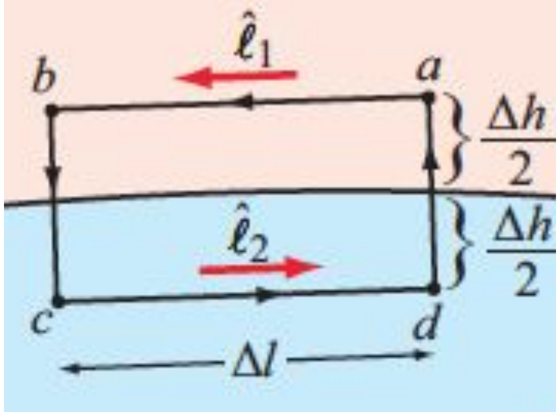
instead we must use:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

6-8 Electromagnetic Boundary Conditions

Using this for determining the boundary conditions:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$



We then apply this over the rectangular region that spans the 2 media.

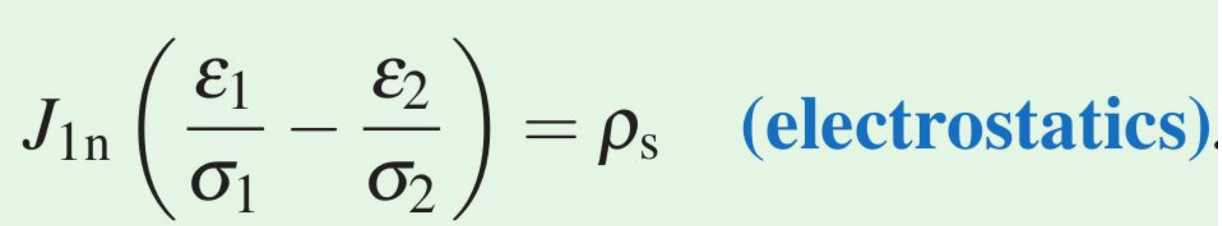
The process involves taking the limit *as the area goes to zero.*
Hence, the right-hand integral goes to zero.
Resulting in the same boundary condition as before.

6-8 Electromagnetic Boundary Conditions

The same thing happens for the magnetostatic boundary conditions.

Hence, the static boundary conditions for **E, D, H, B** can be used for dynamic electromagnetic problems as well.

However, the condition that assumed continuity of the normal component of the current density:

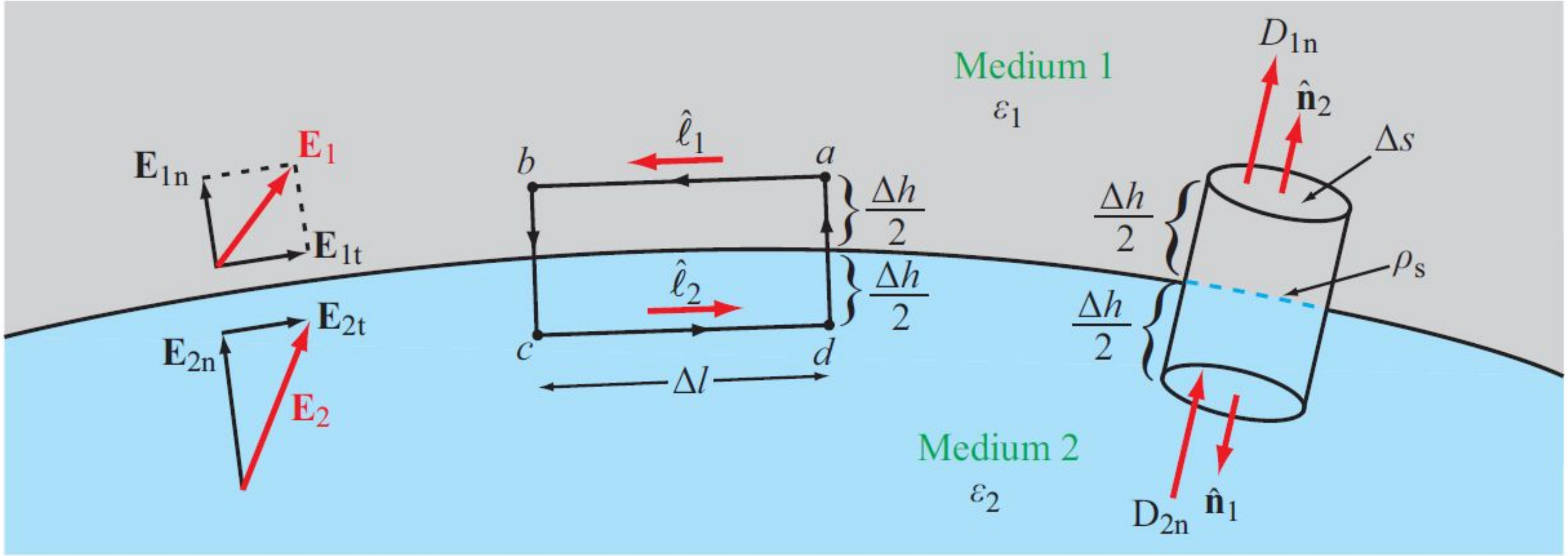


$$J_{1\mathrm{n}}\left(\frac{\varepsilon_1}{\sigma_1}-\frac{\varepsilon_2}{\sigma_2}\right)=\rho_{\mathrm{s}} \quad \textbf{(electrostatics)}.$$

