EECS 230
ENGINEERING ELECTROMAGNETICS
Leland Pierce

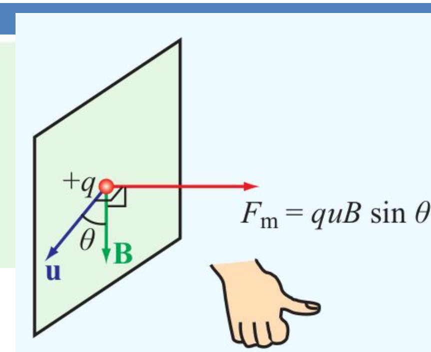
Magnetostatics 4

University of Michigan - Fall 2024

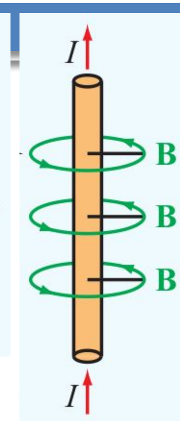
Chapter 5 Overview

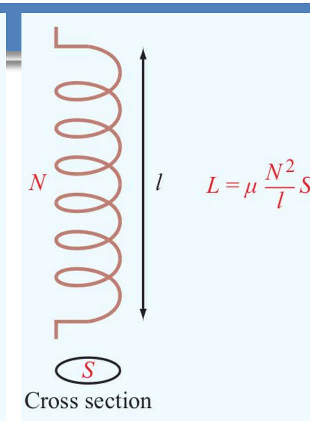
- Maxwell's Equations
  - Magnetostatics
- Magnetic Force
- Magnetic Torque
- Magnetic field from currents
- Gauss's Law for Magnetism
- Ampere's Law
- Magnetic Vector Potential Fi
  - Poisson's eqn
  - Magnetic Flux
- Magnetic Permeability
  - Hysteresis
- Magnetic Boundary Conditions
- Inductance
- Magnetic Energy

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J},$$



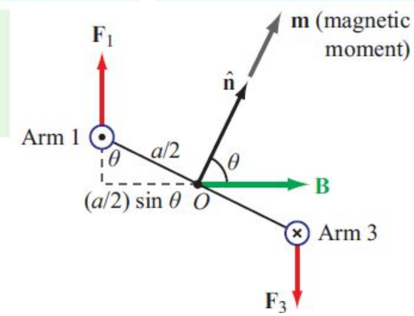
$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$$





$$\nabla \cdot \mathbf{B} = 0 \quad \longleftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0.$$

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint \mathbf{H} \cdot d\boldsymbol{\ell} = I,$$

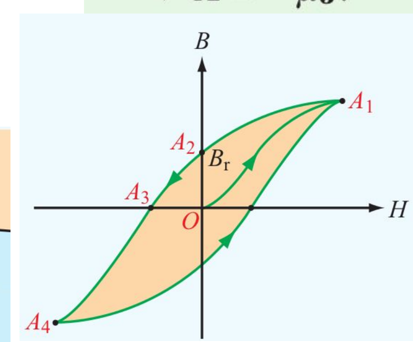


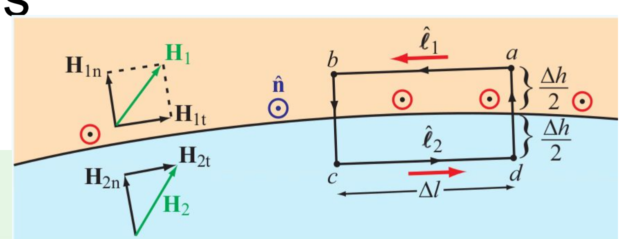
$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (\text{Wb/m}^2),$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}.$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad (\text{Wb}).$$

$$\mathbf{B} = \mu \mathbf{H},$$





$$w_m = \frac{W_m}{v} = \frac{1}{2}\mu H^2 \qquad (\text{J/m}^3).$$

Lecture Coverage

Today's lecture:

Review of Sections 5-1 through 5-6 of the book:

5-1: Magnetostatics: Magnetic Forces and Torques
5-2: **H** due to a steady current (Biot-Savart Law)
5-3: Magnetic Field from Currents: Ampere's Law
5-4: Magnetic Vector Potential Field, Poisson's eqn, Magnetic Flux
5-5: Magnetic Permeability, Hysteresis
5-6: Magnetic Boundary Conditions

Sections 5-7 through 5-8 of the book:

5-7: Inductance
5-8: Magnetic Energy

Chapter 5 Review

Static Conditions:

$$\frac{\partial \mathbf{J}}{\partial t} = 0$$

Magnetostatics:

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

magnetic flux density $\mathbf{B}$

magnetic field intensity $\mathbf{H}$

$$\mathbf{B} = \mu \mathbf{H}.$$

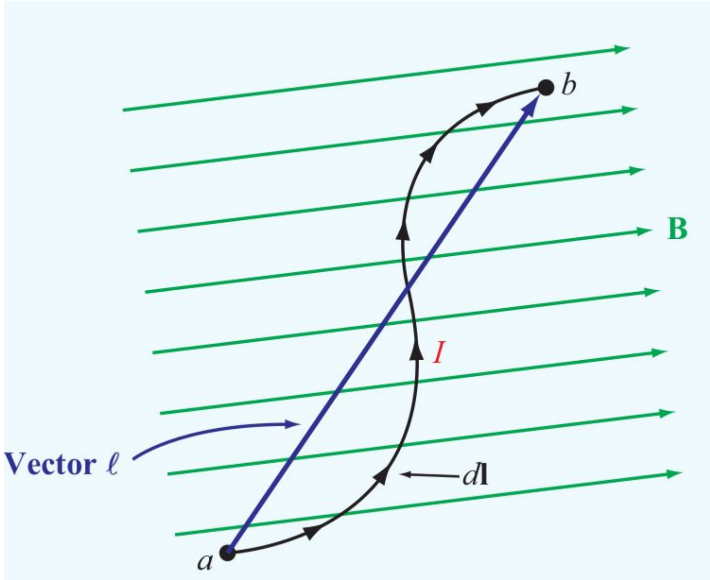
J is the current density

Chapter 5 Review

Magnetic force $$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}$$

Force on a current in a wire:
If **part** of wire is in uniform **B**:

$$\mathbf{F}_m = I\left(\int_\ell d\mathbf{l}\right) \times \mathbf{B} = I\boldsymbol{\ell} \times \mathbf{B},$$



Chapter 5 Review

Torque for a loop with *N* turns, and surface normal $\hat{\mathbf{n}}$ at angle θ relative to *B* direction:

$$T = N I A B_0 \sin\theta.$$

magnetic moment of the loop:

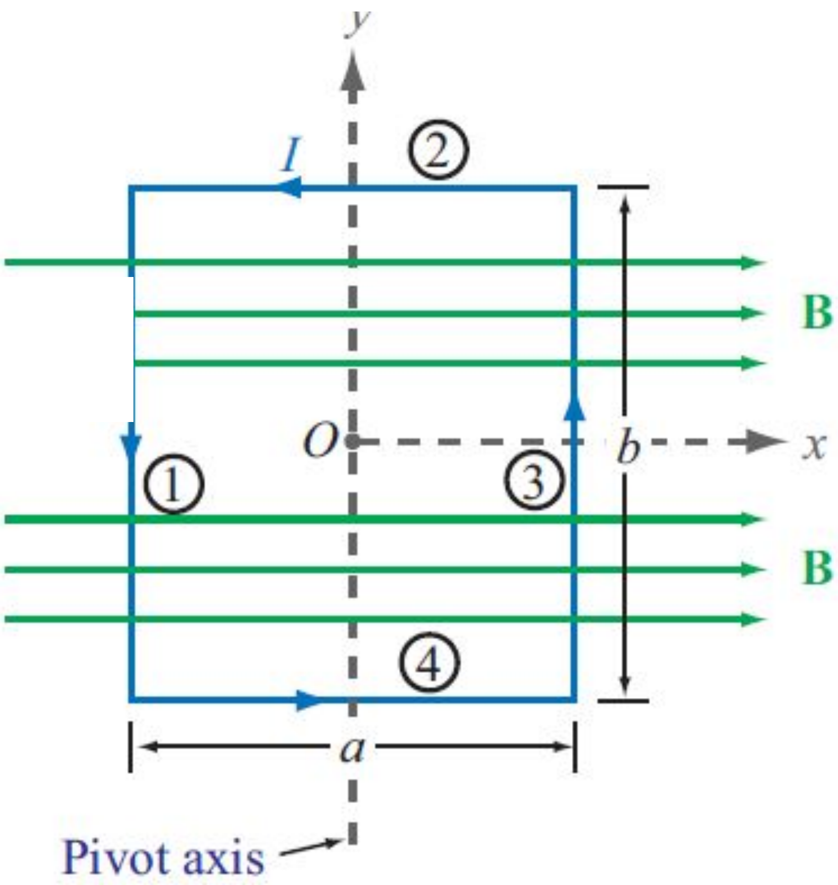
$$m = N I A$$

define:

$$\mathbf{m} = \hat{\mathbf{n}} N I A = \hat{\mathbf{n}} m \qquad (\text{A}\cdot\text{m}^2),$$

so:

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m}).$$

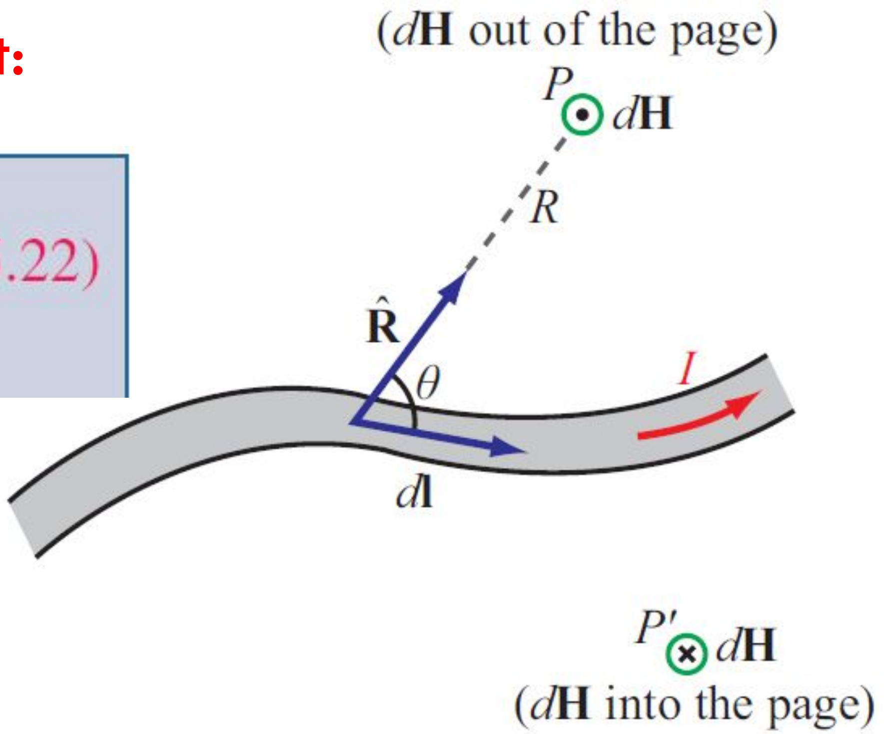


Chapter 5 Review

Biot-Savart Law:

Magnetic field induced by a current:

$$\mathbf{H} = \frac{I}{4\pi} \int_{l} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}, \qquad (5.22)$$



Chapter 5 Review

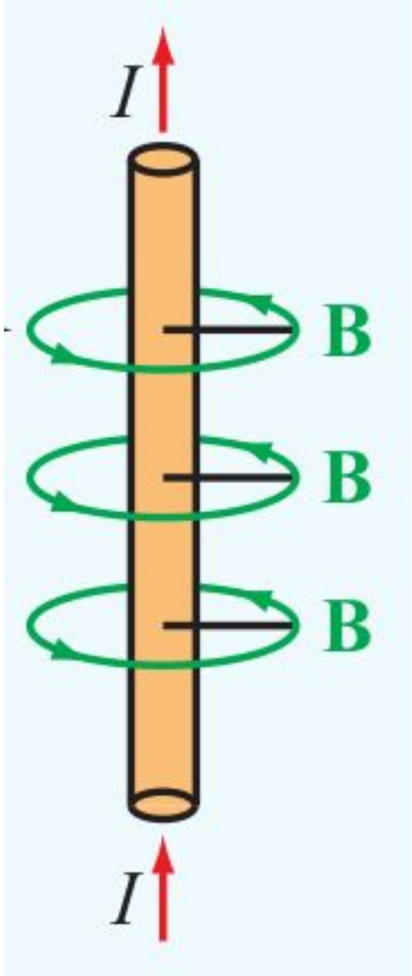
B due to current in straight wire:

$$\mathbf{B} = \mu_0 \mathbf{H} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I l}{2\pi r \sqrt{4r^2 + l^2}}$$

For a very long wire:
In the limit as $l \to \infty$

$$4r^2 + l^2 \to l^2$$

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$



Chapter 5 Review

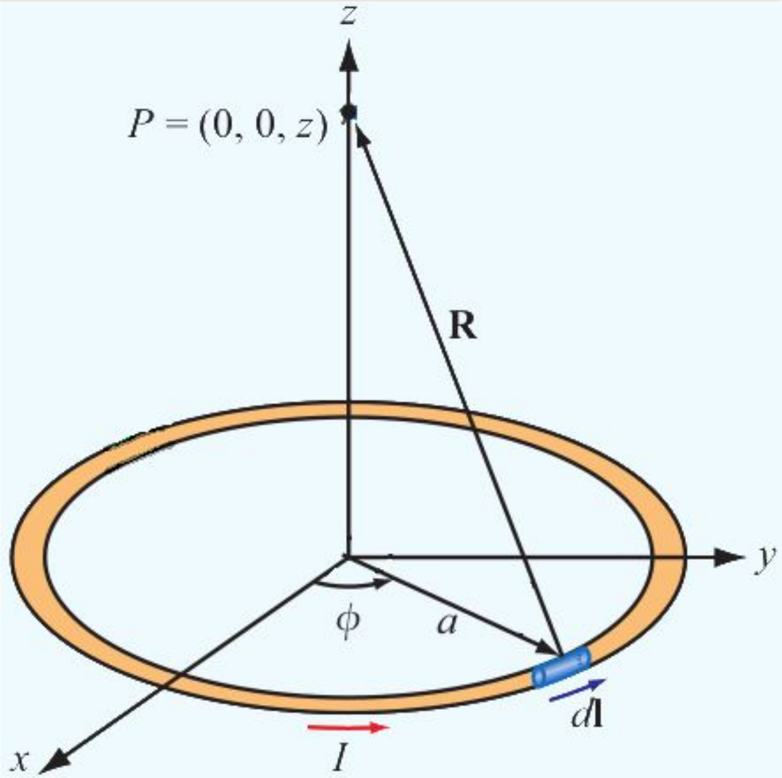
B along *z*-axis, due to circular current loop:

Since the magnetic moment of **any** loop in the *x*-*y* plane is:

$$\mathbf{m} = \hat{\mathbf{z}} I A$$

the magnetic moment of **this** loop is:

$$\mathbf{m} = \hat{\mathbf{z}} I \pi a^2$$



so:

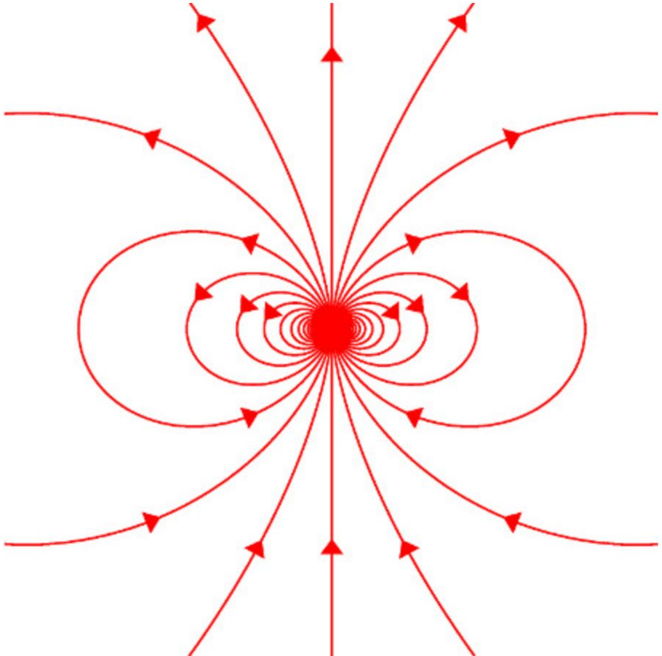
$$\mathbf{H} = \frac{I a^2}{2R^3}\hat{\mathbf{z}} = \frac{\mathbf{m}}{2\pi R^3}\hat{\mathbf{z}}$$

Chapter 5 Review

Magnetic Dipole:

Solving for the fields **everywhere** far from a current loop results in:

$$\mathbf{H} = \frac{m}{4\pi R^3}(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta) \qquad (\text{for } R \gg a).$$



(physics.stackexchange.com)

Chapter 5 Review

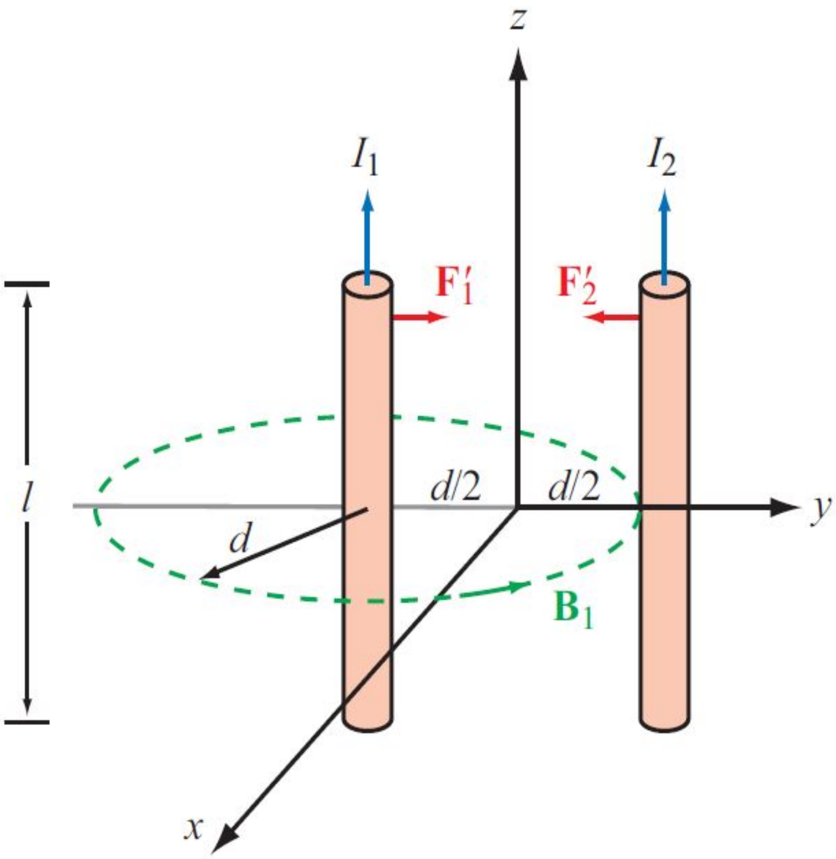
Force due to Parallel Currents:

So the force per unit length fro wire1 on wire2 is:

$$\mathbf{F}'_2 = -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}$$

similarly, from wire2 on wire1:

$$\mathbf{F}'_1 = \hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}$$



This force pulls the two wires together.

