EECS 230
ENGINEERING ELECTROMAGNETICS

Leland Pierce

Magnetostatics 1

University of Michigan - Fall 2024

Announcements

Exam2: Monday Nov 11, 5-6:30PM
rooms EECS1331, EECS1500

Alternate Exam2: Tuesday, Nov 12, 5:30-7PM
rooms: GFL107

Coverage: lectures/homeworks: 11-20

Equation sheet posted on canvas

Can bring 2 "cheat-sheets", calculator, ...

Chapter 5 Overview

Maxwell's Equations

Magnetostatics

Magnetic Force

Magnetic Torque

Magnetic field from currents

Gauss's Law for Magnetism

Ampere's Law

Magnetic Vector Potential Fi

- Poisson's eqn
- Magnetic Flux

Magnetic Permeability

- Hysteresis

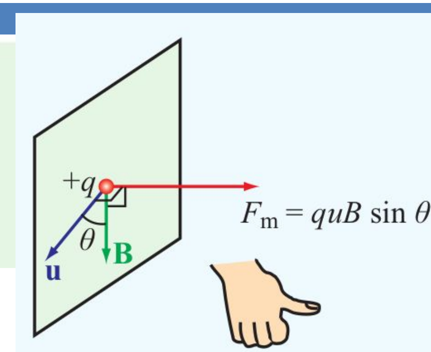
Magnetic Boundary Conditions

Inductance

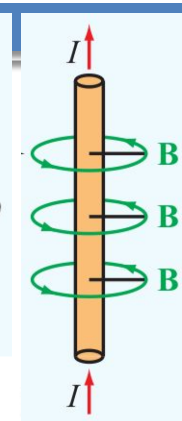
Magnetic Energy

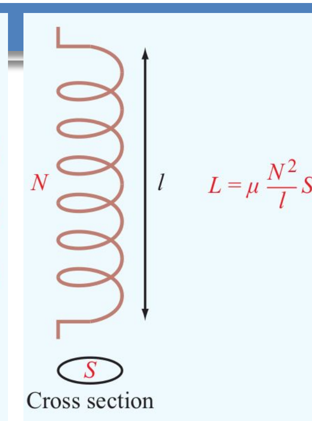
$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J},$$



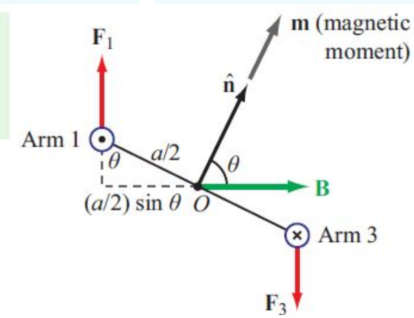
$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$$





$$\nabla \cdot \mathbf{B} = 0 \quad \longleftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0.$$

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint \mathbf{H} \cdot d\boldsymbol{\ell} = I,$$

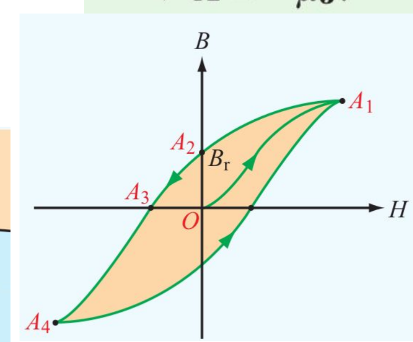


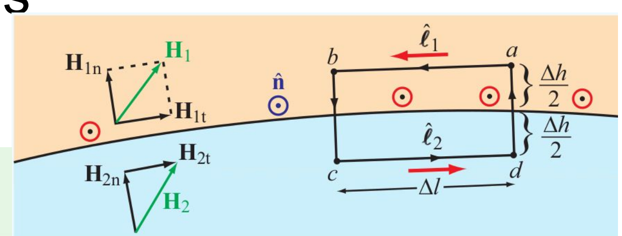
$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (\text{Wb/m}^2),$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}.$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad (\text{Wb}).$$

$$\mathbf{B} = \mu \mathbf{H},$$





$$w_m = \frac{W_m}{\upsilon} = \frac{1}{2}\mu H^2 \qquad (\text{J/m}^3).$$

Lecture Coverage

Today's lecture:

Sections 5-1 through 5-2 of the book:

5-1: Magnetostatics:
Magnetic Forces and Torques
5-2: **H** due to a steady current (Biot-Savart Law)

5-1 Magnetostatics

Static Conditions:

$$\frac{\partial \rho_v}{\partial t} = 0$$

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_v,$$
$$\nabla \times \mathbf{E} = 0.$$

$$\frac{\partial \mathbf{J}}{\partial t} = 0$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J}.$$

Electric and Magnetic Fields are decoupled.

5-1 Magnetostatics

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

magnetic flux density **B**

magnetic field intensity **H**

$$\mathbf{B} = \mu \mathbf{H}.$$

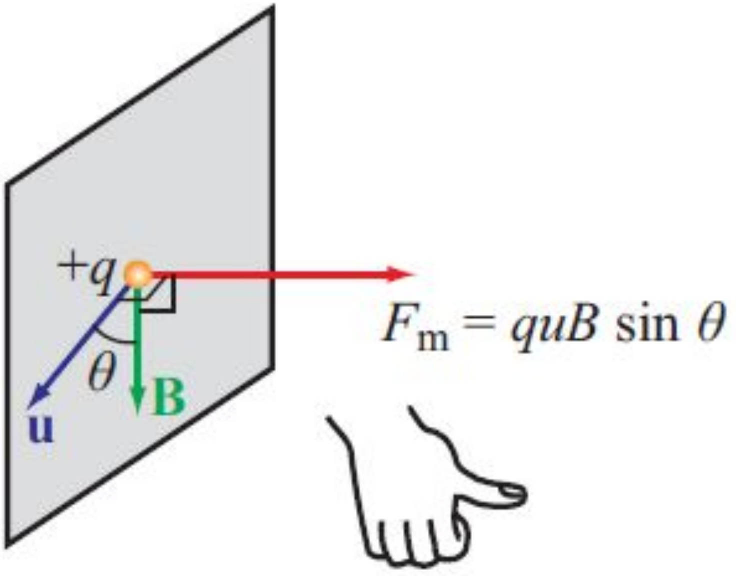
J is the current density

5-1 Electric & Magnetic Forces

Magnetic force

$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$$

$$F_m = q\,u\,B\,\sin\theta$$



(a)

q = charge
u = velocity

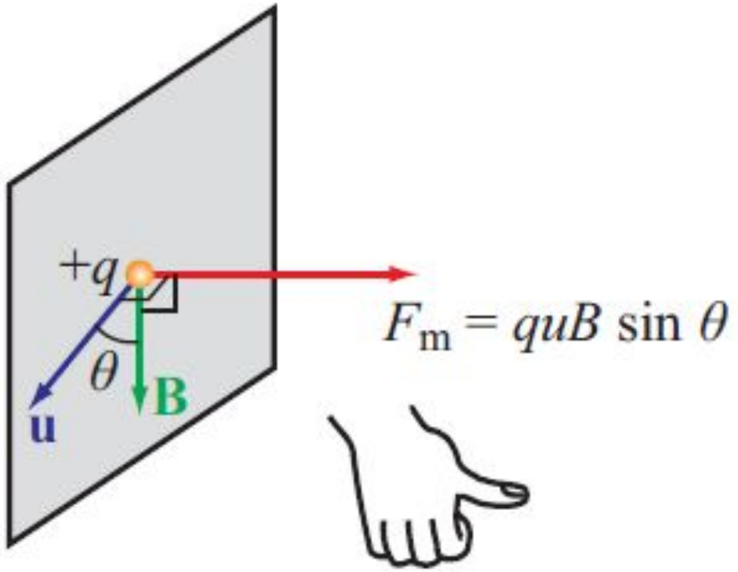
Units for **B**: Newtons / (Coulombs m / sec)
Tesla

5-1 Electric & Magnetic Forces

Magnetic force

$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B}$$

$$F_m = q\,u\,B\,\sin\theta$$



(a)

Right-hand rule:

with 2 vectors drawn so that they extend from same origin.

curl fingers of right hand from first to second vector.

thumb points in direction of the force.

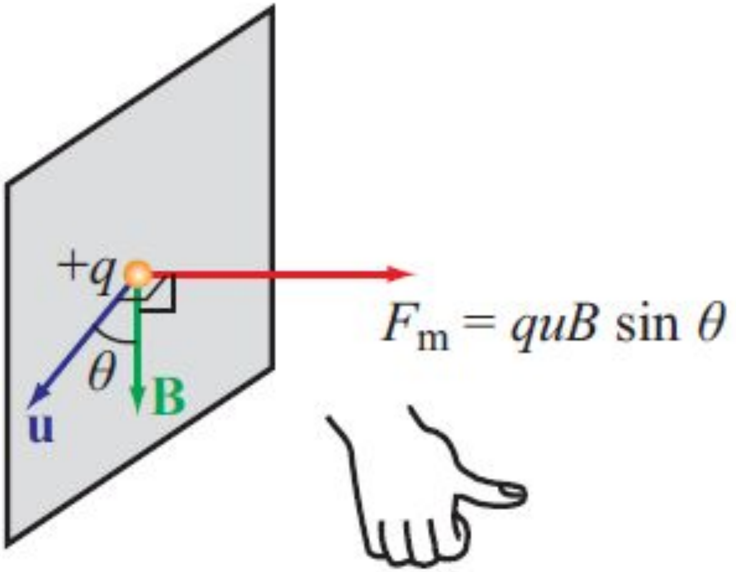
5-1 Electric & Magnetic Forces

Magnetic force

$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$$

$$F_m = q\,u\,B\,sin\theta$$

(a)



Right-hand rule:

curl fingers of right hand from first to second vector.
thumb points in direction of the force.

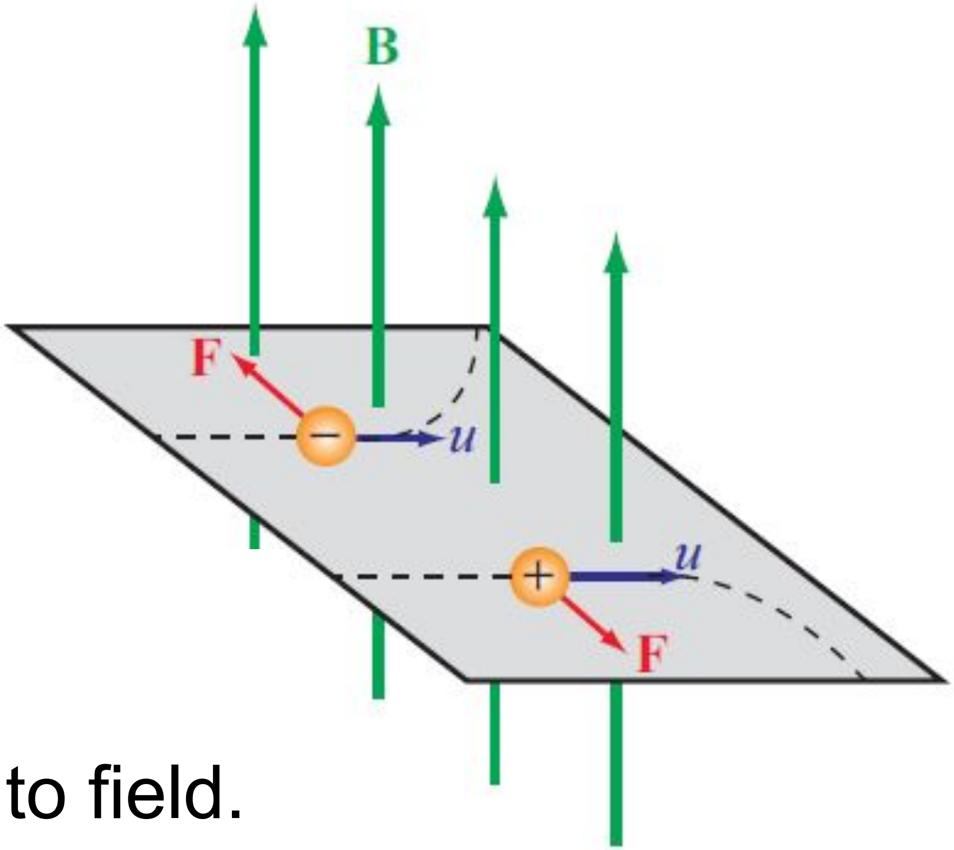
Opposite direction when charge < 0.

5-1 Electric & Magnetic Forces

Magnetic force

$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B}$$

$$F_m = q\, u\, B\, sin\theta$$



Moving charges in a magnetic field:
curve in plane normal to field.

5-1 Electric & Magnetic Forces

Electromagnetic (Lorentz) force

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

Comparison:

- $\mathbf{F}_e$ is parallel to **E**.
- $\mathbf{F}_e$ acts on any charge.
- $\mathbf{F}_e$ does work on charge

- $\mathbf{F}_m$ is perpendicular to **B**.
- $\mathbf{F}_m$ acts on moving charge only.
- $\mathbf{F}_m$ does no work on charge.
It does not change the kinetic energy of charge, only its direction of motion:

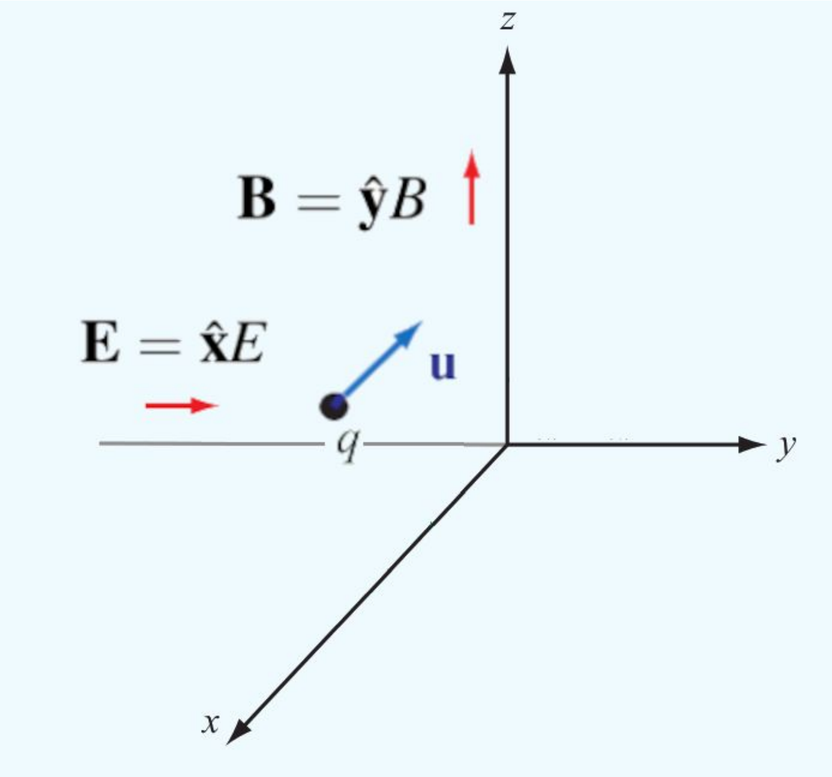
$$dW = \mathbf{F}_m \cdot d\mathbf{l} = (\mathbf{F}_m \cdot \mathbf{u})\, dt = 0$$

Exercise 5.3

Given: a charged particle with unknown velocity, **u**

fields: $\mathbf{E} = \hat{\mathbf{x}}E \quad \mathbf{B} = \hat{\mathbf{y}}B.$

Find: **u** so no net force on the particle.



