EECS 230

ENGINEERING ELECTROMAGNETICS

Leland Pierce

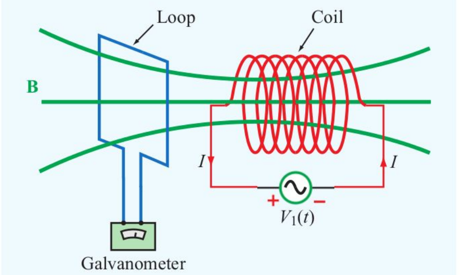
Time-Varying Fields 1

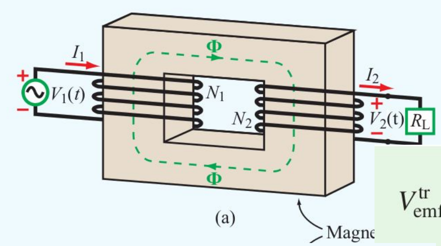
Chapter 6 Overview

- Time-Varying Fields
- Faraday's Law
- Stationary Loop in time-varying field
- Ideal Transformer
- Moving conductor in static field
- The Generator
- Moving conductor in time-varying field
- Displacement Current
- Boundary Conditions
- Charge-Current Continuity
- Free-Charge dissipation in a conductor
- Potentials

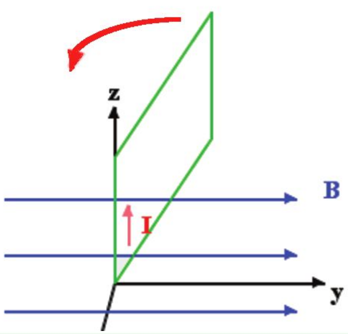
$$\nabla \cdot \mathbf{D} = \rho_v,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

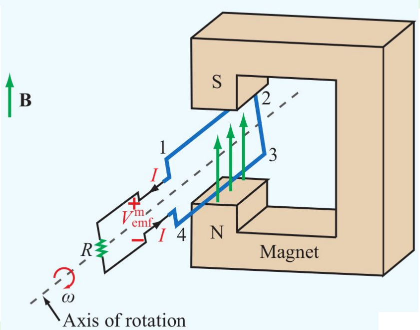
$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

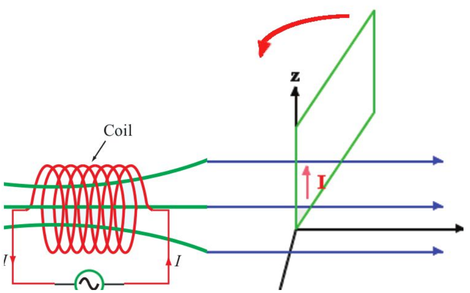




$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}, \quad \textbf{(transformer emf)}$$



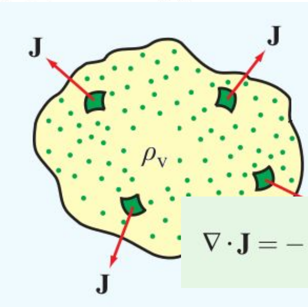




$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}. \quad \textbf{(motional emf)}$$

$$V_{\text{emf}} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s},$$



$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

$$\widetilde{V}(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \frac{\tilde{\rho}_v(\mathbf{R}_i)\, e^{-jkR'}}{R'}\, d\upsilon'$$

$$\rho_v(t) = \rho_{vo} e^{-(\sigma/\varepsilon)t}$$

Lecture Coverage

Today's lecture:

Sections 6-1 through 6-3 of the book:

6-1: Faraday's Law
6-2: Stationary Loop in time-varying **B** Field
6-3: Ideal Transformer

6.1 Maxwell's Equations

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Empirically derived from many measurements

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}.$$

E: Electric Field
H: Magnetic Field
J: Current Density
ρ_v: Charge Density

6.1 Maxwell's Equations

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

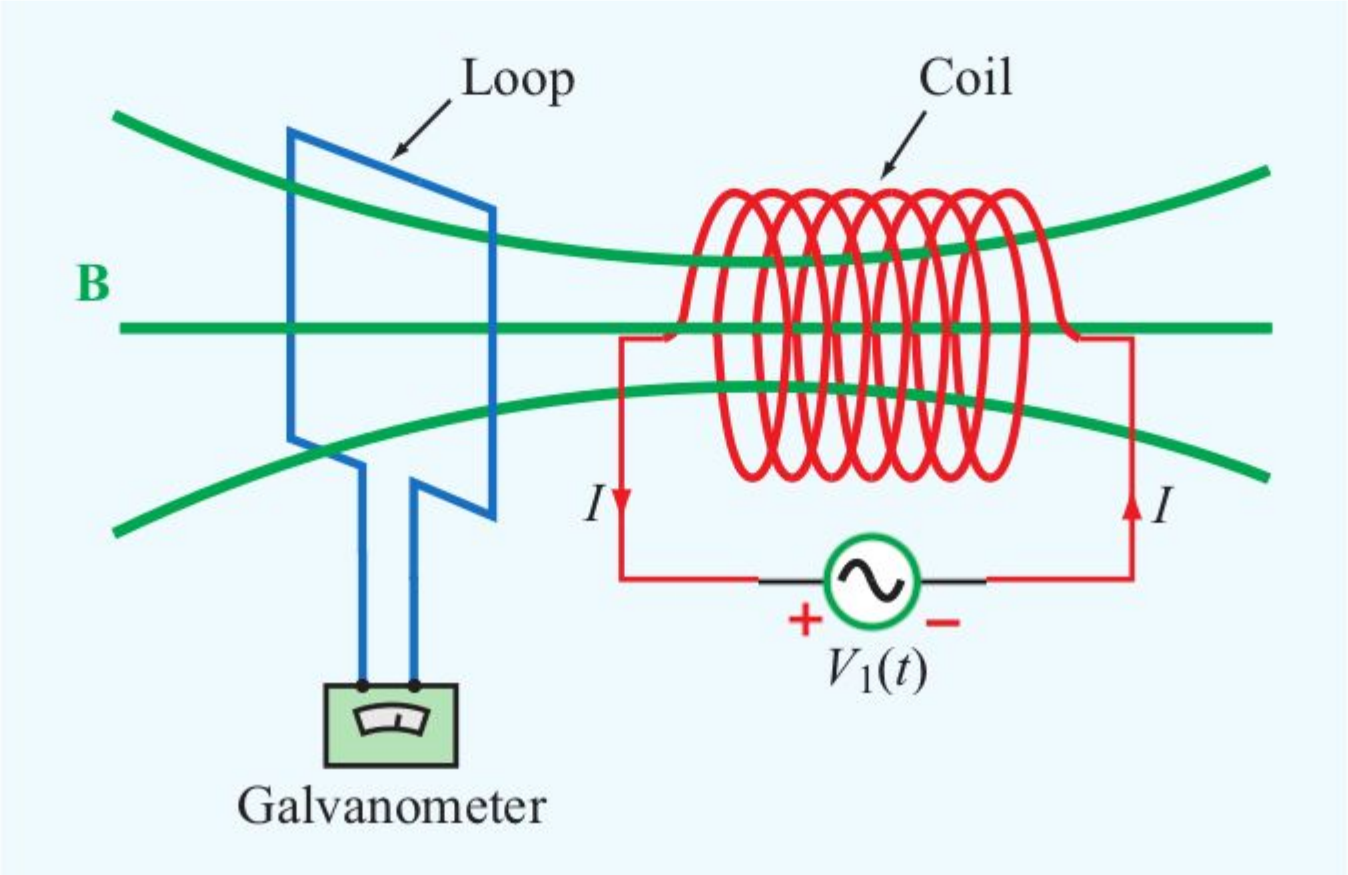
Empirically derived from many measurements

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}.$$

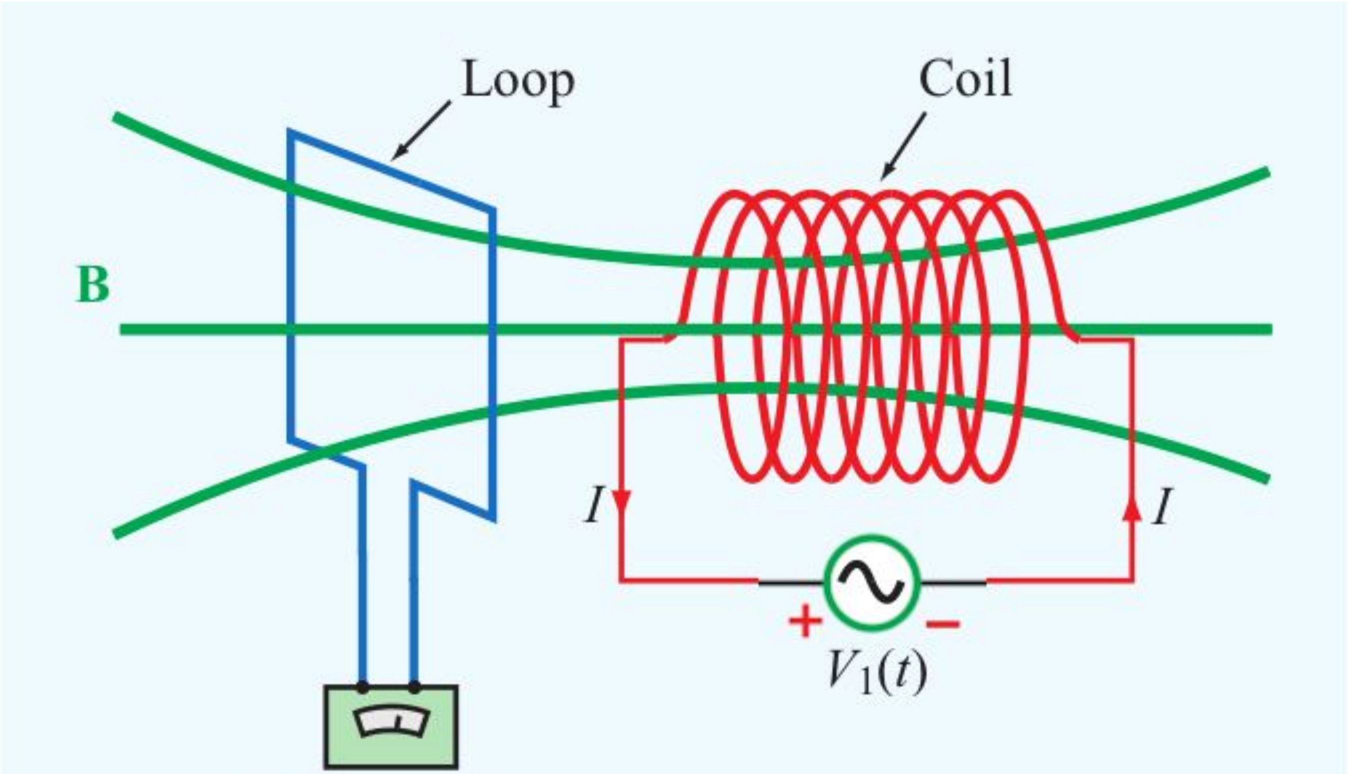
E: Electric Field
H: Magnetic Field
J: Current Density
ρ_v: Charge Density

6.1 Faraday's Law



Time-Varying Magnetic field, **B,** produces a current in a nearby wire loop

6.1 Faraday's Law



$$V_{\text{emf}} = -N\,\frac{d\Phi}{dt} = -N\,\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$$

6.1 Faraday's Law

$$V_{\text{emf}} = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$$

d/dt can be due to:

1. time-varying **B** field, stationary loop
2. static **B** field, moving loop,
3. **both** varying with time.