Chapter 5 Review

Ampère's Law:

In Electrostatics: $\nabla \times \mathbf{E} = 0 \quad \leftrightarrow \quad \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$

In Magnetostatics: $\nabla \times \mathbf{H} = \mathbf{J} \quad \leftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I,$

I is the current **crossing** the surface of the contour *C*

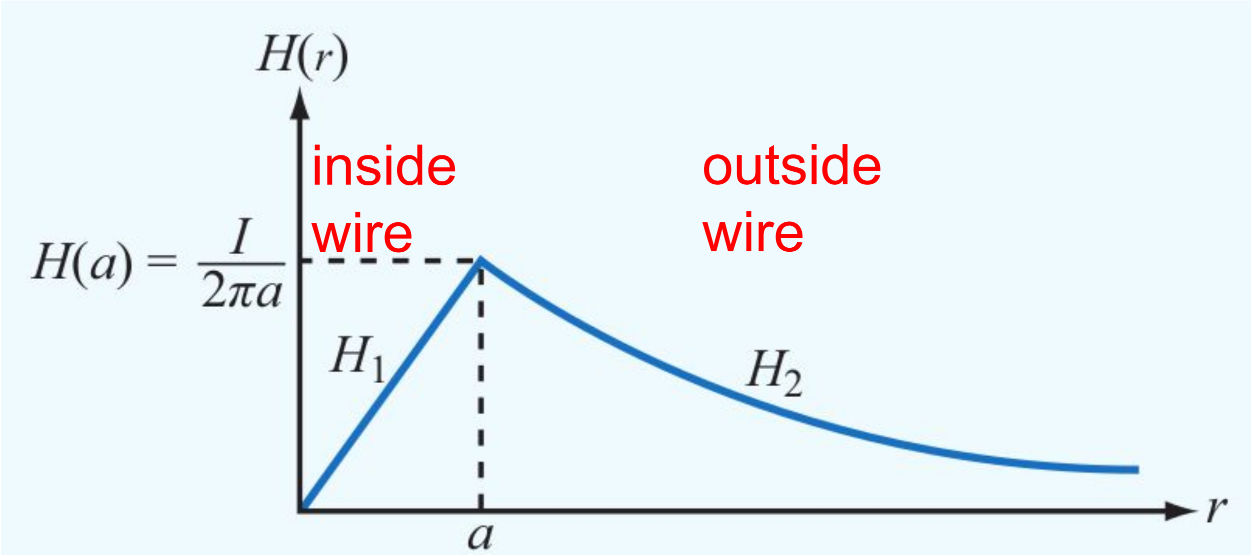
Using the right-hand rule, with the thumb pointing along the direction of C:
Current is positive if points along the direction of the fingers.

Chapter 5 Review

Magnetic Field of current in long straight wire

Complete solution:

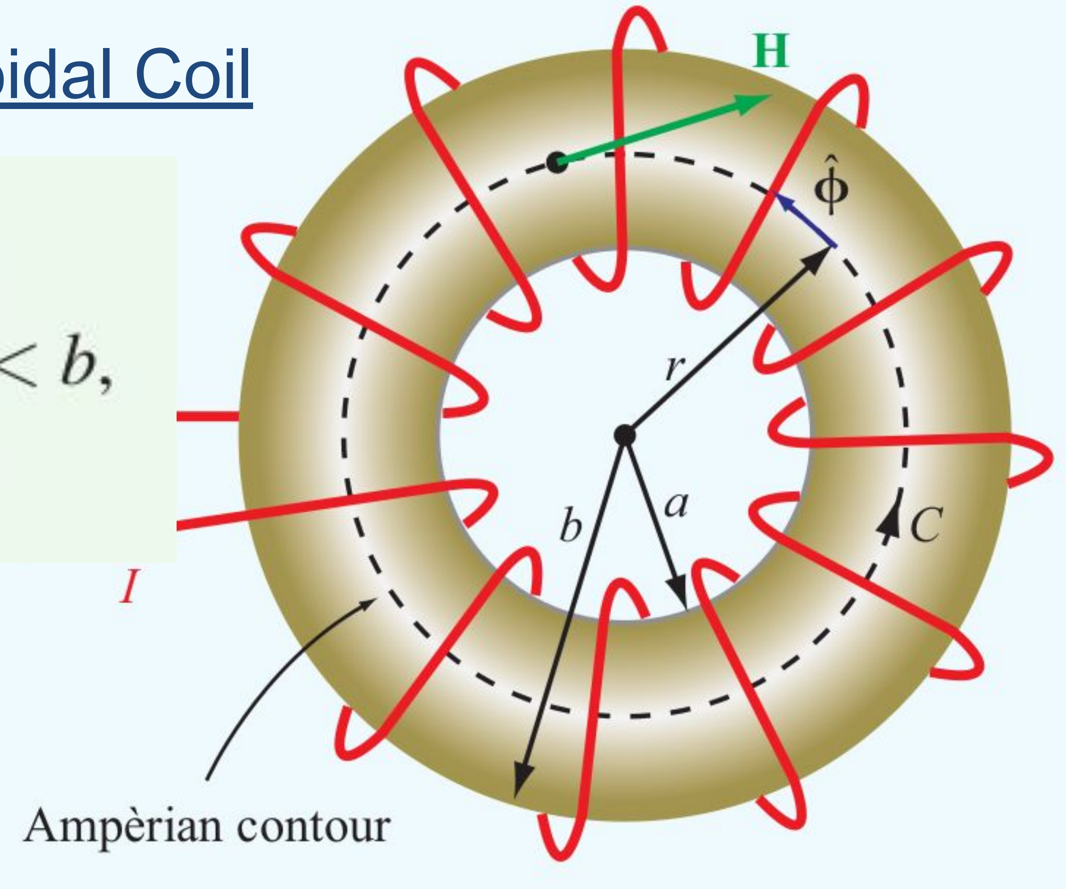
$$\mathbf{H}(r) = \begin{cases} \hat{\boldsymbol{\phi}}\,\dfrac{r}{2\pi a^2} I & 0 < r \le a \\ \hat{\boldsymbol{\phi}}\,\dfrac{1}{2\pi r} I & a \le r \end{cases}$$



Chapter 5 Review

Magnetic Field inside Toroidal Coil

$$\mathbf{H} = \begin{cases} 0 & \text{for } r < a, \\ -\hat{\boldsymbol{\phi}}\,\dfrac{NI}{2\pi r} & \text{for } a < r < b, \\ 0 & \text{for } r > b. \end{cases}$$



Field is concentrated inside the toroid.

Chapter 5 Review

Magnetic Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Vector Poisson's Equation:

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{\upsilon'} \frac{\mathbf{J}}{R'}\, d\upsilon'$$

Chapter 5 Review

Magnetic Materials:

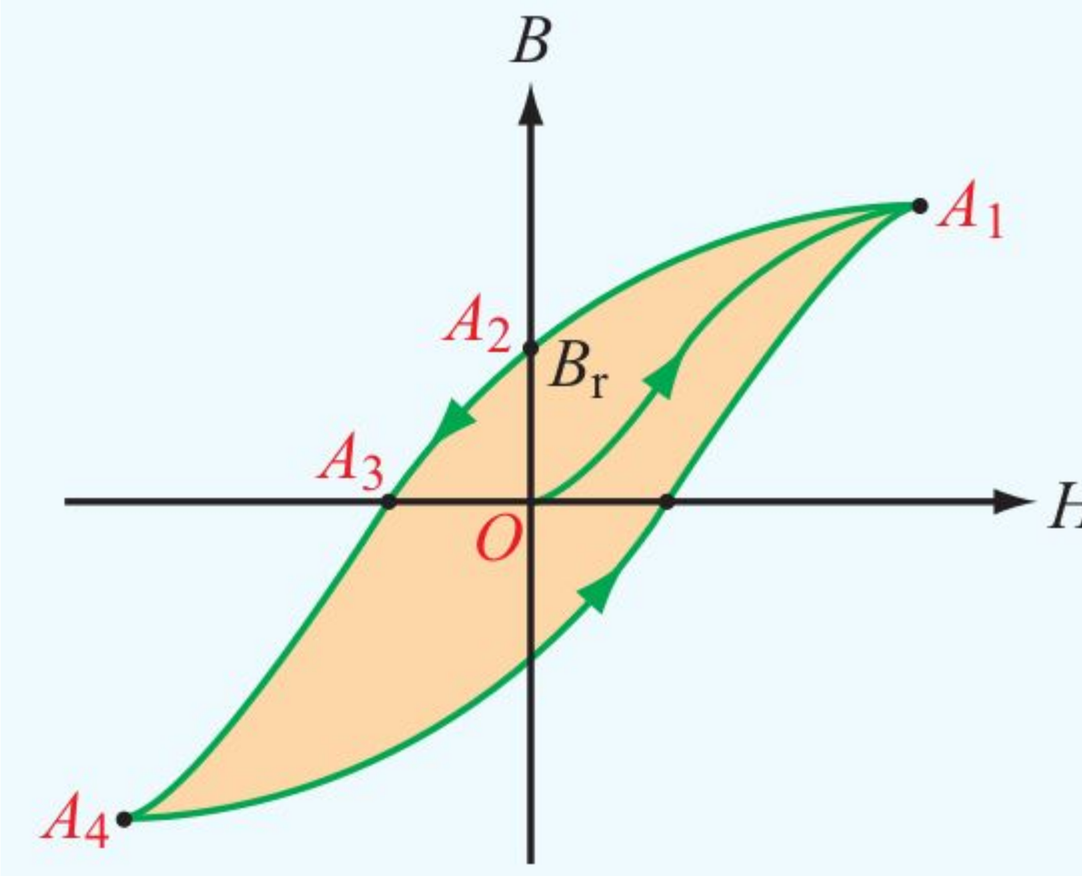
$$\mathbf{B} = \mu \mathbf{H},$$

μ is called the permeability.

Chapter 5 Review

Ferromagnetic Materials:

Hysteresis enables the creation of permanent magnets

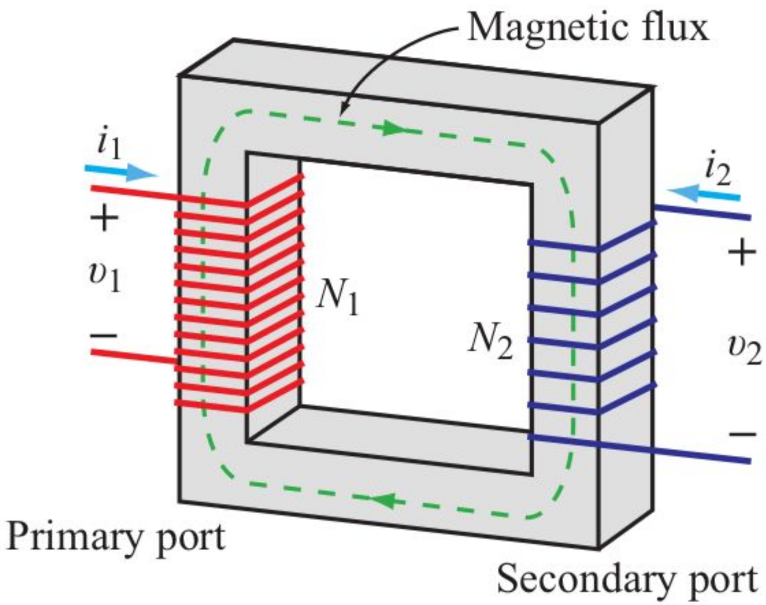


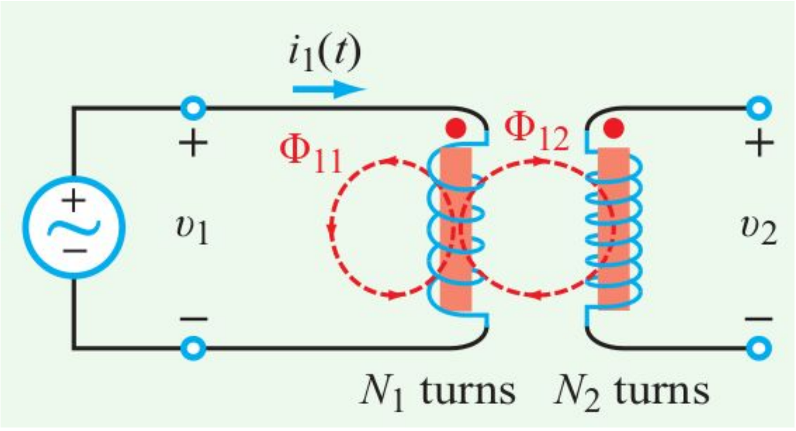
Chapter 5 Review

Iron-Core Transformers

Ferromagnetic: $\mu_r \approx 10^5$.

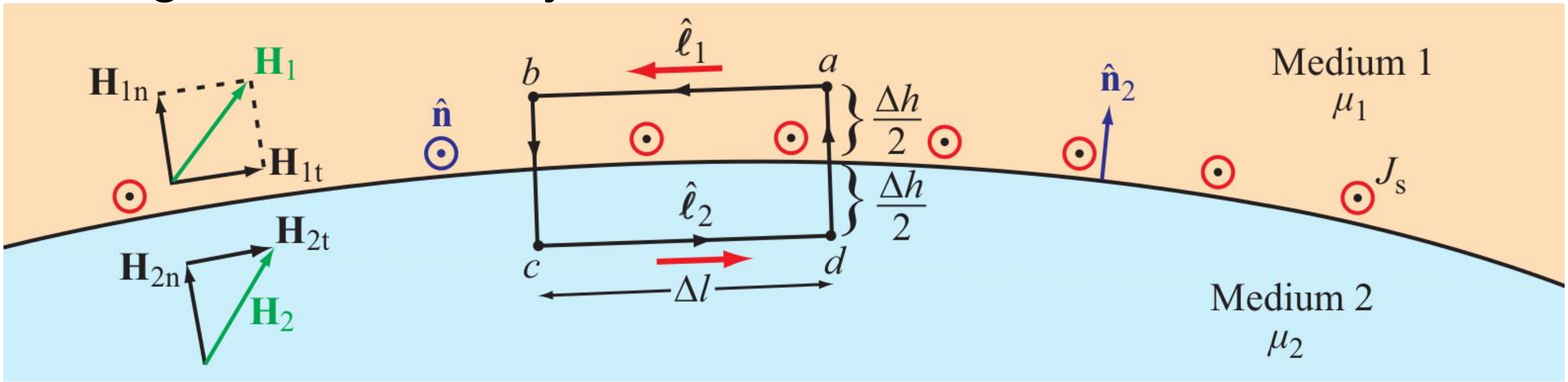
Concentrates the magnetic flux





Chapter 5 Review

Magnetic Boundary Conditions:



$$H_{1t} = H_{2t}.$$

$$\mathbf{B}_{1n} = \mathbf{B}_{2n}$$

Tangential **H** is continuous (finite conductivities)

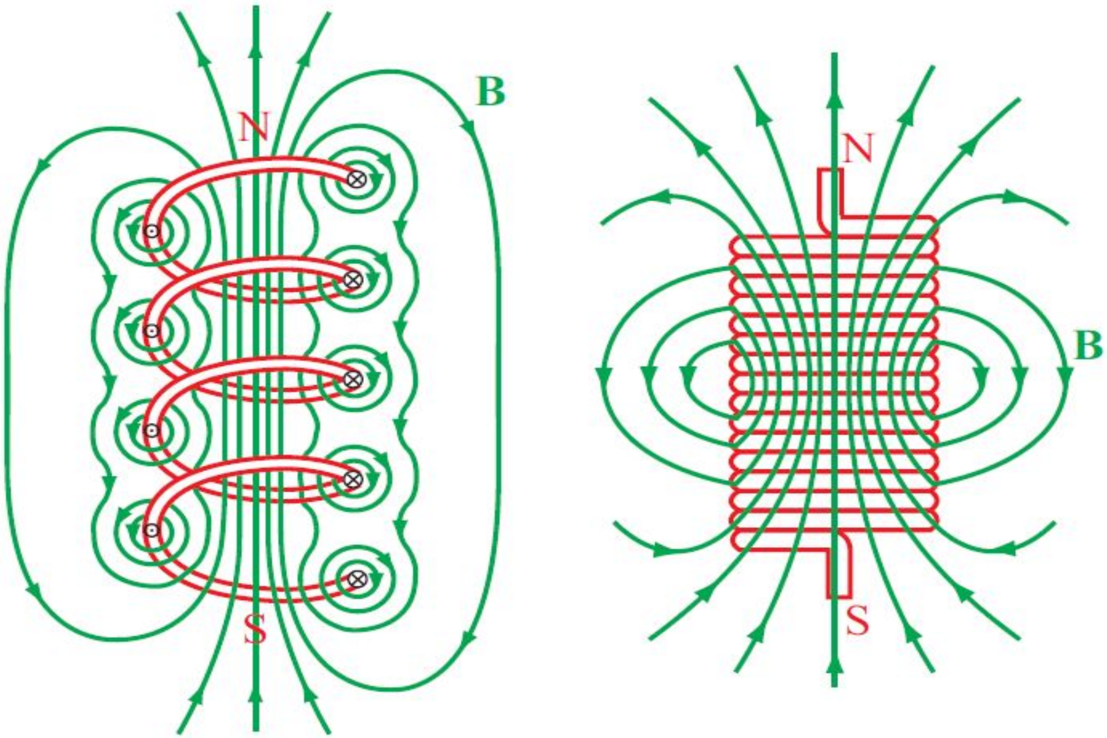
Normal **B** is continuous

5-7 Inductance

Typical inductor is a **solenoid:**

Current induces a magnetic field.

The field inside is reinforced by the closely-spaced wires.



(a) Loosely wound solenoid

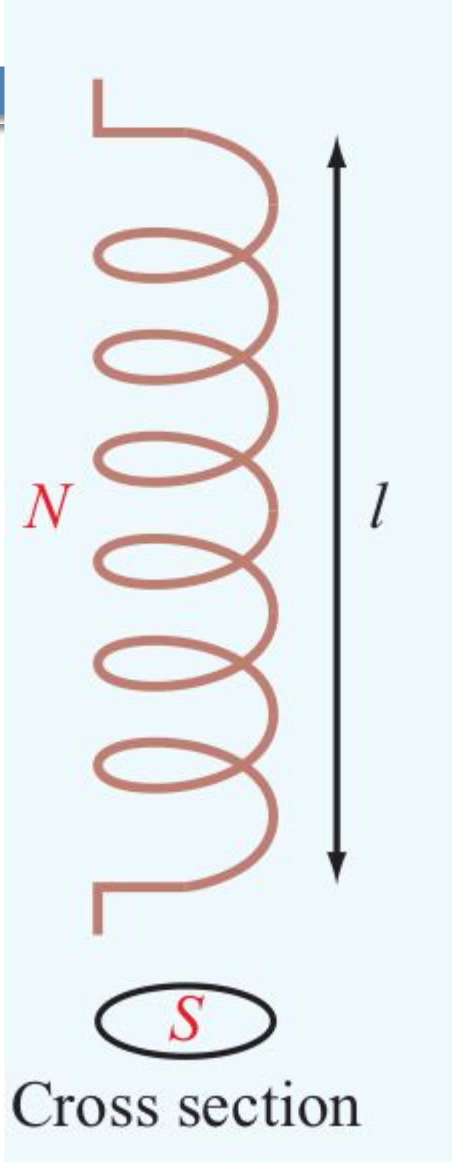
(b) Tightly wound solenoid

5-7 Inductance

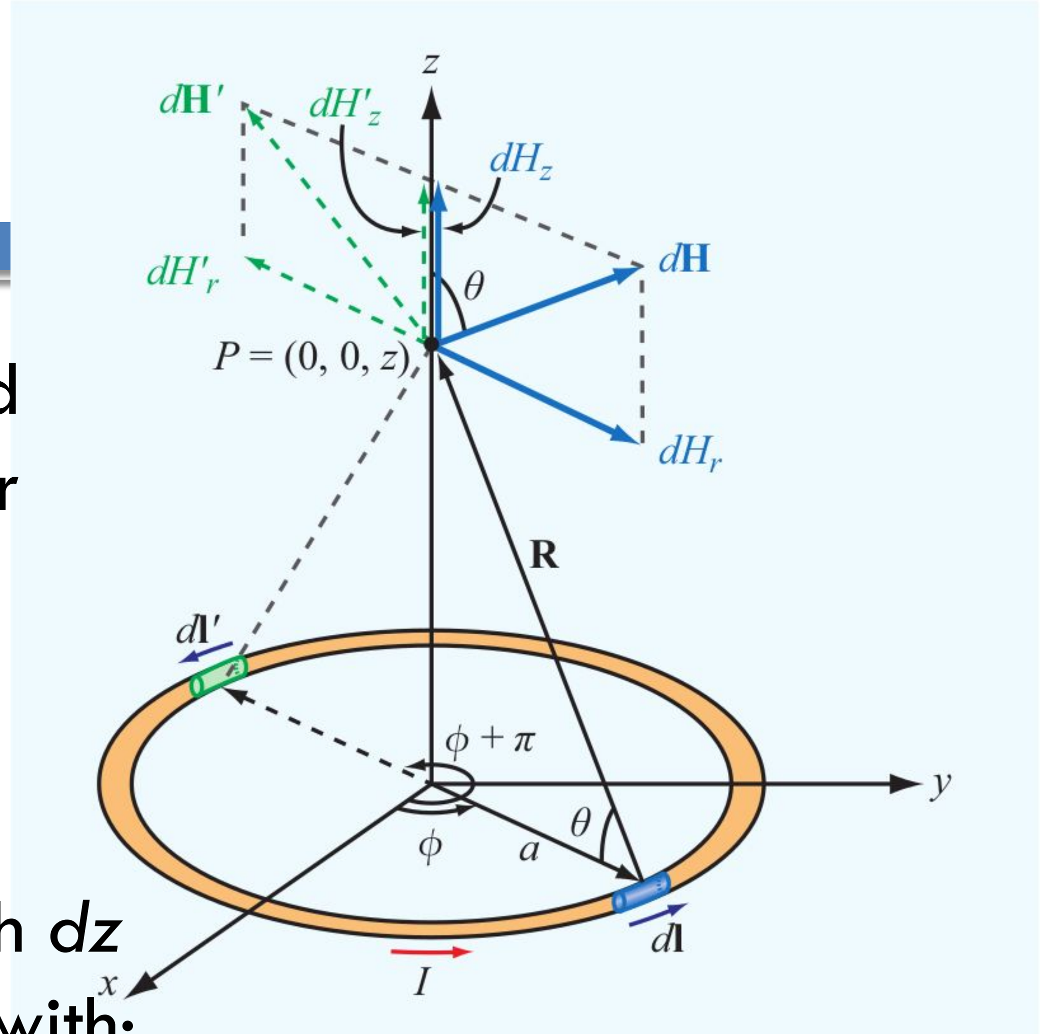
Solenoid geometry:
length: l
radius: a
number of "turns": N