Solution:

$$\mathbf{F}_{\mathrm{e}} = q\mathbf{E} = \hat{\mathbf{x}}qE$$

$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B} = q(\mathbf{u} \times \hat{\mathbf{y}}B)$$

Exercise 5.3

Solution:

$$\mathbf{F}_{\mathrm{e}} = q\mathbf{E} = \hat{\mathbf{x}}qE$$

$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B} = q(\mathbf{u} \times \hat{\mathbf{y}}B)$$

Zero net force:

$$\mathbf{F}_e + \mathbf{F}_m = 0$$

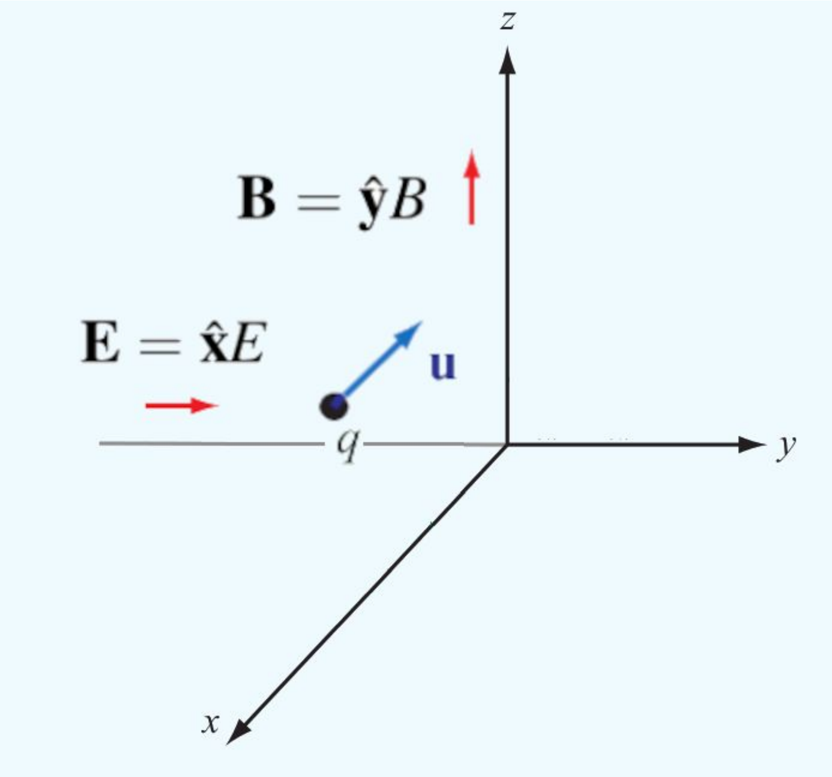
$$\mathbf{F}_e = -\mathbf{F}_m = 0$$

$$\hat{\mathbf{x}}qE = -q\mathbf{u} \times \hat{\mathbf{y}}B$$

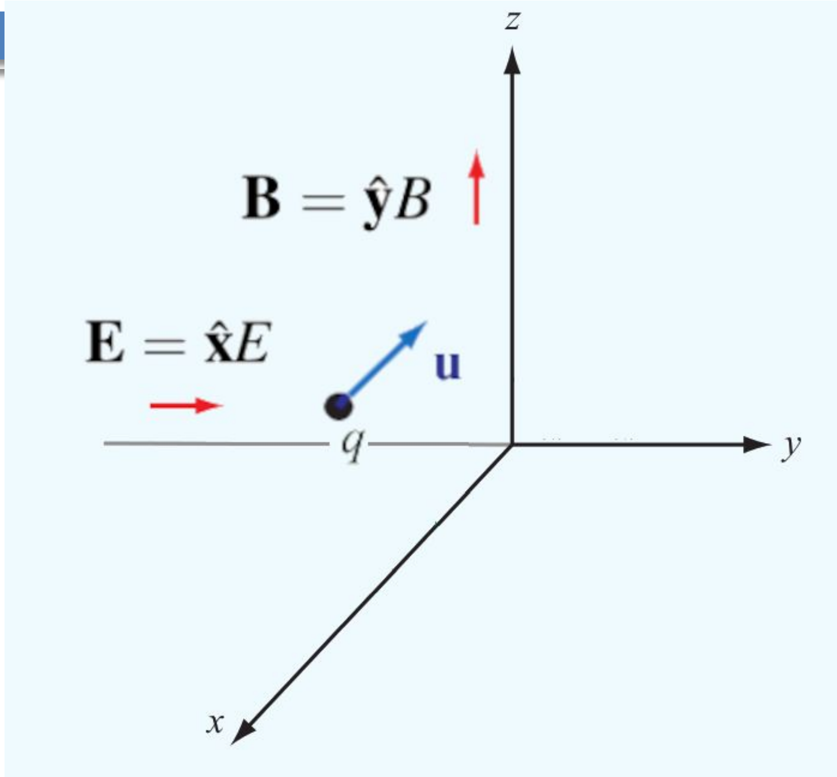
so:

$$\mathbf{u} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}}$$

$$\hat{\mathbf{u}} = \hat{\mathbf{z}}$$



Exercise 5.3



Solution:

$$\hat{\mathbf{x}}qE = -q\mathbf{u} \times \hat{\mathbf{y}}B$$

$$\hat{\mathbf{u}} = \hat{\mathbf{z}}$$

hence:

$$qE = quB$$

$$u = E/B$$

$$\boxed{\mathbf{u} = \hat{\mathbf{z}}E/B}$$

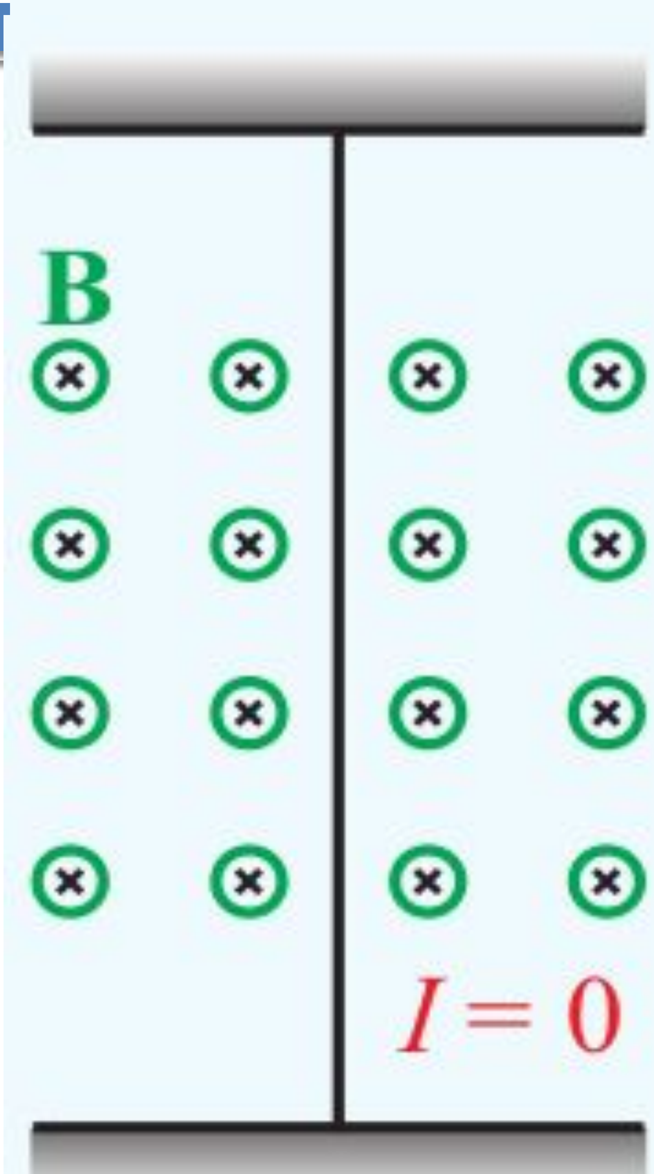
Note: any *y*-component of **u** will have no effect on $\mathbf{F}_m$

5-1 Magnetic Force on a Current Element

Flexible wire

Uniform **B** field perpendicular to wire
(into page)

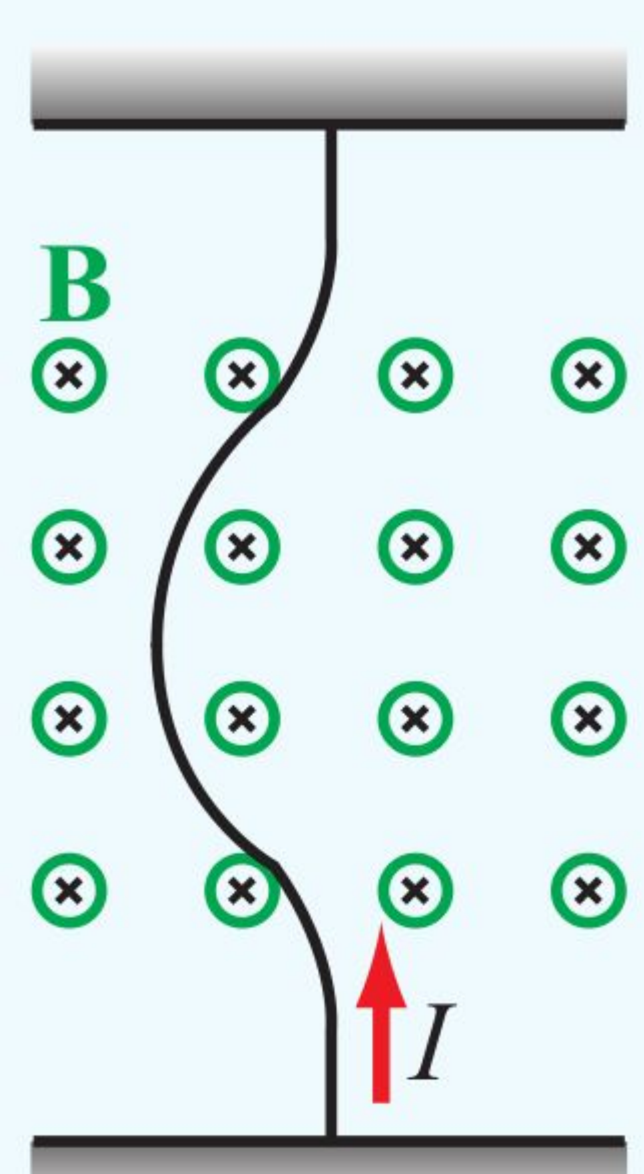
Apply current to the wire



5-1 Magnetic Force on a Current Element

Flexible current-carrying wire
Uniform **B** field perpendicular to wire
(into page)

Wire is deformed by $\mathbf{F}_m$

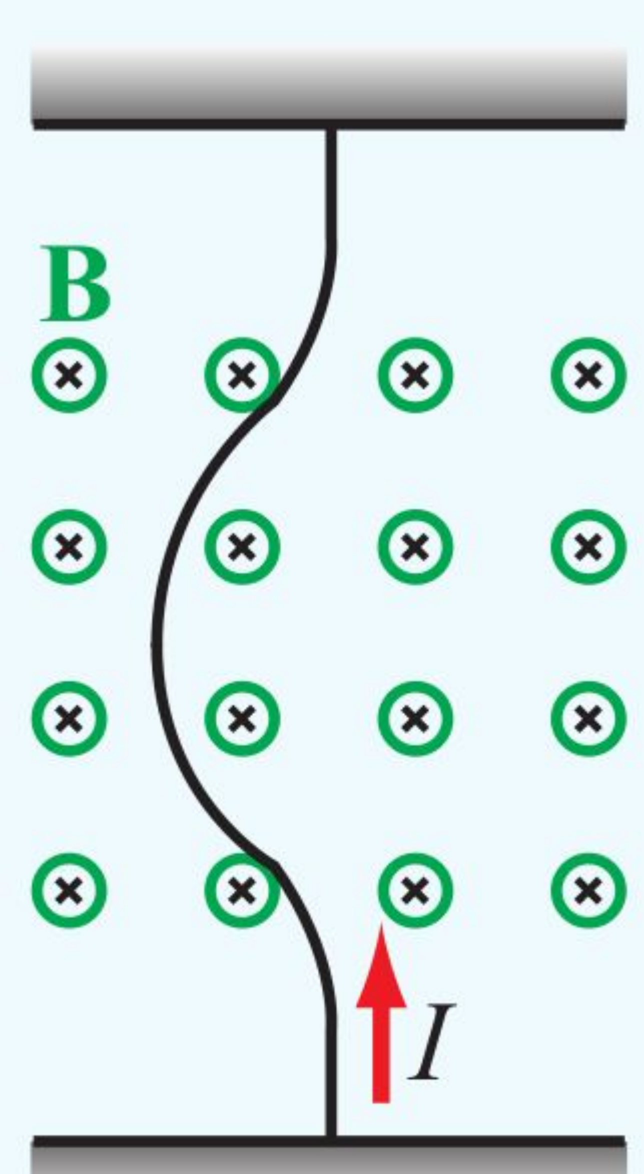


5-1 Magnetic Force on a Current Element

Flexible current-carrying wire

Uniform **B** field perpendicular to wire (into page)

Let's derive a mathematical model for this situation.