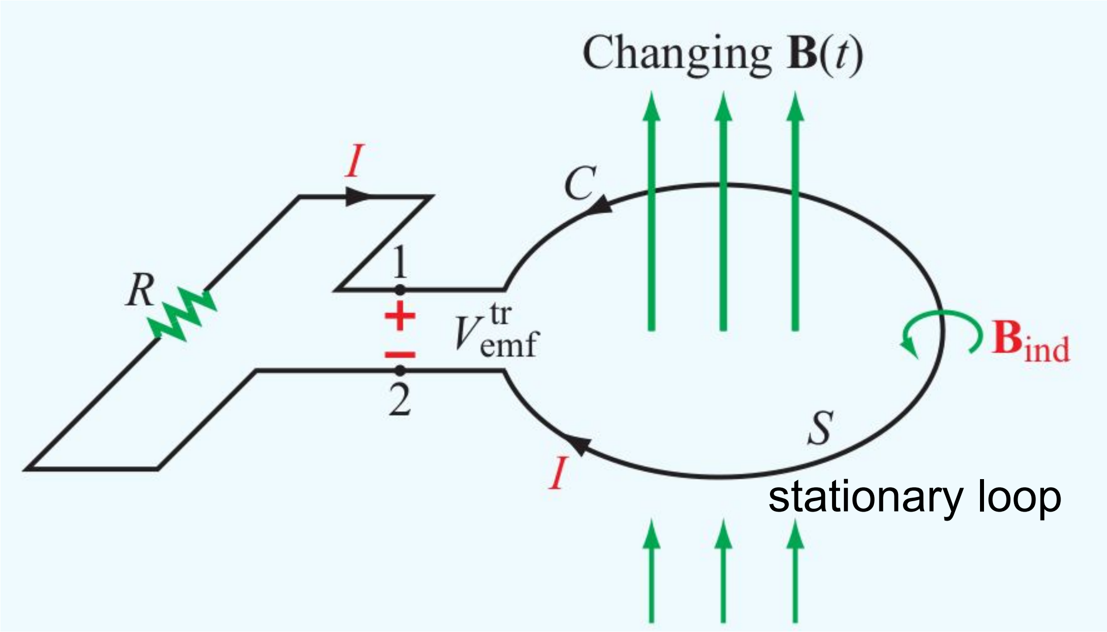
V_{emf} = transformer-emf + motional-emf

$$V_{\text{emf}} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}$$

6.2 Transformer EMF

d/dt is due to:

1. time-varying **B** field, stationary loop
2. static **B** field, moving loop,
3. **both** varying with time.

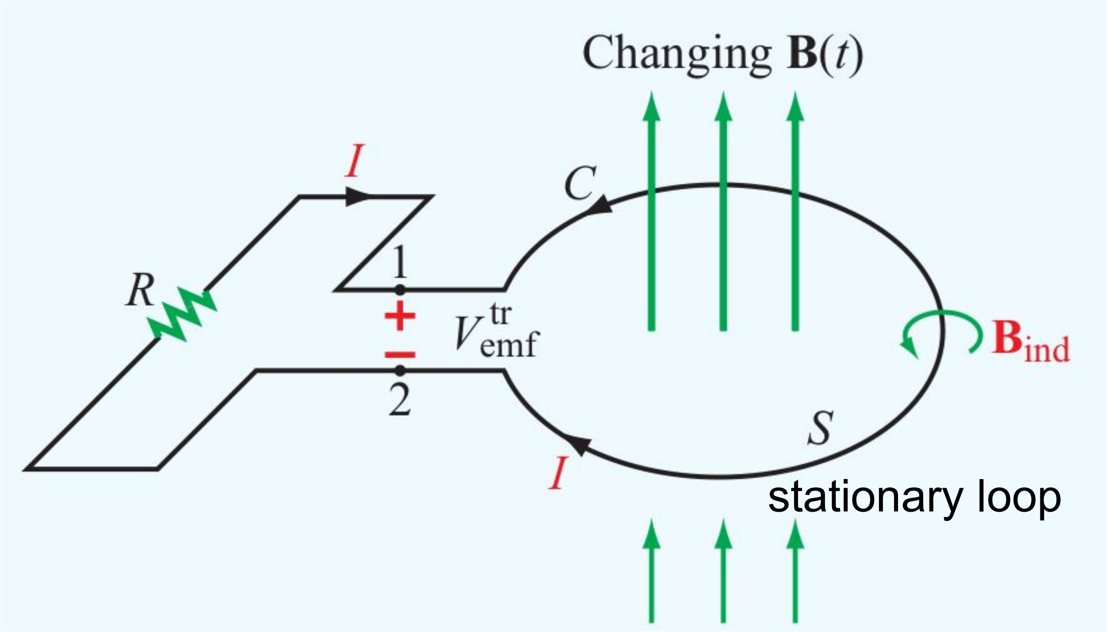


6.2 Transformer EMF

Time-varying **B** field, Stationary loop

$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}, \quad \textbf{(transformer emf)}$$

6.2 Transformer EMF



Lenz's Law:

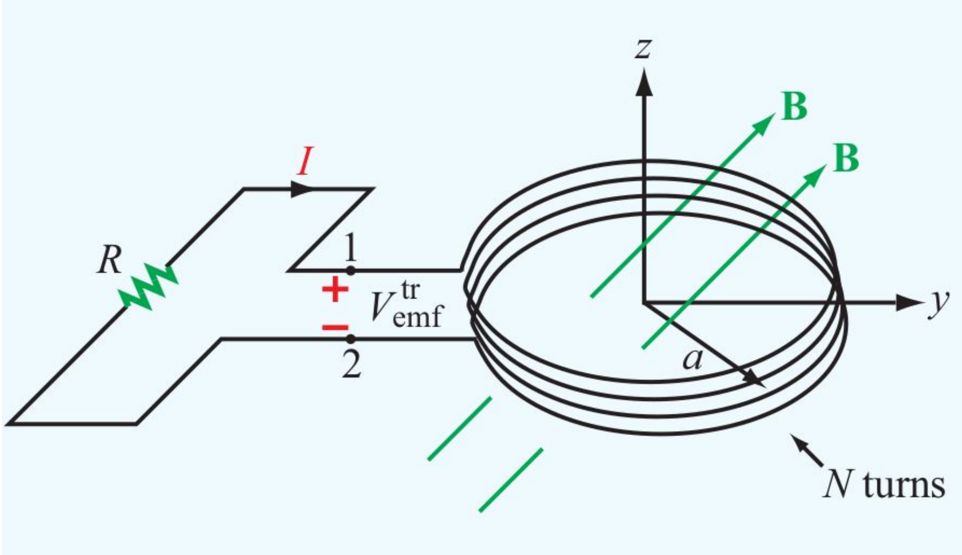
The induced current creates a magnetic field $\mathbf{B}_{ind}$ that opposes the *change* in the magnetic flux.

Example 6.1 Transformer EMF

Given:

Loop with N turns, radius *a*
in x-y plane,
centered at origin

$$\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$$

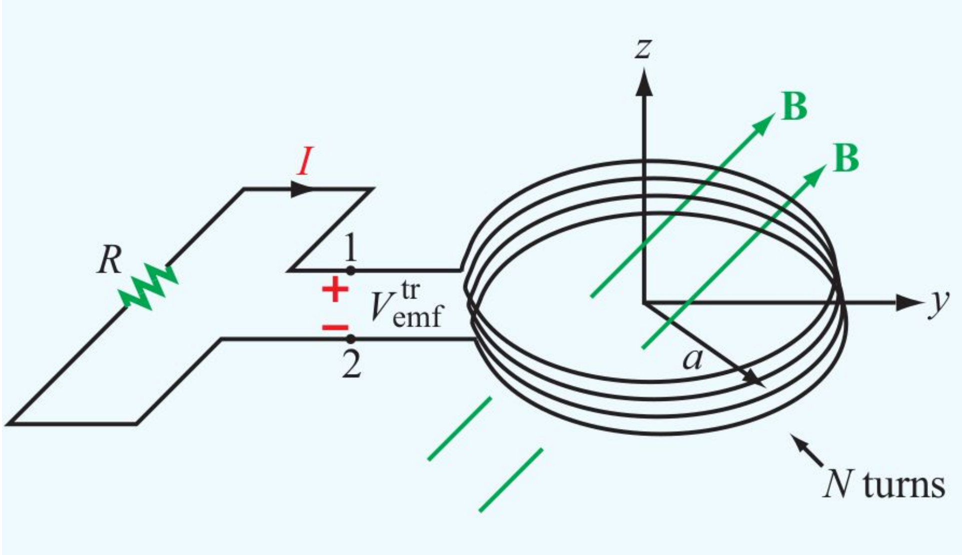


Find: $V_{\text{emf}}^{\text{tr}}$

Example 6.1 Transformer EMF

Solution:

$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s},$$



$d\mathbf{s} = \hat{\mathbf{z}}\, ds$

$$\frac{\partial}{\partial t}\left[B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t\right] = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\omega\cos\omega t$$

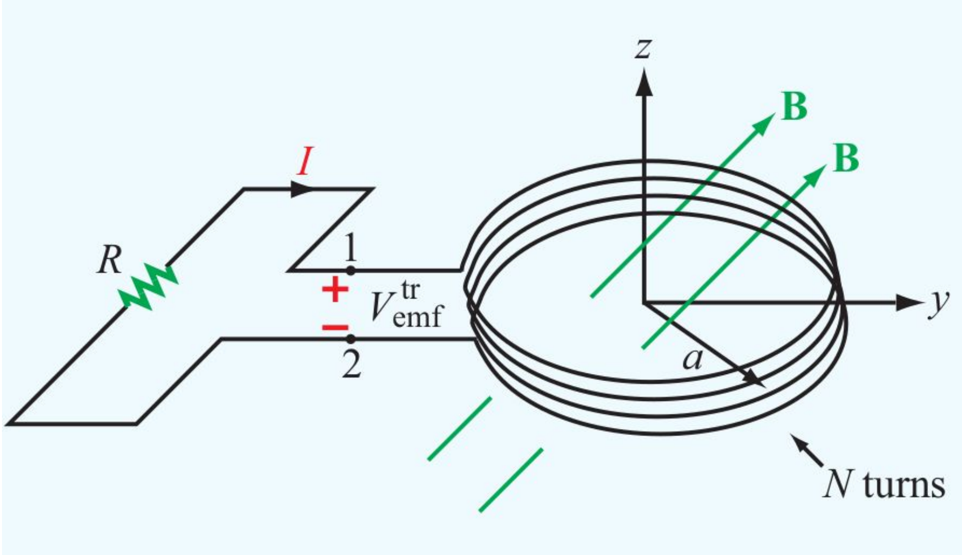
Example 6.1 Transformer EMF

$$
\begin{aligned}
\mathrm{V}_{\mathrm{emf}}^{\mathrm{tr}} &= -N \int_S B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\, \omega \cos \omega t \cdot \hat{\mathbf{z}}\, ds \\
&= -N B_0 \int_S 3\omega \cos \omega t \, ds \\
&= -3 N B_0 \, \omega \cos \omega t \int_S ds \\
&= \boxed{-3 N B_0 \pi a^2 \omega \cos \omega t}
\end{aligned}
$$

Example 6.1 Transformer EMF

Apply Lenz's Law

The induced current creates a magnetic field $\mathbf{B}_{ind}$ that opposes the *change* in the magnetic flux.



+B is in direction of d**s**: **+z** direction
dΦ/dt ∝ +cos(t): positive at t=0+
opposing that: means -B, so **Bind** in **-z-**dir inside the loop means current going in the direction shown.
Ohm's law: V_R<0, so: V_{emf}<0, as given previously.

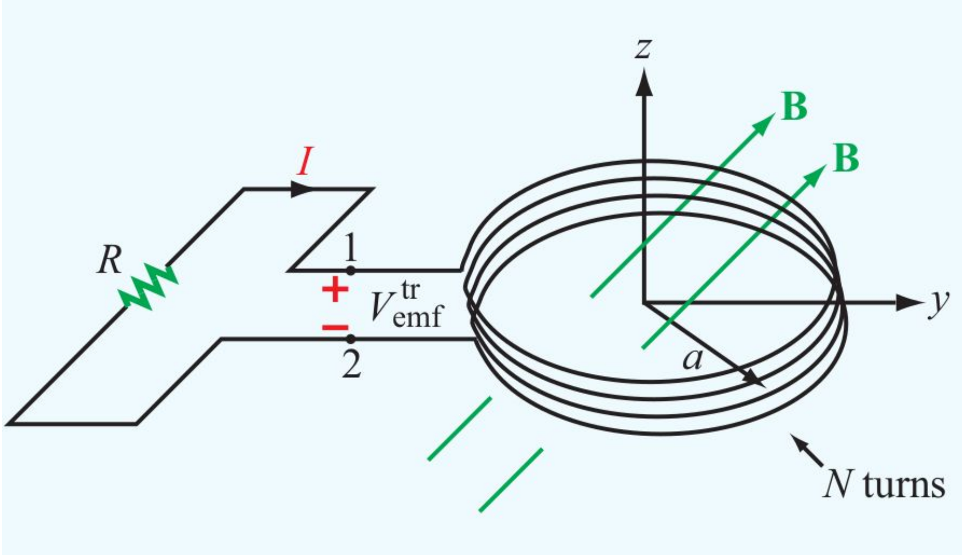
Example 6.1 Transformer EMF

What if we'd chosen d**s** differently?

$$d\mathbf{s} = -\hat{\mathbf{z}}\, ds$$

leads to:

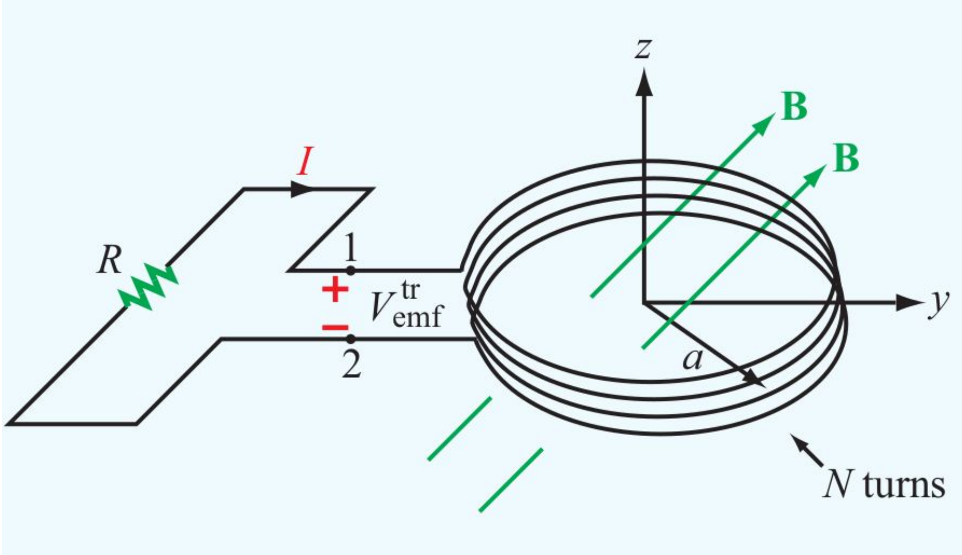
$$V_{\mathrm{emf}}^{\mathrm{tr}} = +3NB_0\pi a^2\omega\cos\omega t$$



Example 6.1 Transformer EMF

Apply Lenz's Law

The induced current creates a magnetic field $\mathbf{B}_{ind}$ that opposes the *change* in the magnetic flux.