For a tightly-wound solenoid:
each loop is nearly a circular loop

So, we make the approximation:
a collection of circular loops



5-7 Inductance



We already know the field along the axis of a circular loop:

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I' a^2}{2(a^2 + z^2)^{3/2}}$$

Treat an incremental length *dz* of the solenoid as a loop, with:

n=N/l turns per unit length

then: *n dz* is number of turns per z increment

Hence: current per z increment: $I' = I\, n\, dz$

5-7 Inductance

$$d\mathbf{B} = \mu\, d\mathbf{H}$$

$$= \hat{\mathbf{z}} \frac{\mu n I a^2}{2(a^2 + z^2)^{3/2}}\, dz.$$

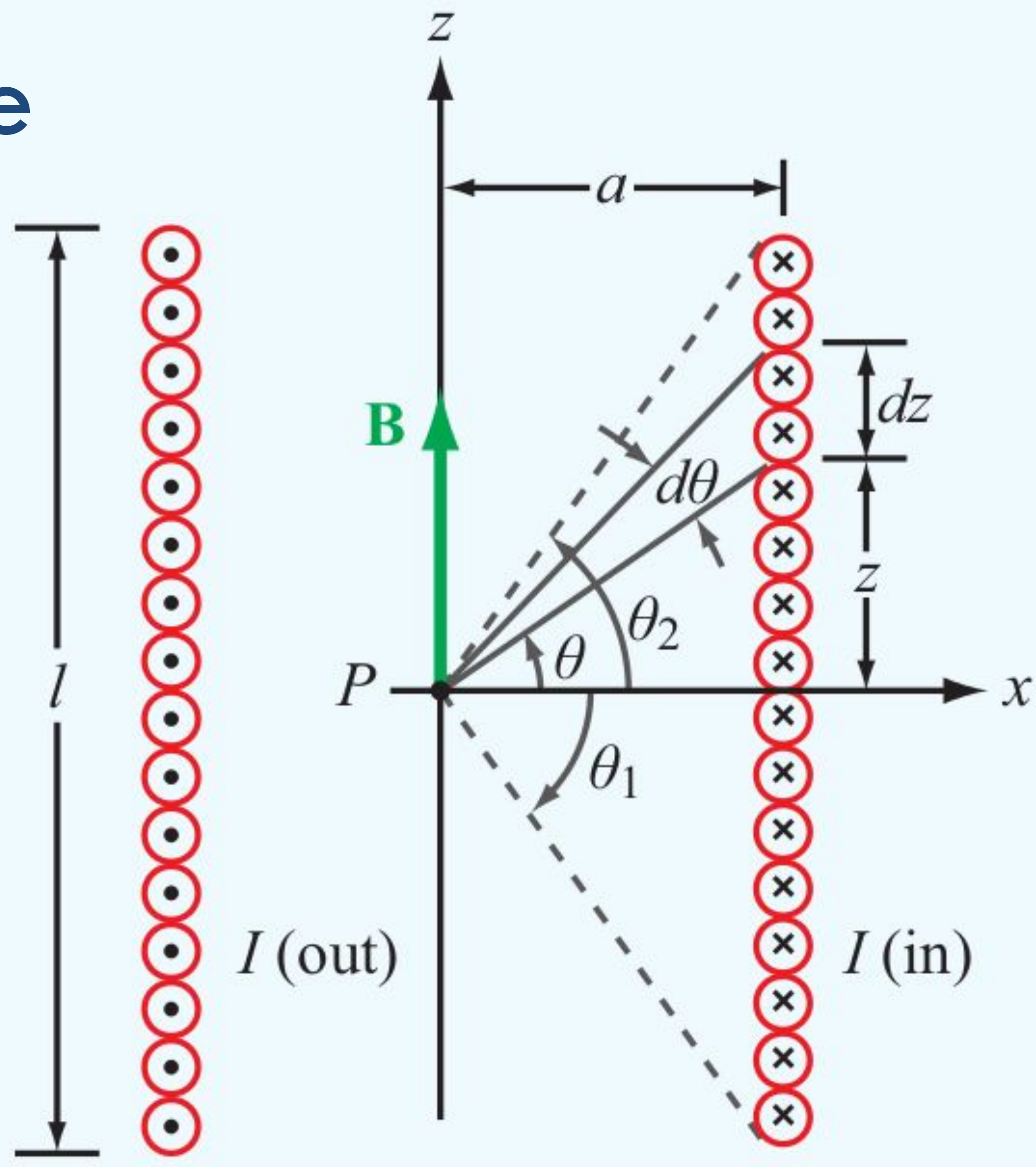
Get the field at the center point *P*:

Integrate $d\mathbf{B}$ over the solenoid's length

Plan:

Express z in terms of θ

Then integrate over θ



(a) Cross section

5-7 Inductance

tan θ = z/a

so: $z = a \tan\theta$

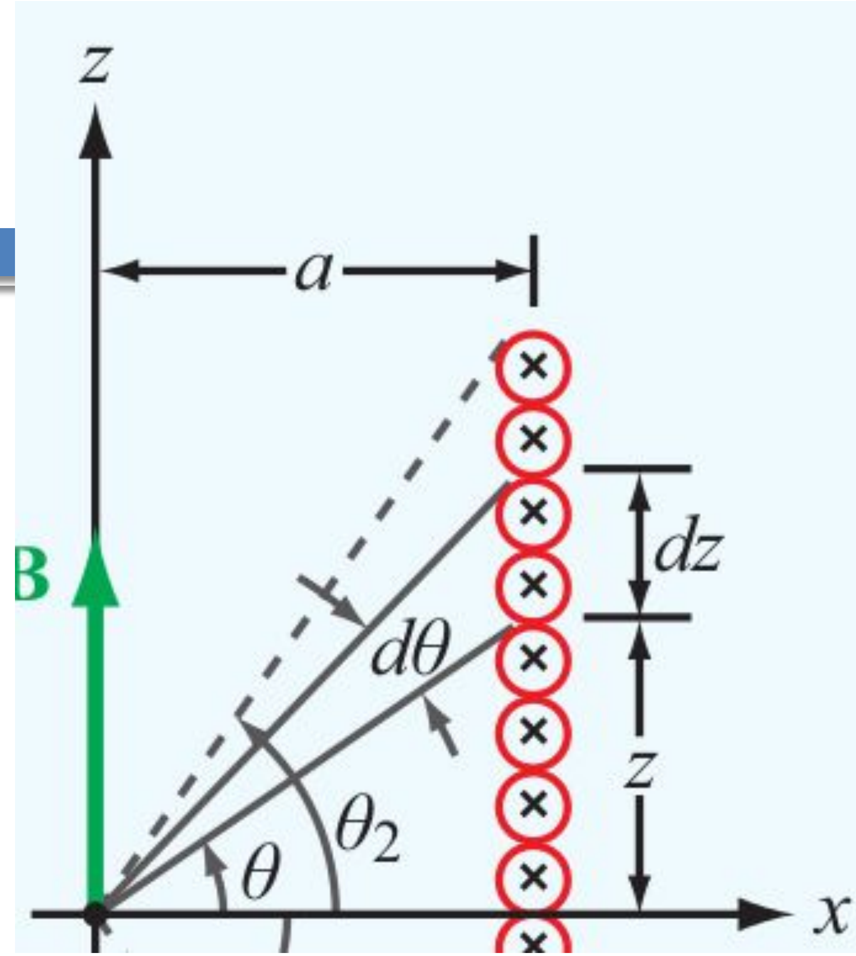
$a^2+z^2 = a^2 + a^2 \tan^2\theta$

$= a^2(1+\tan^2\theta)$

$a^2+z^2 = a^2 \sec^2\theta$

$dz = a\, d(\tan\theta)\, d\theta$

$dz = a \sec^2\theta\, d\theta$



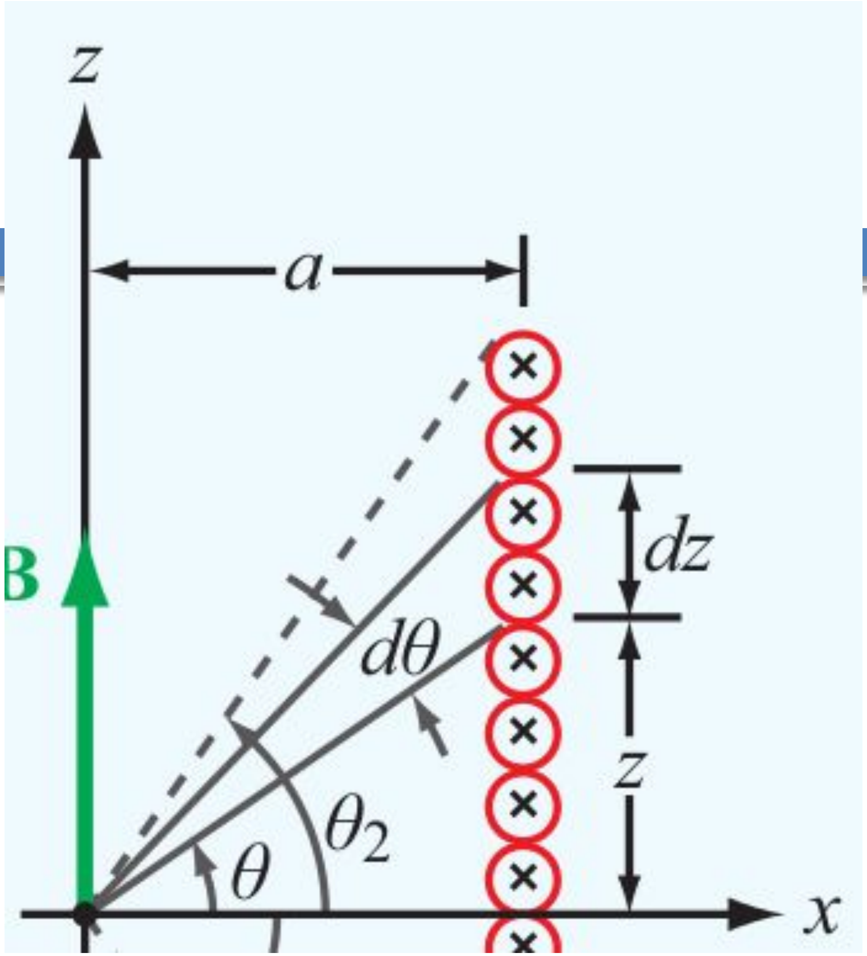
5-7 Inductance

$a^2+z^2 = a^2 \sec^2\theta$

$dz = a \sec^2\theta \, d\theta$

$$d\mathbf{B} = \hat{\mathbf{z}} \frac{\mu n I a^2}{2(a^2+z^2)^{3/2}} \, dz.$$

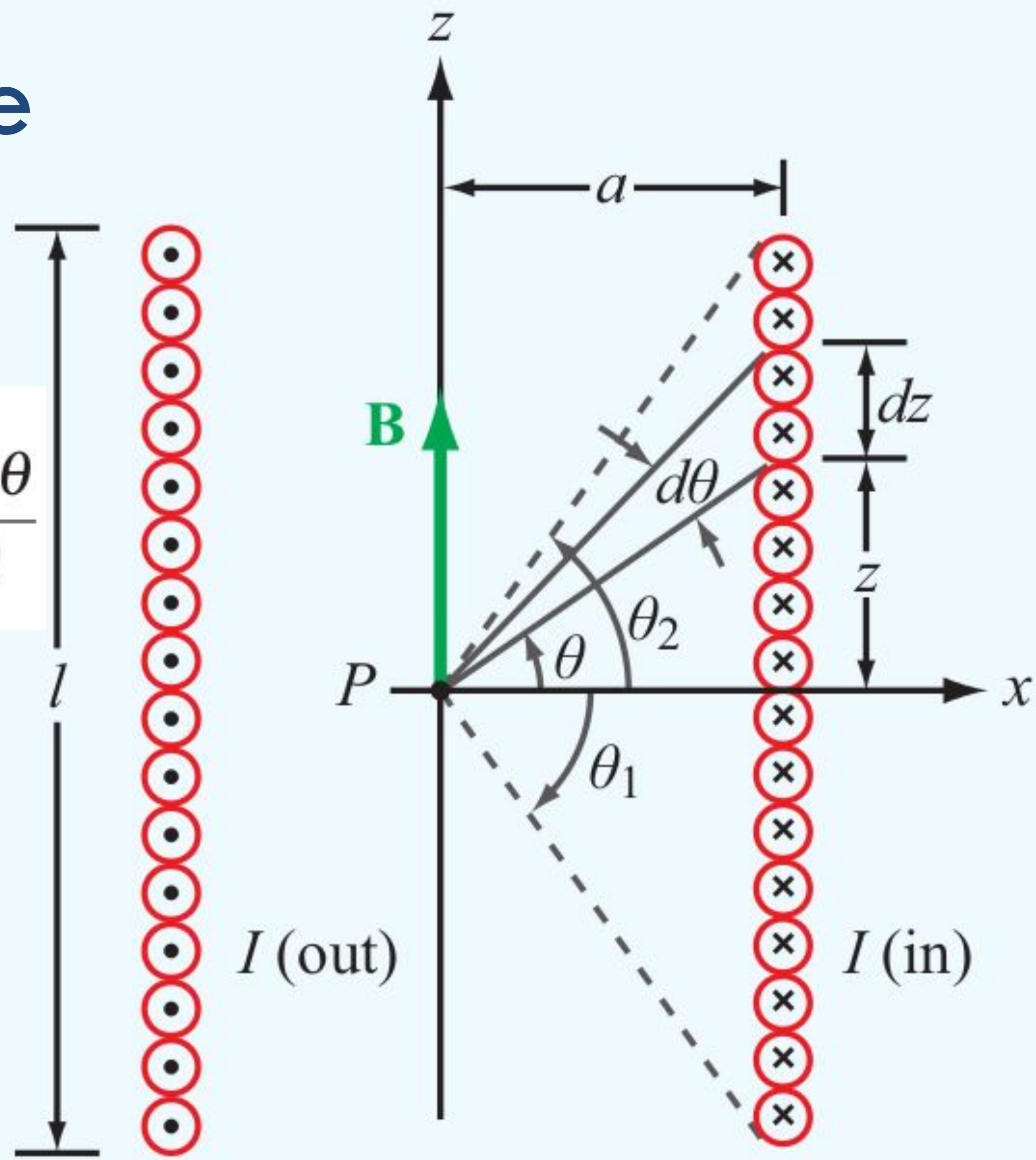
$$\mathbf{B} = \hat{\mathbf{z}} \frac{\mu n I a^2}{2} \int_{\theta_1}^{\theta_2} \frac{a \sec^2\theta \, d\theta}{a^3 \sec^3\theta}$$



5-7 Inductance

$$\mathbf{B} = \hat{\mathbf{z}}\,\frac{\mu n I a^2}{2}\int_{\theta_1}^{\theta_2}\frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta}$$

where we are integrating from θ_1 to θ_2: angles from the ends of the solenoid to the center.



(a) Cross section

5-7 Inductance

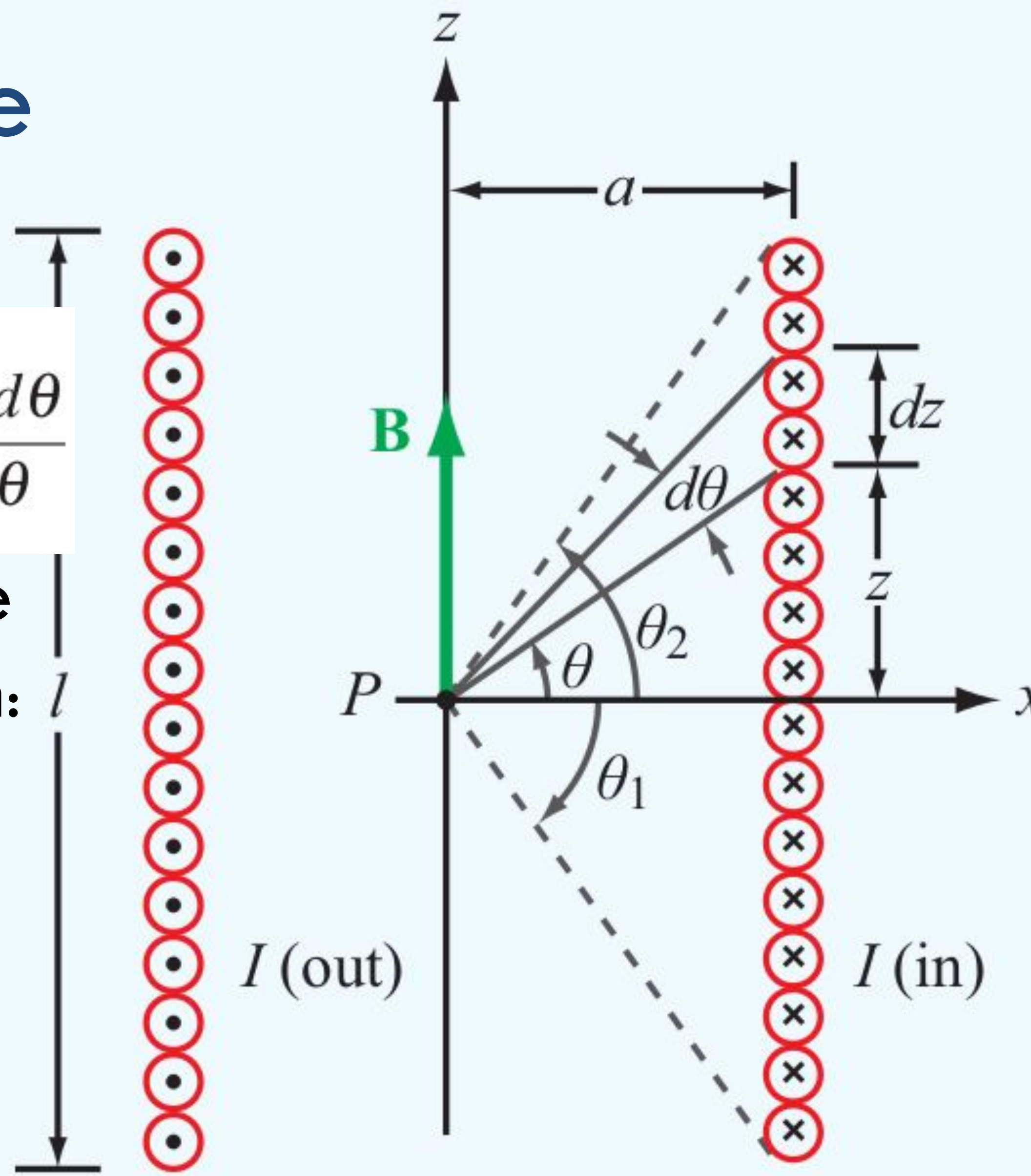
$$\mathbf{B} = \hat{\mathbf{z}} \frac{\mu n I a^2}{2} \int_{\theta_1}^{\theta_2} \frac{a \sec^2\theta \, d\theta}{a^3 \sec^3\theta}$$

move the a's outside the integral, integral is then:

$$= \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sec\theta}$$

$$= \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta$$

$$= \sin\theta_2 - \sin\theta_1$$



(a) Cross section

5-7 Inductance

$$\mathbf{B} = \hat{\mathbf{z}} \frac{\mu n I}{2} (\sin\theta_2 - \sin\theta_1)$$