5-1 Magnetic Force on a Current Element

Flexible current-carrying wire

Uniform **B** field perpendicular to wire (into page)

Number free electrons per volume: N_e

Free charge density in wire:

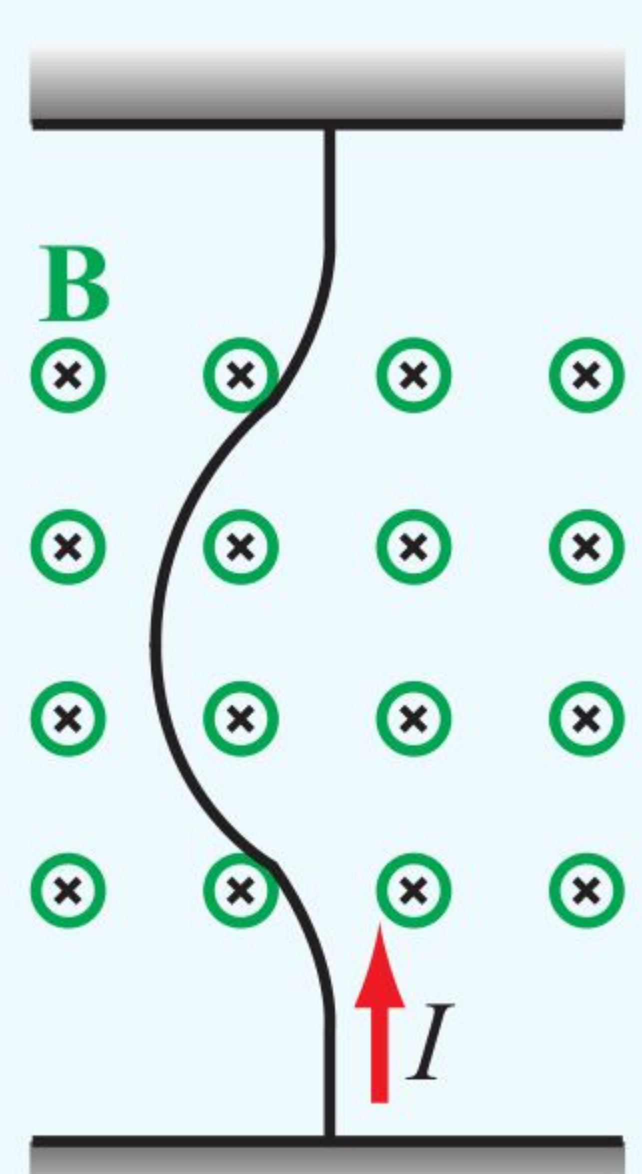
$$\rho_{ve} = -N_e e.$$

Free charge in wire element:

$$dQ = \rho_{ve} A\, dl = -N_e e A\, dl,$$

Hence, magnetic force:

$$d\mathbf{F}_m = dQ\, \mathbf{u}_e \times \mathbf{B} = -N_e e A\, dl\, \mathbf{u}_e \times \mathbf{B},$$



5-1 Magnetic Force on a Current Element

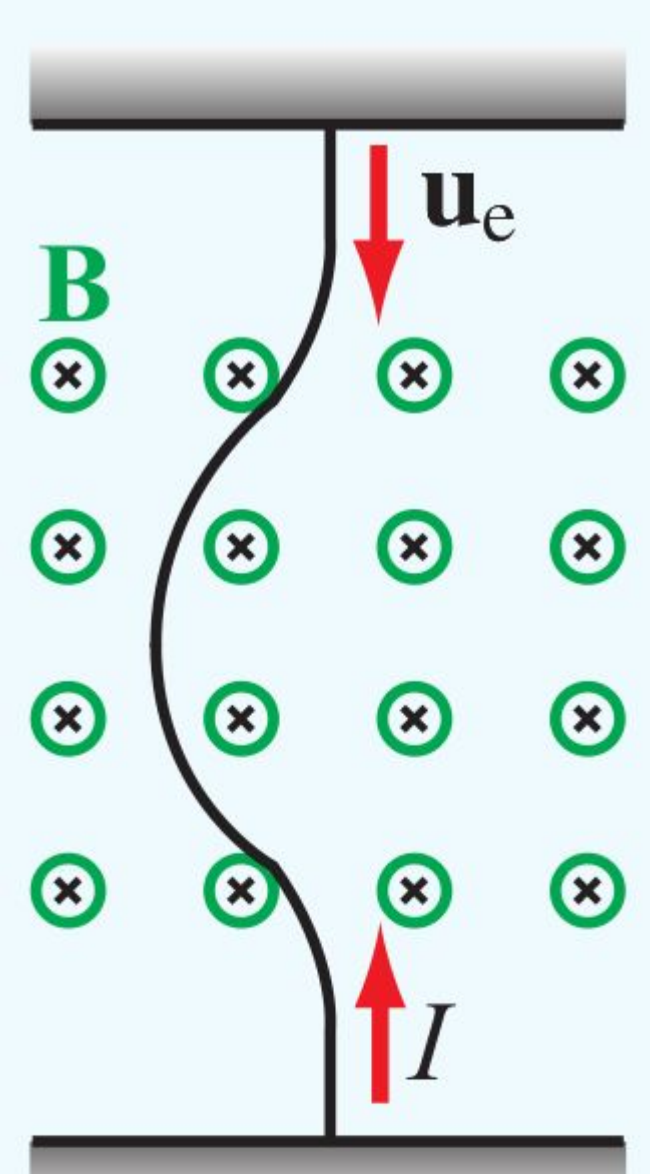
Choose d**l** directed along **I**.
Electron velocity, $\mathbf{u}_e$, is parallel to d**l** but opposite in direction:

$$dl\,\mathbf{u}_e = -d\mathbf{l}\,u_e$$

so get:

$$d\mathbf{F}_m = N_e e A u_e \, d\mathbf{l} \times \mathbf{B}.$$

units: N_e: free electrons/m^3
e: Coulombs/electron
A: m^2
u_e: m/sec
hence: C/sec or Amps



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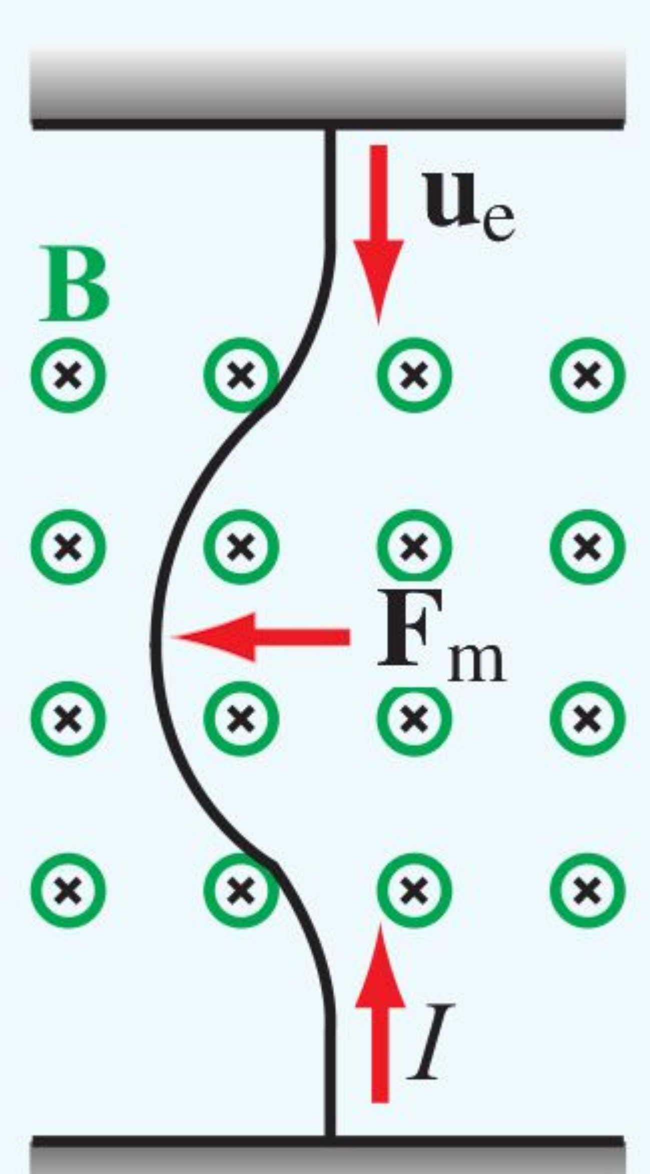
Choose d**l** directed along **I**.
Electron drift velocity, $\mathbf{u}_e$, is parallel to d**l** but opposite in direction:

$$dl\,\mathbf{u}_e = -d\mathbf{l}\,u_e$$

so get:

$$d\mathbf{F}_m = N_e e A u_e\, d\mathbf{l} \times \mathbf{B}.$$

$$d\mathbf{F}_m = I\, d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}.$$



5-1 Magnetic Force on a Current Element

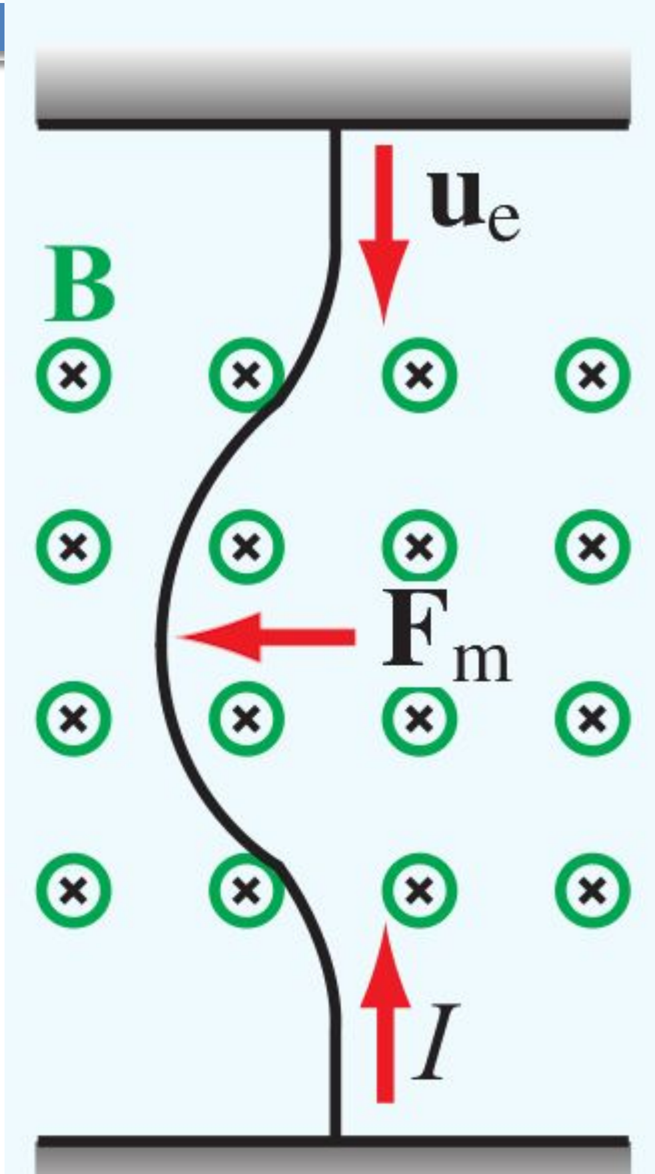
$$d\mathbf{F}_\mathrm{m} = N_\mathrm{e} e A u_\mathrm{e}\, d\mathbf{l} \times \mathbf{B}.$$

$$d\mathbf{F}_\mathrm{m} = I\, d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}.$$

Integrated over the *part* of the wire in the magnetic field:

$$\mathbf{F}_m = I \int_C d\mathbf{l} \times \mathbf{B}$$

Integrated over the straight wire: gives force *before* wire is deformed.



5-1 Magnetic Force on a Current Element

If **all** of wire is in uniform **B**:

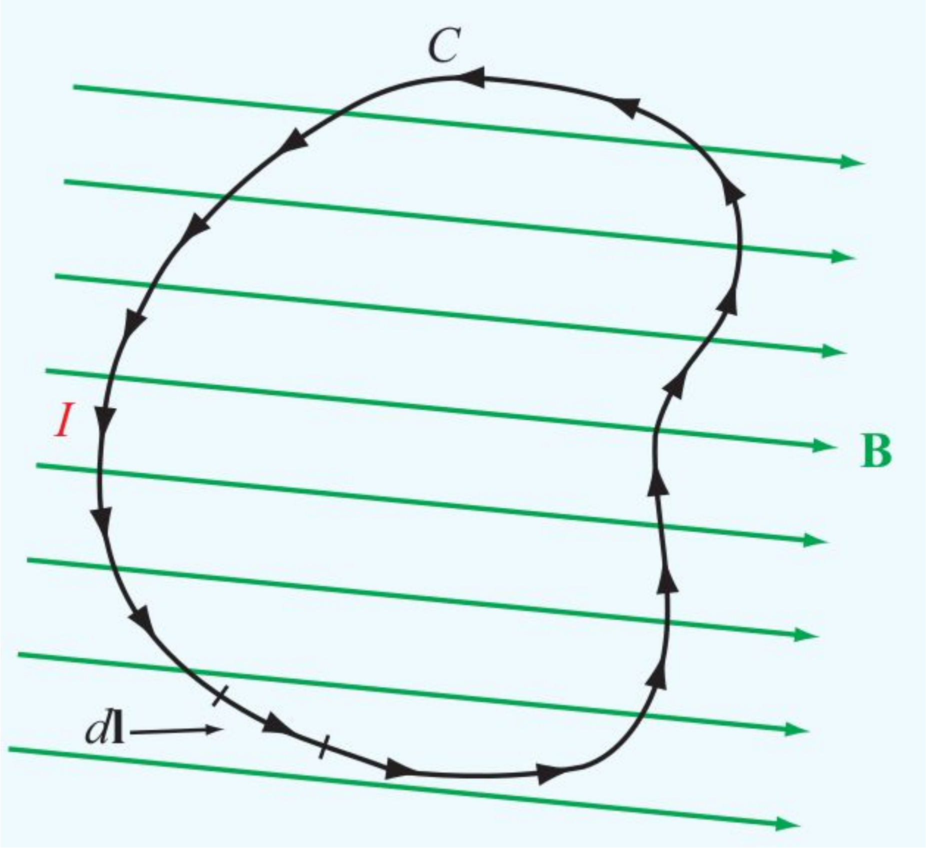
$$\mathbf{F}_m = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}.$$

$$\mathbf{F}_m = I \left(\oint_C d\mathbf{l} \right) \times \mathbf{B} = 0.$$

Can take **B** out of integral because **B** is constant along the path.

Still have non-zero force on portions of wire.