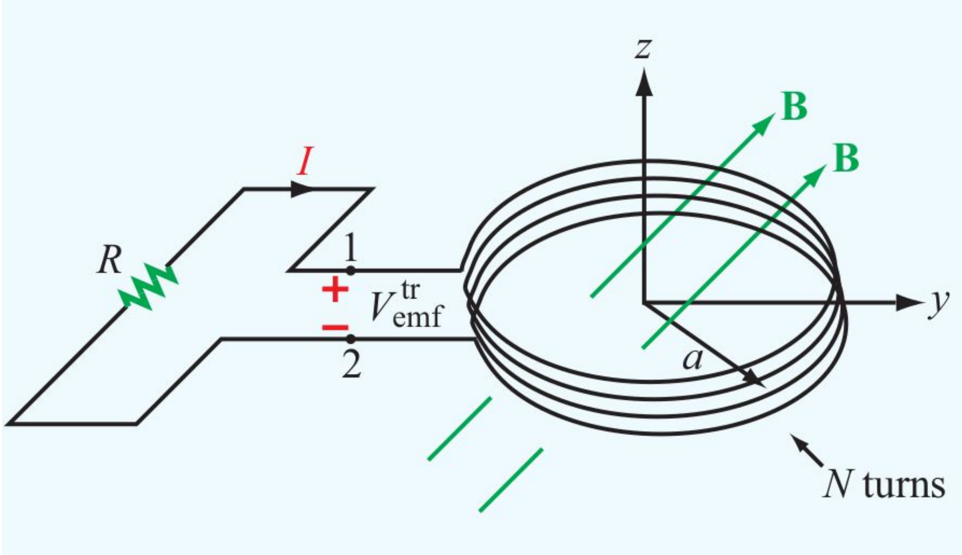
+Φ is in direction of d**s**: **-z** direction
dΦ/dt ∝ -cos(t): negative at t=0+
opposing that: means +Φ, so **B** in **-z**-dir inside the loop means current going in the direction shown.
Ohm's law: $V_R<0$, so: $V_{emf}<0$: need to change the sign.

Example 6.1 Transformer EMF

Apply Lenz's Law

So no matter our choice of the direction of d**s**:

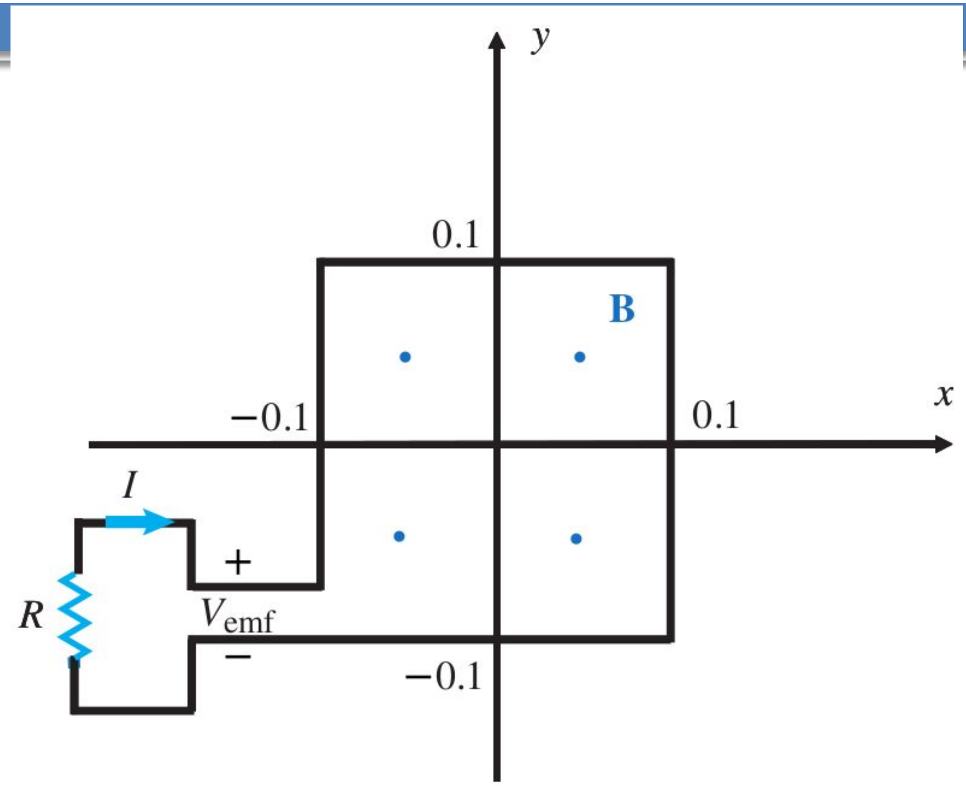
as long as we apply Lenz's Law we get the same sign for V_{EMF}, and the current.



Exercise 6.2: Transformer EMF

Given:

Loop with N=10 turns,
in x-y plane,
centered at origin



$$\mathbf{B} = \hat{\mathbf{z}} B_0 x^2 \cos 10^3 t$$

$$B_0 = 100 \text{ T}$$

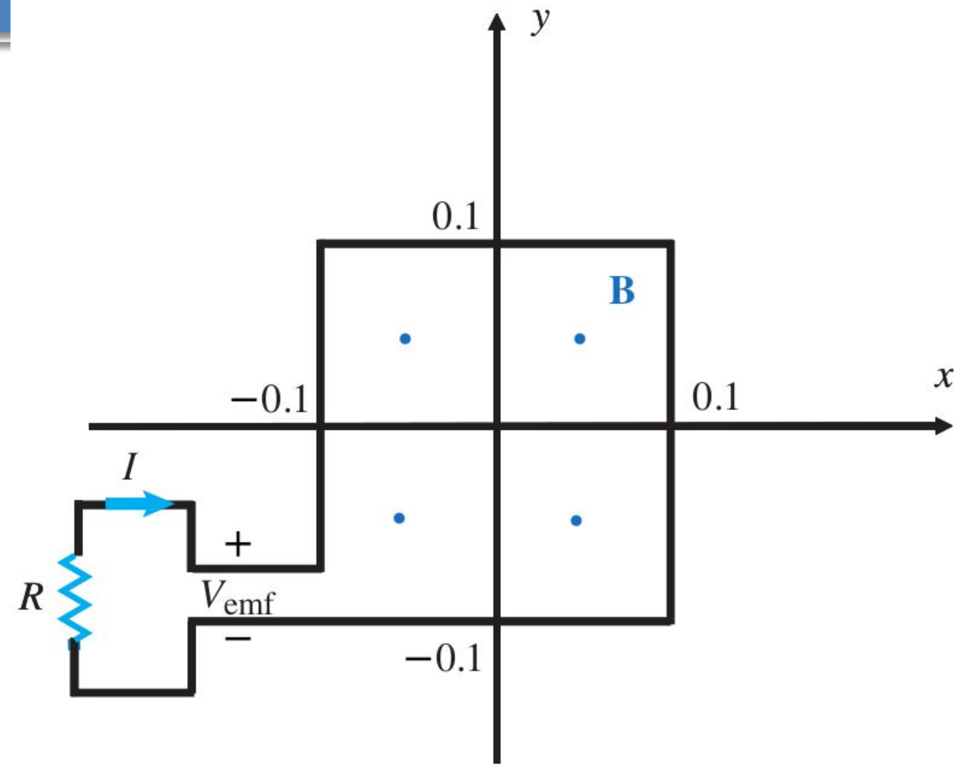
$$R = 1 K\Omega$$

Find: Current, I

Exercise 6.2: Transformer EMF

Solution:

1. Solve for Φ
2. Solve for V_{emf}
3. $|I| = |V_{emf}/R|$
4. Use Lenz's law to get the sign.



Exercise 6.2: Transformer EMF

1. Solve for Φ

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

$$= \int_{x=-0.1}^{0.1} \int_{y=-0.1}^{0.1} (\hat{\mathbf{z}}\, 100x^2 \cos 10^3 t) \cdot \hat{\mathbf{z}}\, dx\, dy$$

$$= (100 \cos 10^3 t) \times 0.2 \int_{-0.1}^{0.1} x^2\, dx$$

$$= 20 \cos 10^3 t \left. \frac{x^3}{3} \right|_{-0.1}^{0.1}$$

Exercise 6.2: Transformer EMF

1. Solve for Φ

$$\Phi = 20\cos 10^3 t \left.\frac{x^3}{3}\right|_{-0.1}^{0.1}$$

$$\Phi = \frac{20}{3}\cos 10^3 t((0.1)^3 + (0.1)^3) = 13.3 \times 10^{-3}\cos 10^3 t.$$

Exercise 6.2: Transformer EMF

2. Solve for V_{emf}

$$
\begin{aligned}
V_{\text{emf}} &= -N\frac{d\Phi}{dt} \\
&= -10\frac{d}{dt}(13.3 \times 10^{-3}\ \cos 10^3 t) \\
&= 10(13.3 \times 10^{-3})10^3 \sin 10^3 t) \\
&= 133 \sin 10^3 t \ \ \text{V}
\end{aligned}
$$

Exercise 6.2: Transformer EMF

3. Solve for the current

$$|I| = \left|\frac{V_{\text{emf}}}{R}\right|$$

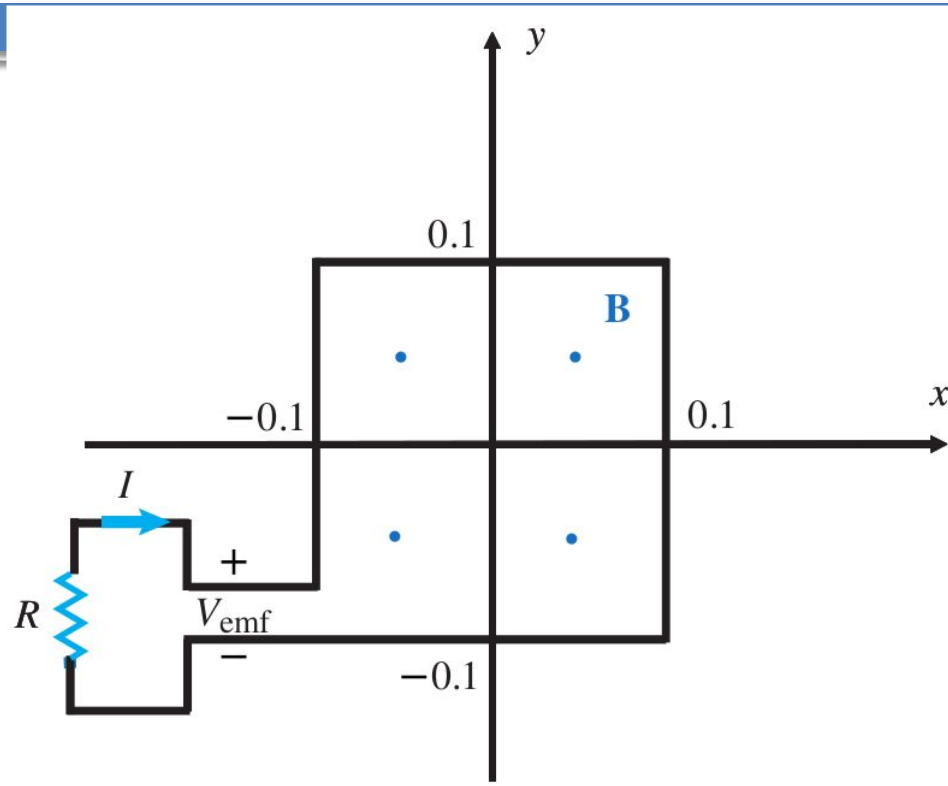
$$= \frac{133 \sin 10^3 t \text{ V}}{1000\,\Omega}$$