is **not true** for dynamic electromagnetics problems.

6-8 Electromagnetic Boundary Conditions

Boundary Conditions between two dielectric materials:



$$\mathbf{E}_{1t} = \mathbf{E}_{2t}$$

$$D_{1n} - D_{2n} = \rho_s \qquad (\mathrm{C/m^2}).$$

$$H_{1t} = H_{2t}.$$

$$\mathbf{B}_{1n} = \mathbf{B}_{2n}$$

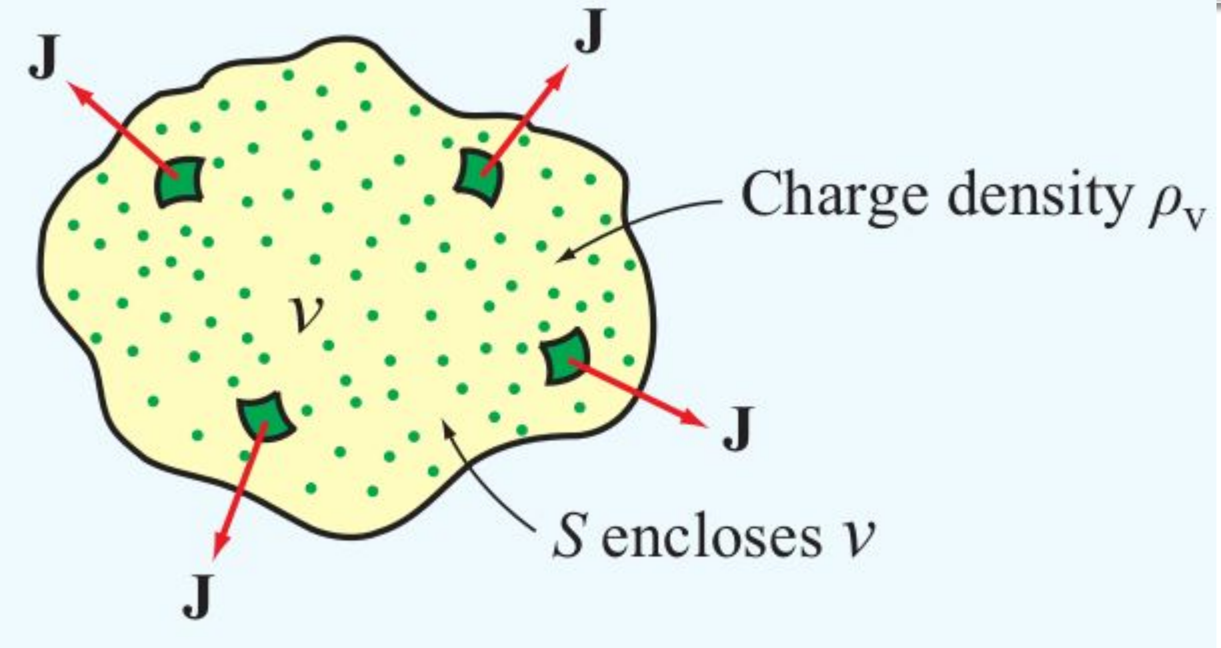
6-9 Charge-Current Continuity Relation

Statics:
Current density: **J**, **independent** of charge density: ρ_v

Time-Varying:
No longer true.

Let's derive the relationship between these parameters in the time-varying case.