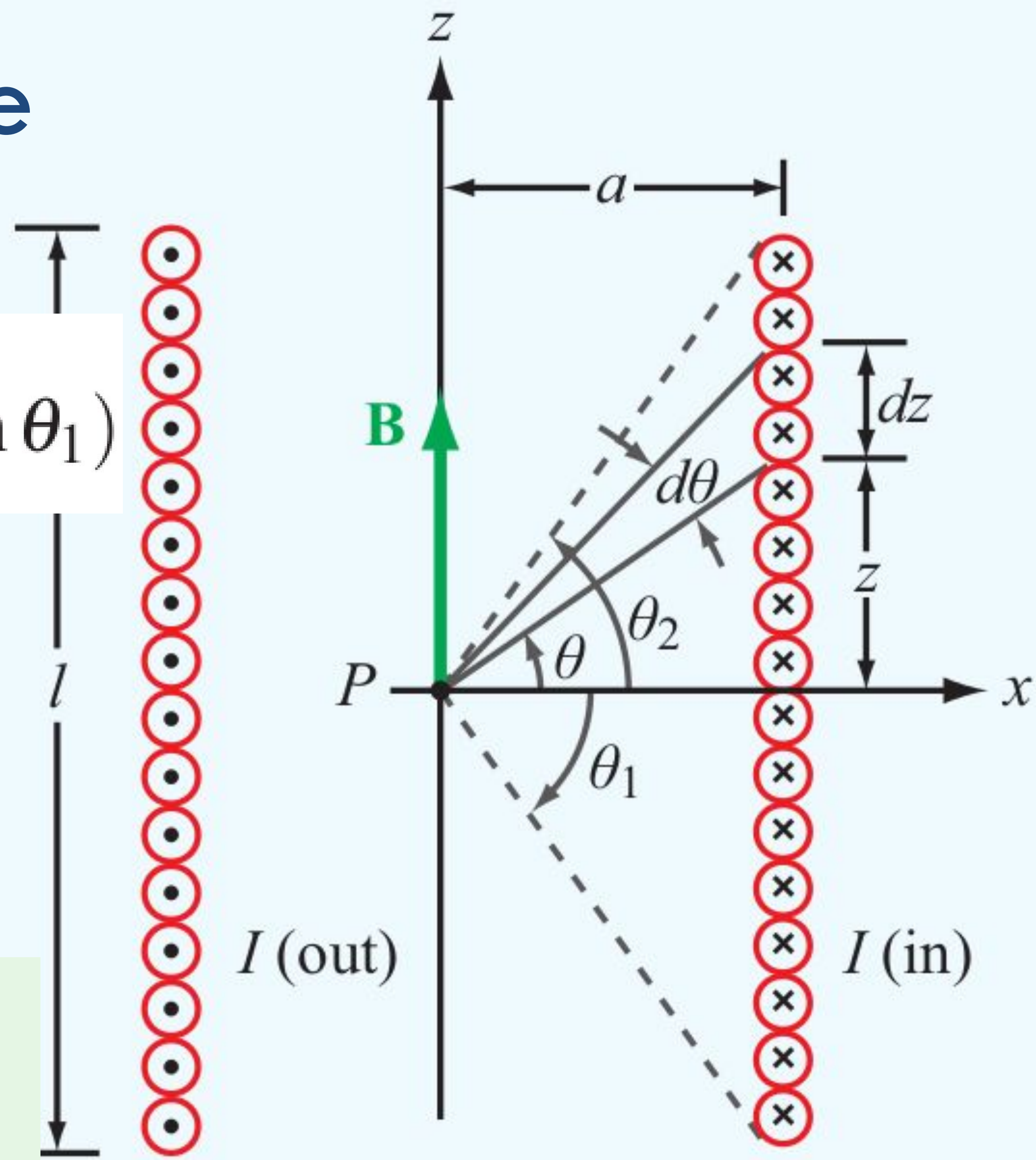
If assume solenoid is long, $l >> a$, then:

θ_1 = -90°

θ_2 = +90°

$$\mathbf{B} \approx \hat{\mathbf{z}} \mu n I = \frac{\hat{\mathbf{z}} \mu N I}{l}$$

(long solenoid with $l/a \gg 1$).



(a) Cross section

(Field at CENTER of solenoid)

5-7 Inductance

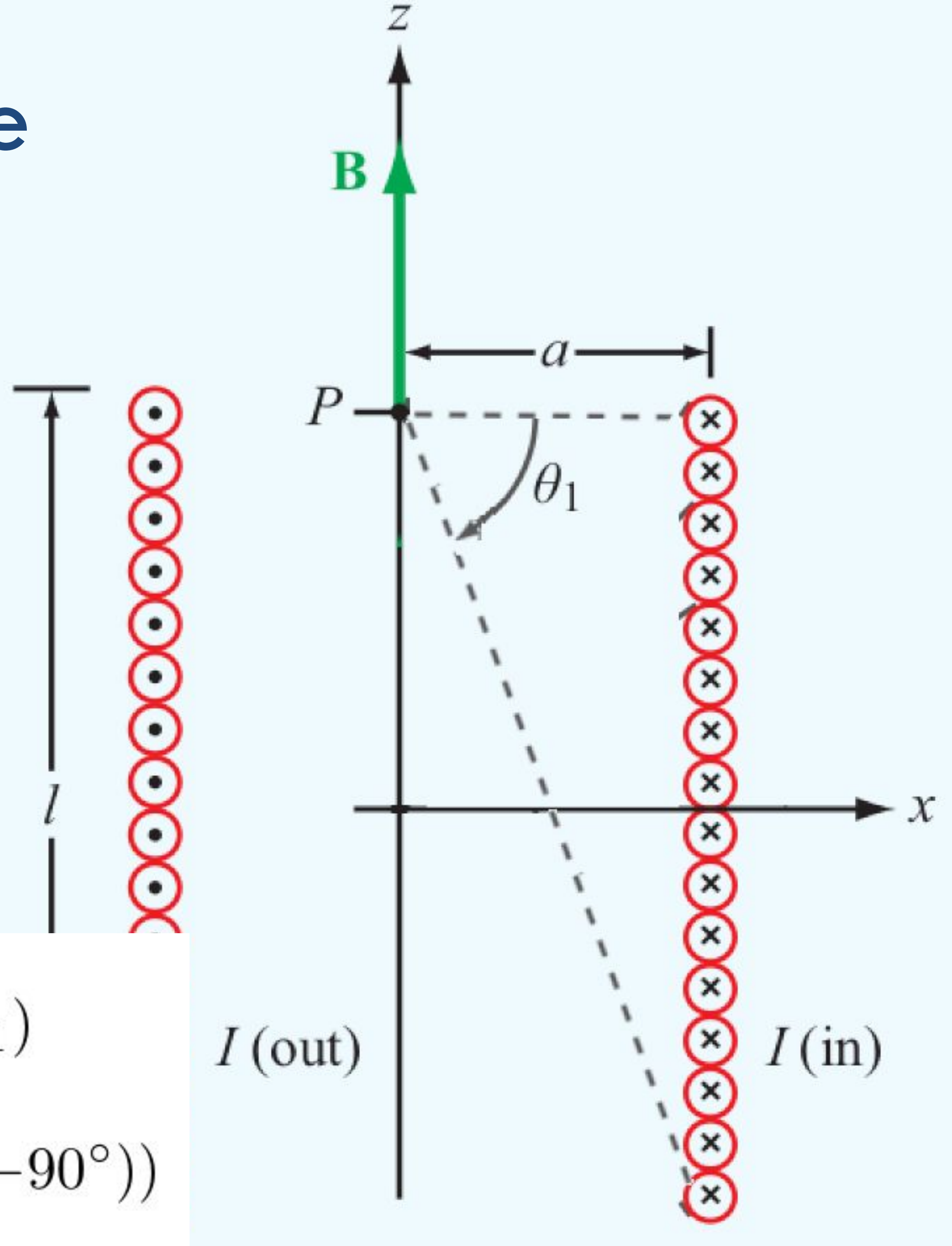
What is **B** at an edge of a long solenoid?

This changes the integration limits, so that:

$\theta_1 = -90°$
$\theta_2 = 0°$

$$\mathbf{B} = \hat{\mathbf{z}}\frac{\mu n I}{2}(\sin\theta_2 - \sin\theta_1)$$

$$\mathbf{B} = \hat{\mathbf{z}}\frac{\mu n I}{2}(\sin(0) - \sin(-90°))$$



5-7 Inductance

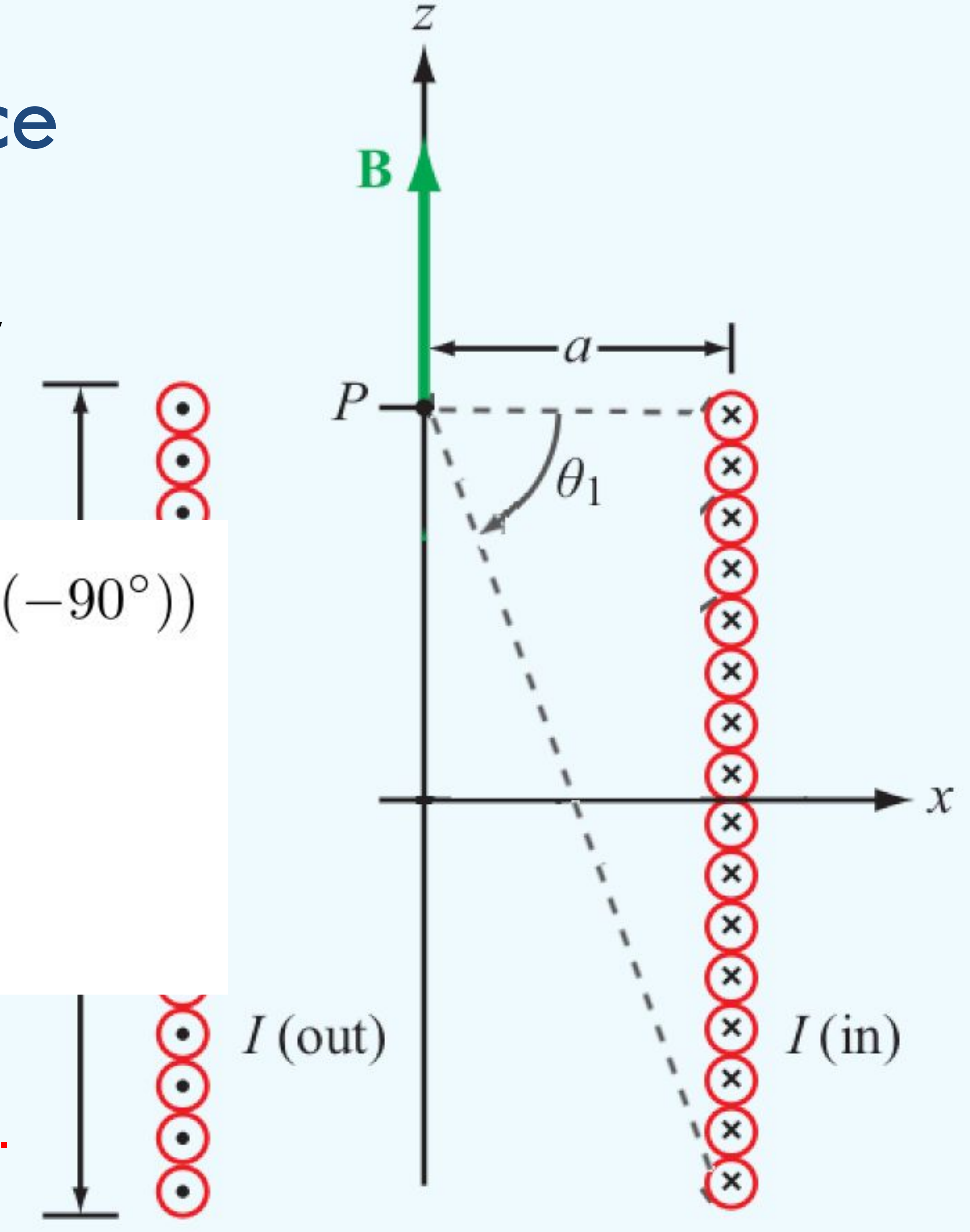
What is **B** at an edge of a long solenoid?

$$\mathbf{B} = \hat{\mathbf{z}}\frac{\mu n I}{2}(\sin(0) - \sin(-90^\circ))$$

$$\mathbf{B} = \hat{\mathbf{z}}\frac{\mu n I}{2}(0 - (-1))$$

$$\mathbf{B} = \hat{\mathbf{z}}\frac{\mu n I}{2}$$

Half the magnitude of the field at the center.



5-7 Inductance

What is **B** at any *z*?

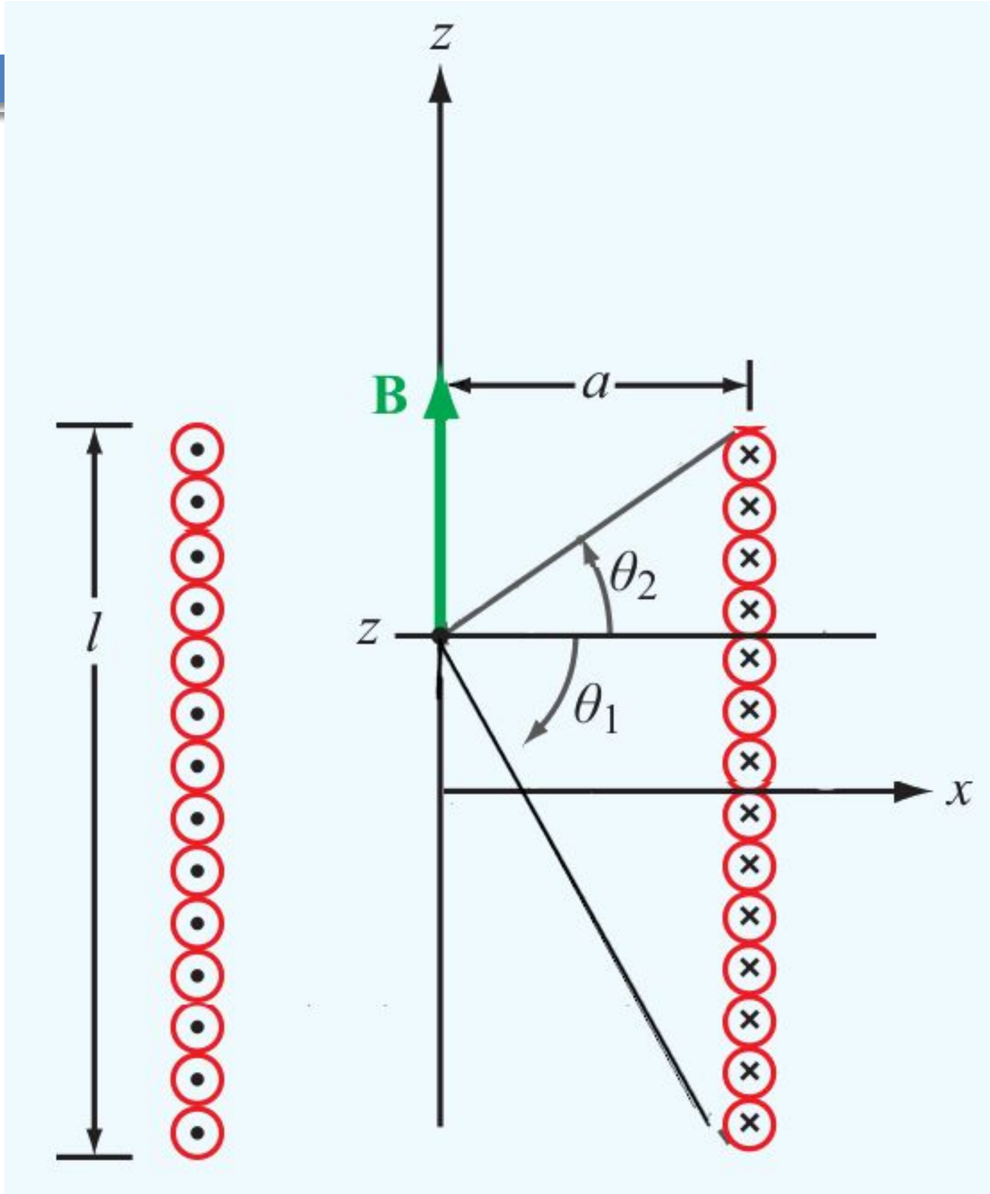
1. Choose an arbitrary *z*.

2. Draw angles

3. Use expression for **B**

4. Determine expressions for $\sin\theta_1$ and $\sin\theta_2$

5. Plug in.



5-7 Magnetic Flux

Recall definition of Magnetic Flux "Linking" a surface S:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb)}.$$

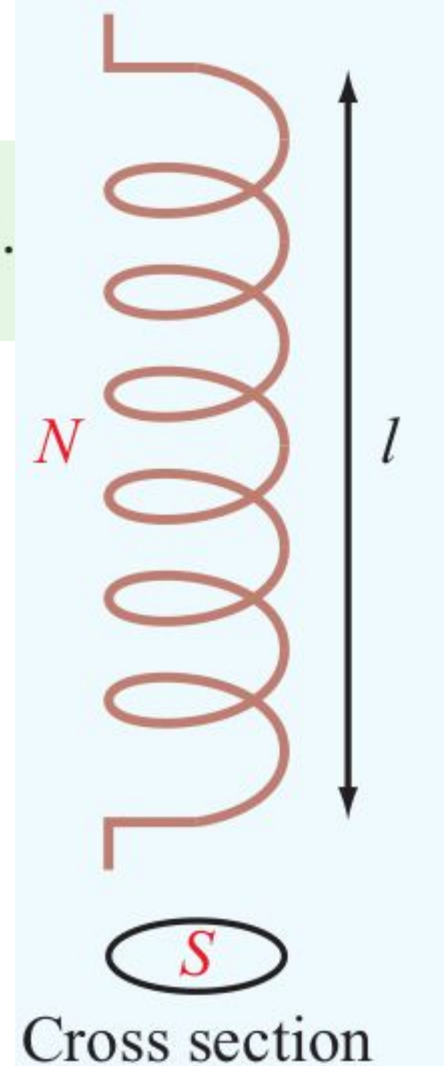
For a *single loop* of a long solenoid:

$$\Phi = \int_S \hat{\mathbf{z}} \left(\mu \frac{N}{l} I \right) \cdot \hat{\mathbf{z}} \, ds = \mu \frac{N}{l} IS$$

Total Magnetic Flux Linkage for a solenoid:

$$\Lambda = N\Phi = \mu \frac{N^2}{l} IS \qquad \text{(Wb)}$$

assuming **B** is uniform. *S*=Area



5-7 Self Inductance

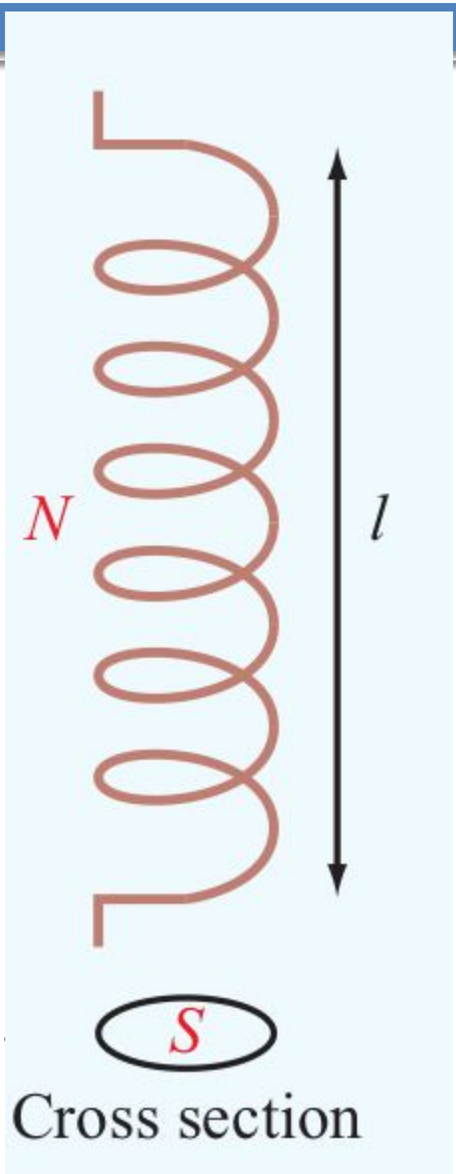
Definition of Self Inductance:

$$L = \frac{\Lambda}{I} \qquad \text{(H).}$$

For a solenoid:

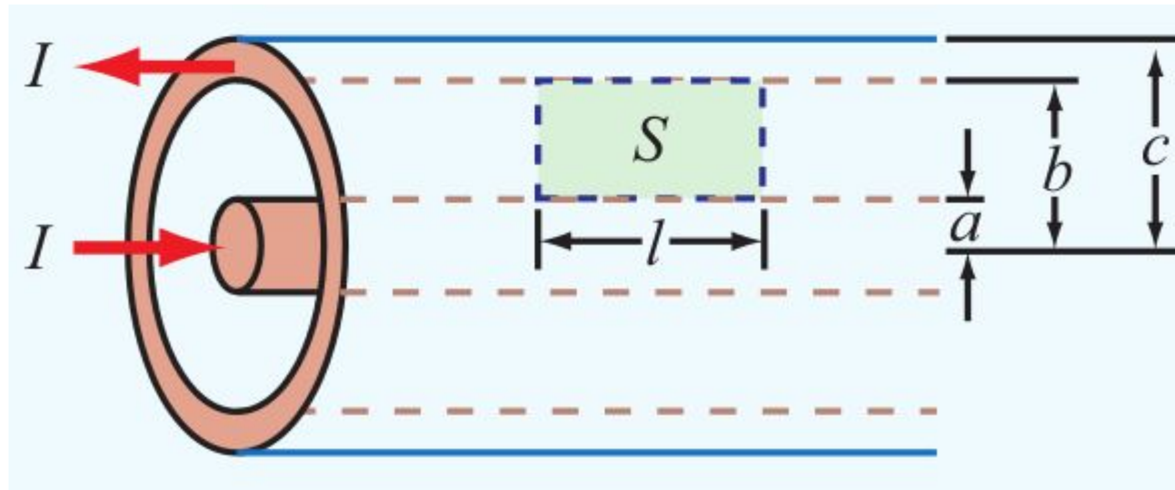
$$L = \mu \frac{N^2}{l} S$$

So, if a solenoid is used as an inductor, this would be the value of its inductance.