$$|I| = 133 \sin 10^3 t \ \text{ mA}$$

Exercise 6.2: Transformer EMF

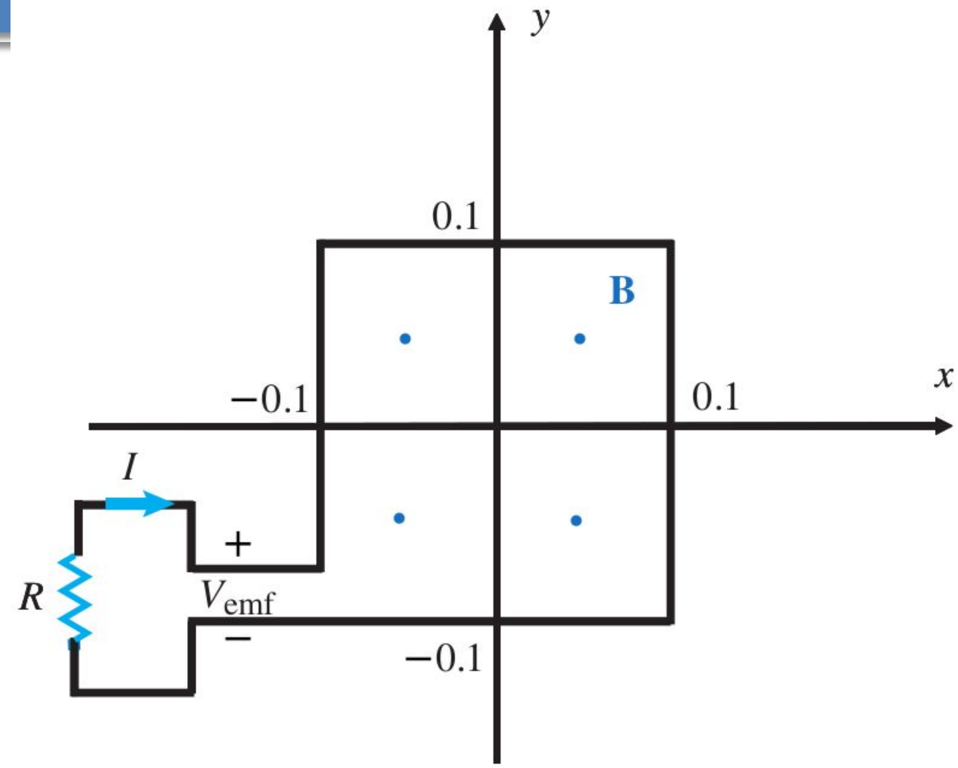
4. Lenz's Law

The induced current creates a magnetic field $\mathbf{B}_{ind}$ that opposes the *change* in the magnetic flux.



+Φ is in +**z** direction
dΦ/dt ∝-sin(t): negative at t=0+
opposing that: means +Φ, so **B** in **+z-**dir inside the loop
means current going in opposite direction to that shown

Exercise 6.2: Transformer EMF



4. Lenz's Law

means current going in opposite direction to that shown:

$$I = -133 \sin 10^3 t \quad \text{(mA)}.$$

And Ohm's law: V = -IR

So the voltage should be +, which it is:

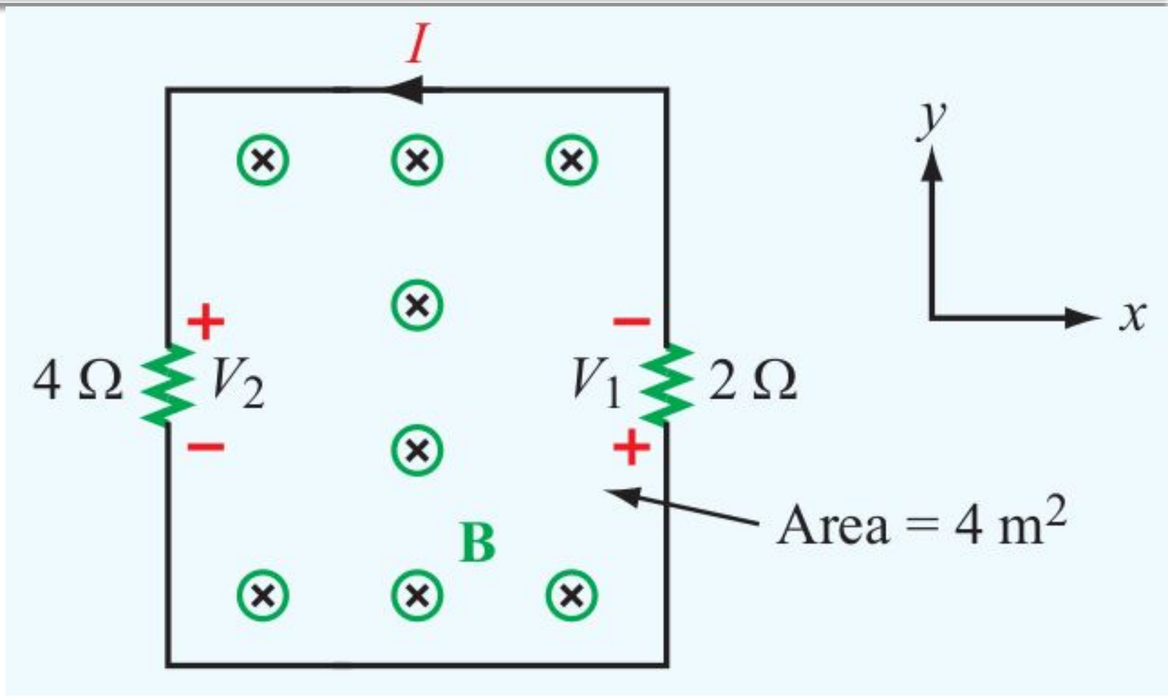
$$V_{emf} = 133 \sin 10^3 t \ \text{V}$$

Example 6-2

Given:
loop in x-y plane
loop area = $4 m^2$
wire resistance=0Ω

$$\mathbf{B} = -\hat{\mathbf{z}}0.3t \text{ (T)}$$

Find: V_1 and V_2.



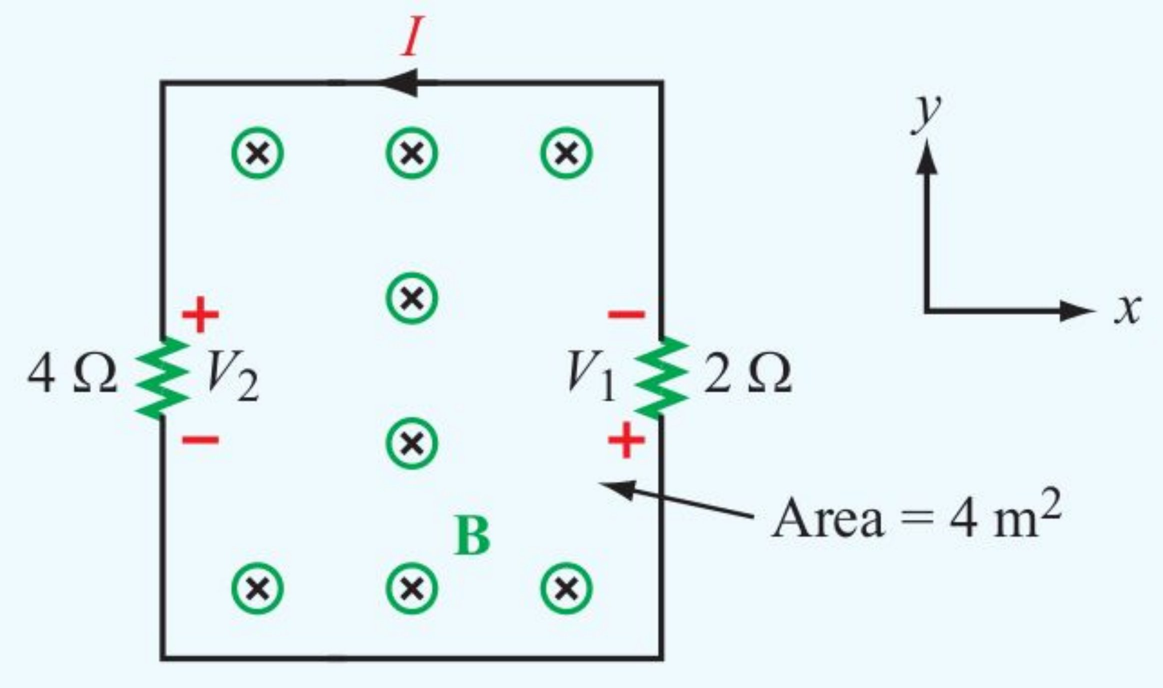
Solution: 1. Solve for Φ
2. Solve for V_{emf}
3. $|I| = |V_{emf}/R|$, using Lenz's law for the sign.
4. Find V_1 and V_2.

Example 6-2

1. Solve for Φ

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

S is the surface of the loop, directed in the +z-direction



$$d\mathbf{s} = \hat{\mathbf{z}}\, dx\, dy$$

B does not depend on x or y

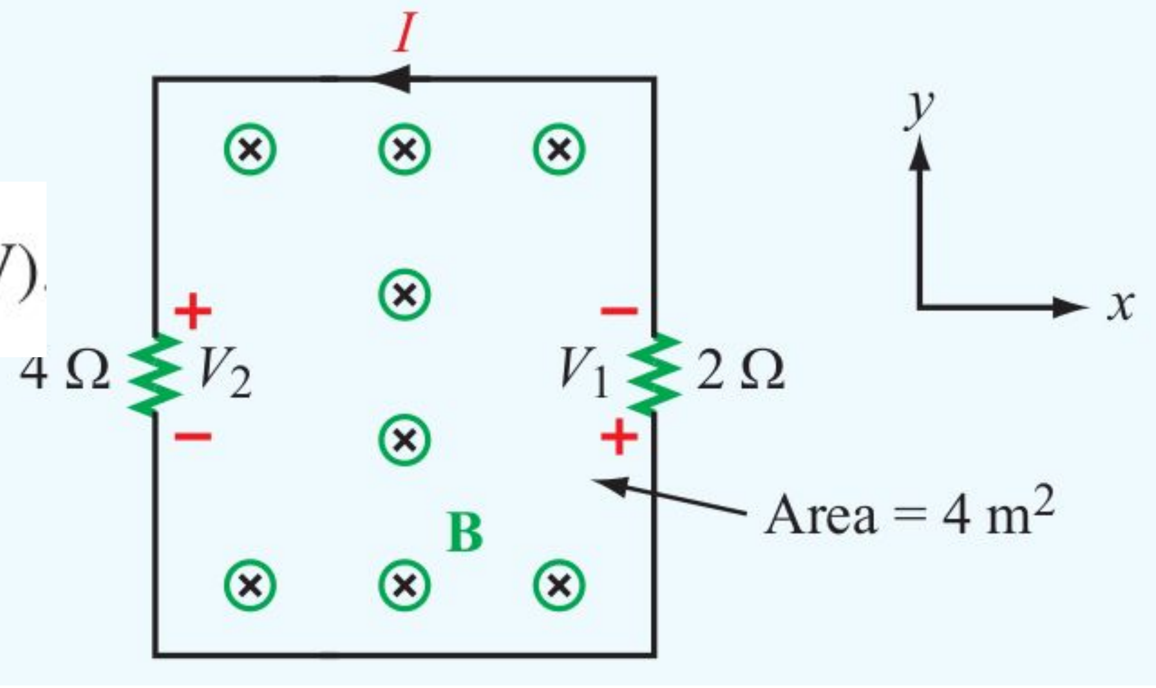
$$\Phi = \int_S (-\hat{\mathbf{z}} 0.3t) \cdot \hat{\mathbf{z}}\, ds = -0.3t \times 4 = -1.2t \quad \text{(Wb)}$$