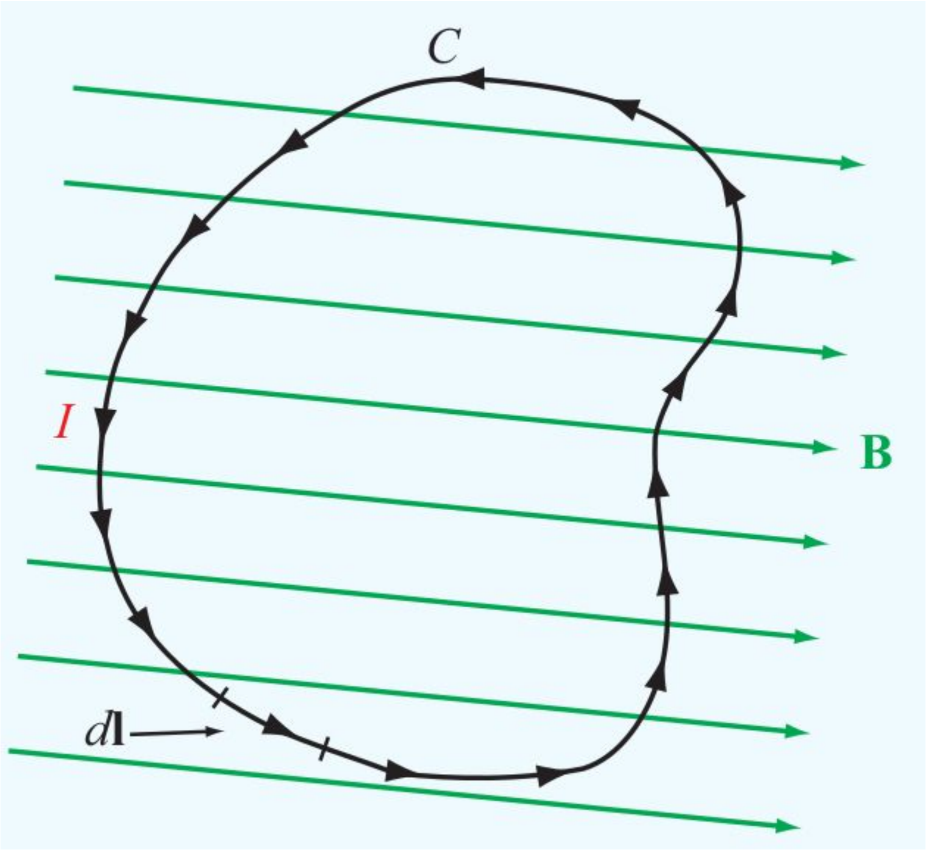
The vector sum of forces is zero.



5-1 Magnetic Force on a Current Element

In magnetostatics all currents must flow in closed paths.

The alternative is that the current flows in a path with an endpoint.



5-1 Magnetic Force on a Current Element

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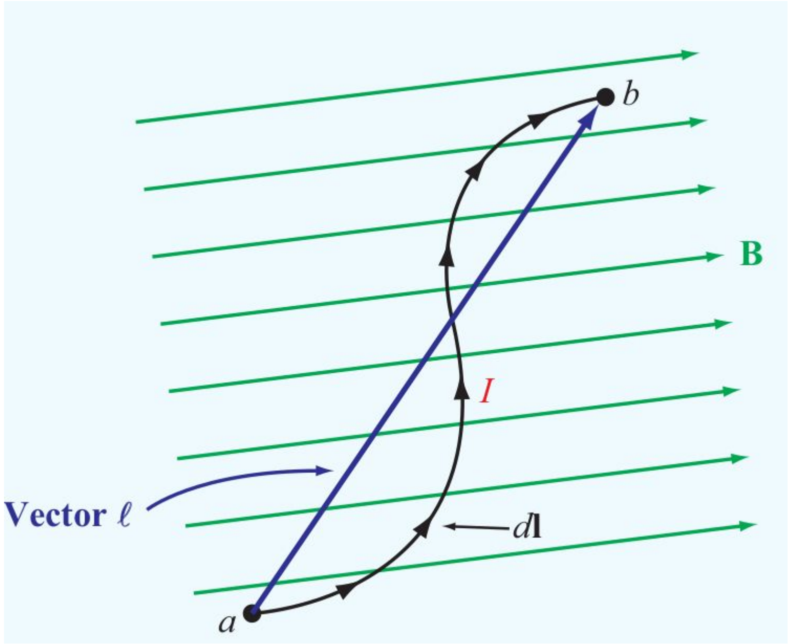


The alternative is that the current flows in a path with an endpoint.

Hence, charges would accumulate at the endpoint:

$$Q(b) = f(t)$$

so:

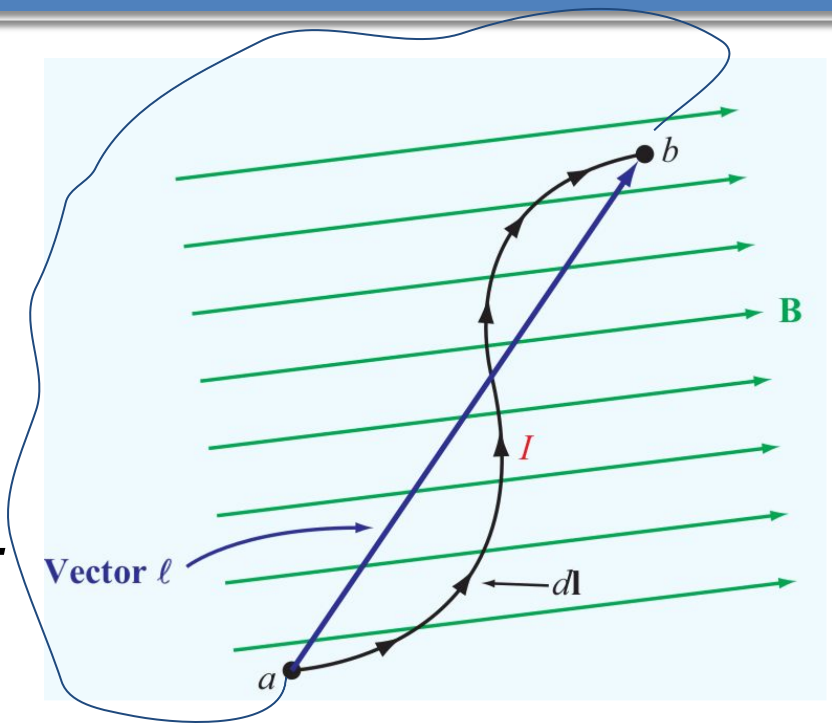
$$\frac{\partial Q}{\partial t} \neq 0$$

violating our assumptions

5-1 Magnetic Force on a Current Element

If **part** of wire is in uniform **B**:

$$\mathbf{F}_{\mathrm{m}} = I\left(\int_{\ell} d\mathbf{l}\right) \times \mathbf{B} = I\boldsymbol{\ell} \times \mathbf{B},$$



Where $\boldsymbol{\ell}$ is the vector from *a* to *b*.

Note: the ***Path*** from *a* to *b is* not important: the integral gives same result for *any* path from *a* to *b*. As long as the path is entirely within the uniform **B** field.

Example 5-1 Force on Semi-Circular Conductor

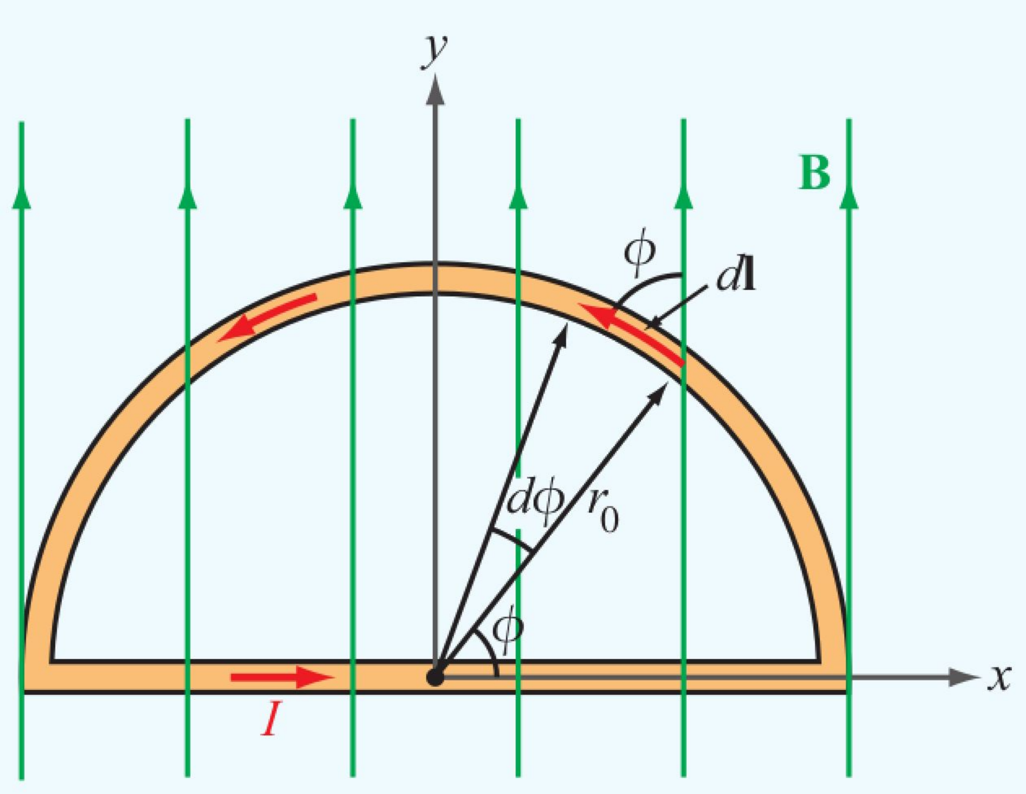
Given: Current, I, in loop:

$$\mathbf{B} = \hat{\mathbf{y}} B_0$$

radius r_0

Find: $\mathbf{F}_1$ on straight part

$\mathbf{F}_2$ on curved part



Solution:

$$\mathbf{F}_1 = I\boldsymbol{\ell} \times \mathbf{B}$$

$$\boldsymbol{\ell} = \hat{\mathbf{x}} 2r_0$$

Example 5-1 Force on Semi-Circular Conductor

Solution:

$$\mathbf{B} = \hat{\mathbf{y}} B_0$$

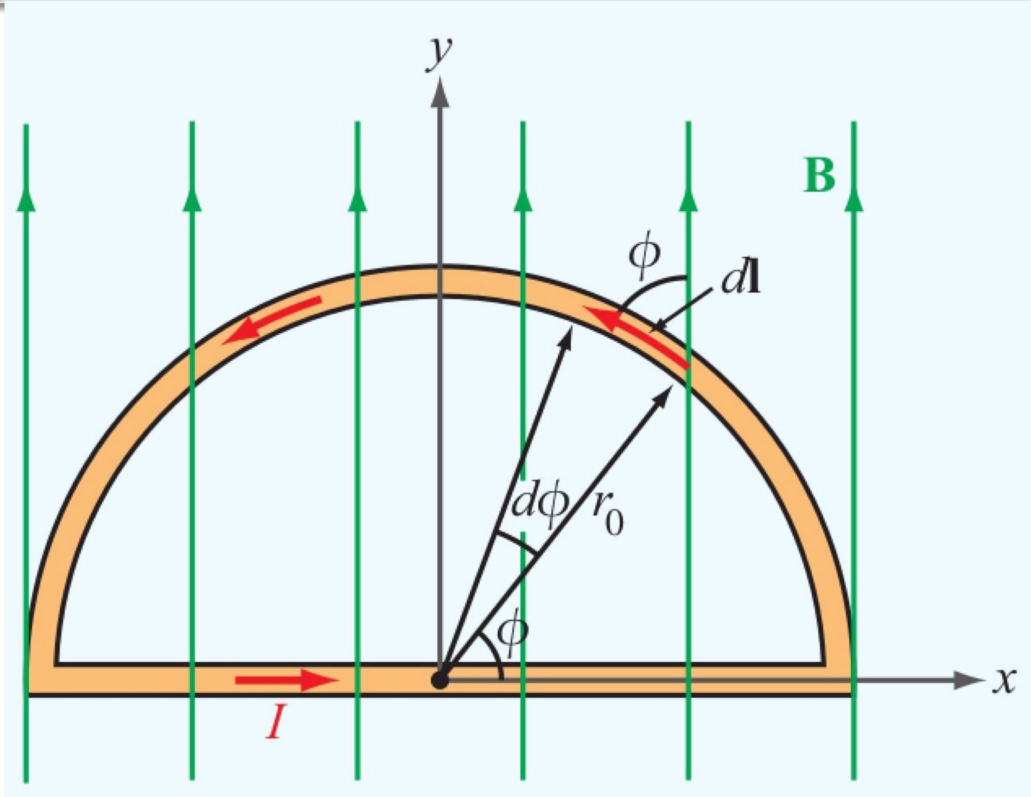
$$\mathbf{F}_1 = I\boldsymbol{\ell} \times \mathbf{B}$$

$$\boldsymbol{\ell} = \hat{\mathbf{x}} 2r_0$$

so:

$$\mathbf{F}_1 = I\hat{\mathbf{x}} 2r_0 \times \hat{\mathbf{y}} B_0$$

$$\boxed{\mathbf{F}_1 = 2IB_0r_0\hat{\mathbf{z}}}$$



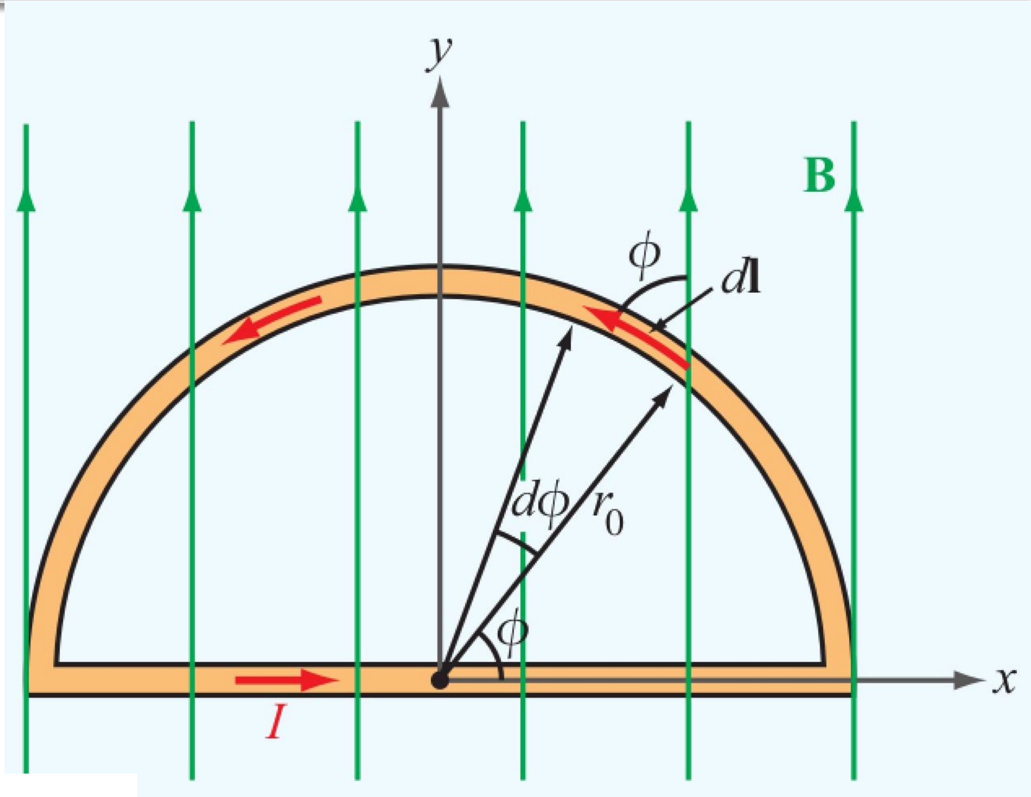
Example 5-1 Force on Semi-Circular Conductor