5-7 Coax Self Inductance

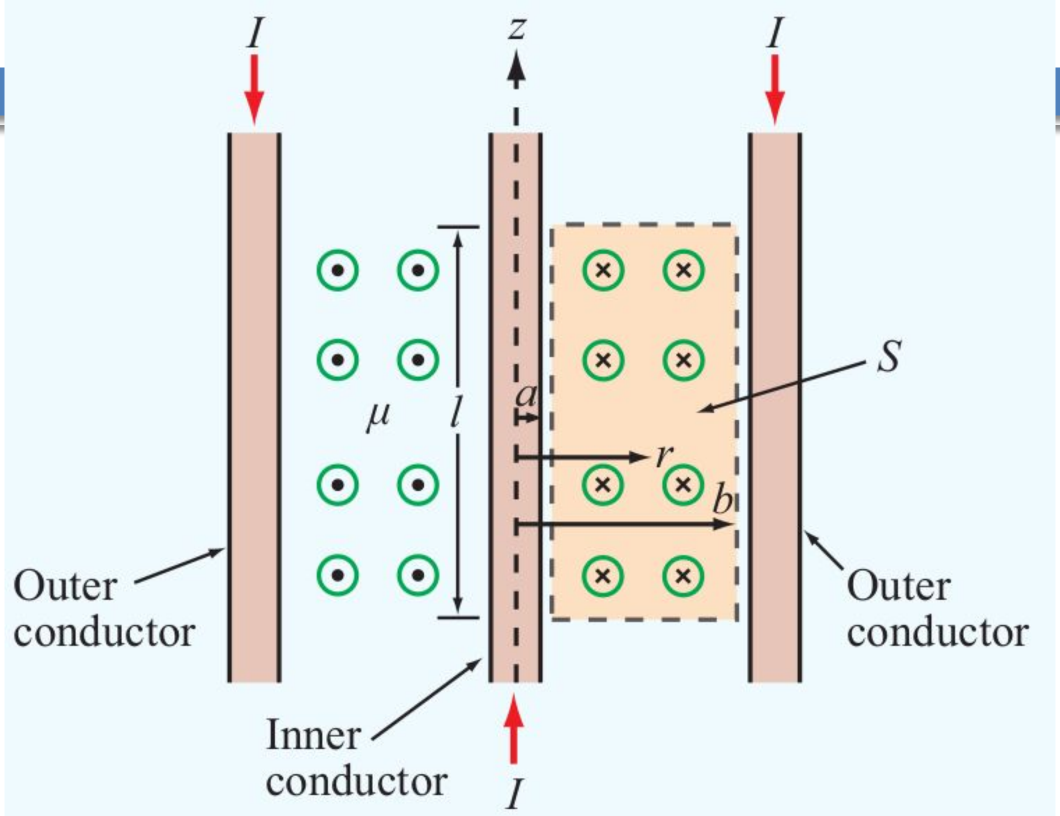
For a coax:
The magnetic field is directed in the $\hat{\phi}$ direction, so:



Choose the area, **S**, to be perpendicular to the field, and within the dielectric.

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I}\int_S \mathbf{B}\cdot d\mathbf{s}.$$

5-7 Coax Self Inductance



For a coax:

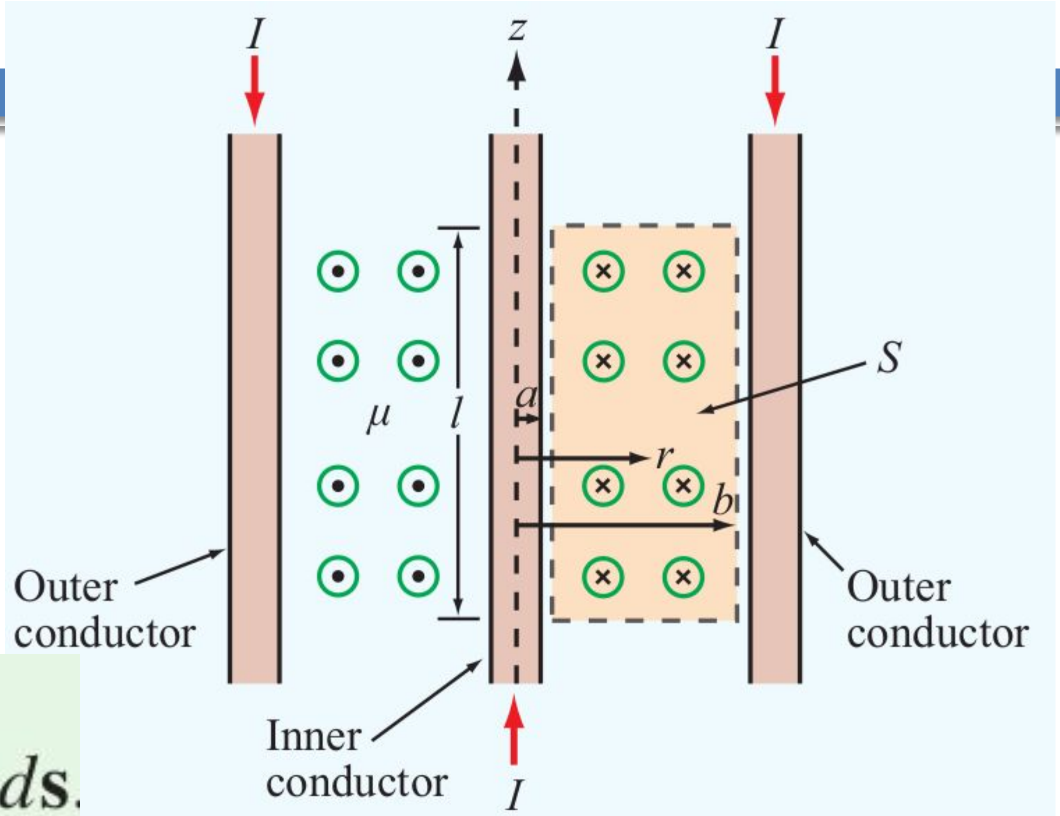
The magnetic field is due to both:

1. the current in the **inner** conductor
2. the current in the **outer** conductor.

Turns out: **B**(outer conductor) << **B**(inner conductor)

So: we can ignore it.

5-7 Coax Self Inductance



For a coax:

Approximately:

$$\mathbf{B} = \hat{\phi}\, \frac{\mu I}{2\pi r}$$

Plug into:

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s}$$

with: $d\mathbf{s} = \hat{\phi}\, dr\, dz$

since **B** does not depend on *z*, integral over *z* is just l

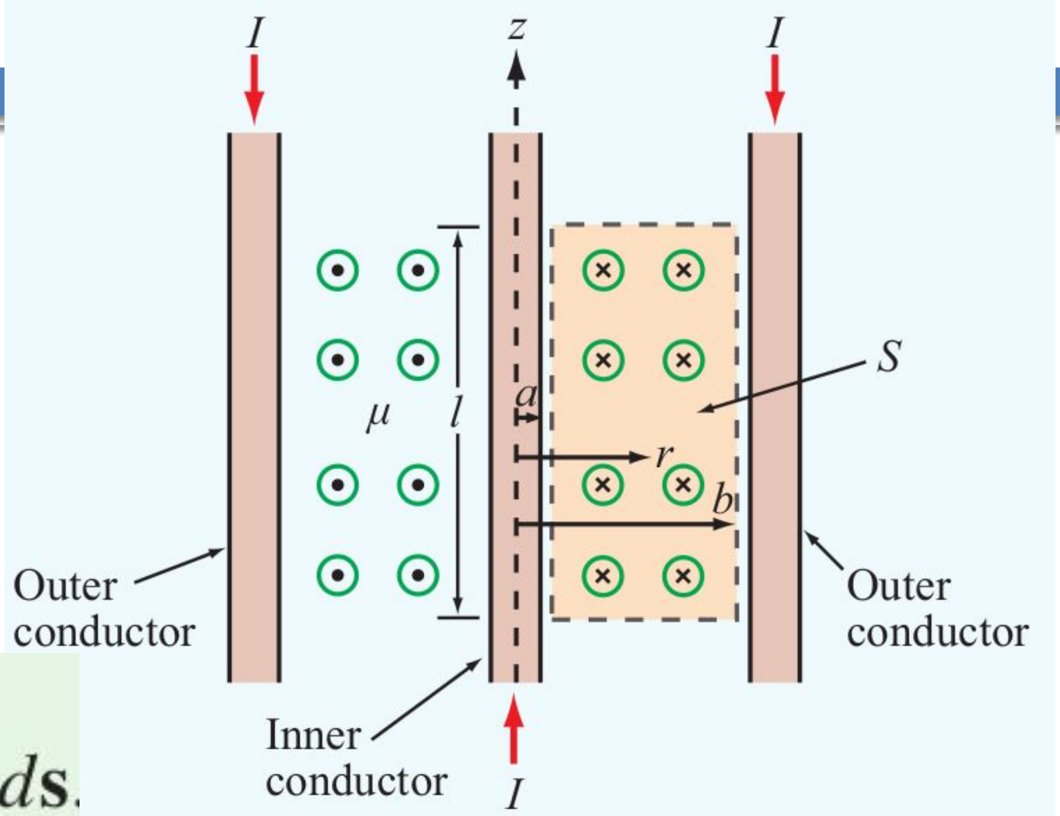
length

5-7 Coax Self Inductance

For a coax:

Approximately:

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\, \frac{\mu I}{2\pi r}$$



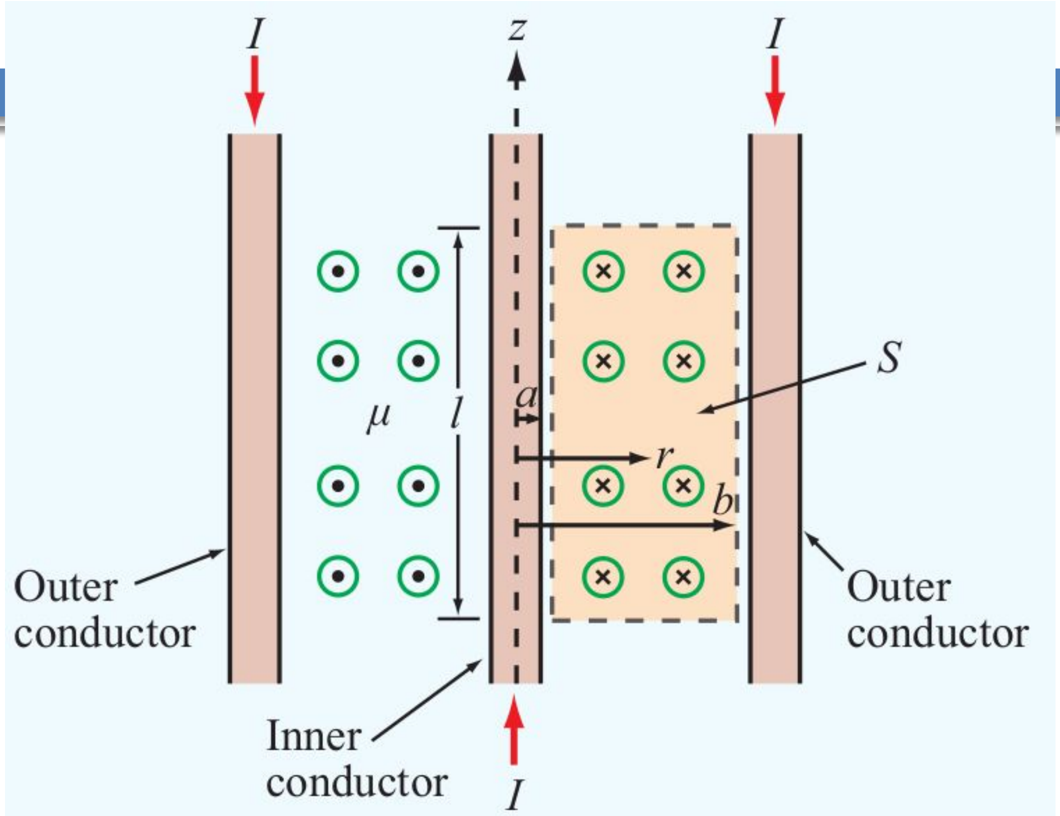
Plug into:

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s}.$$

get:

$$\Phi = l \int_a^b B\, dr = l \int_a^b \frac{\mu I}{2\pi r}\, dr = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

5-7 Coax Self Inductance



For a coax:

Since:

$$\Phi = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

get:

$$L' = \frac{L}{l} = \frac{\Phi}{lI} = \boxed{\frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)}$$

(As given in chapter 2)

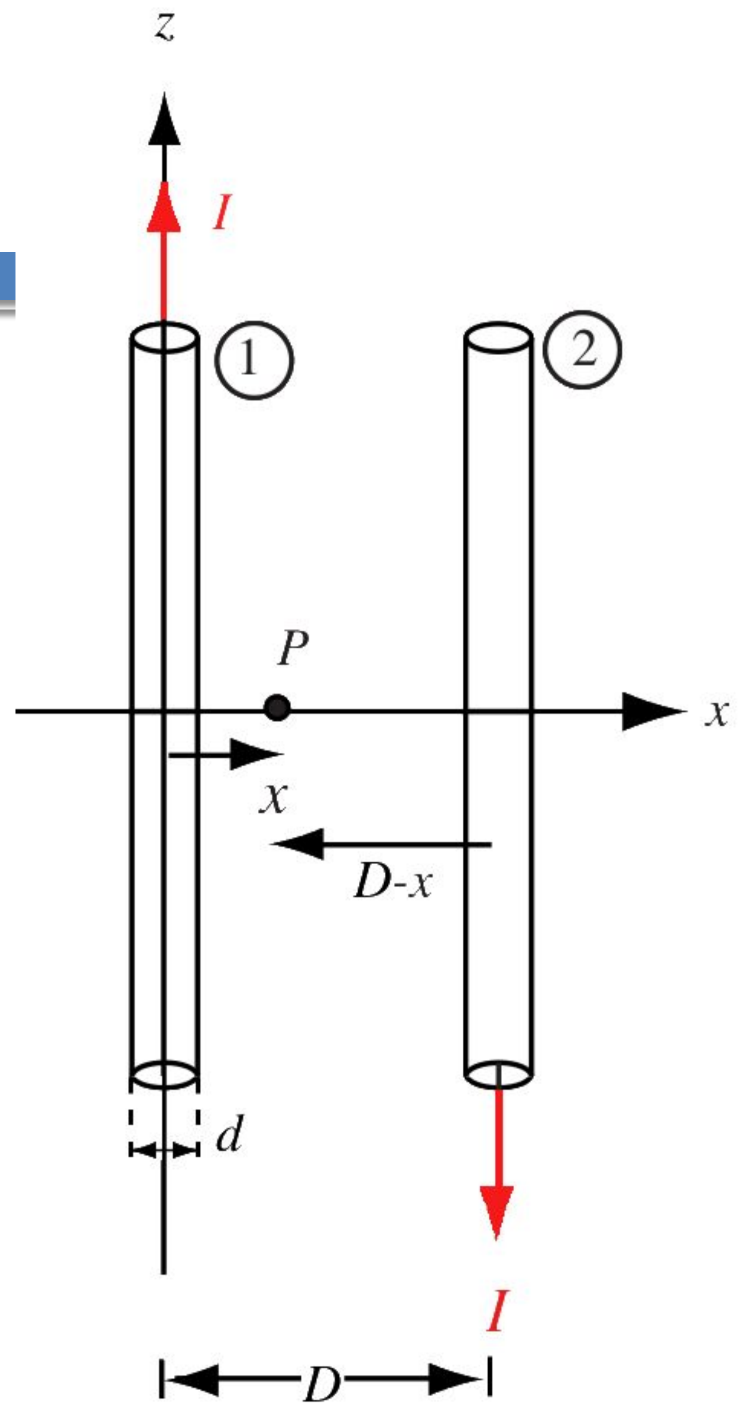
5-7 Two-Wire Self Inductance

For a 2-wire transmission-line:
wire diameter: *d*
wire center separation: *D*

Choose to:
orient wires along *z*-axis,
separate wires along *x*-axis,
y-axis into page

Use superposition to find **B** field at point *P*, between the wires

B = **B**(wire1) + **B**(wire2)