Example 6-2

2. Solve for *Vemf*

$$V_{\text{emf}}^{\text{tr}} = -\frac{d\Phi}{dt} = 1.2 \qquad \text{(V)}.$$

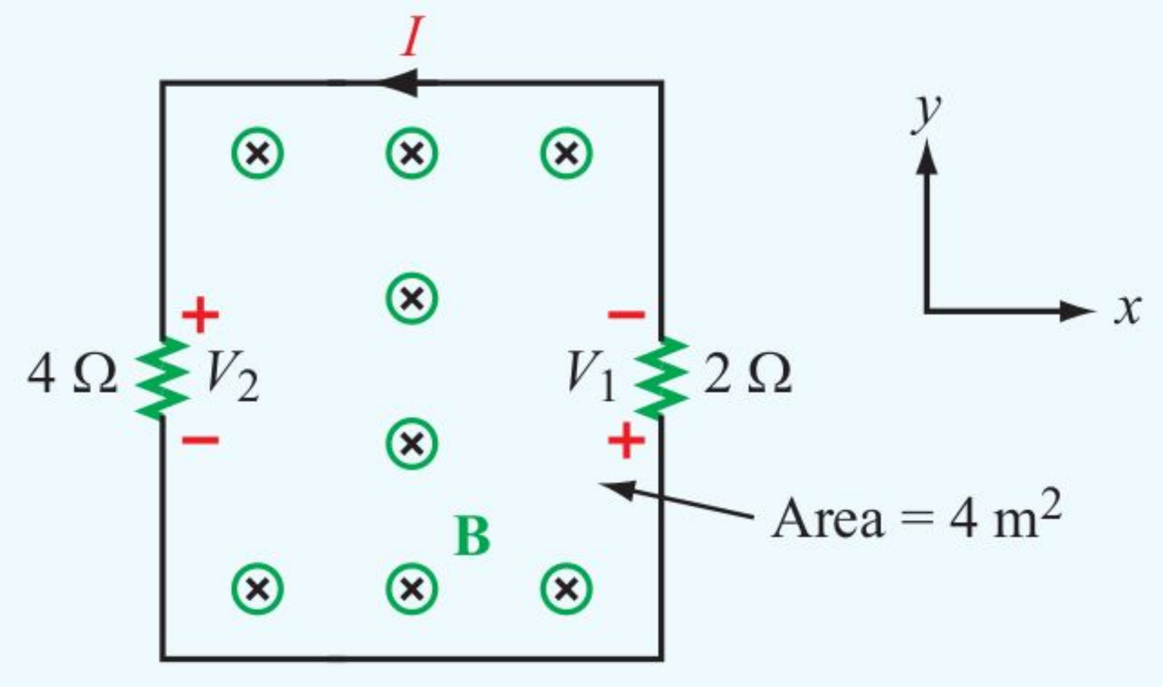
3. $|I| = |V_{emf}/R|$,
   using Lenz's law
   for the sign.



R_{equiv} = 6Ω (in series)
$|I|$ = 1.2V/6Ω = 0.2 Amps

Example 6-2

3. $|I| = |V_{emf}/R|$, using Lenz's law for the sign.



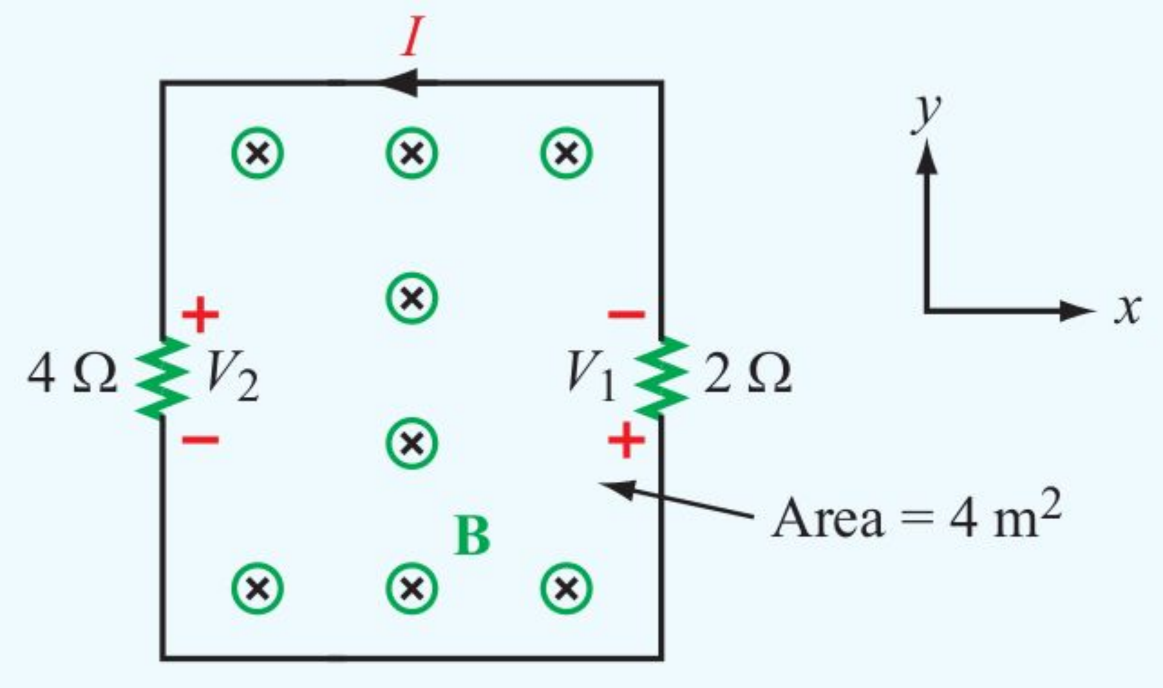
$|I|$ = 1.2V/6Ω = 0.2 A

+Φ is in +**z** direction
dΦ/dt ∝ -const: negative at t=0+
opposing that: means +Φ, so **B** in **+z-**dir inside the loop means current going in same direction to that shown:

I = +0.2 A

Example 6-2

4. Find V_1 and V_2



Ohm's Law:

$V_1 = +I R_1 = 0.2\text{A}\cdot 2\Omega = 0.4\text{ V} = V_1$
$V_2 = +I R_2 = 0.2\text{A}\cdot 4\Omega = 0.8\text{ V} = V_2$

Transformers: Example 4

Given: Transformer:

primary: a vertical wire:

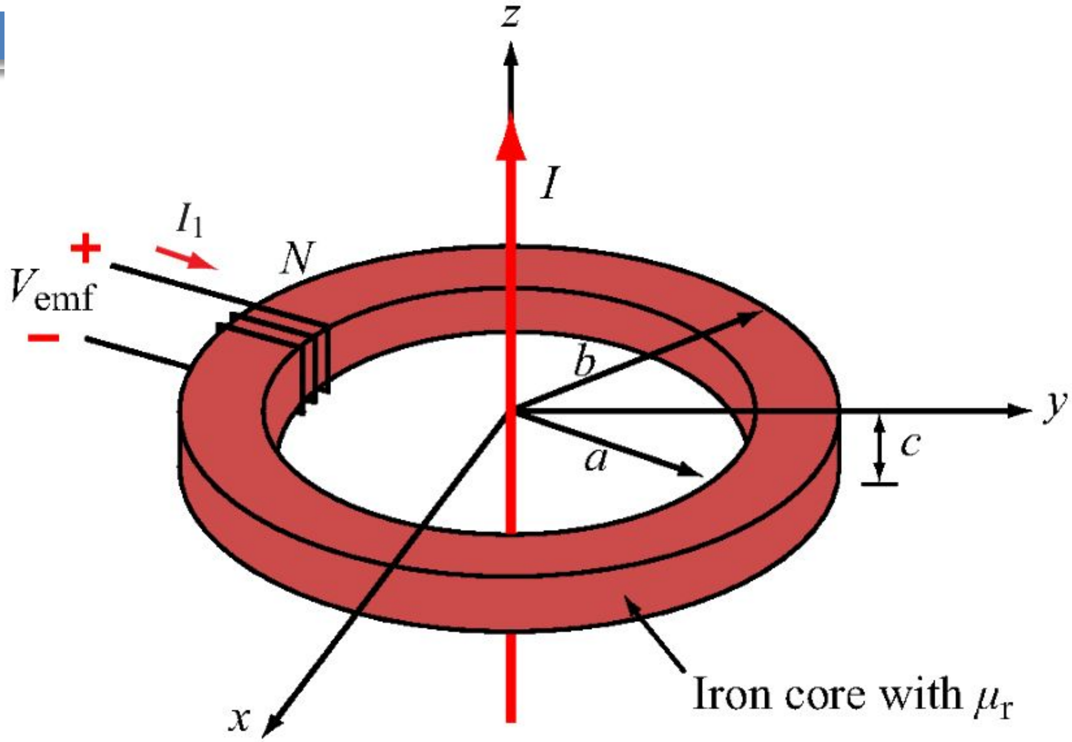
current $I=I_0 \cos \omega t$

secondary:

toroidal core: μ_r

centered at origin

N-turn coil

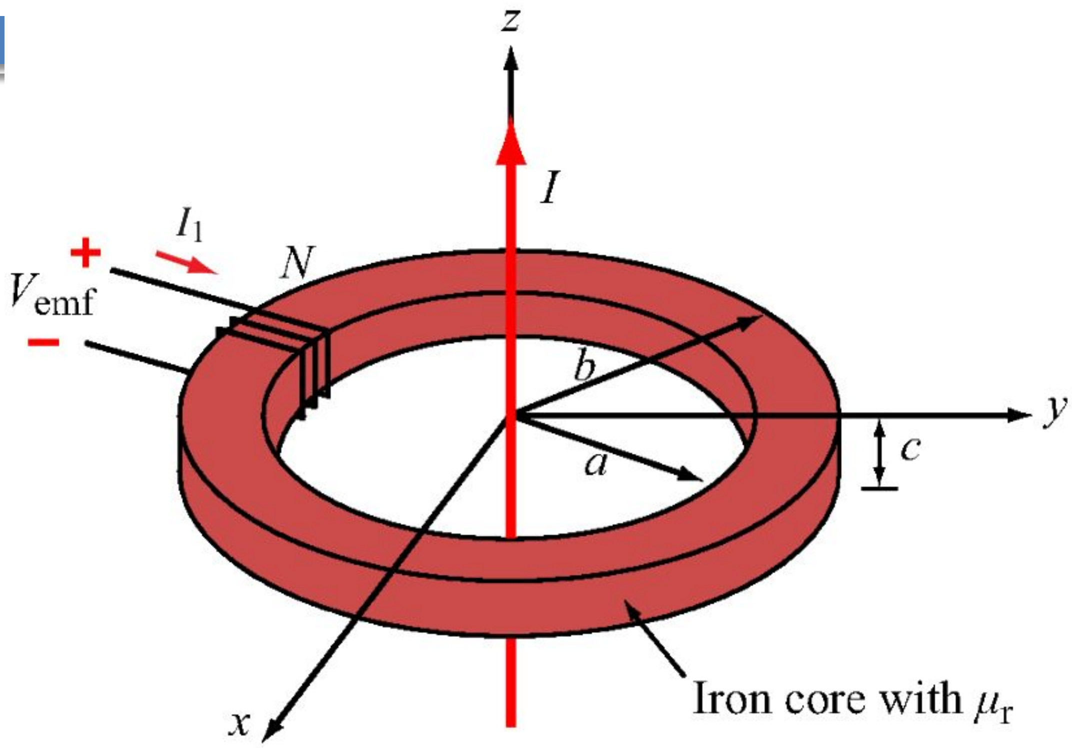


Find: V_{emf}

Transformers: Example 4

Solution:

1. Solve for Φ
2. Solve for V_{emf}
3. Apply Lenz's Law



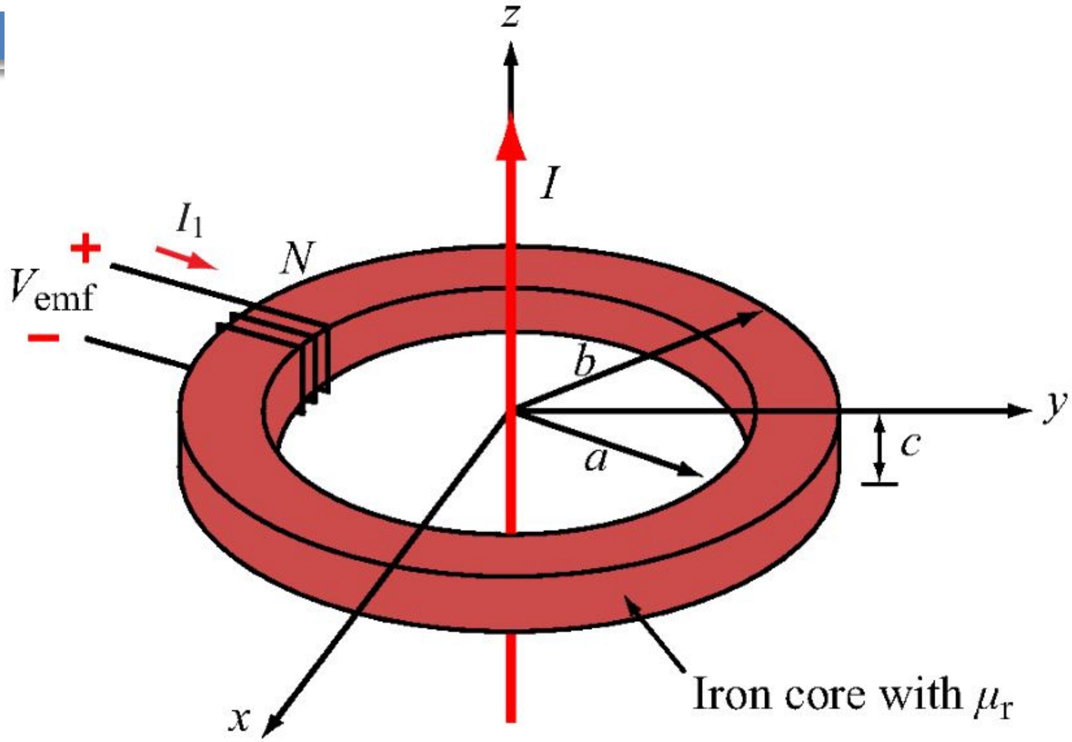
Transformers: Example 4

1. Solve for Φ

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

B due to vertical wire:

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu I}{2\pi r}$$



*d***s** is area of wire loops, normal is in ϕ-direction:

$$d\mathbf{s} = \hat{\boldsymbol{\phi}}\, dr\, dz$$

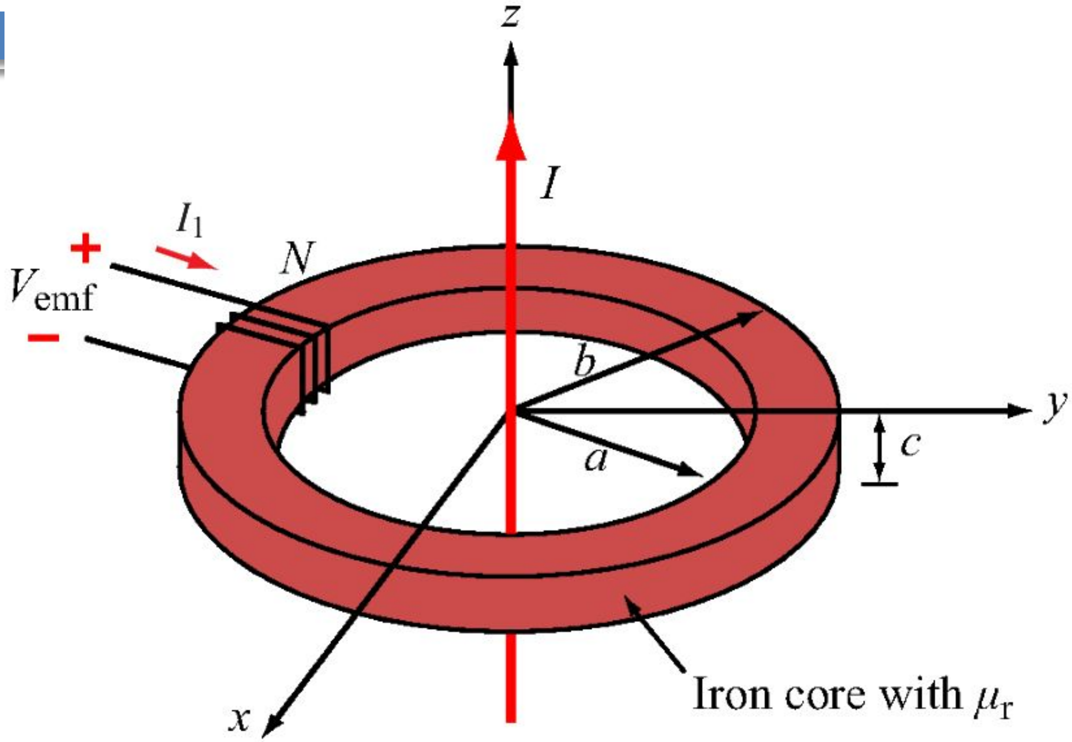
Transformers: Example 4

1. Solve for Φ

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu I}{2\pi r}$$

$$d\mathbf{s} = \hat{\boldsymbol{\phi}}\, dr\, dz$$



$$\Phi = \int_{z=-c/2}^{c/2} \int_{r=a}^{b} \hat{\boldsymbol{\phi}} \frac{\mu I}{2\pi r} \cdot \hat{\boldsymbol{\phi}}\, dr\, dz$$

Transformers: Example 4

1. Solve for Φ

$$\Phi = \int_{z=-c/2}^{c/2} \int_{r=a}^{b} \hat{\boldsymbol{\phi}} \frac{\mu I}{2\pi r} \cdot \hat{\boldsymbol{\phi}} \, dr \, dz$$

$$= \frac{\mu I c}{2\pi} \int_{r=a}^{b} \frac{1}{r} \, dr$$

$$= \frac{\mu I c}{2\pi} \ln r \Big|_{a}^{b}$$

$$= \frac{\mu I c}{2\pi} (\ln b - \ln a)$$

$$= \frac{\mu I c}{2\pi} \ln \left(\frac{b}{a} \right)$$

Transformers: Example 4

2. Solve for V_{emf}

$$\Phi = \frac{\mu I c}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$V_{\text{emf}} = -N\frac{d\Phi}{dt}$$

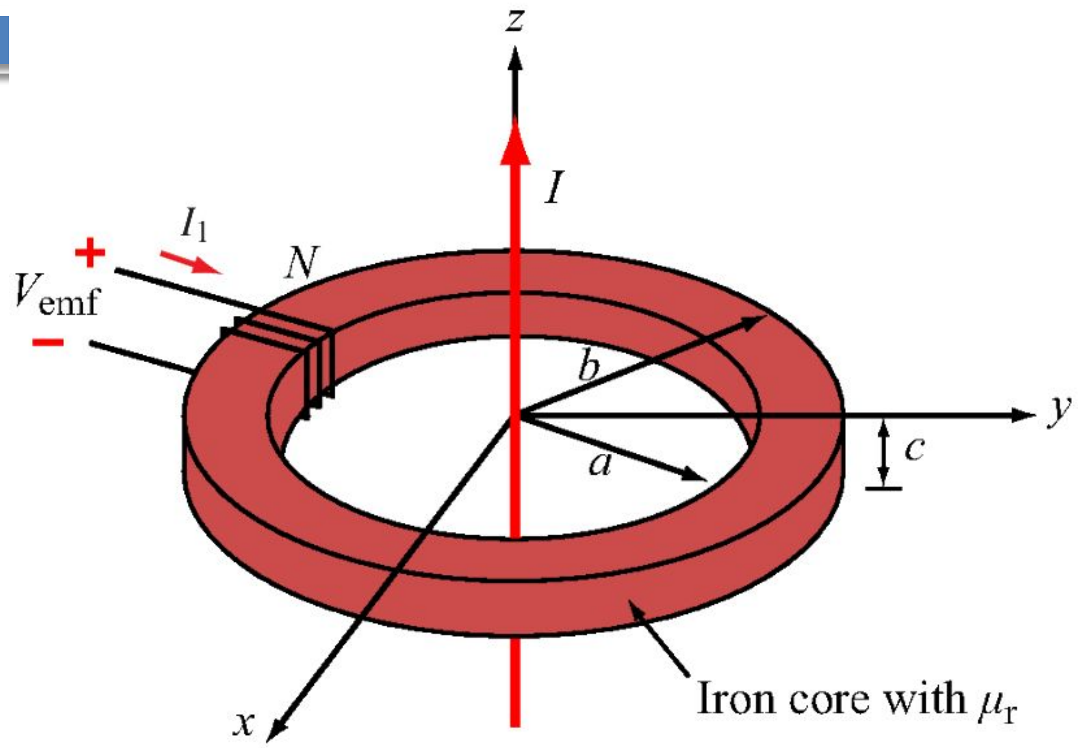
$$= \frac{-N\mu c}{2\pi} \ln\left(\frac{b}{a}\right) \frac{dI}{dt}$$

$$= \frac{-N\mu c}{2\pi} \ln\left(\frac{b}{a}\right) \frac{d(I_0 \cos\omega t)}{dt}$$

$$V_{\text{emf}} = \frac{N\mu I_0 c}{2\pi} \ln\left(\frac{b}{a}\right) \omega \sin\omega t$$

Transformers: Example 4

3. Lenz's Law:



+Φ is in +$\boldsymbol{\phi}$ direction

dΦ/dt ∝ -sin(): negative at t=0+

opposing that: means +Φ, so **B** in **+$\boldsymbol{\phi}$**-dir inside the loop

means current I_1 going in opposite direction to that shown

So V_{EMF} has the correct polarity

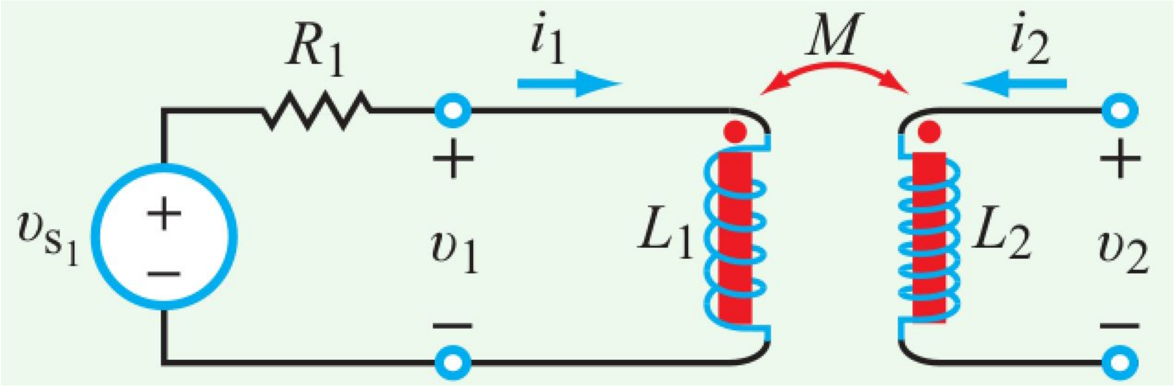
Transformers: Example 5

Given: Plug in your cell-phone charger, but don't attach your cell phone

Find: Current in primary side of transformer.

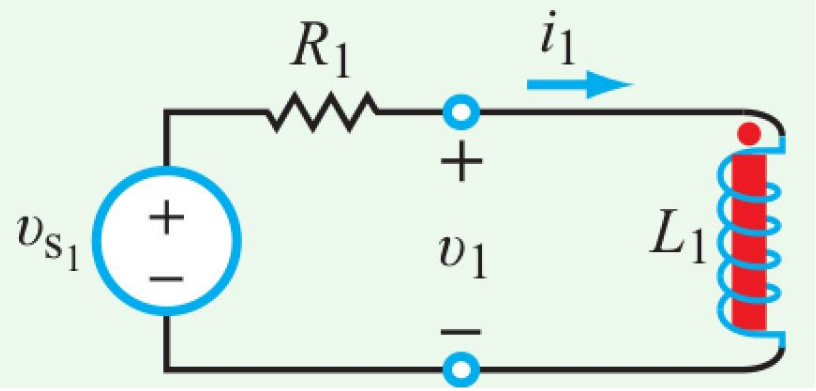
Solution:

A simplified model for the relevant part of the charger uses the following circuit:



Transformers: Example 5

Since I_2=0, can simplify the circuit further:

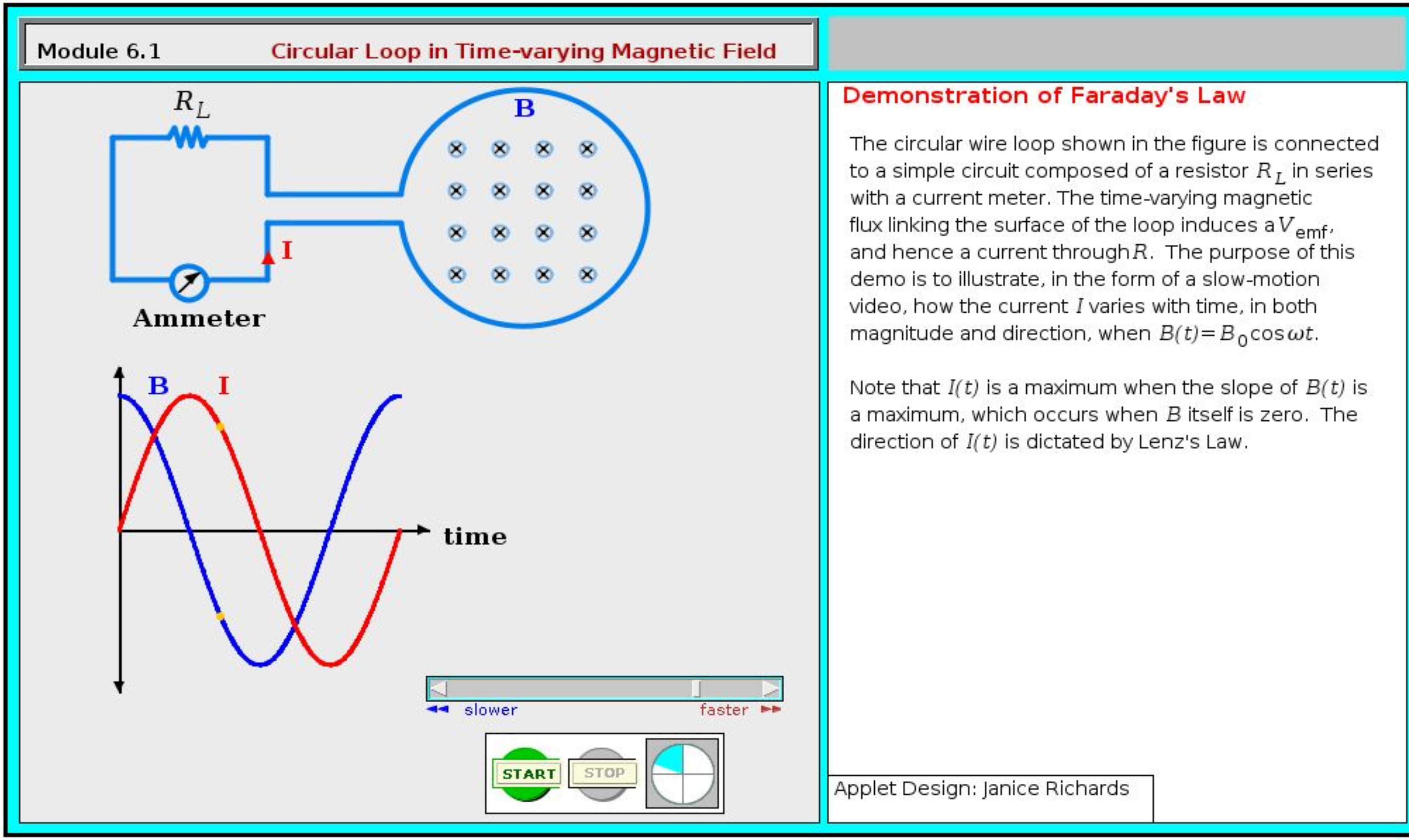


And use EECS 215 methods to solve it:

$$i_1 = v_{s1} / (R_1 + j\omega L_1)$$

For many chargers: this current is small: < 10mA

Module 6.1

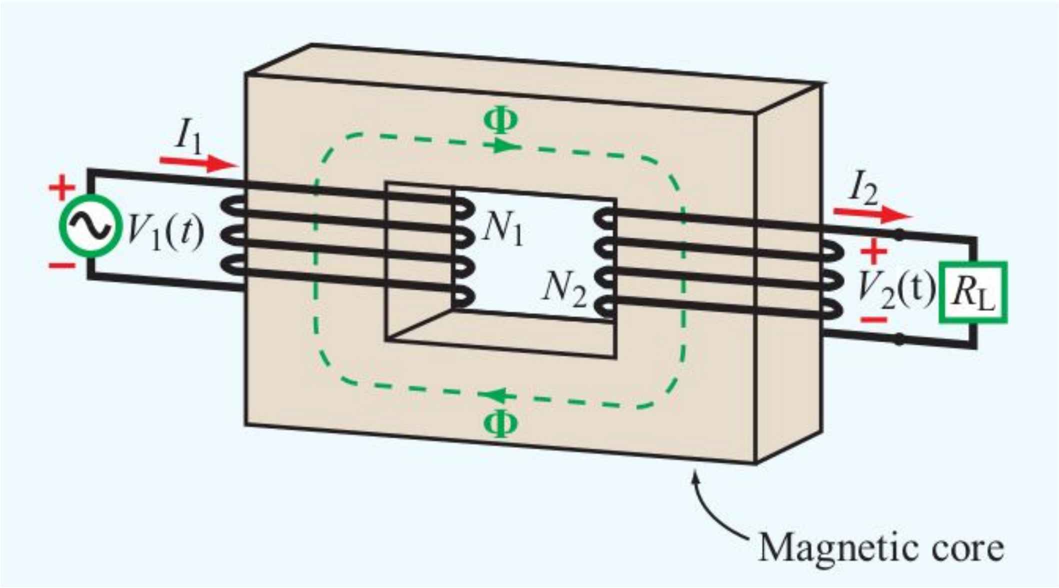


Module 6.1
Circular Loop in Time-varying Magnetic Field
R_L
B
I
Ammeter
B
I
time
slower
faster
START
STOP
Demonstration of Faraday's Law
The circular wire loop shown in the figure is connected to a simple circuit composed of a resistor R_L in series with a current meter. The time-varying magnetic flux linking the surface of the loop induces a V_{emf}, and hence a current through R. The purpose of this demo is to illustrate, in the form of a slow-motion video, how the current I varies with time, in both magnitude and direction, when $B(t) = B_0 \cos \omega t$.
Note that $I(t)$ is a maximum when the slope of $B(t)$ is a maximum, which occurs when B itself is zero. The direction of $I(t)$ is dictated by Lenz's Law.
Applet Design: Janice Richards

6-3 The Ideal Transformer

Time-varying **B** field, Stationary loop

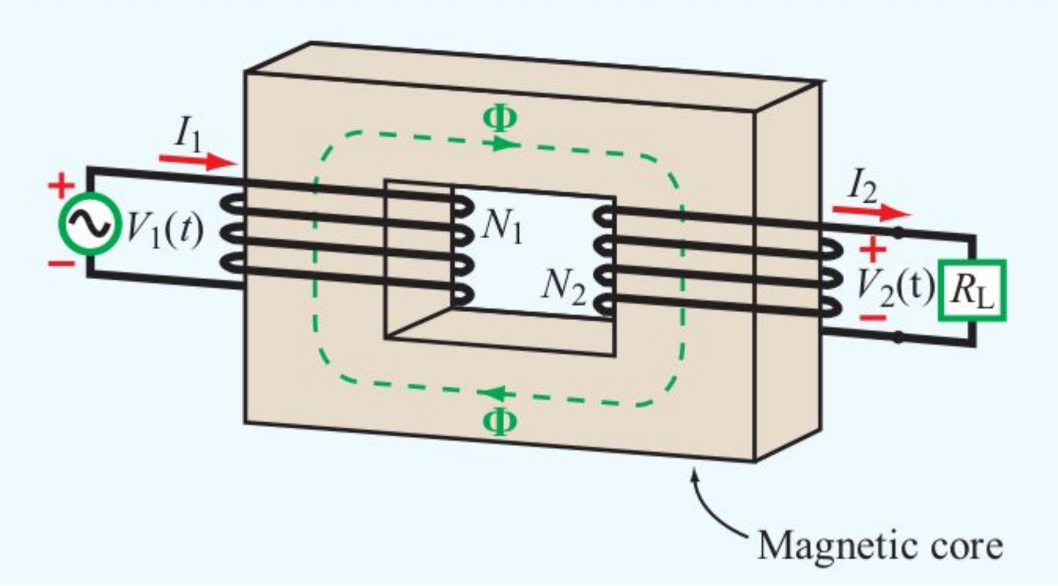
Real-world application: Transformer



6-3 The Ideal Transformer

Real-world application: Transformer

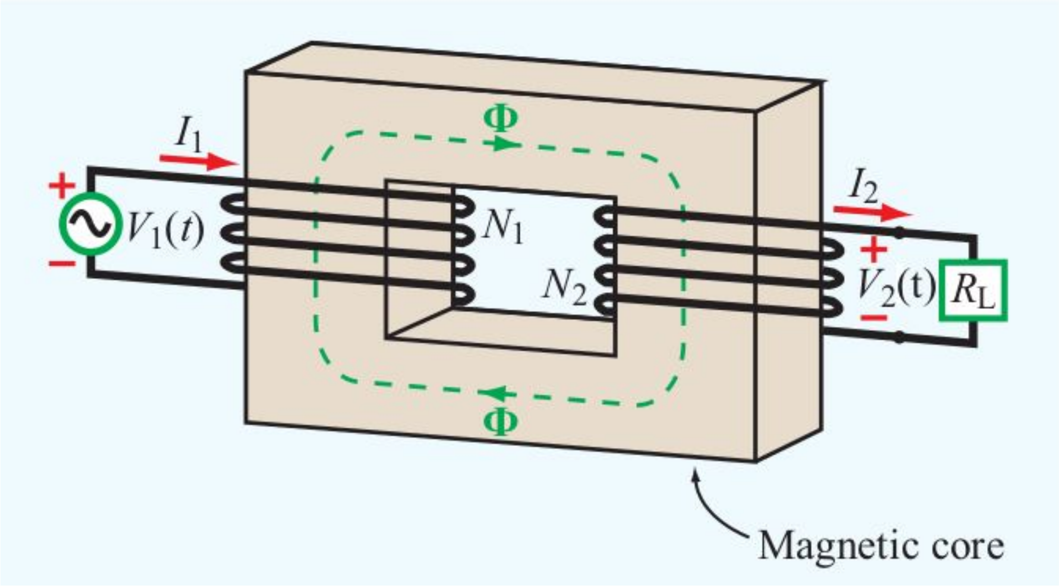
Time-varying source
Flux from coil1 induces current in coil2



6-3 The Ideal Transformer

Real-world application: Transformer

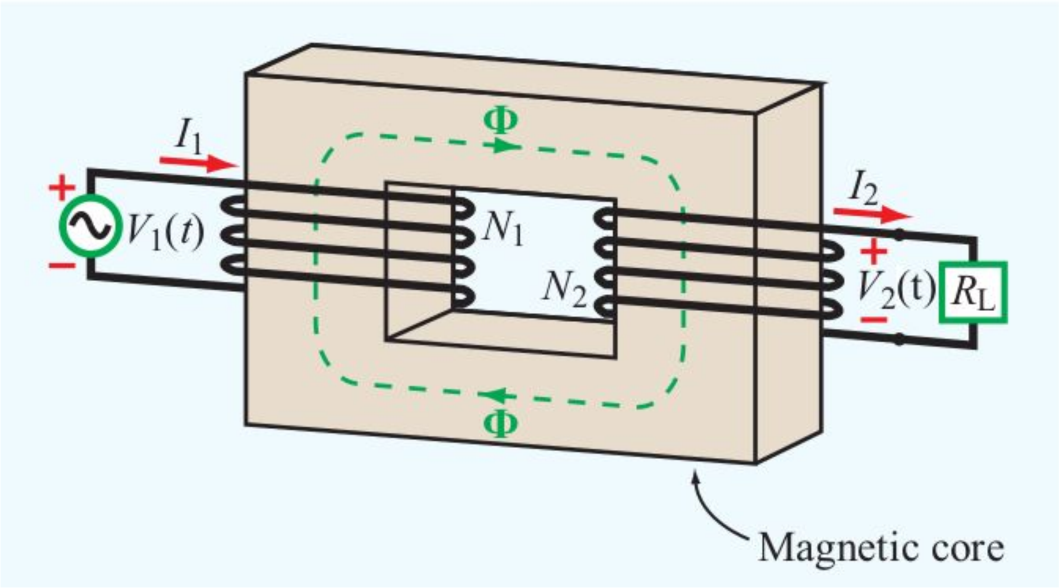
Currents defined so that induced current, I_2, generates **B** opposing increasing flux from I_1.