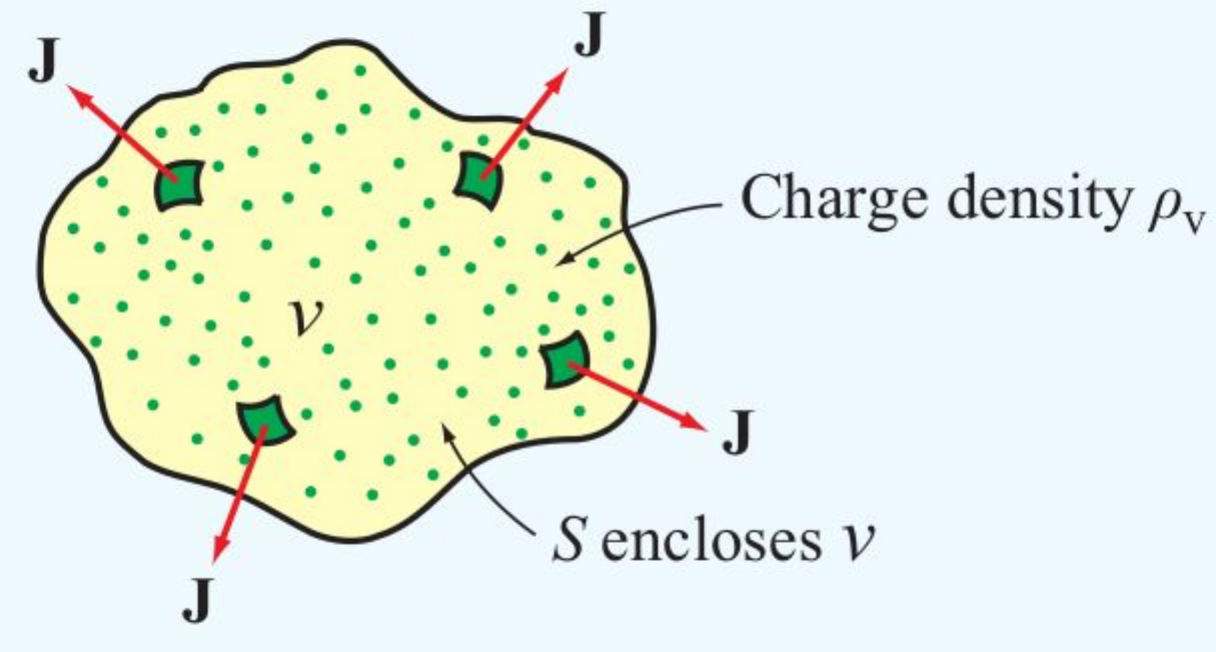
6-9 Charge-Current Continuity Relation



Have a volume with some time-varying charge density. This includes the case of charges entering or leaving the volume, crossing *S*.

These flows are current densities: **J**

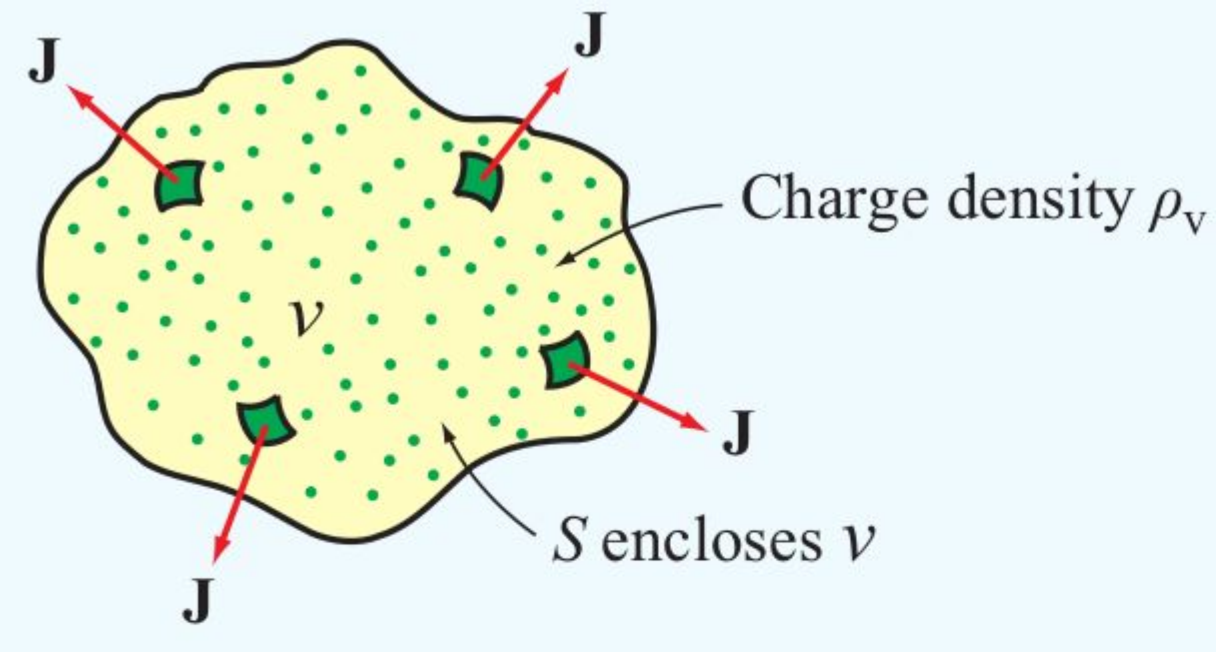
6-9 Charge-Current Continuity Relation



Define I to be the net current flowing across S out of $\mathcal{V}$. So I is the negative rate of change of the enclosed charge, Q:

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_v \, d\mathcal{V}$$

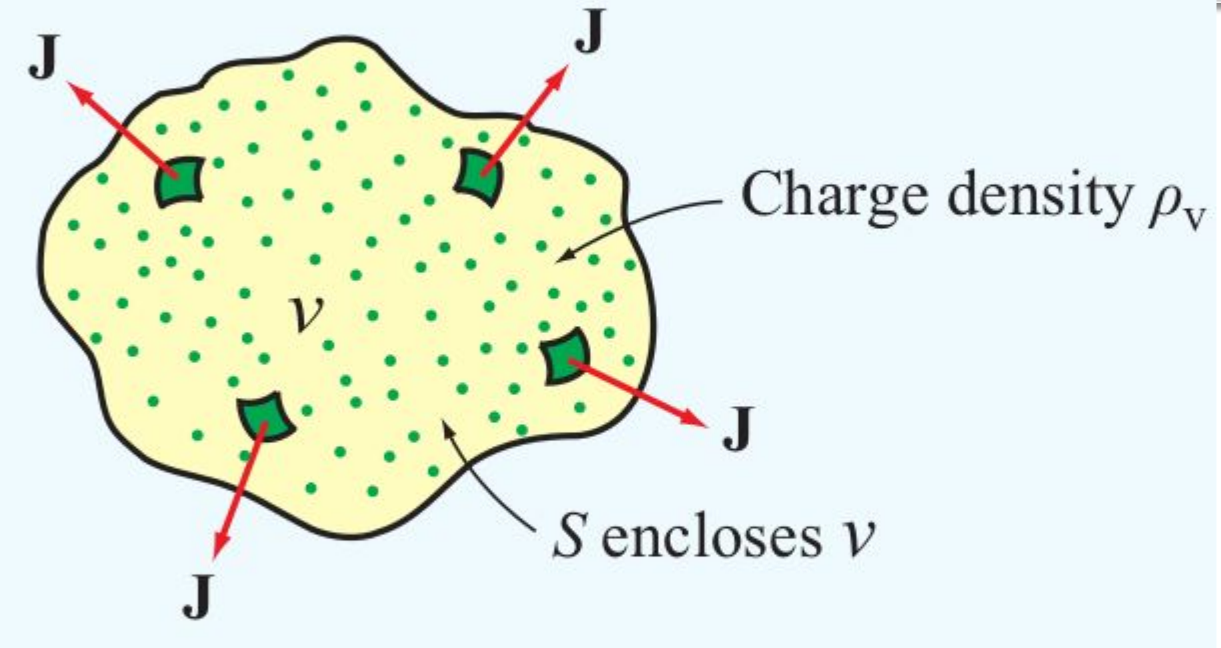
6-9 Charge-Current Continuity Relation



$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{\upsilon} \rho_v \, d\upsilon$$

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{d}{dt}\int_{\upsilon} \rho_v \, d\upsilon.$$