5-7 Two-Wire Self Inductance

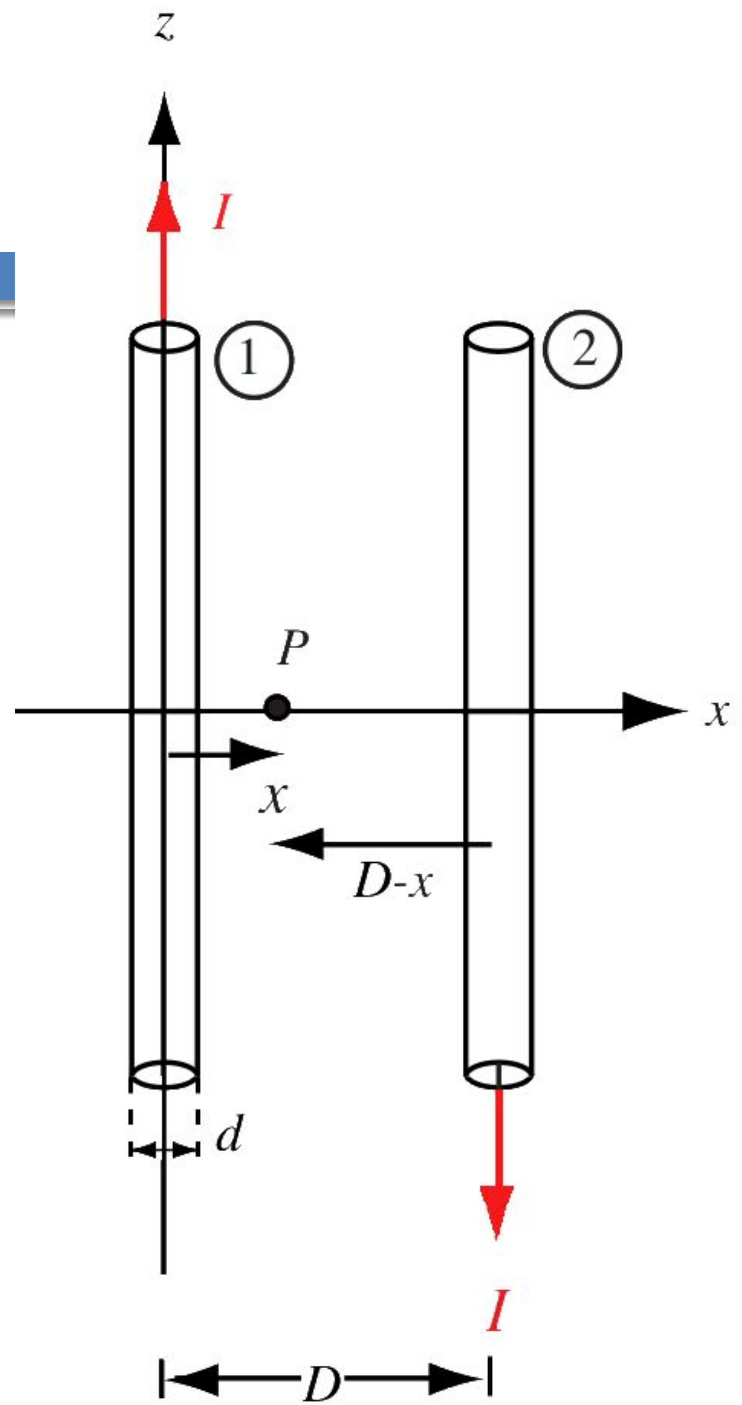
For a 2-wire transmission-line:
previously solved for a wire with current going up *z*-axis:

$$\mathbf{B} = \hat{\phi}\frac{\mu I}{2\pi r}$$

adapt to this situation:

$$\mathbf{B}(\text{wire1}) = \hat{\mathbf{y}}\frac{\mu I}{2\pi x}$$

$$\mathbf{B}(\text{wire2}) = \hat{\mathbf{y}}\frac{\mu I}{2\pi(D-x)}$$



5-7 Two-Wire Self Inductance

For a 2-wire transmission-line:

Hence:

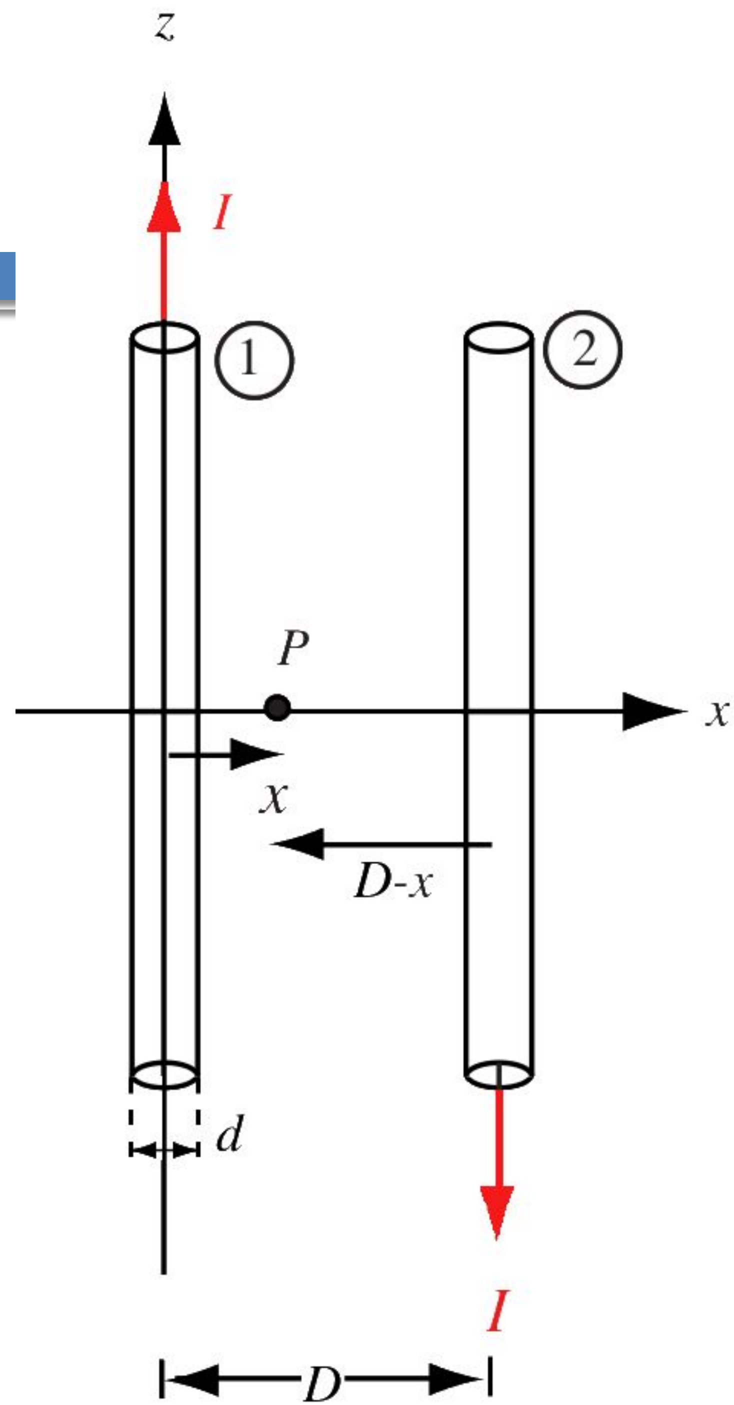
$$\mathbf{B} = \hat{\mathbf{y}}\frac{\mu I}{2\pi x} + \hat{\mathbf{y}}\frac{\mu I}{2\pi(D-x)}$$

$$\mathbf{B} = \hat{\mathbf{y}}\frac{\mu I D}{2\pi x(D-x)}$$

Since the self-inductance:

$$L' = \frac{L}{l} = \frac{\Phi}{lI}$$

Need to find Φ



5-7 Two-Wire Self Inductance

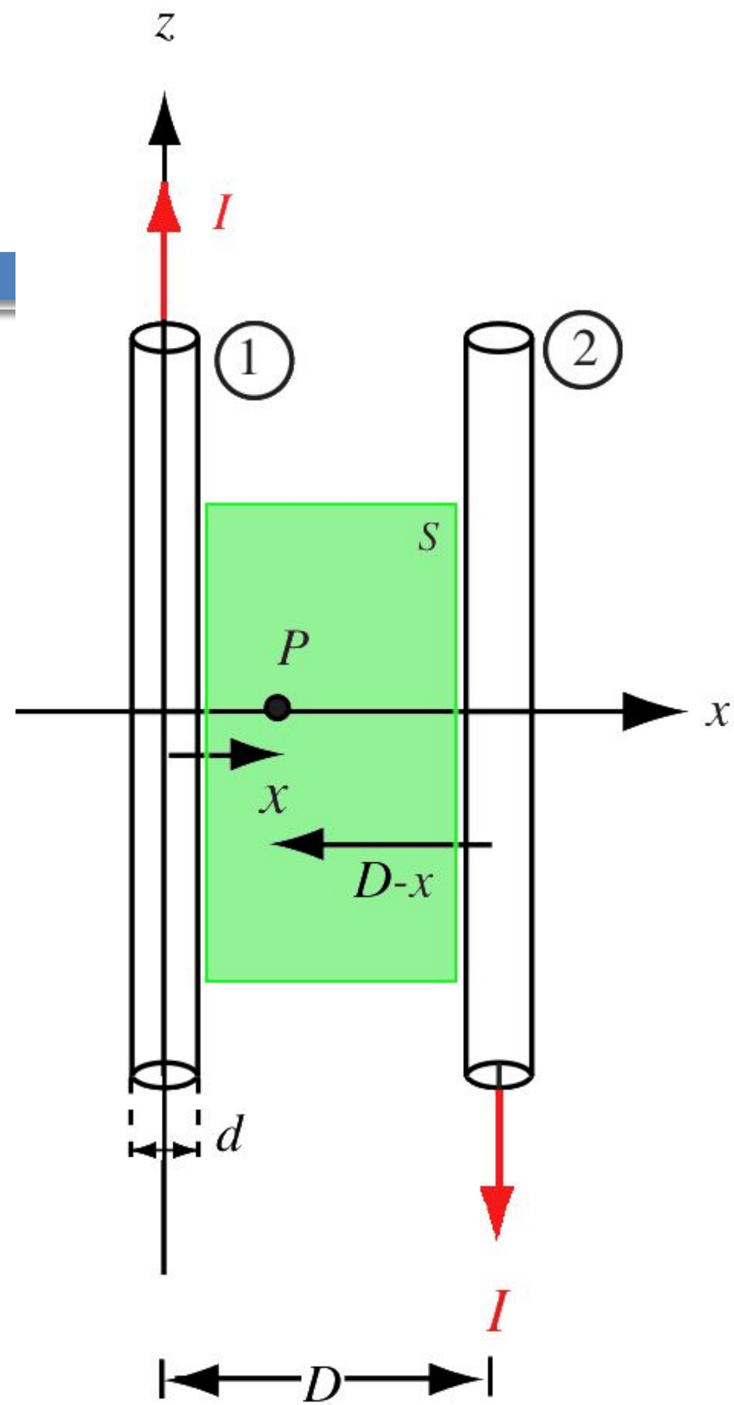
For a 2-wire transmission-line:

Calculating Φ:

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s}$$

Choose *S* to be a region between the 2 wires: Normal to **B.**

$$d\mathbf{s} = \hat{\mathbf{y}}\, dx\, dz$$



5-7 Two-Wire Self Inductance

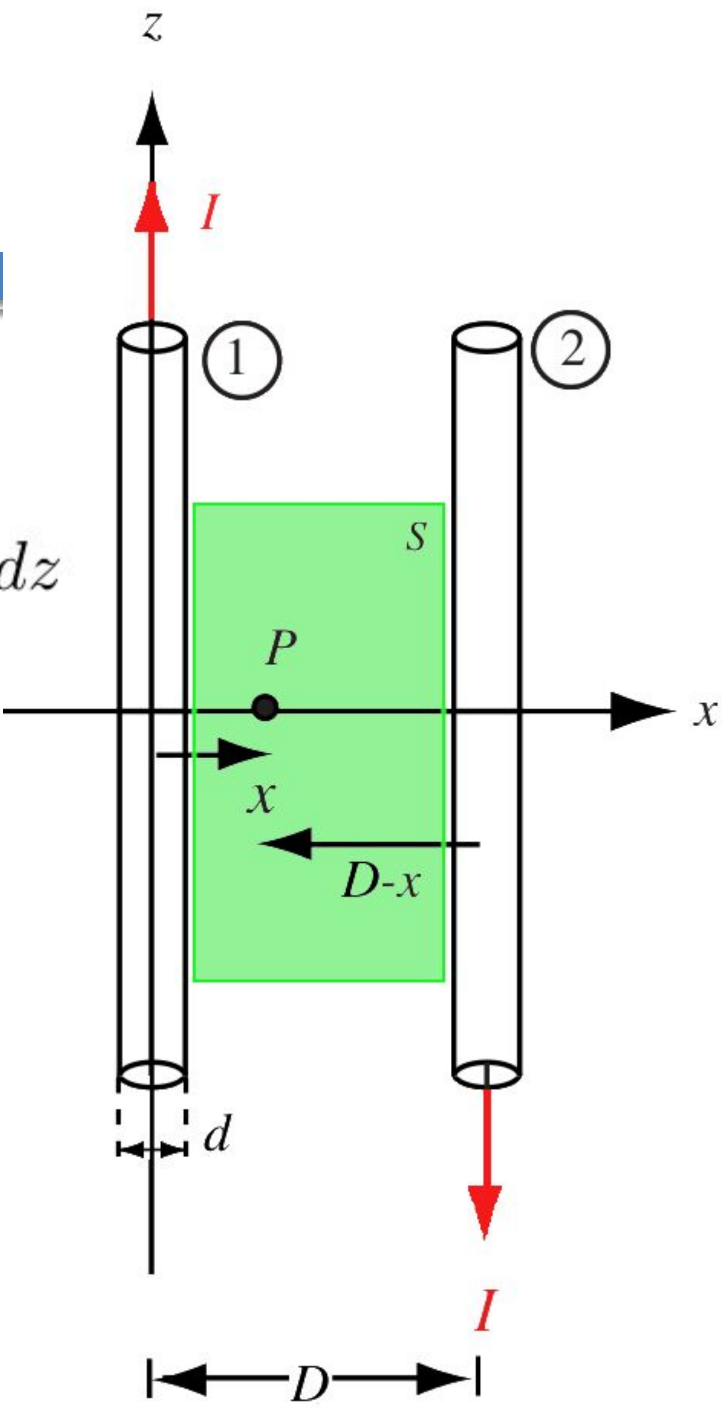
For a 2-wire transmission-line:

$$\Phi = \int_{z=0}^{l} \int_{x=d/2}^{D-d/2} \hat{\mathbf{y}} \frac{\mu I D}{2\pi x(D-x)} \cdot \hat{\mathbf{y}} dx\, dz$$

integrate over a length *l* of the transmission line, and along *x* between the 2 wires.

Simplify:

$$\Phi = \frac{\mu I D}{2\pi} \int_{x=d/2}^{D-d/2} \frac{dx}{x(D-x)} \int_{z=0}^{l} dz$$



5-7 Two-Wire Self Inductance

For a 2-wire transmission-line:

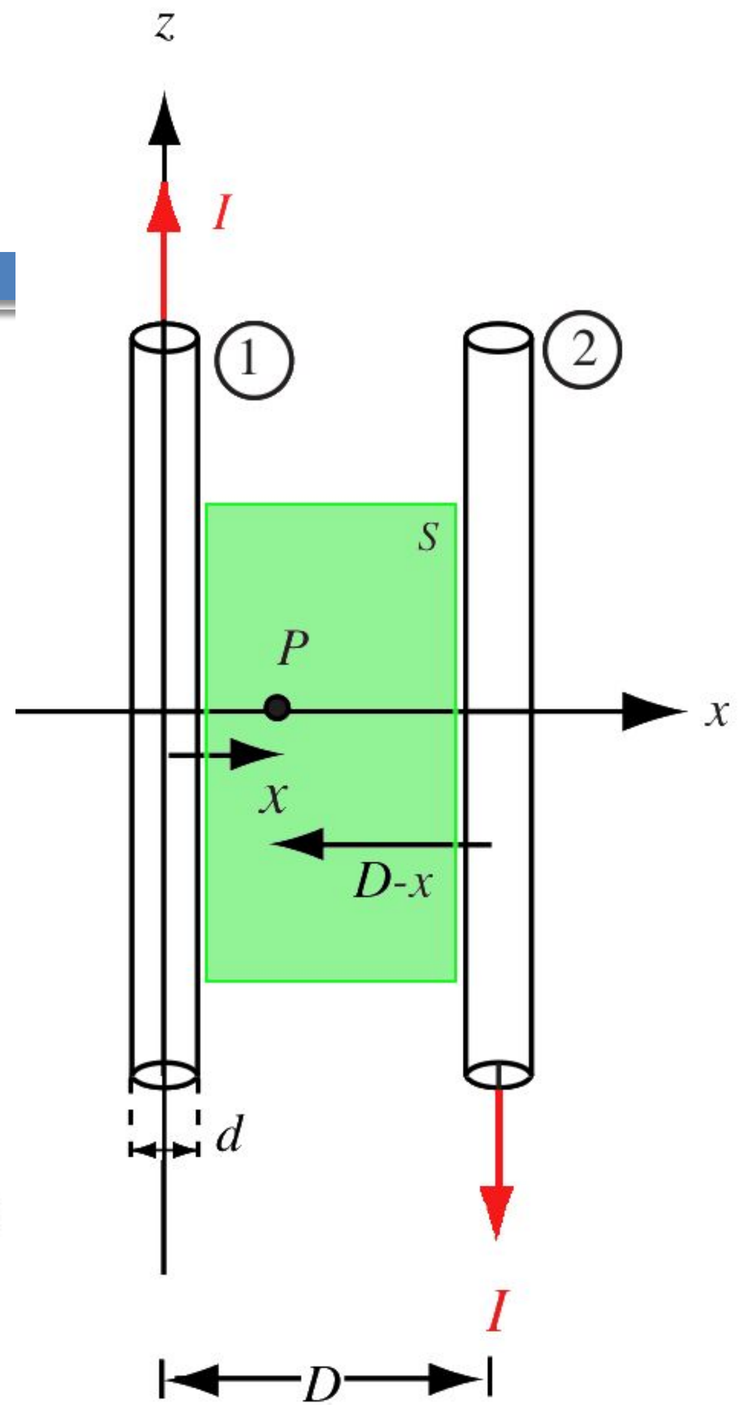
$$\Phi = \frac{\mu I D}{2\pi} \int_{x=d/2}^{D-d/2} \frac{dx}{x(D-x)} \int_{z=0}^{l} dz$$

From an integral table:

$$\int \frac{dx}{x(a+bx)} = -\frac{1}{a} \ln \frac{a+bx}{x}$$

with *a*=*D*, *b*=-1, giving:

$$\Phi = \frac{\mu I D l}{2\pi} \left[-\frac{1}{D} \ln \frac{D-x}{x} \right] \Bigg|_{x=d/2}^{D-d/2}$$



5-7 Two-Wire Self Inductance

For a 2-wire transmission-line:

$$\Phi = \frac{\mu I D l}{2\pi}\left[-\frac{1}{D}\ln\frac{D-x}{x}\right]\Bigg|_{x=d/2}^{D-d/2}$$

$$\Phi = \frac{\mu I D l}{2\pi}\frac{1}{D}\ln\frac{x}{D-x}\Bigg|_{x=d/2}^{D-d/2}$$

$$\Phi = \frac{\mu I l}{2\pi}\ln\frac{x}{D-x}\Bigg|_{x=d/2}^{D-d/2}$$

$$\Phi = \frac{\mu I l}{2\pi}\left[\ln\frac{D-d/2}{D-D+d/2} - \ln\frac{d/2}{D-d/2}\right]$$

5-7 Two-Wire Self Inductance

For a 2-wire transmission-line:

$$\Phi = \frac{\mu I l}{2\pi}\left[\ln\frac{D-d/2}{D-D+d/2} - \ln\frac{d/2}{D-d/2}\right]$$

$$\Phi = \frac{\mu I l}{2\pi}\left[\ln\frac{D-d/2}{d/2} + \ln\frac{D-d/2}{d/2}\right]$$

$$\Phi = \frac{\mu I l}{2\pi}\left[2\ln\frac{D-d/2}{d/2}\right]$$

assume D>>d:

$$\Phi = \frac{\mu I l}{\pi}\left[\ln\frac{2D}{d}\right]$$

5-7 Two-Wire Self Inductance

For a 2-wire transmission-line:

$$\Phi = \frac{\mu I l}{\pi}\left[\ln\frac{2D}{d}\right]$$

recall:

$$L' = \frac{L}{l} = \frac{\Phi}{lI}$$

so:

$$L' = \frac{\mu}{\pi}\ln\left(\frac{2D}{d}\right)$$

(D>>d)