Solution:

$$\mathbf{B} = \hat{\mathbf{y}} B_0$$

$$\mathbf{F}_2 = I \int_\ell d\ell \times \mathbf{B}$$

$$d\ell = \hat{\boldsymbol{\phi}} r_0 \, d\phi$$



Plug in:

$$\mathbf{F}_2 = I \int_{\phi=0}^{\pi} \hat{\boldsymbol{\phi}} r_0 \times \hat{\mathbf{y}} B_0 \, d\phi$$

Example 5-1 Force on Semi-Circular Conductor

Solution:

$$\mathbf{F}_2 = I \int_{\phi=0}^{\pi} \hat{\boldsymbol{\phi}} r_0 \times \hat{\mathbf{y}} B_0 \, d\phi$$

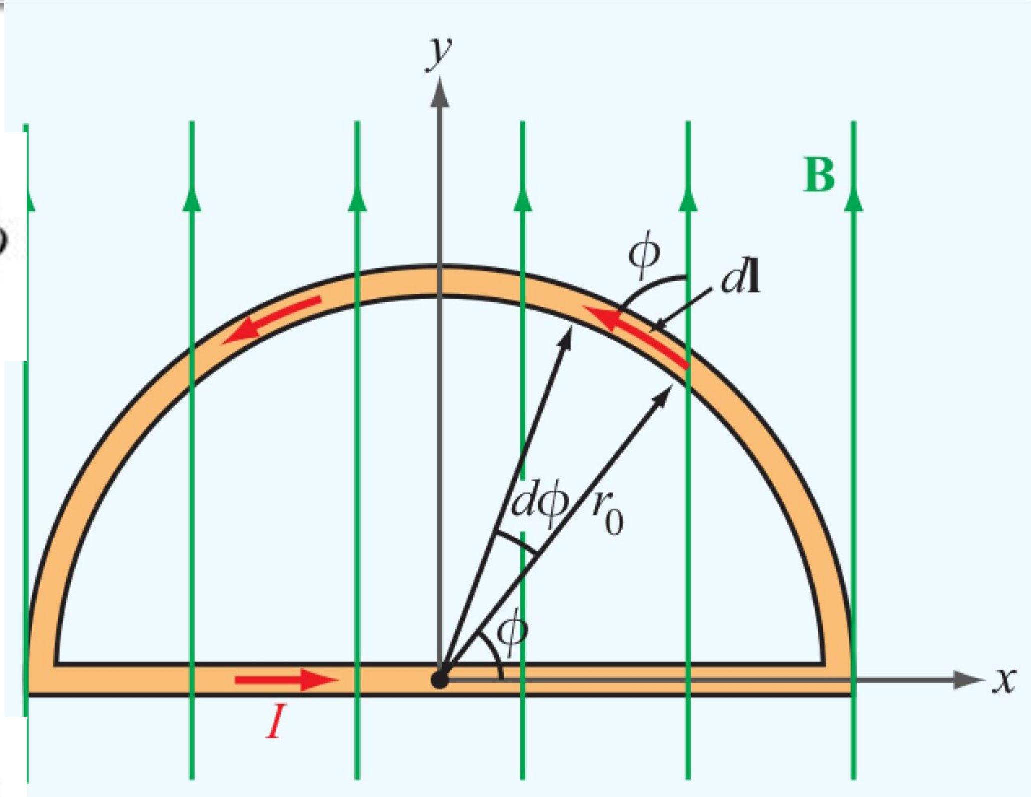
we know:

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$$

so:

$$\mathbf{F}_2 = -I B_0 r_0 \hat{\mathbf{z}} \int_{\phi=0}^{\pi} \sin\phi \, d\phi$$

$$\mathbf{F}_2 = I B_0 r_0 \hat{\mathbf{z}} \Big[\cos\phi \Big]_{\phi=0}^{\pi}$$



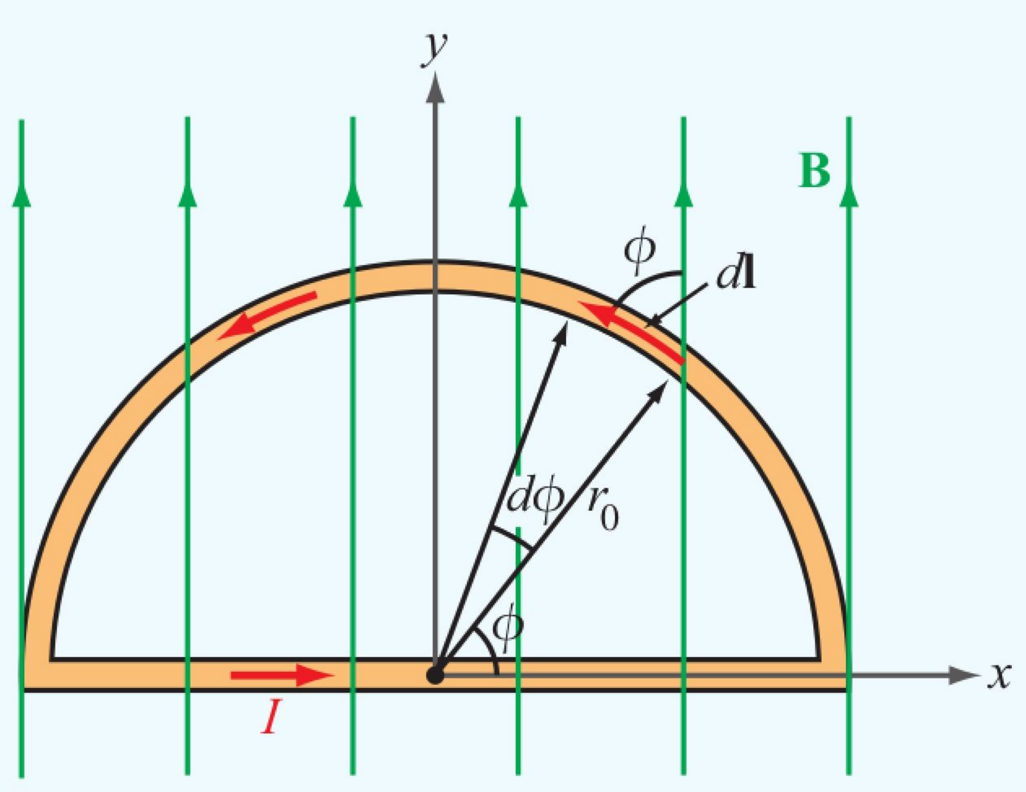
Example 5-1 Force on Semi-Circular Conductor

Solution:

$$\mathbf{F}_2 = IB_0 r_0 \hat{\mathbf{z}} \left[\cos\phi \right]_{\phi=0}^{\pi}$$

$$\mathbf{F}_2 = IB_0 r_0 \hat{\mathbf{z}} \left[-1 - 1 \right]$$

$$\boxed{\mathbf{F}_2 = -2IB_0 r_0 \hat{\mathbf{z}}}$$



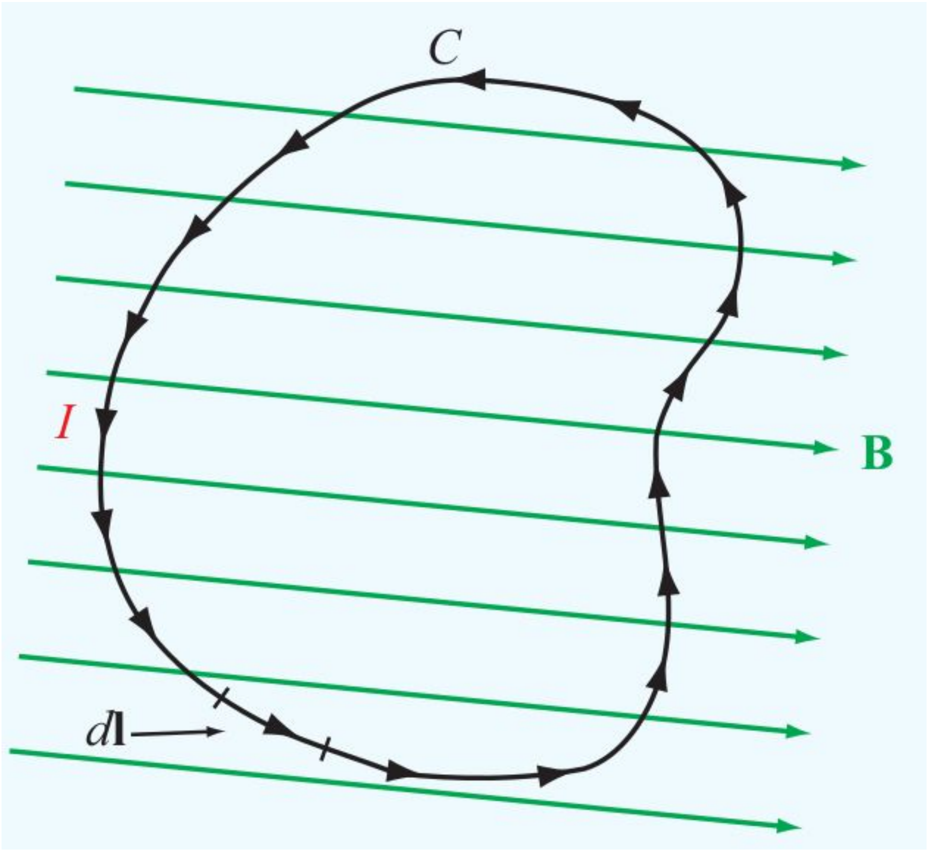
Notice that $\mathbf{F}_1 + \mathbf{F}_2 = 0$ as it should for a closed path.

5-1 Torque

Recall that the *net* force on a loop in a uniform B field is zero.

But that there are non-zero forces on *portions* of the wire.

In general, these forces may deform, or even rotate, the wire.



5-1 Torque

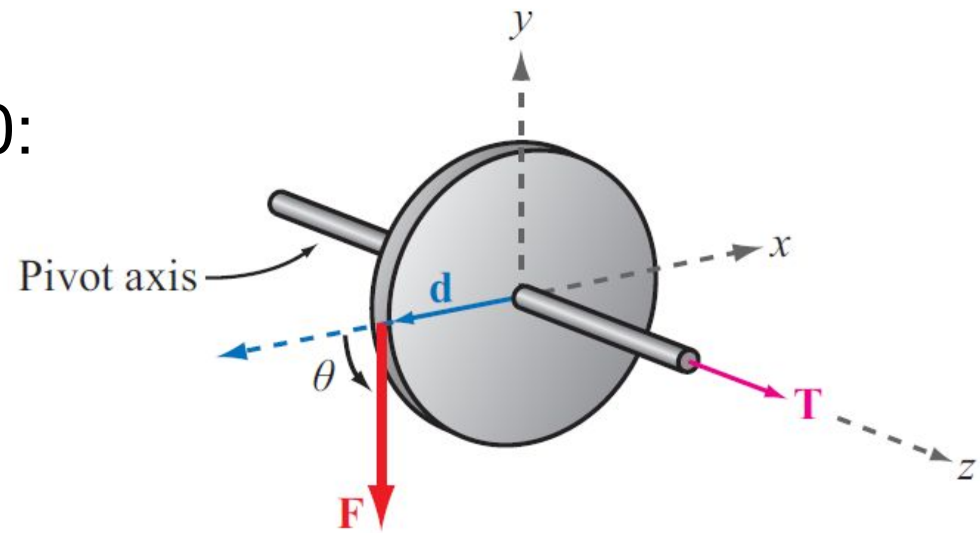
Recall from Physics 140:

$$\mathbf{T} = \mathbf{d} \times \mathbf{F} \qquad (\text{N} \cdot \text{m})$$

d = moment arm

F = force

T = torque



Torque is normal to the moment-arm, **d**, and the applied force, **F**.
Right-hand rule applies.

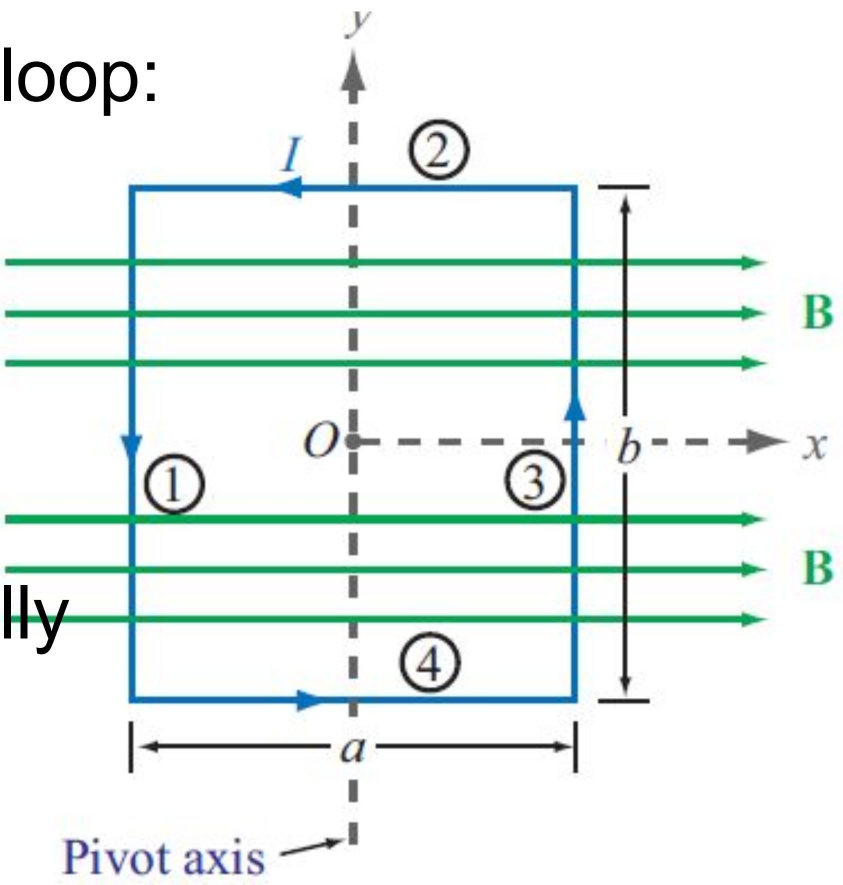
5-1 Magnetic Torque on Current Loop

Given: A rectangular closed loop:

Uniform current I
Uniform magnetic field:

$$\mathbf{B} = \hat{\mathbf{x}} B_0$$

Arranged so can mechanically rotate about vertical *y*-axis, through the middle



Find: Torque

5-1 Magnetic Torque on Current Loop

Solution: First, determine forces on each portion:

$$\mathbf{F}_m = I\left(\int_\ell d\ell\right) \times \mathbf{B}$$

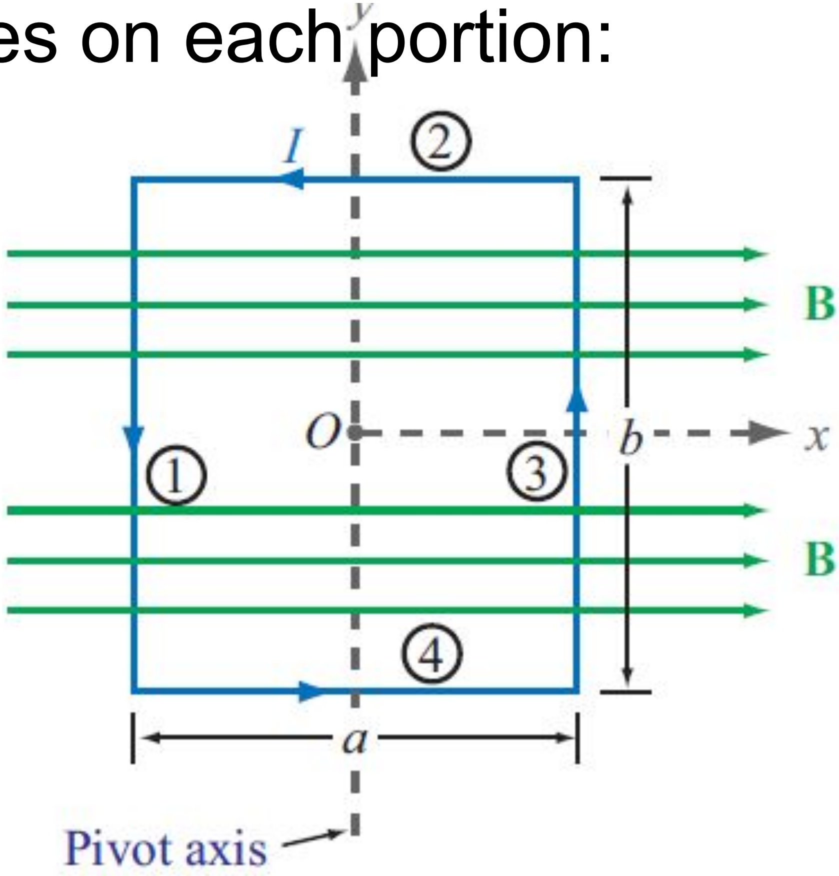
$$\mathbf{F}_m = I\left(\ell_1 + \ell_2 + \ell_3 + \ell_4\right) \times \mathbf{B}$$

$$\mathbf{F}_1 = I(-\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = \hat{\mathbf{z}}IbB_0$$

$$\mathbf{F}_2 = I(-\hat{\mathbf{x}}a) \times (\hat{\mathbf{x}}B_0) = 0$$

$$\mathbf{F}_3 = I(+\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = -\hat{\mathbf{z}}IbB_0$$

$$\mathbf{F}_4 = I(+\hat{\mathbf{x}}a) \times (\hat{\mathbf{x}}B_0) = 0$$



Sum of forces = 0

But the forces rotate the loop

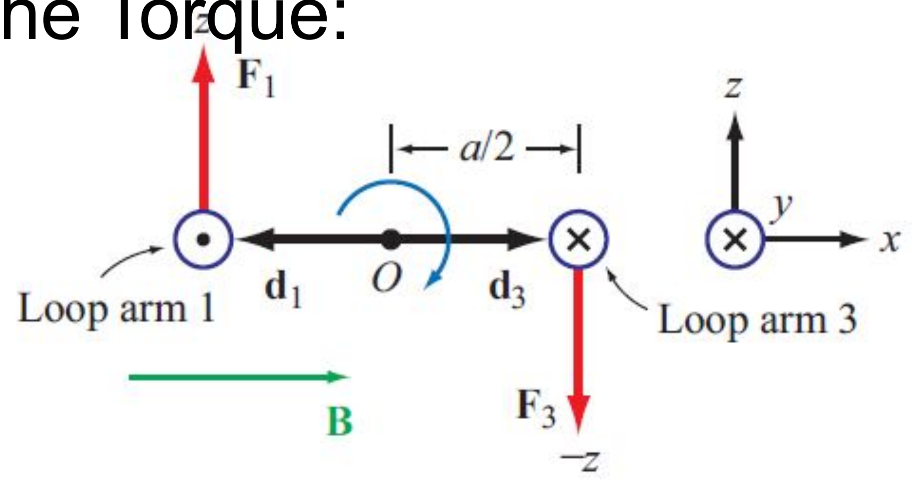
5-1 Magnetic Torque on Current Loop

Solution: Next, determine the Torque:

$$\mathbf{T} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3$$

$$= \left(-\hat{\mathbf{x}}\,\frac{a}{2}\right) \times \left(\hat{\mathbf{z}} I b B_0\right) + \left(\hat{\mathbf{x}}\,\frac{a}{2}\right) \times \left(-\hat{\mathbf{z}} I b B_0\right)$$

$$\mathbf{T} = \hat{\mathbf{y}} I a b B_0 = \hat{\mathbf{y}} I A B_0,$$



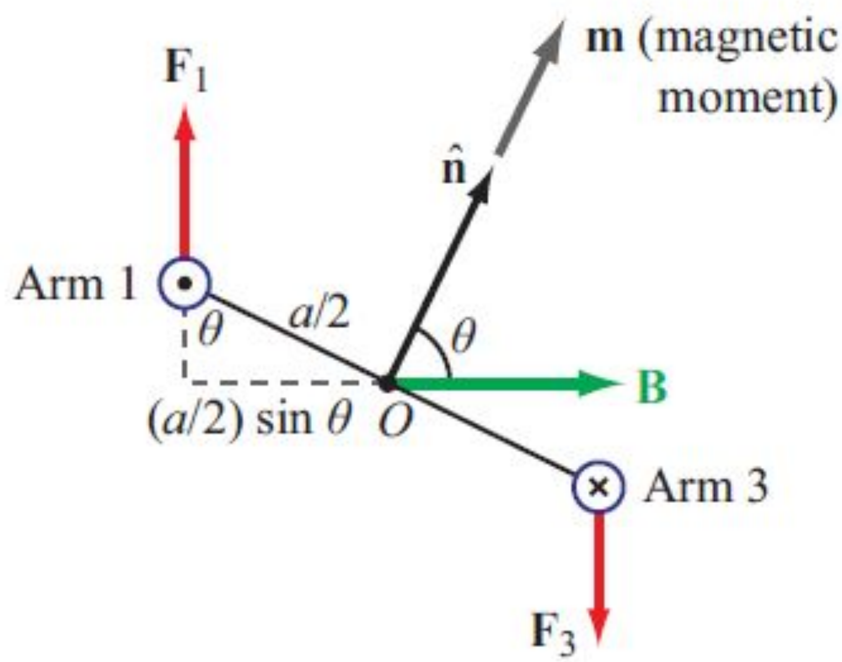
Looking edge-on

Torque rotates loop clockwise

5-1 Magnetic Torque on Current Loop

Given: Inclined Loop:
Find: Torque
Solution:
Forces are the same.
Moment arms different.

θ is the angle between the normal to the loop and the direction of **B**, so:



Looking edge-on

$$\mathbf{d}_1 = -\hat{\mathbf{x}}\frac{a}{2}\sin\theta$$

$$\mathbf{d}_3 = +\hat{\mathbf{x}}\frac{a}{2}\sin\theta$$

5-1 Magnetic Torque on Current Loop

Solution: Next, determine the Torque:

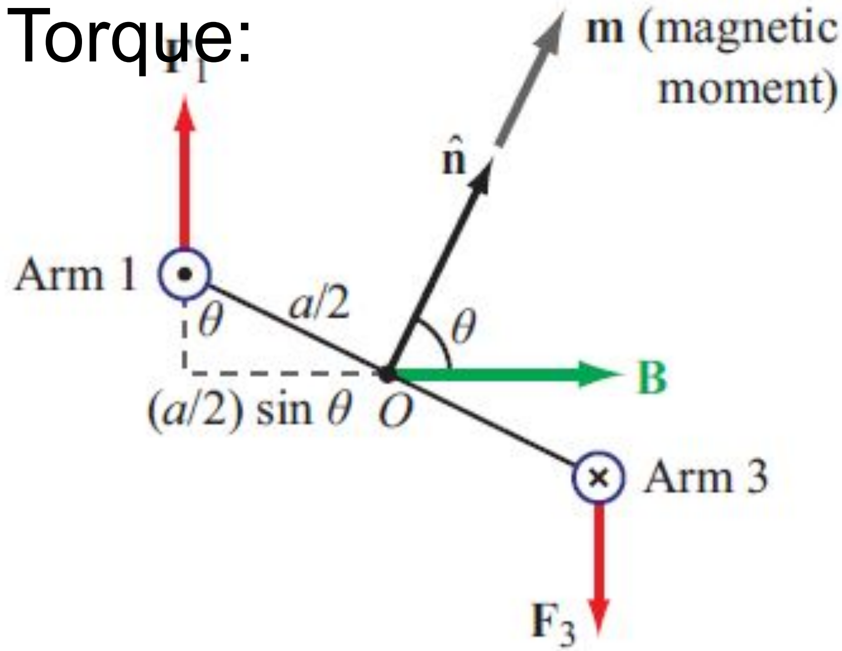
$$\mathbf{T} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3$$

$$\mathbf{T} = \left(-\hat{\mathbf{x}}\frac{a}{2}\sin\theta\right) \times (\hat{\mathbf{z}}IbB_0)$$

$$+ \left(\hat{\mathbf{x}}\frac{a}{2}\sin\theta\right) \times (-\hat{\mathbf{z}}IbB_0$$

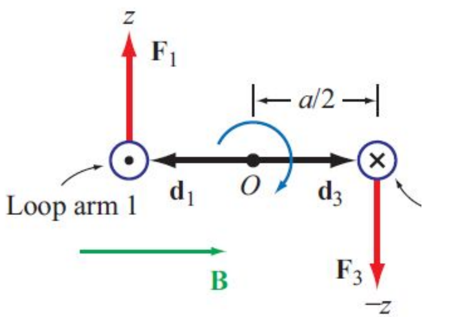
$$\mathbf{T} = \hat{\mathbf{y}}IabB_0\sin\theta$$

$$\boxed{\mathbf{T} = \hat{\mathbf{y}}IAB_0\sin\theta}$$

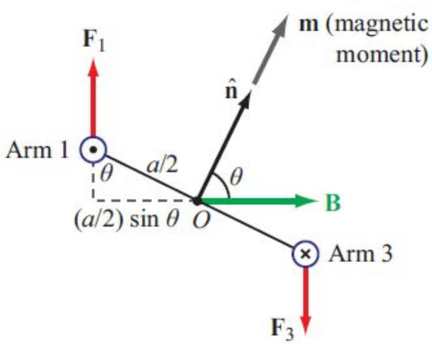


5-1 Magnetic Torque on Current Loop

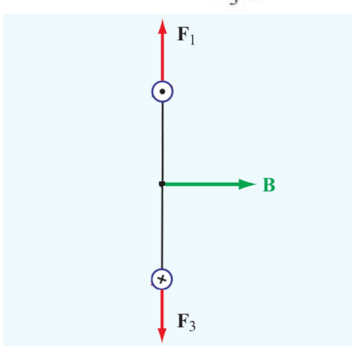
Special cases:



$$\theta = 90^\circ \quad \mathbf{T} = \hat{\mathbf{y}} I A B_0 \sin 90^\circ = \hat{\mathbf{y}} I A B_0$$



$$\theta = 45^\circ \quad \mathbf{T} = \hat{\mathbf{y}} I A B_0 \sin 45^\circ = \hat{\mathbf{y}} I A B_0 \left(\frac{\sqrt{2}}{2}\right)$$



$$\theta = 0^\circ \quad \mathbf{T} = \hat{\mathbf{y}} I A B_0 \sin 0^\circ = 0$$

5-1 Inclined Loop

For a loop with N turns, and surface normal $\hat{\mathbf{n}}$ at angle θ relative to B direction:

$$T = N I A B_0 \sin\theta.$$

magnetic moment of the loop:

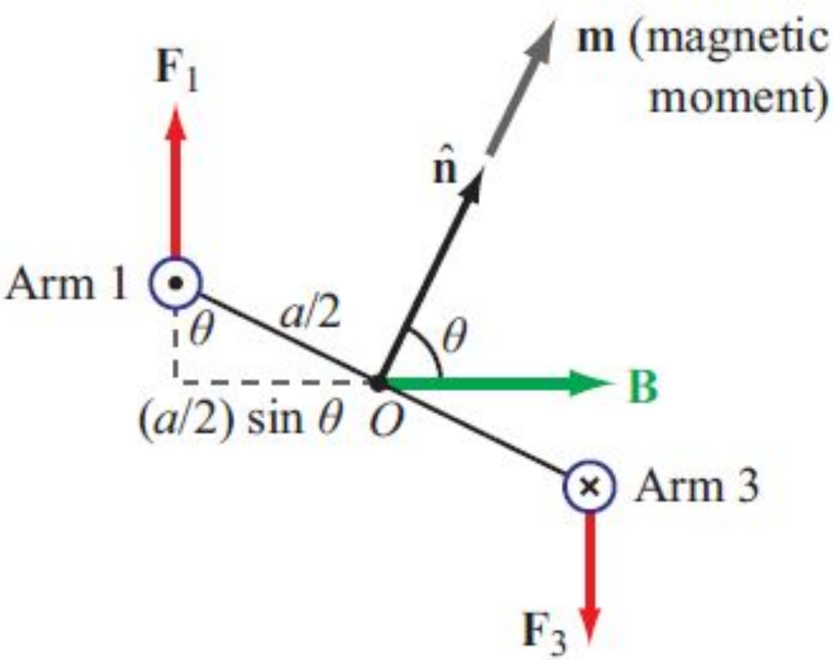
$$m = N I A$$

define:

$$\mathbf{m} = \hat{\mathbf{n}} N I A = \hat{\mathbf{n}} m \qquad (\text{A}\cdot\text{m}^2),$$

so:

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m}).$$



Looking edge-on

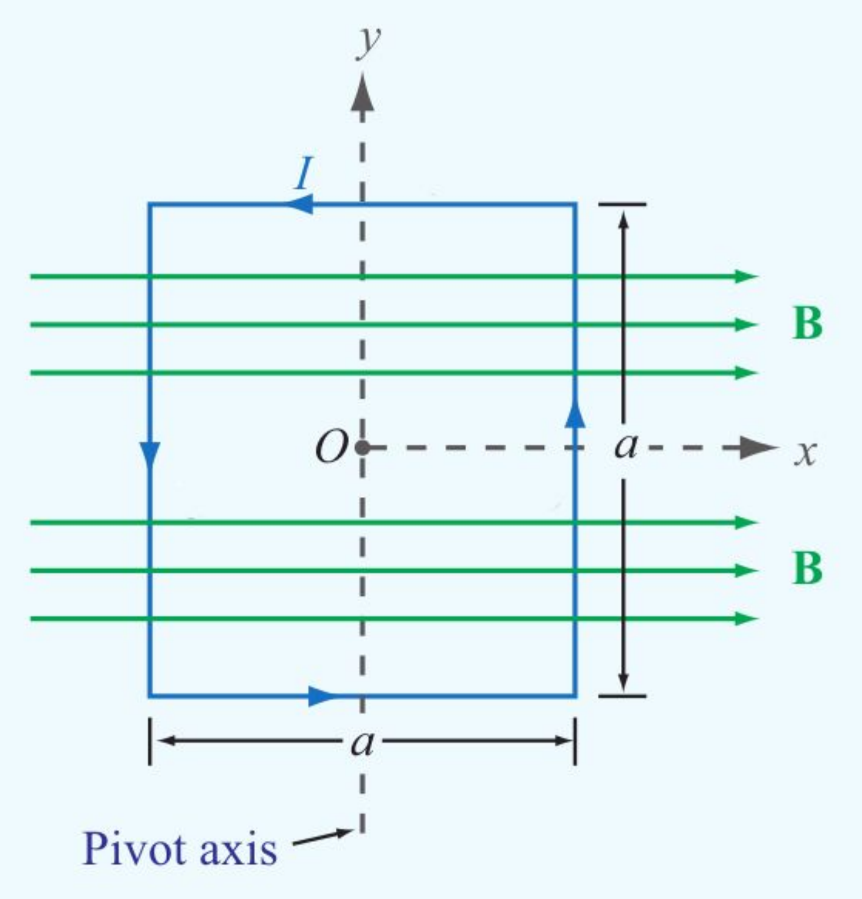
Exercise 5-5 Square Coil

Given: square coil: 100 turns

a=0.5m, B_0=0.2T

max Torque = 0.04 Nm

Find: Current, I



Solution: $T_{\max} = NIAB_0$

$$I = \frac{T_{\max}}{NAB_0}$$

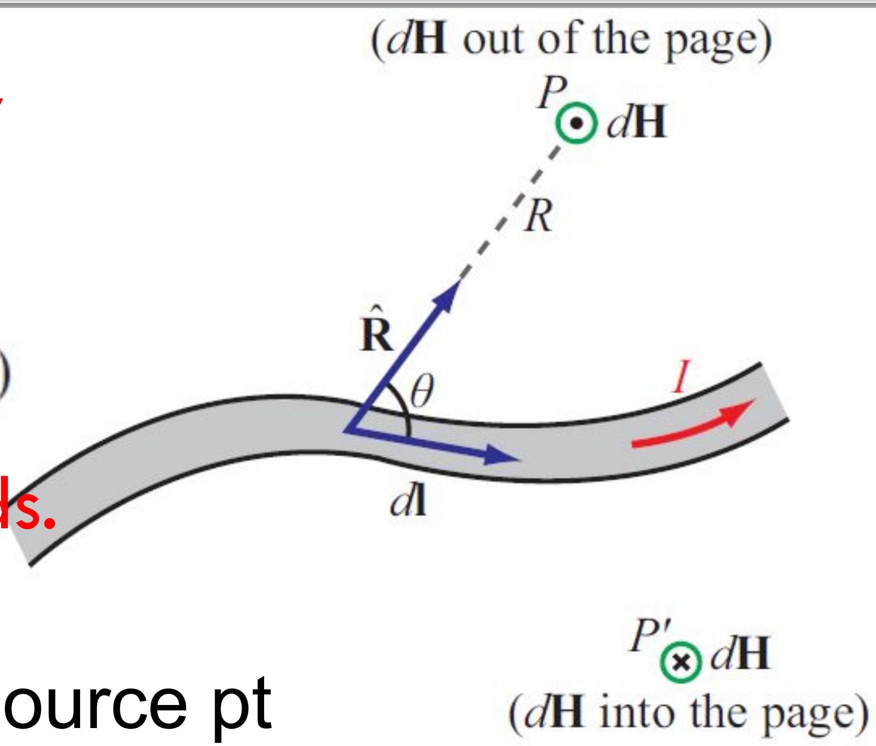
$$= \frac{4 \times 10^{-2}}{100 \times (0.5)^2 \times 0.2} = 8 \quad \text{(mA)}$$

5-2 Biot-Savart Law

Magnetic field induced by a differential current:

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

$\hat{\mathbf{R}}$ NOT in spherical coords.



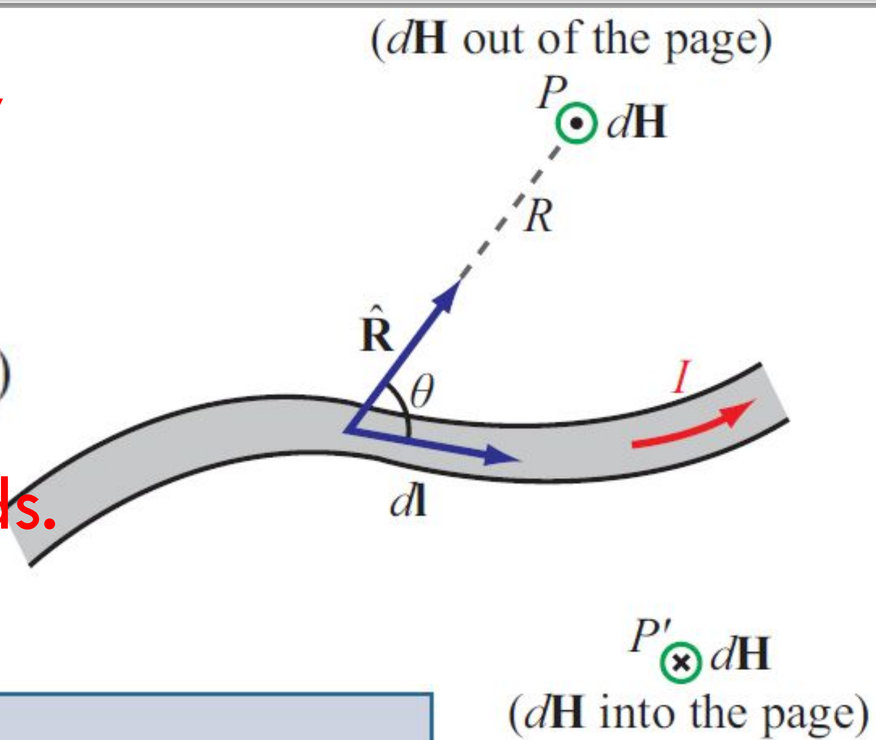
R is the vector from the source pt to the observation point.

5-2 Biot-Savart Law

Magnetic field induced by a differential current:

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

$\hat{\mathbf{R}}$ NOT in spherical coords.



For the entire length:

$$\mathbf{H} = \frac{I}{4\pi} \int_{l} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}, \qquad (5.22)$$

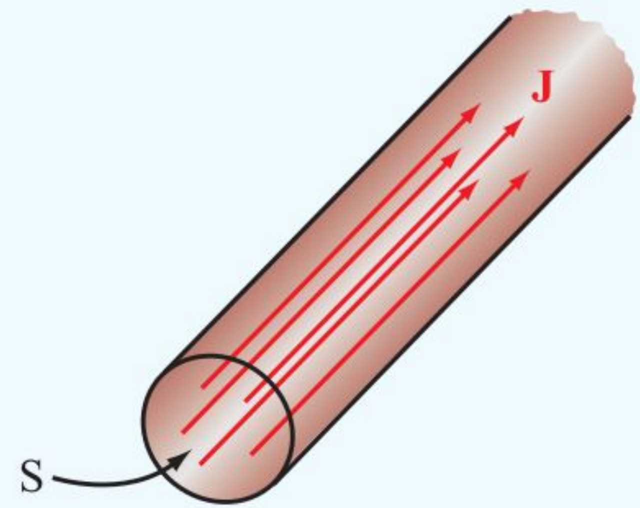
where l is the line path along which I exists.

5-2 Magnetic Field due to Current Densities

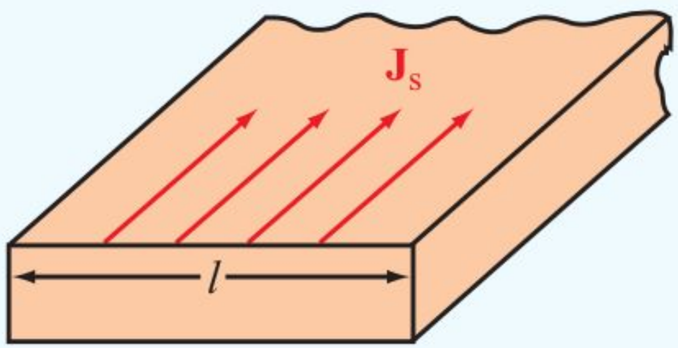
Generalize to Current Densities:

$$\mathbf{H} = \frac{1}{4\pi} \int_S \frac{\mathbf{J}_s \times \hat{\mathbf{R}}}{R^2} \, ds \quad \text{(surface current)},$$

$$\mathbf{H} = \frac{1}{4\pi} \int_V \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R^2} \, dV \quad \text{(volume current)}.$$



(a) Volume current density **J** in A/m^2



(b) Surface current density $\mathbf{J}_s$ in A/m

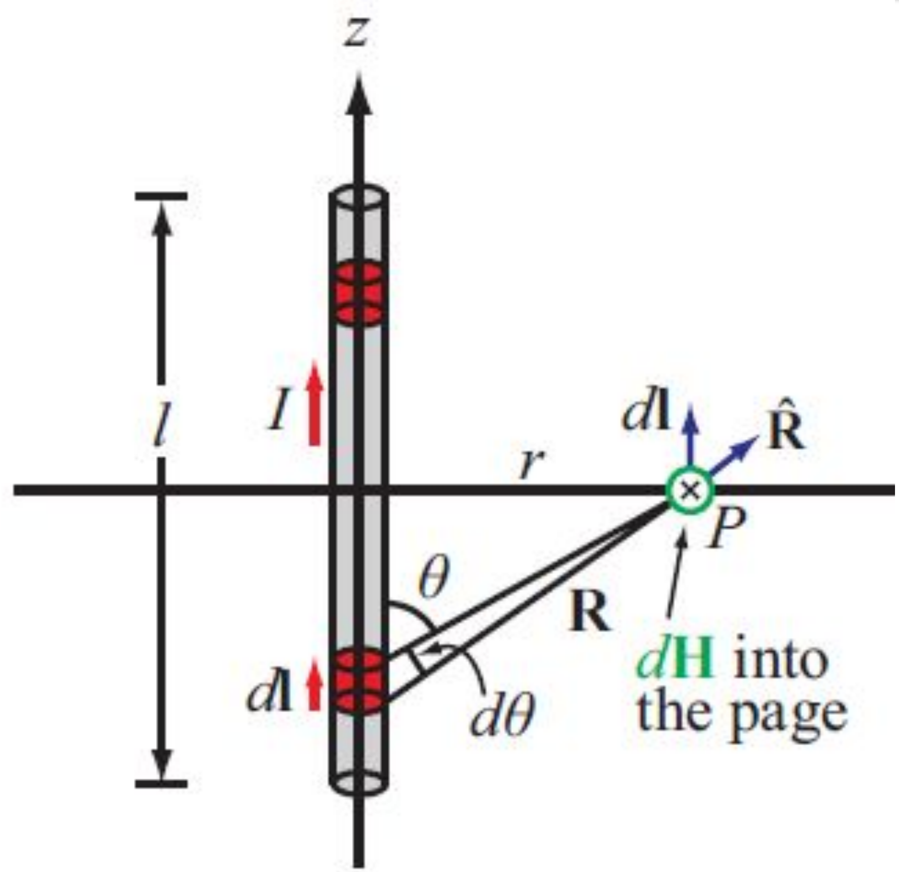
Example 5-2: Straight Current

Given:

A straight, vertical wire along z-axis length l with current I in $+z$-direction

Find:

B in x-y plane at distance r from the wire



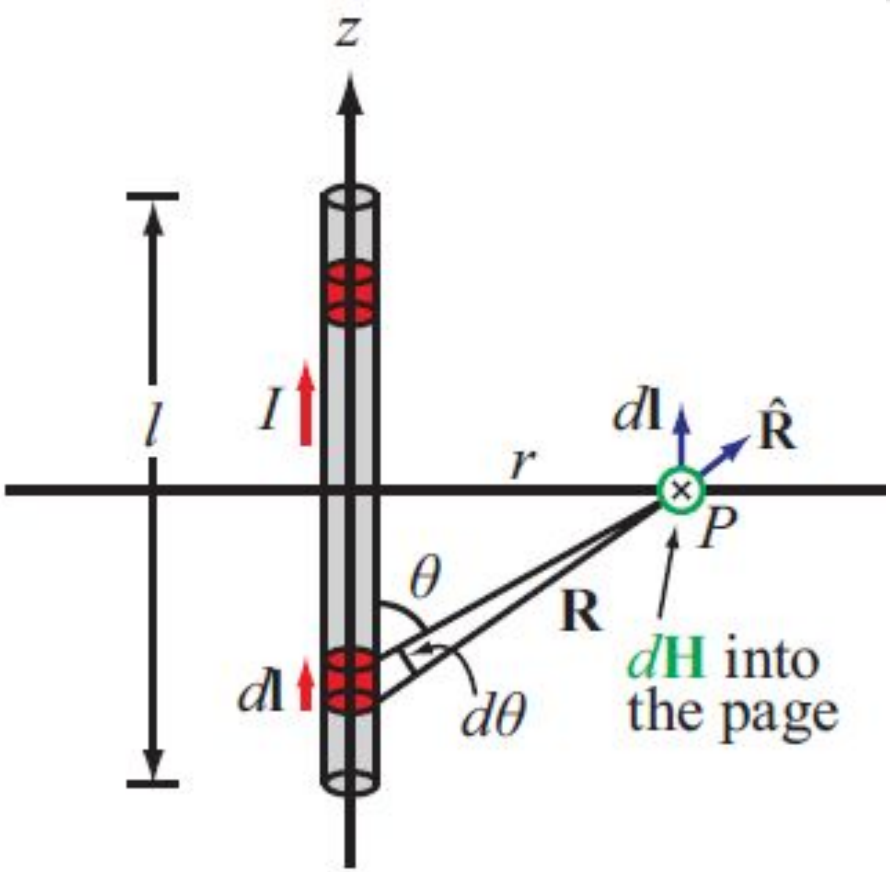
Cont.