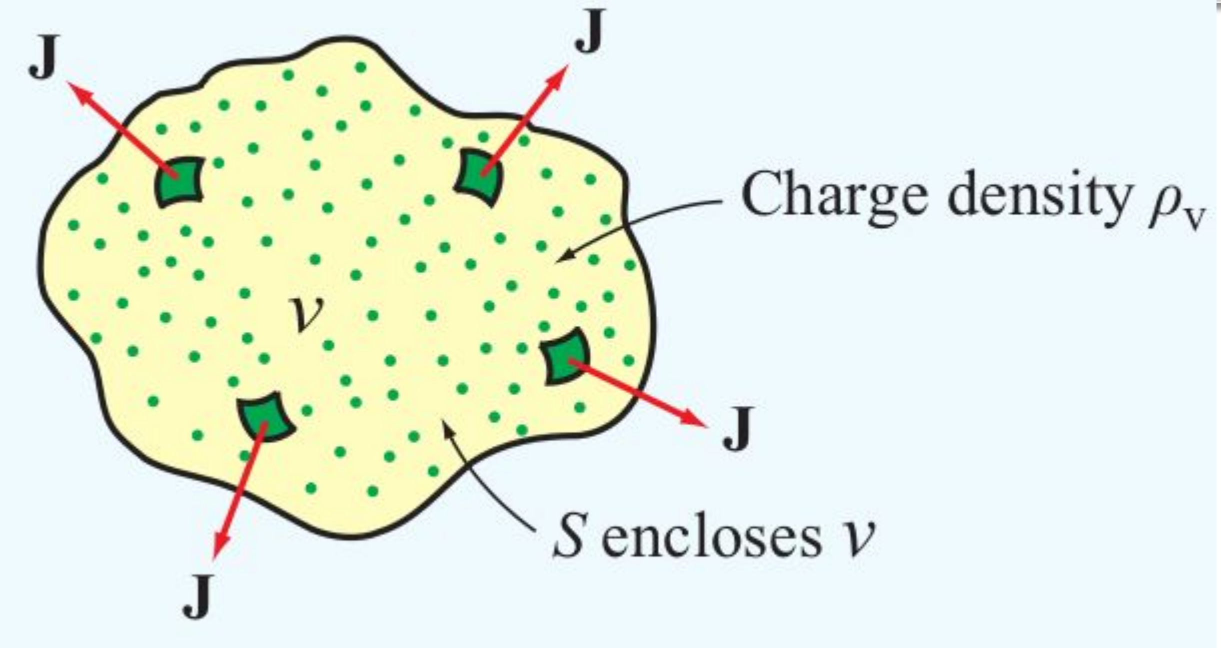
6-9 Charge-Current Continuity Relation



From the divergence theorem:

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = \int_\upsilon \nabla \cdot \mathbf{J} \, d\upsilon = -\frac{d}{dt} \int_\upsilon \rho_v \, d\upsilon$$

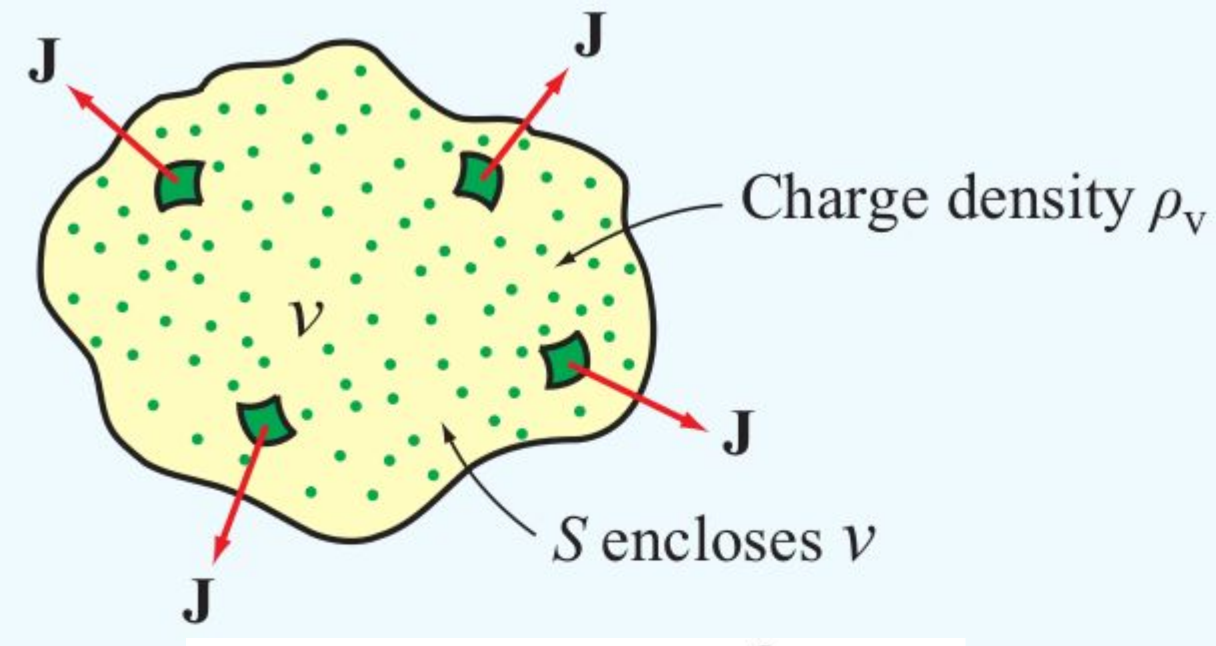
6-9 Charge-Current Continuity Relation



Assume S and $\mathcal{V}$ don't change with time:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\mathcal{V} = -\int_{\mathcal{V}} \frac{\partial \rho_v}{\partial t}\, d\mathcal{V}$$