5-7 Mutual Inductance

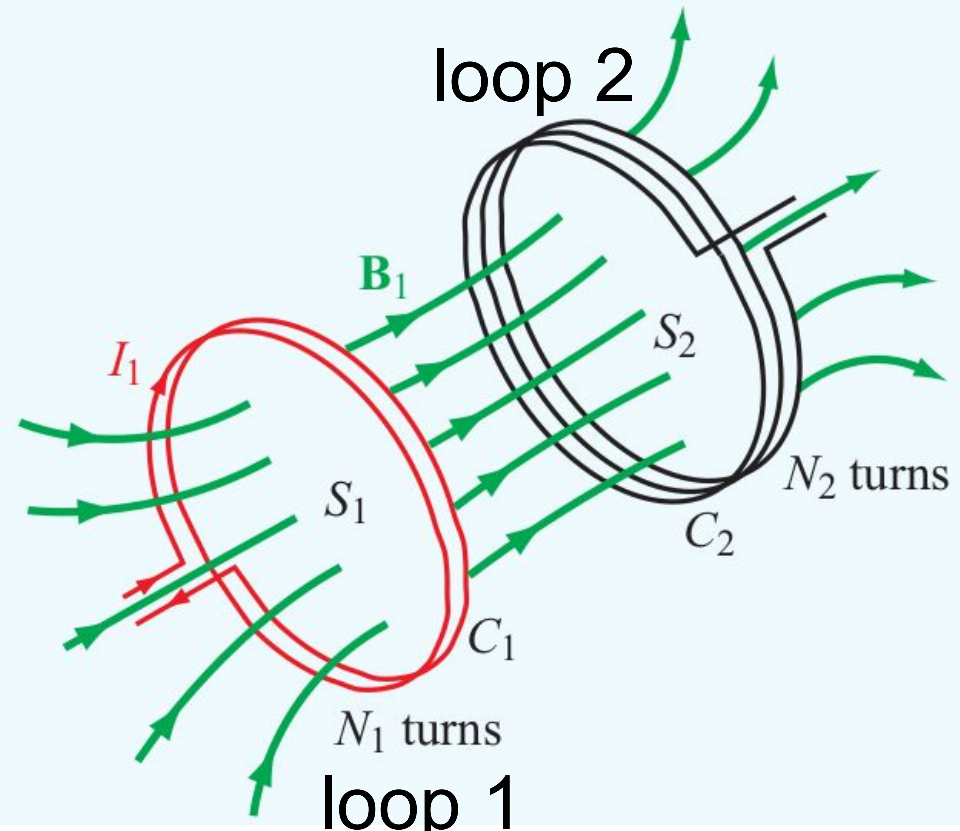
For 2 loops near each other:

No current in Loop2.
Current in Loop1
produces field B_1
which interacts with Loop2

Using:

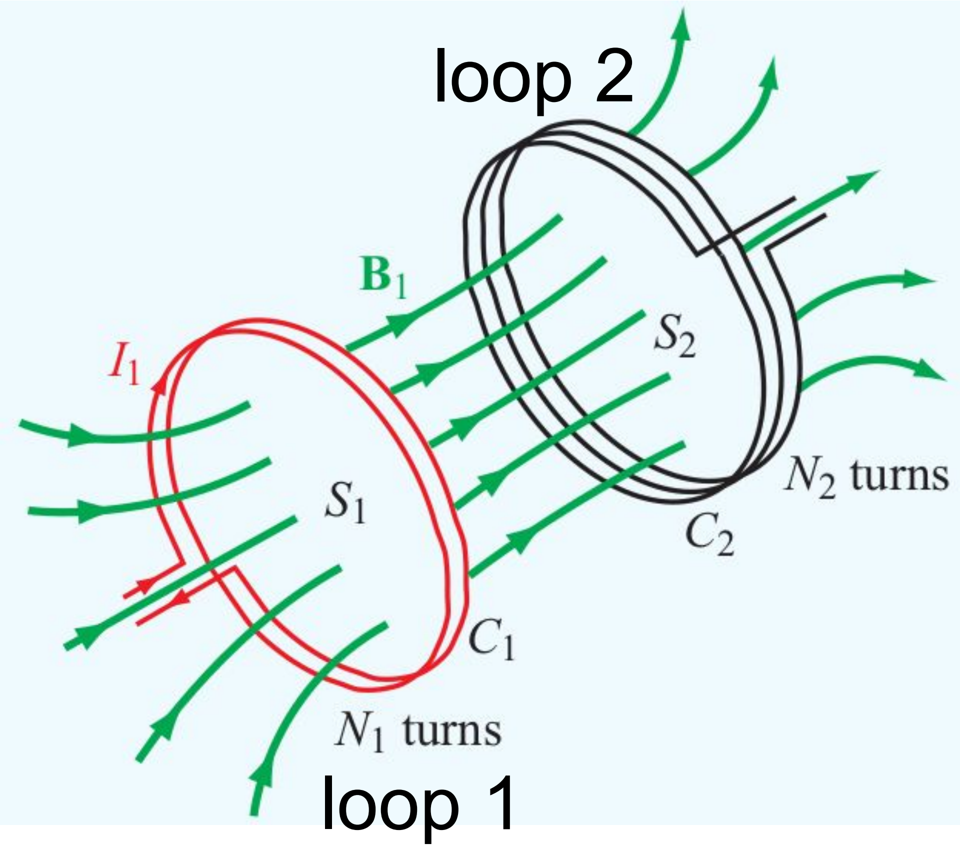
$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb)}.$$

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}.$$



Φ_{12}: Flux "linking" one turn of Loop2 due to:
field produced by Loop1

5-7 Mutual Inductance



For 2 loops near each other:
Flux "linking" one turn of Loop2 due to field produced by Loop1:

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}.$$

Total Magnetic Flux Linkage:

$$\Lambda_{12} = N_2 \Phi_{12} = N_2 \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}.$$

5-7 Mutual Inductance

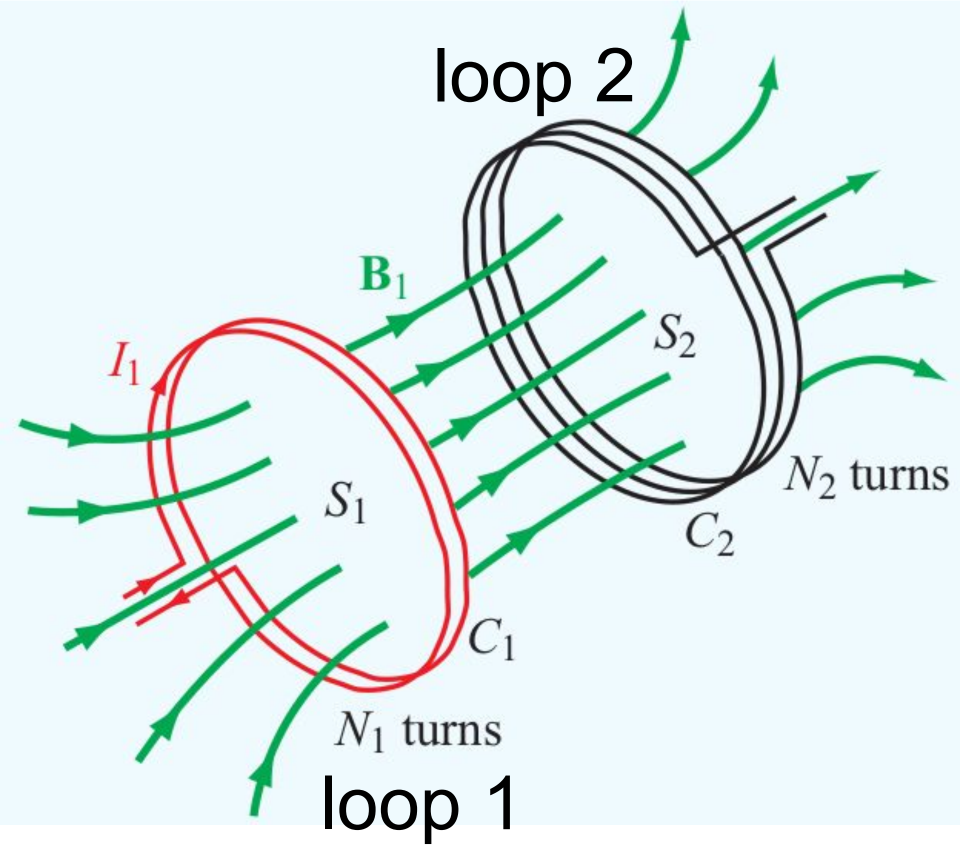
For 2 loops near each other:

Total Magnetic Flux Linkage:

$$\Lambda_{12} = N_2 \Phi_{12} = N_2 \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}.$$

Mutual Inductance:

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}$$



Example 5-9: Mutual Inductance

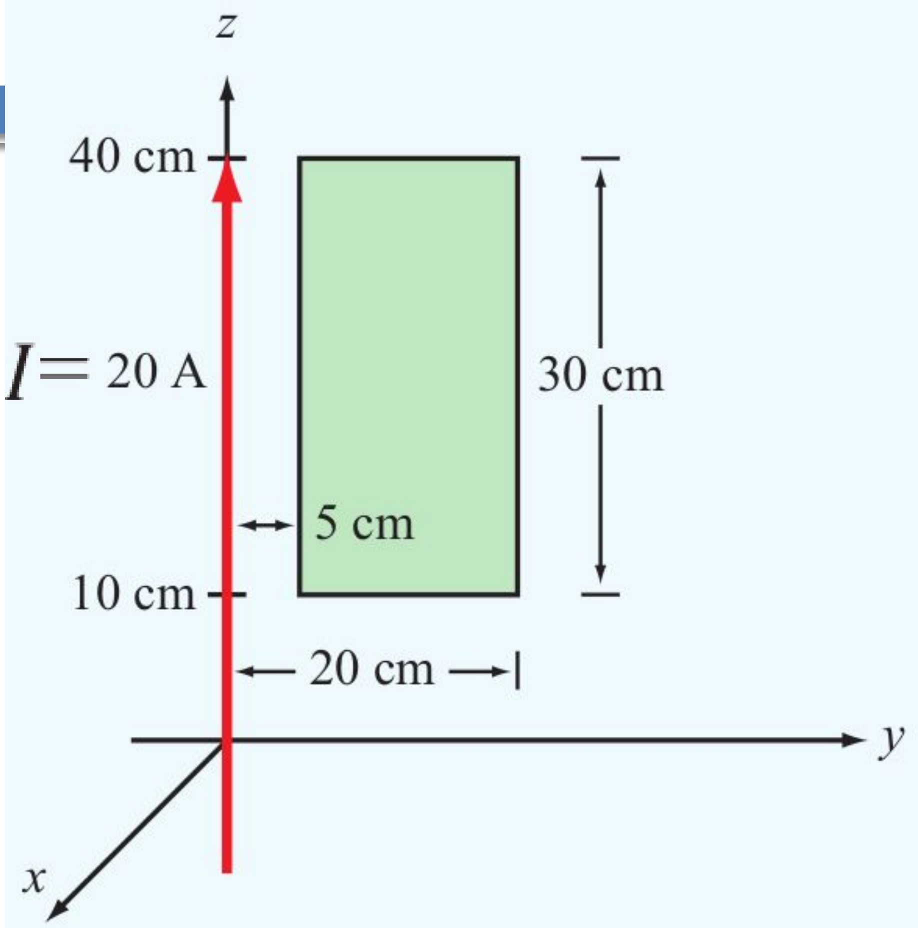
For a wire near a loop:
Current in wire,
not in loop.

General formula:

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}$$

Field at loop due to wire:

$$\mathbf{B}_1 = -\hat{\mathbf{x}} \frac{\mu_0 I}{2\pi y}$$



Differential area: normal points in same direction as **B**:

$$d\mathbf{s} = -\hat{\mathbf{x}}\, dy\, dz$$

Example 5-9: Mutual Inductance

For a wire near a loop:
Current in wire,
not in loop.

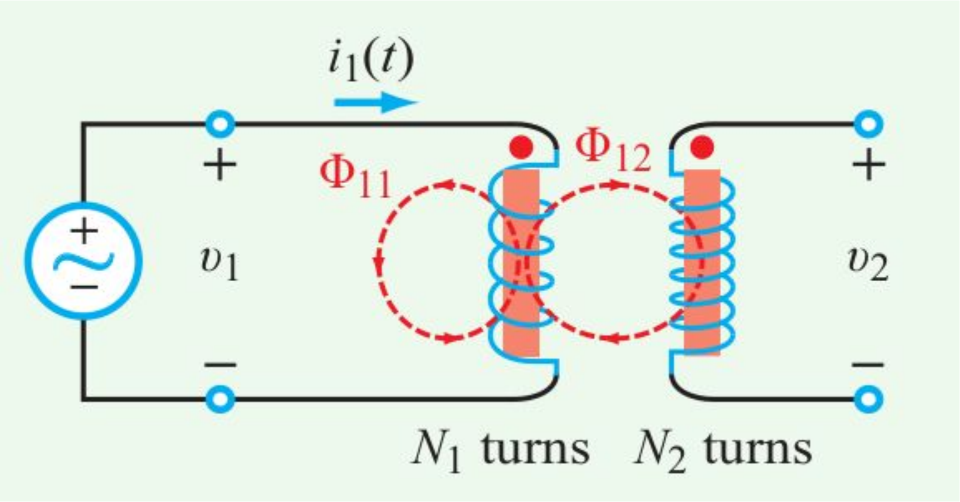
$$L_{12} = \frac{1}{I}\int_{y=0.05}^{0.20}\int_{z=0.1}^{0.4}\left(-\hat{\mathbf{x}}\,\frac{\mu_0 I}{2\pi y}\right)\cdot(-\hat{\mathbf{x}}\,dy\,dz)$$

$$= \frac{\mu_0}{2\pi}\left(\int_{0.05}^{0.20}\frac{dy}{y}\right)\left(\int_{0.1}^{0.4}dz\right)$$

$$= \frac{0.3\mu_0}{2\pi}\,(\ln y)\Big|_{0.05}^{0.20} = \frac{0.3\mu_0}{2\pi}\ln\left(\frac{0.2}{0.05}\right) = \boxed{83\text{ nH}}.$$

Lab5: Mutual Inductance in Circuits

2 nearby coils of wire:



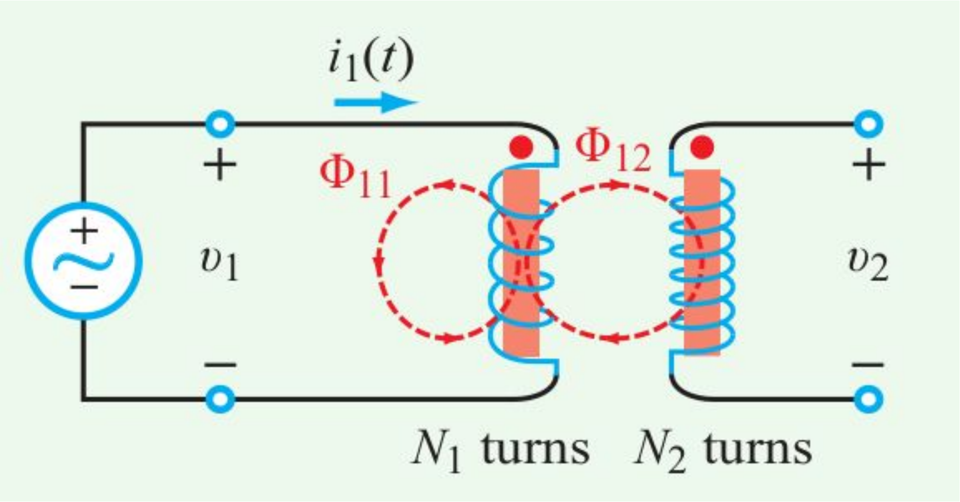
Current i_1 induces magnetic flux Φ_{11} linking coil 1 alone and induces magnetic flux Φ_{12} linking both coils.

Total magnetic flux through coil 1:

$$\Phi_1 = \Phi_{11} + \Phi_{12}.$$

Lab5: Mutual Inductance in Circuits

2 nearby coils of wire:



Self inductance of coil 1:

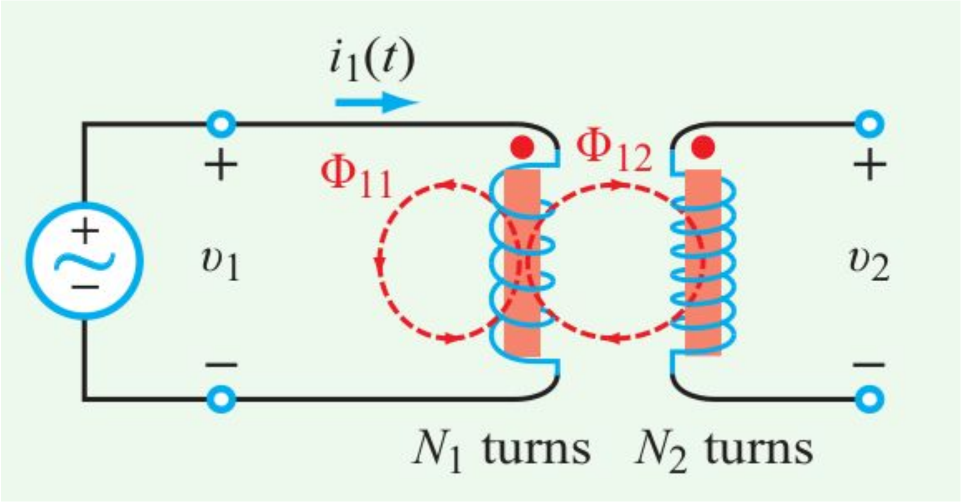
$$L_1 = \frac{\Lambda_1}{i_1}$$

induced voltage across coil 1:

$$v_1 = \frac{d\Lambda_1}{dt} = L_1 \frac{di_1}{dt}$$

Lab5: Mutual Inductance in Circuits

2 nearby coils of wire:



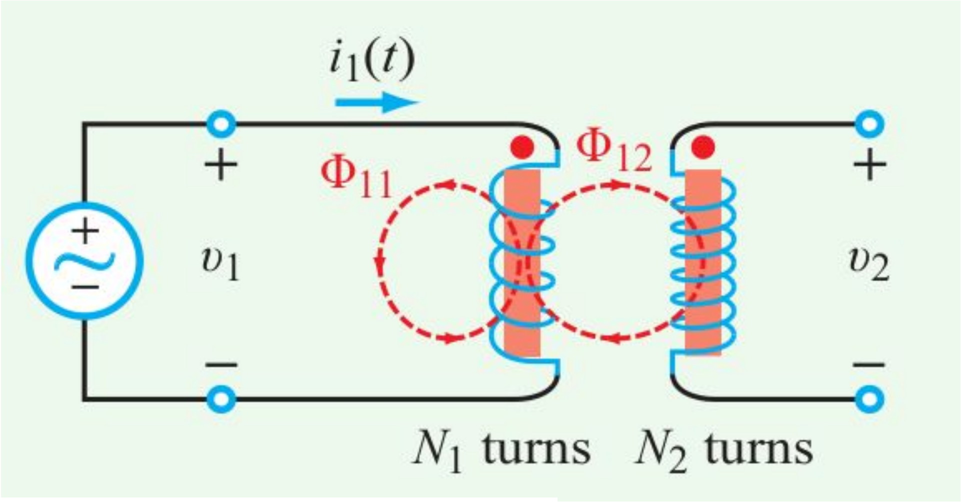
Similarly for coil 2:

$$v_2 = \frac{d\Lambda_2}{dt} = N_2 \frac{d\Phi_{12}}{dt}$$

How does v_2 depend on di_1/dt?

Lab5: Mutual Inductance in Circuits

2 nearby coils of wire:



Rewrite:

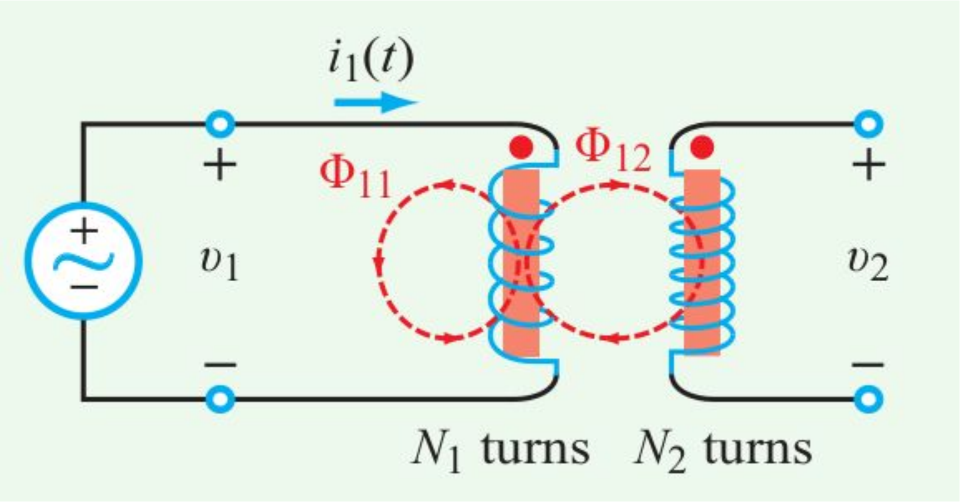
$$v_2 = N_2 \frac{d\Phi_{12}}{di_1} \times \frac{di_1}{dt}$$

Define Mutual Inductance:

$$M_{21} = N_2 \frac{d\Phi_{12}}{di_1}$$

Lab5: Mutual Inductance in Circuits

2 nearby coils of wire:



So:

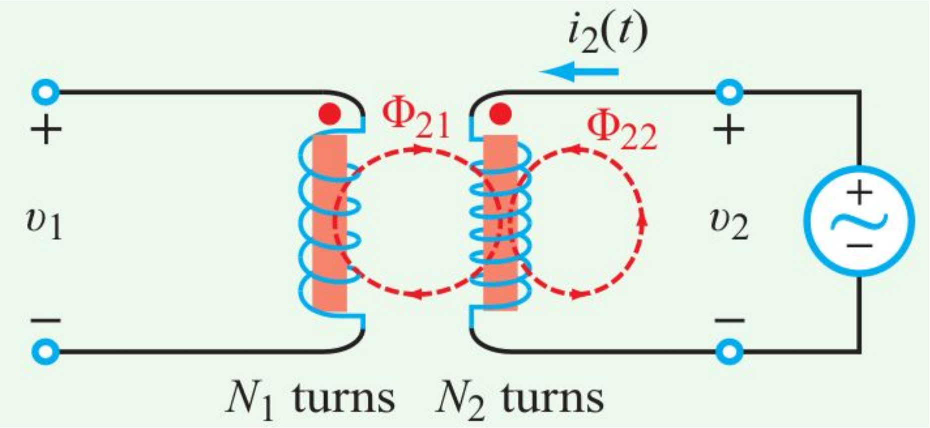
$$v_2 = \pm M_{21} \frac{di_1}{dt}$$

the sign depends on current direction and coil winding direction.