6-3 The Ideal Transformer

Real-world application: Transformer

Ideal: Lossless, and core has $\mu=\infty$
All flux confined within the core



6-3 The Ideal Transformer

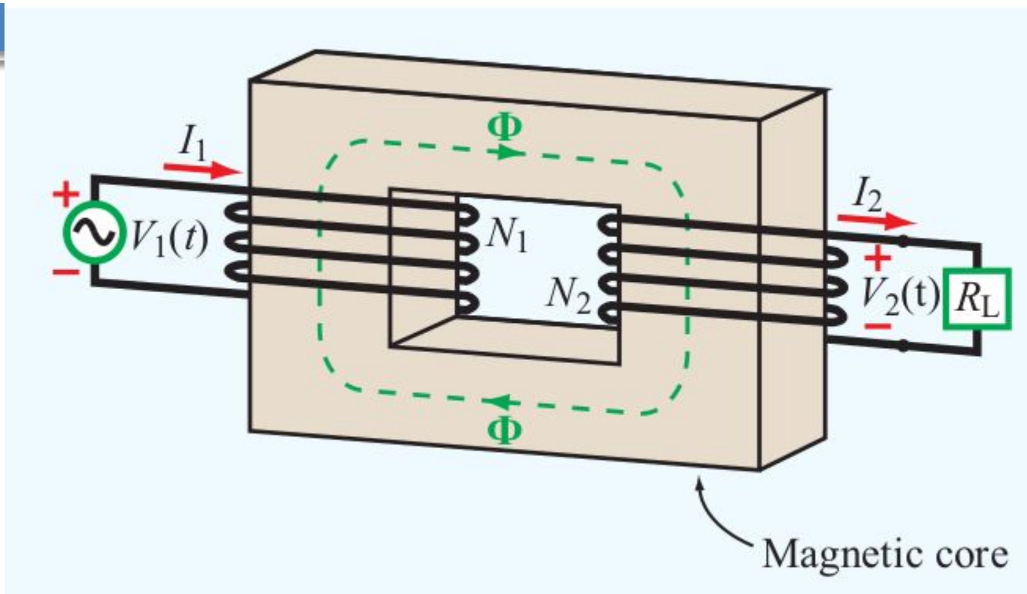
Apply Faraday's Law:

$$V_1 = -N_1 \frac{d\Phi}{dt}$$

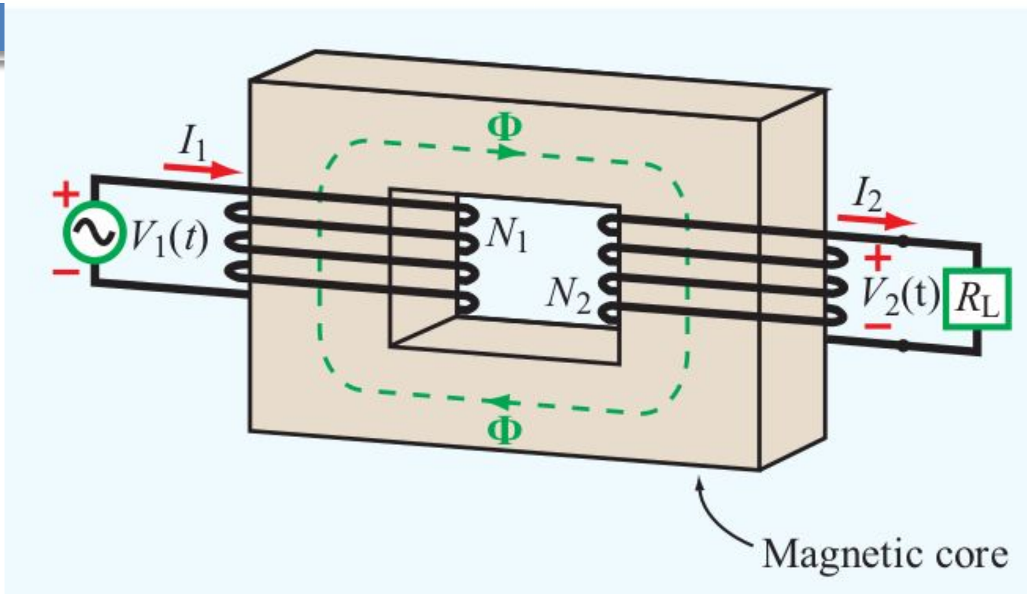
$$V_2 = -N_2 \frac{d\Phi}{dt}$$

combine:

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$



6-3 The Ideal Transformer



because it's lossless:

$$P_1 = P_2.$$

and so:

$$I_1 V_1 = I_2 V_2$$

$$V_1/V_2 = I_2/I_1$$

so:

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

compared to:

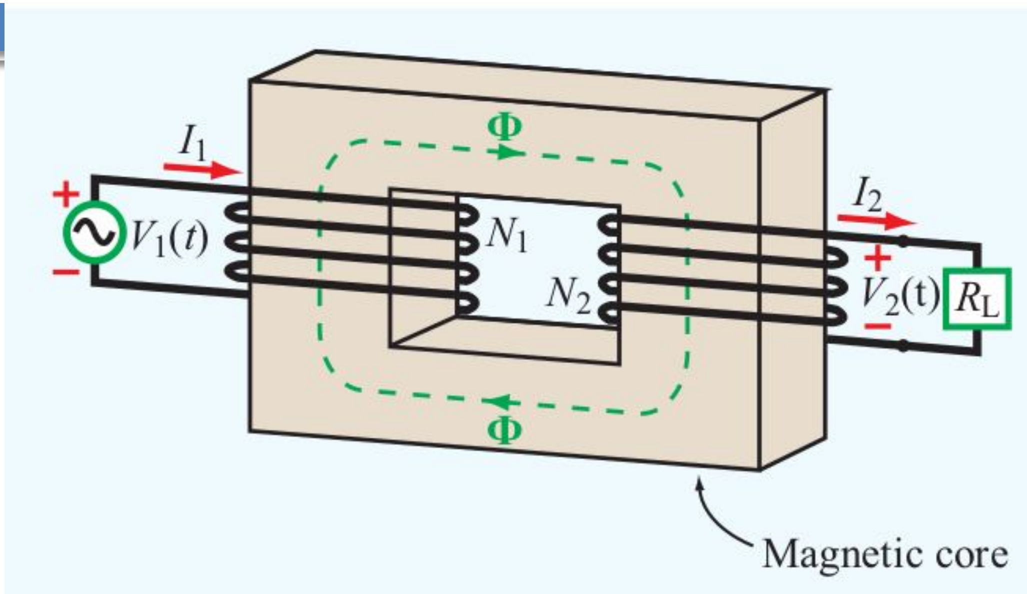
$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

Current: inverse relationship compared to the Voltages.

6-3 The Ideal Transformer

For example, if $N_2 = 10N_1$,

$V_2 = 10\ V_1$
$I_2 = 0.1\ I_1$



$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

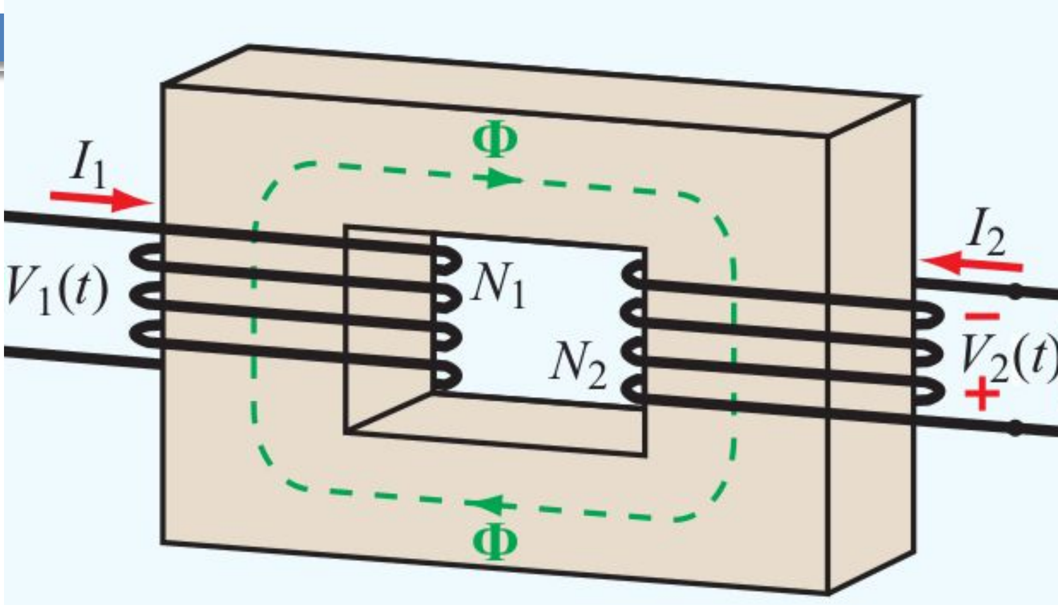
$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

6-3 The Ideal Transformer

If secondary wound in the opposite direction:

V_2 and I_2 are defined opposite to those in previous case.

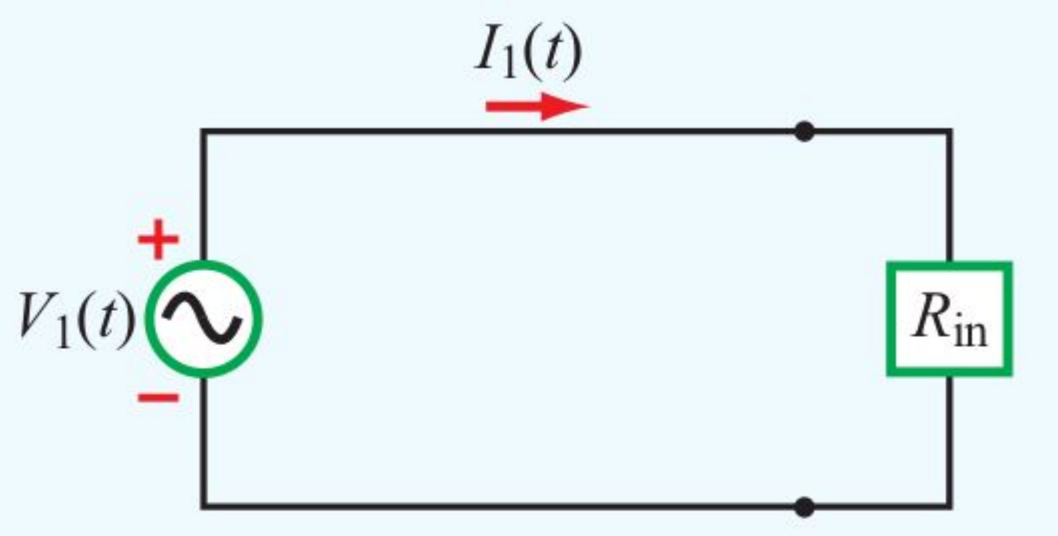
Positive I_2 creates flux that opposes change of flux due to positive dI_1/dt



$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

6-3 The Ideal Transformer



Equivalent Circuit:
from the "primary" side:

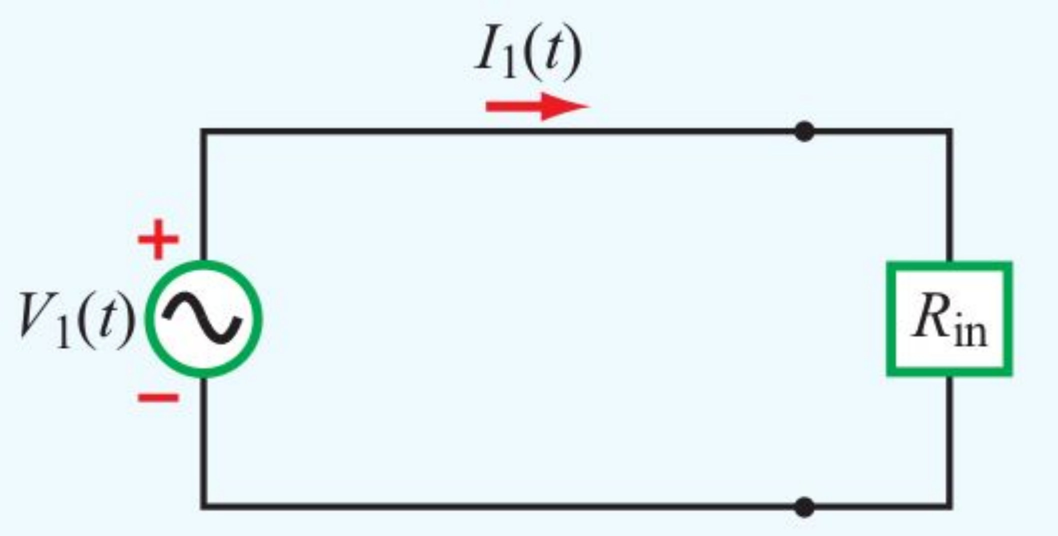
$$R_{in} = \frac{V_1}{I_1}$$
$$= \frac{V_2 \, N_1/N_2}{I_2 \, N_2/N_1}$$
$$= \frac{V_2}{I_2}\left(\frac{N_1}{N_2}\right)^2$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} \qquad \frac{V_1}{V_2} = \frac{N_1}{N_2}$$

Since: $V_2 = I_2 R_L$:

$$R_{in} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

6-3 The Ideal Transformer



Equivalent Circuit:

$$R_{in} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

Phasor domain:

$$Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

Homework

Homework 22 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Sections 6-4 through 6-6:

Moving Loop in static **B** Field

The Electromagnetic Generator

Moving Loop in time-varying **B** Field