6-9 Charge-Current Continuity Relation

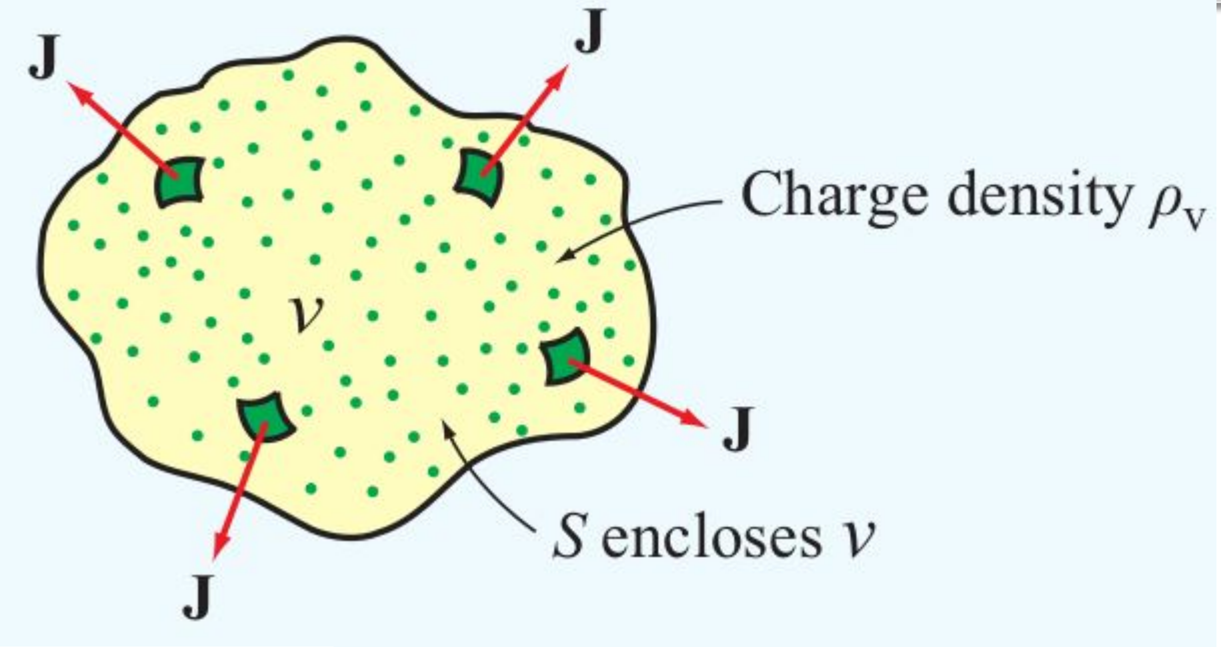


$$\int_v \nabla \cdot \mathbf{J}\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv$$

To be true for any volume, integrands must be equal:

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}}$$

6-9 Charge-Current Continuity Relation

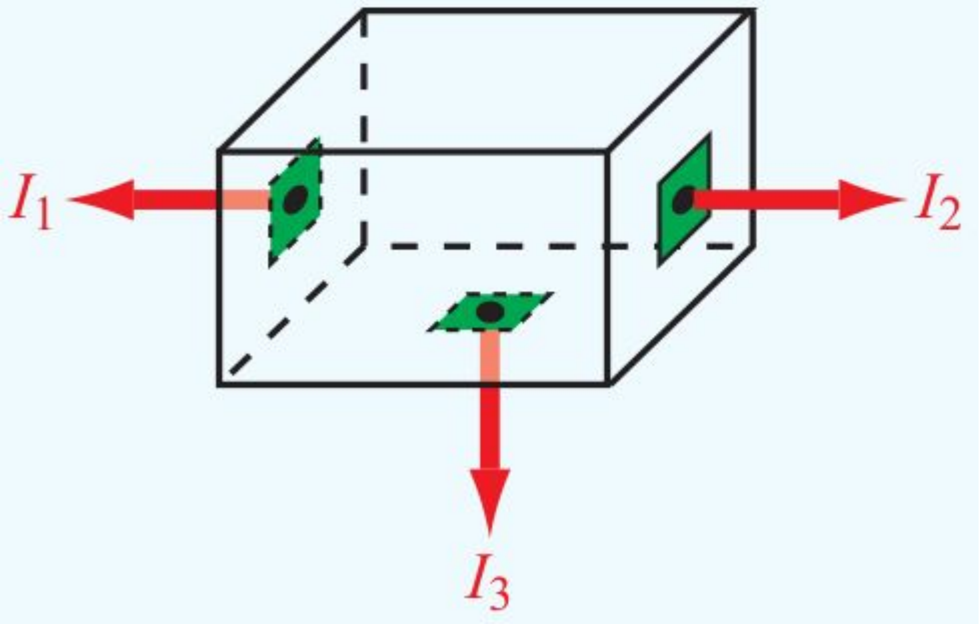


If $\partial\rho_v/\partial t = 0$:

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = 0. \quad \textbf{(Kirchhoff's current law)}$$

6-9 Charge-Current Continuity Relation

In an electrical circuit, enclose a node in a surface:



Integral becomes a summation:

$$\sum_i I_i = 0,$$ **(Kirchhoff's current law)**

Example 5

Given: The current density in a conducting medium is:

$$\mathbf{J}(x,y,z;t) = (\hat{\mathbf{x}}z^2 - \hat{\mathbf{y}}4y^2 + \hat{\mathbf{z}}2x)\cos\omega t$$

Find: $\rho_\mathrm{v}(x,y,z;t)$

Solution: we know: $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho_\mathrm{v}}{\partial t}$

so:

$$\nabla \cdot \mathbf{J} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (\hat{\mathbf{x}}z^2 - \hat{\mathbf{y}}4y^2 + \hat{\mathbf{z}}2x)\cos\omega t$$

$$= -4\frac{\partial}{\partial y}(y^2\cos\omega t) = -8y\cos\omega t$$

Example 5

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

$$-\nabla \cdot \mathbf{J} dt = d\rho_v$$

$$-\int \nabla \cdot \mathbf{J} dt = \int d\rho_v$$

$$-\int \nabla \cdot \mathbf{J} dt = \rho_v$$

Example 5

$$\rho_v = \int 8y \cos(\omega t) dt$$

$$\boxed{\rho_v = \frac{8y}{\omega} \sin(\omega t) + C}$$

where *C* is a constant of integration.

Example 6

Given: in cylindrical coordinates:

$$\mathbf{J} = \hat{\mathbf{r}} J_{\mathrm{r}}(r). \qquad \rho_{\mathrm{v}} = \rho_0 r \cos \omega t \quad (\mathrm{C/m^3})$$

Find: $J_r(r)$

Solution: we know: $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho_{\mathrm{v}}}{\partial t}$

for any vector function **A**:

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r}(r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

in this case: $\nabla \cdot \mathbf{J} = \dfrac{1}{r} \dfrac{\partial}{\partial r}(r J_{\mathrm{r}})$

Example 6

$$\nabla \cdot \mathbf{J} = \frac{1}{r}\frac{\partial}{\partial r}(rJ_{\mathrm{r}})$$

We also need to calculate:

$$-\frac{\partial \rho_{\mathrm{v}}}{\partial t} = -\frac{\partial}{\partial t}(\rho_0 r \cos \omega t) = \rho_0 r \omega \sin \omega t \, .$$

set equal:

$$\frac{1}{r}\frac{\partial}{\partial r}(rJ_{\mathrm{r}}) = \rho_0 r \omega \sin \omega t \, ,$$

$$\frac{\partial}{\partial r}(rJ_{\mathrm{r}}) = \rho_0 r^2 \omega \sin \omega t \, .$$

Example 6

So:

$$\frac{\partial}{\partial r}(rJ_{\mathrm{r}}) = \rho_0 r^2 \omega \sin \omega t$$

$$\int \partial(rJ_r) = \rho_0 \omega \sin \omega t \int_0^r r^2 dr$$

$$rJ_r = (\rho_0 \omega \sin \omega t)\frac{r^3}{3}\Big|_0^r$$

$$\boxed{J_r = \frac{\rho_0 \omega r^2}{3} \sin \omega t}$$

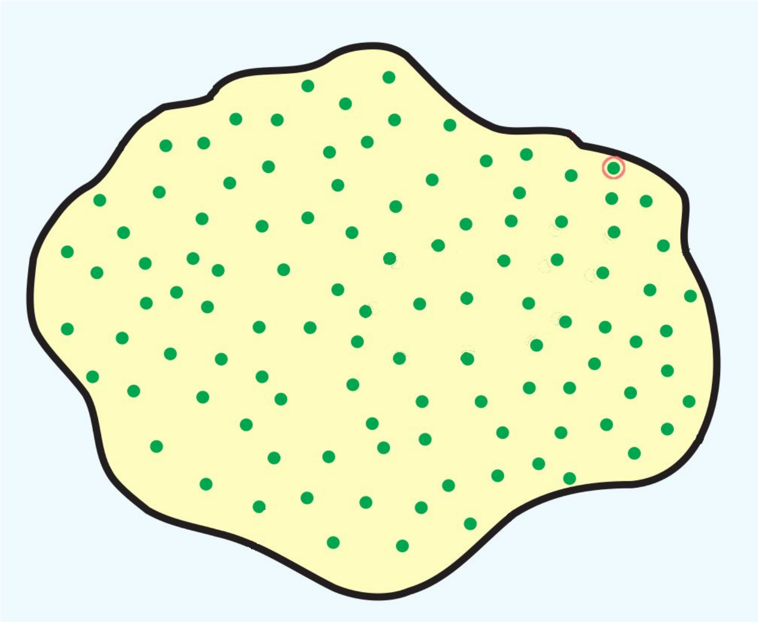
6-10 Free Charge Dissipation in a Conductor

What happens if you place a certain amount of free charge INSIDE a material?