+ in this situation.

Lab5: Mutual Inductance in Circuits

Putting voltage source on other side:



Get:

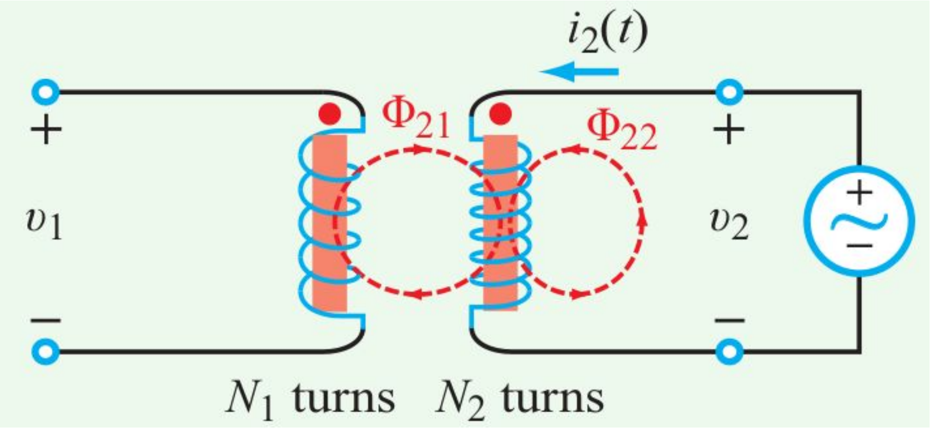
$$v_1 = \pm M_{12} \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt}$$

It turns out that $M_{12} = M_{21} = M$

Lab5: Mutual Inductance in Circuits

Putting voltage source on other side:



So:

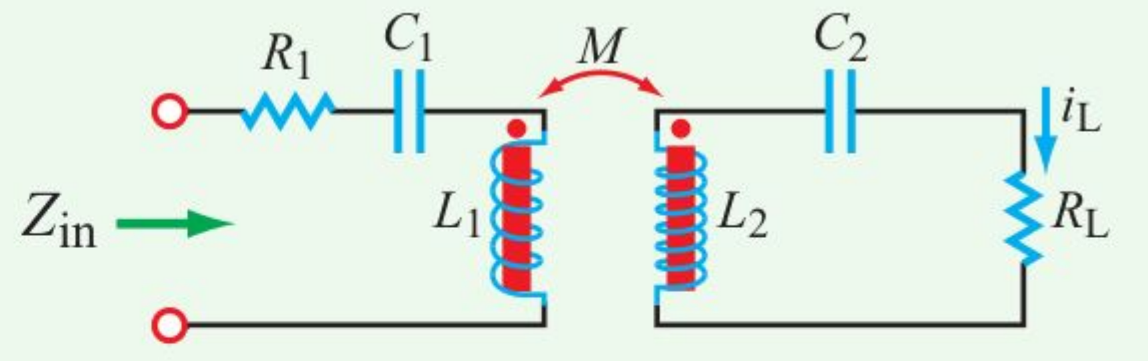
The mutual inductance can be treated like an inductor, but with a different current.

Can see all details in Circuits book, chapter 11.

Example 1

Given: The circuit:

Find: Z_{in}



(a) Time domain

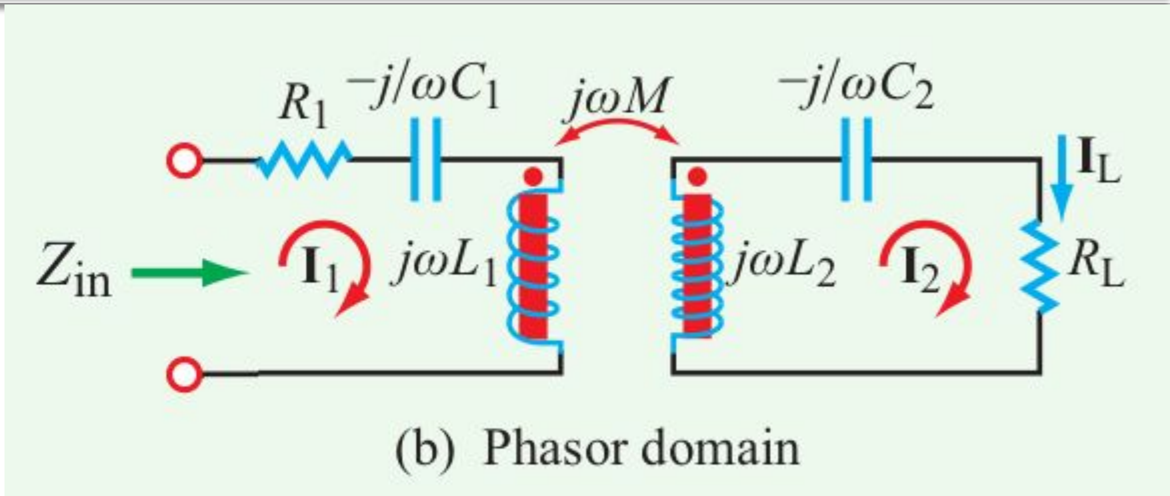
Solution:

1. Convert to phasor domain

Example 1

Given: The circuit:

Find: Z_{in}



(b) Phasor domain

Solution:

1. Convert to phasor domain

2. KVL:

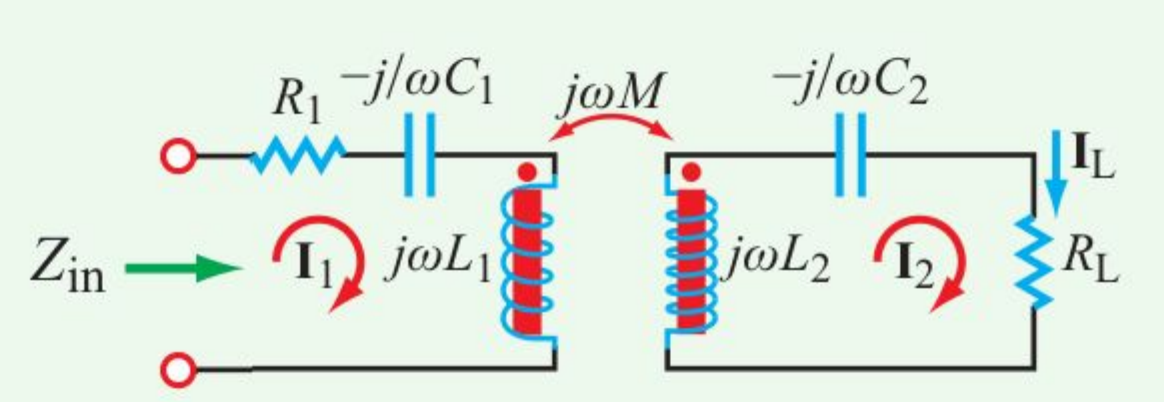
$-I_1 Z_{in} + I_1 R_1 + I_1[-j/(\omega C_1)] + I_1[j\omega L_1] - I_2 j\omega M = 0$ (eqn1)

$I_2[j\omega L_2] + I_2[-j/(\omega C_2)] + I_2 R_L - I_1 j\omega M = 0$ (eqn2)

Example 1

Given: The circuit:

Find: Z_{in}



(b) Phasor domain

Solution:

3. solve for I_2:

$$I_2[\,j\omega L_2] + I_2[-j/(\omega C_2)] + I_2 R_L - I_1\, j\omega M = 0 \qquad \text{(eqn2)}$$

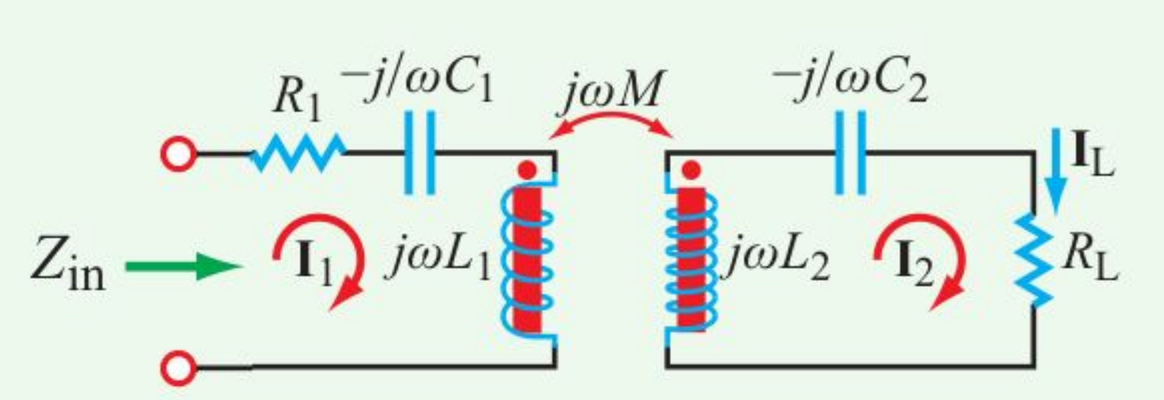
$$I_2[\,j\omega L_2 - j/(\omega C_2) + R_L\,] = I_1\, j\omega M$$

$$I_2 = \frac{I_1\, j\omega M}{j\omega L_2 - j/(\omega C_2) + R_L}$$

Example 1

Given: The circuit:

Find: Z_{in}



(b) Phasor domain

Solution:

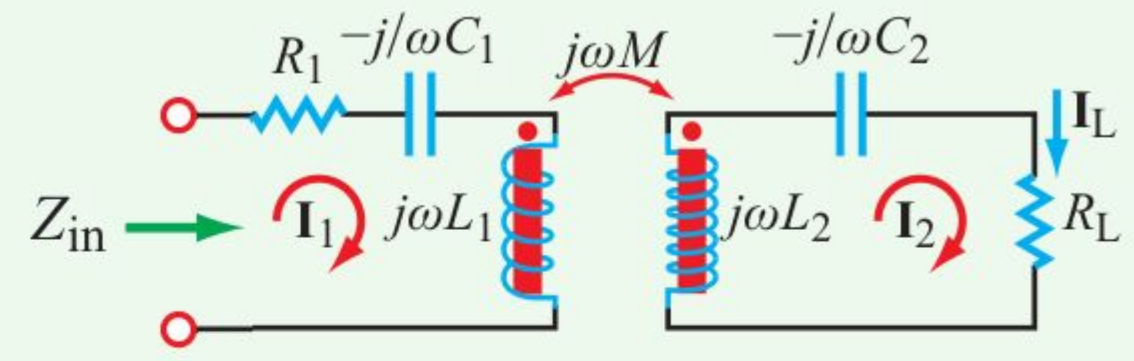
4. plug into eqn1:

$$-I_1 Z_{in} + I_1 R_1 + I_1[-j/(\omega C_1)] + I_1[j\omega L_1] + \frac{I_1\, j\omega M}{j\omega L_2 - j/(\omega C_2) + R_L}\, j\omega M = 0$$

Example 1

Given: The circuit:

Find: Z_{in}



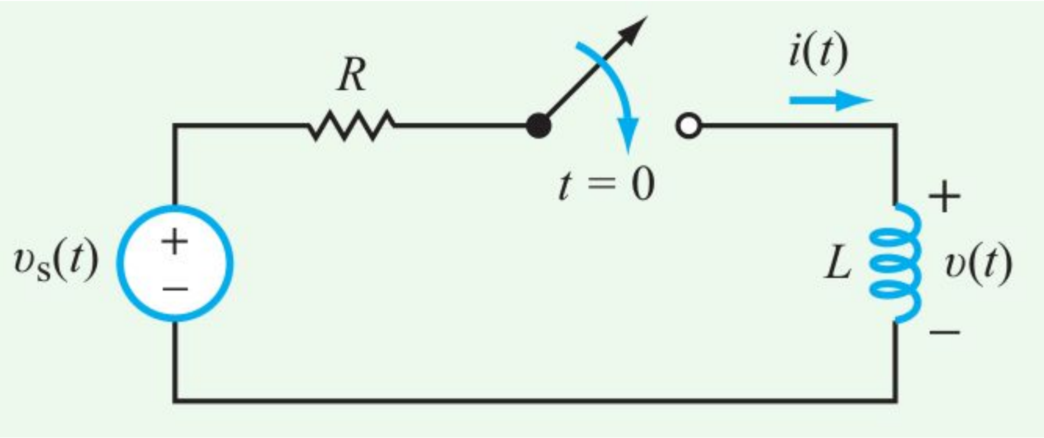
(b) Phasor domain

Solution:

4. solve for Z_{in}:

$$Z_{in} = R_1 + [-j/(\omega C_1)] + [\,j\omega L_1] + \frac{(\omega M)^2}{j\omega L_2 - j/(\omega C_2) + R_L}$$

5-8 Magnetic Energy



know: v_L = L di/dt so: v dt = L di

and since Power = iv

and Energy = $\int$ Power dt

Get the **Magnetic Energy** stored in an inductor:

$$W_m = \int p\,dt = \int iv\,dt = L\int_0^I i\,di = \frac{1}{2}LI^2 \quad \text{(J)}$$

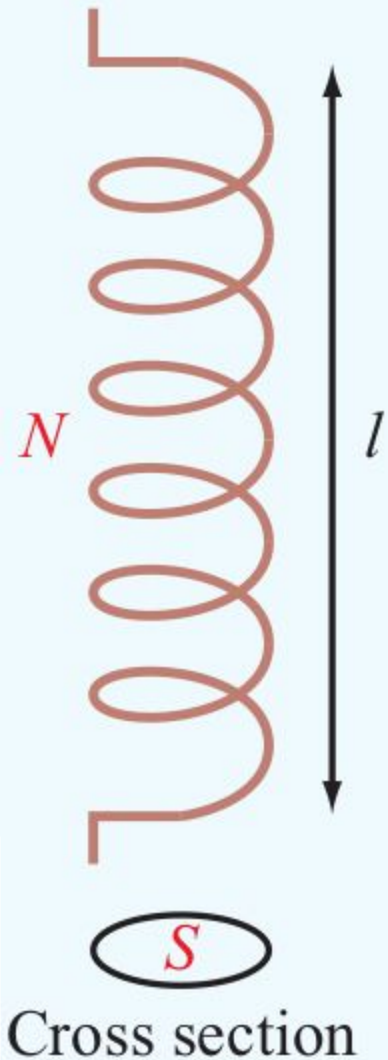
5-8 Magnetic Energy

For a solenoid:

$$L = \mu N^2 S / l$$

$$B = \mu N I / l$$

$$I = B l / (\mu N)$$



plug in:

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2}\left(\mu \frac{N^2}{l} S\right)\left(\frac{Bl}{\mu N}\right)^2$$

$$= \frac{1}{2}\frac{B^2}{\mu}(lS) = \frac{1}{2}\mu H^2 \upsilon$$

5-8 Magnetic Energy

For a solenoid:

Magnetic Energy Density:

$$w_m = \frac{W_m}{\upsilon} = \frac{1}{2}\mu H^2 \qquad (\text{J/m}^3)$$

Turns out this is true for any configuration, not just solenoids.

And so for any volume we get:

$$W_m = \frac{1}{2}\int_{\upsilon} \mu H^2 \, d\upsilon \qquad (\text{J})$$

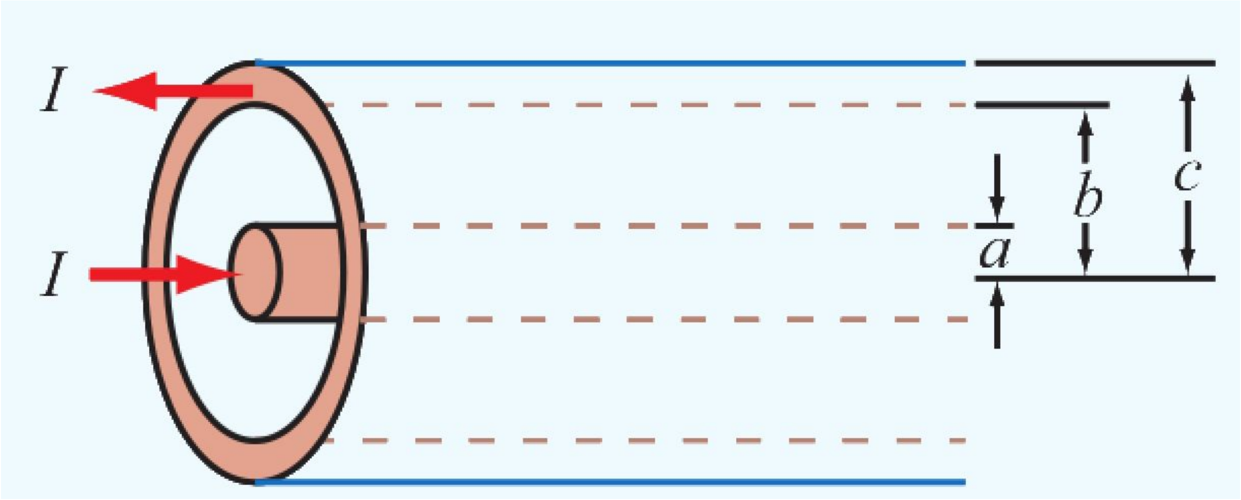
5-8 Magnetic Energy

For a coax:

with steady current I

Determine stored magnetic energy

in a segment of length *d*



5-8 Magnetic Energy

For a coax:

we know:

$$L' = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \qquad \text{(H/m)}.$$

and since:

$$W_m = \frac{1}{2} L I^2$$

get:

$$W_m = \frac{1}{2} \frac{\mu_0 d}{2\pi} \ln\left(\frac{b}{a}\right) I^2$$

5-8 Magnetic Energy

For a straight wire:

uniform current I, infinite length, radius *a*

Find Stored Magnetic energy inside wire

Use: $$W_\mathrm{m} = \frac{1}{2}\int_{\upsilon} \mu H^2 \, d\upsilon \qquad \text{(J)}$$

know: $$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{r}{2\pi a^2} I$$ (inside the wire)

since using cylindrical coords: $d\mathcal{V} = r\, dr\, d\phi\, dz$

5-8 Magnetic Energy

For a straight wire:

$$W_m = \int_{z=0}^{1m} \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \frac{\mu_0}{2} \frac{r^2}{4\pi^2 a^4} I^2 \, r \, dr \, d\phi \, dz$$

$$W_m = \frac{\mu_0 I^2}{8\pi^2 a^4} \int_{z=0}^{z=1m} dz \int_{\phi=0}^{2\pi} d\phi \int_{r=0}^{a} r^3 \, dr$$

$$W_m = \frac{\mu_0 I^2}{4\pi a^4} \left[\frac{r^4}{4} \right]_{r=0}^{a}$$

$$\boxed{W_m = \frac{\mu_0 I^2}{16\pi} \quad \text{Joules}}$$

Homework

Homework 21 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Sections 6-1 through 6-3:

Time-Varying: All 4 Maxwell's Equations:

Voltage in one circuit induced by a time-varying current in another. (Faraday's Law)

Stationary Loop in time-varying Magnetic Field

The Ideal Transformer