The excess charge gives rise to an electric field.

Repels the nearby charges. As they move, they cause charges near them to move.

Continues until the excess charge is on the surface



6-10 Free Charge Dissipation in a Conductor

How fast will this happen?

Good Conductor: $< 1\text{X}10^{-15}$ sec

Good Dielectric: $> 1\text{X}10^{5}$ sec

6-10 Free Charge Dissipation in a Conductor

Calculate how fast this will happen

place a charge density: ρ_{vo} inside a conductor

know: $\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$

since **J** = σ **E**:

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_v}{\partial t}$$

from Maxwell's eqns:

$$\nabla \cdot \mathbf{E} = \rho_v / \varepsilon,$$

6-10 Free Charge Dissipation in a Conductor

Calculate how fast this will happen

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_{\mathrm{v}}}{\partial t} \qquad \nabla \cdot \mathbf{E} = \rho_{\mathrm{v}}/\varepsilon,$$

get:

$$\frac{\partial \rho_{\mathrm{v}}}{\partial t} + \frac{\sigma}{\varepsilon}\rho_{\mathrm{v}} = 0.$$

use initial condition: $\rho_{\mathrm{v}}(t=0) = \rho_{\mathrm{vo}}$

get:

$$\boxed{\rho_{\mathrm{v}}(t) = \rho_{\mathrm{vo}} e^{-(\sigma/\varepsilon)t} = \rho_{\mathrm{vo}} e^{-t/\tau_{\mathrm{r}}} \qquad (\mathrm{C/m^3}),}$$

6-10 Free Charge Dissipation in a Conductor

Calculate how fast this will happen

$$\boxed{\rho_v(t) = \rho_{vo} e^{-(\sigma/\varepsilon)t} = \rho_{vo} e^{-t/\tau_r} \qquad (\text{C/m}^3),}$$

$\tau_r = \varepsilon/\sigma$ is the Relaxation Time Constant

Similar to the time constant when charging or discharging a capacitor, from EECS 215.

6-10 Free Charge Dissipation in a Conductor

Calculate how fast this will happen

$$\boxed{\rho_{\mathrm{v}}(t) = \rho_{\mathrm{vo}} e^{-(\sigma/\varepsilon)t} = \rho_{\mathrm{vo}} e^{-t/\tau_{\mathrm{r}}} \qquad (\mathrm{C/m^3}),}$$

For copper: $\varepsilon \approx \varepsilon_0$ = 8.854 X 10^{-12} F/m
σ= 5.8 X 10^{7} S/m
τ_r= 1.5 X 10^{-19} sec

For mica: ε = $6\varepsilon_0$ = 6 X 8.854 X 10^{-12} F/m
σ= 1 X 10^{-15} S/m
τ_r= 5 X 10^{+4} sec ≈ 15 hours

Example 7

Based on the value for τ_r which of these is the better insulator?

Dry Soil: $\varepsilon = 2.5\varepsilon_0 = 2.5 \text{ X } 8.854 \text{ X } 10^{-12}$ F/m
$\sigma = 1 \text{ X } 10^{-4}$ S/m

Fresh Water: $\varepsilon = 80\varepsilon_0 = 80 \text{ X } 8.854 \text{ X } 10^{-12}$ F/m
$\sigma = 1 \text{ X } 10^{-3}$ S/m

Solution:

Dry Soil: $\tau_r = \varepsilon/\sigma = 2.2 \text{ X } 10^{-7}$ sec
Fresh Water: $\tau_r = \varepsilon/\sigma = 7.1 \text{ X } 10^{-7}$ sec (better insulator)

Exercise 6-6

Given: Quartz: ε_r=5, σ = 1X10^{-17} S/m

Find: relaxation time constant, τ_r

time for charge density to decay

to 1% of initial value

Solution: know: $\tau_r = \varepsilon/\boldsymbol{\sigma}$

$$\tau_r = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_r \varepsilon_0}{\sigma} :$$

$$= \frac{5 \times 8.85 \times 10^{-12}}{10^{-17}} = 4.425 \times 10^6 \text{ s} = \boxed{51.2 \text{ days}}$$

Exercise 6-6

Know:

$$\rho_v(t) = \rho_{v0} e^{-t/\tau_r}$$

plug in:

$$\frac{\rho_v}{\rho_{v0}} = 0.01 = e^{-t/51.2}$$

$$\ln 0.01 = -\frac{t}{51.2}$$

$$t = -51.2 \ln 0.01 = \boxed{236 \text{ days}}$$

Homework

Homework 24 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Sections 7-1 through 7-3:

Time-Harmonic Fields

Plane-Wave Propagation in Lossless Media

Wave Polarization