Example 5-2: Straight Current

Solution:

Use Biot-Savart formula:

$$\mathbf{H} = \frac{I}{4\pi} \int_{l} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2}$$



$$d\mathbf{l} = \hat{\mathbf{z}}\ dz$$

$$d\mathbf{l} \times \hat{\mathbf{R}} = dz\ (\hat{\mathbf{z}} \times \hat{\mathbf{R}})$$

Cont.

Example 5-2: Straight Current

In cylindrical coordinates:

$$\mathbf{R}_1 = r\hat{\mathbf{r}} + |z|\hat{\mathbf{z}}$$

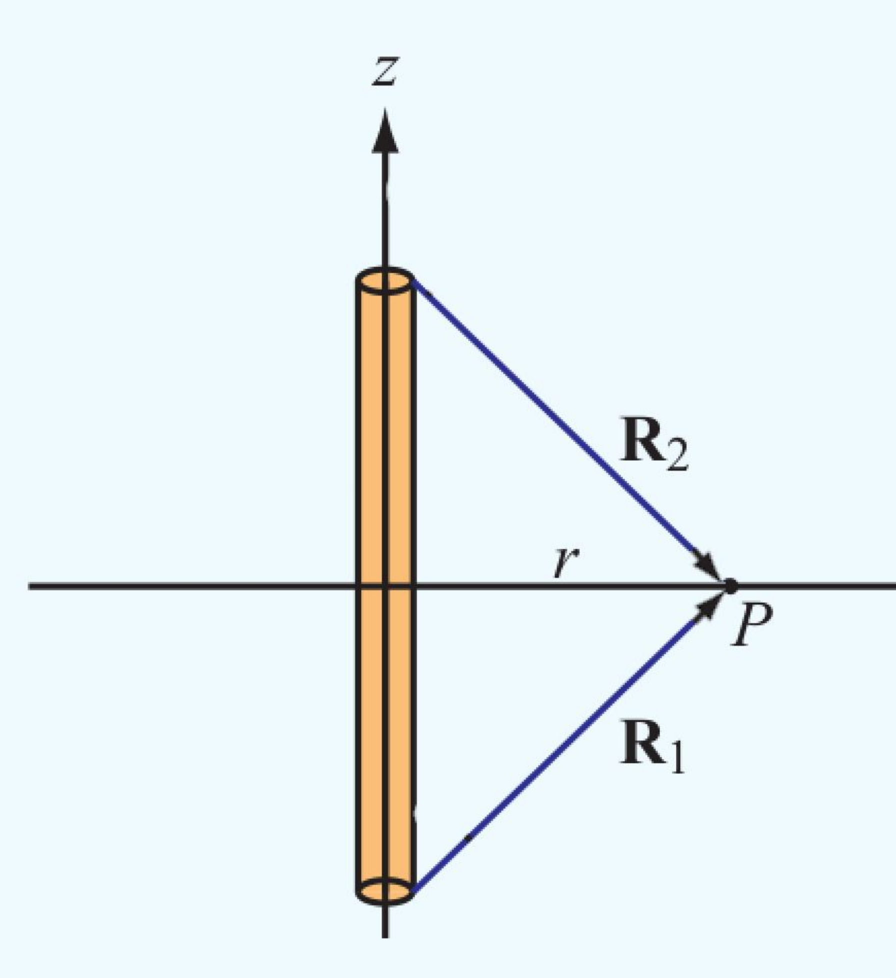
$$\mathbf{R}_1 = r\hat{\mathbf{r}} - z\hat{\mathbf{z}}$$

$$\mathbf{R}_2 = r\hat{\mathbf{r}} - |z|\hat{\mathbf{z}}$$

$$\mathbf{R}_2 = r\hat{\mathbf{r}} - z\hat{\mathbf{z}}$$

so, in general:

$$\mathbf{R} = r\hat{\mathbf{r}} - z\hat{\mathbf{z}}$$



Cont.

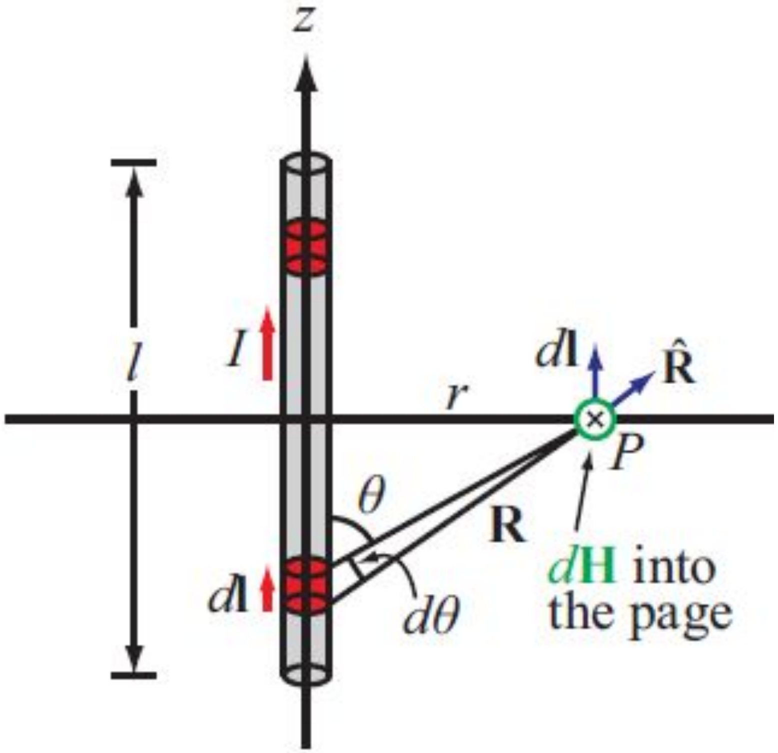
Example 5-2: Straight Current

In cylindrical coordinates:

$$\hat{\mathbf{R}} = \frac{1}{R}(r\hat{\mathbf{r}} - z\hat{\mathbf{z}})$$

$$\begin{aligned} \mathrm{d}\mathbf{l} \times \hat{\mathbf{R}} &= (\mathrm{d}z\hat{\mathbf{z}}) \times \frac{1}{R}(r\hat{\mathbf{r}} - z\hat{\mathbf{z}}) \\ &= \frac{\mathrm{d}z}{R}(r\hat{\mathbf{z}} \times \hat{\mathbf{r}} - z\hat{\mathbf{z}} \times \hat{\mathbf{z}}) \\ &= \frac{\mathrm{d}z}{R} r\hat{\boldsymbol{\phi}} \end{aligned}$$

$$\mathrm{d}\mathbf{l} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\phi}} \sin\theta \, \mathrm{d}z$$

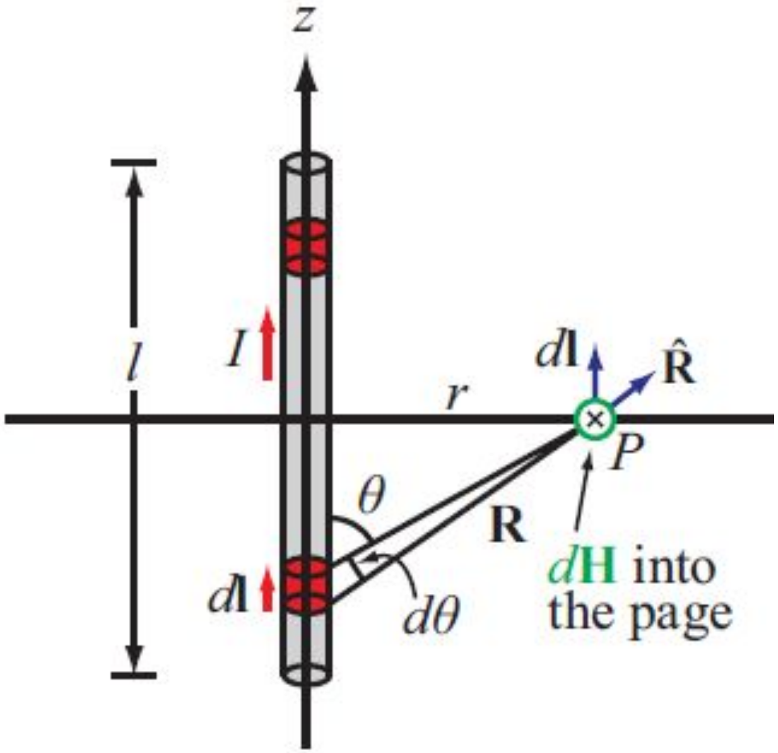


Cont.

Example 5-2: Straight Current

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2}$$

$$d\mathbf{l} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\phi}} \sin\theta \; dz,$$



$$\mathbf{H} = \frac{I}{4\pi} \int_{z=-l/2}^{z=l/2} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \int_{-l/2}^{l/2} \frac{\sin\theta}{R^2} \, dz.$$

Cont.

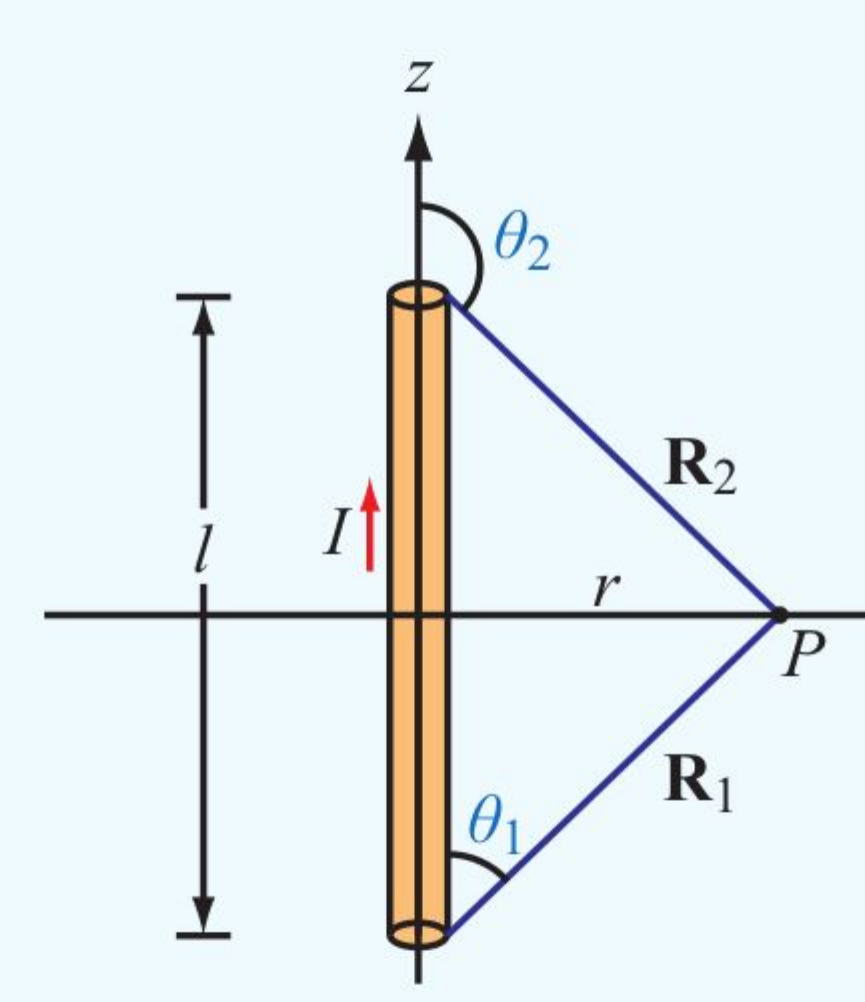
Example 5-2: Straight Current

Need to use trigonometry to express (sinθ/R^2) dz as something we can integrate:

$$\sin\theta = \frac{r}{R} \rightarrow R = \frac{r}{\sin\theta}$$

$$\cos\theta = \frac{-z}{R}$$

$$\text{since for } z < 0, \cos\theta > 0$$



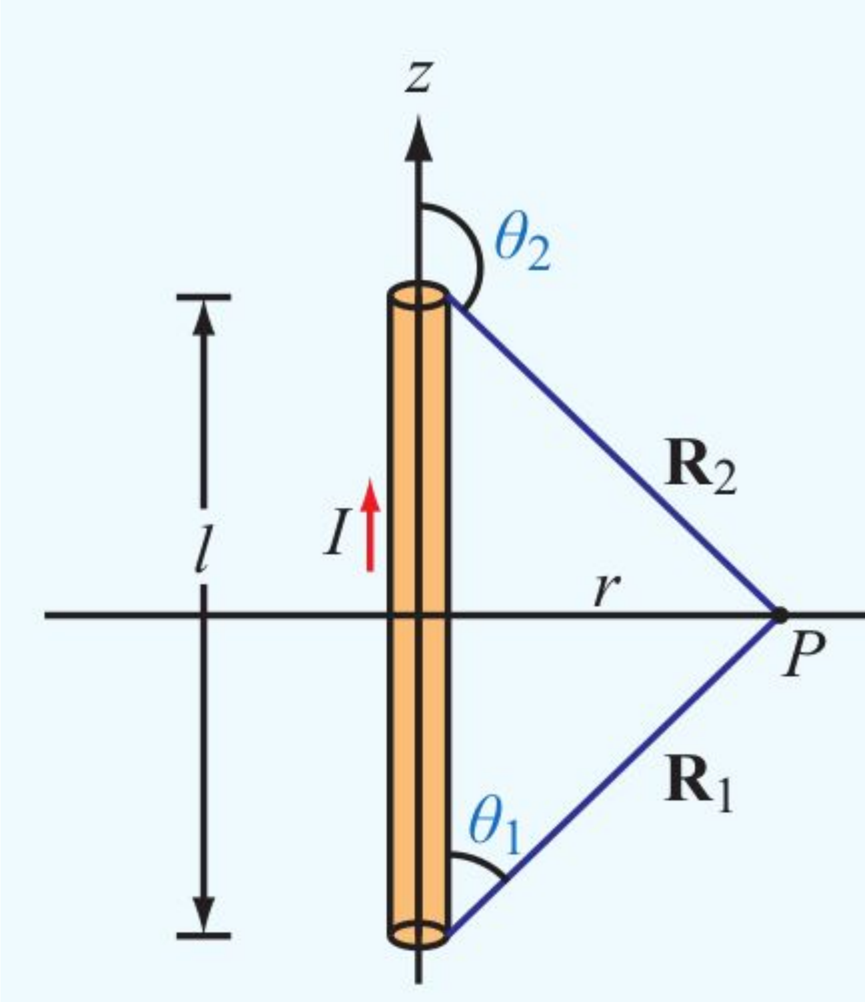
Cont.

Example 5-2: Straight Current

$$z = -R\cos\theta$$

$$z = -r\frac{\cos\theta}{\sin\theta}$$

$$z = -r\frac{1}{\tan\theta}$$



Cont.

Example 5-2: Straight Current

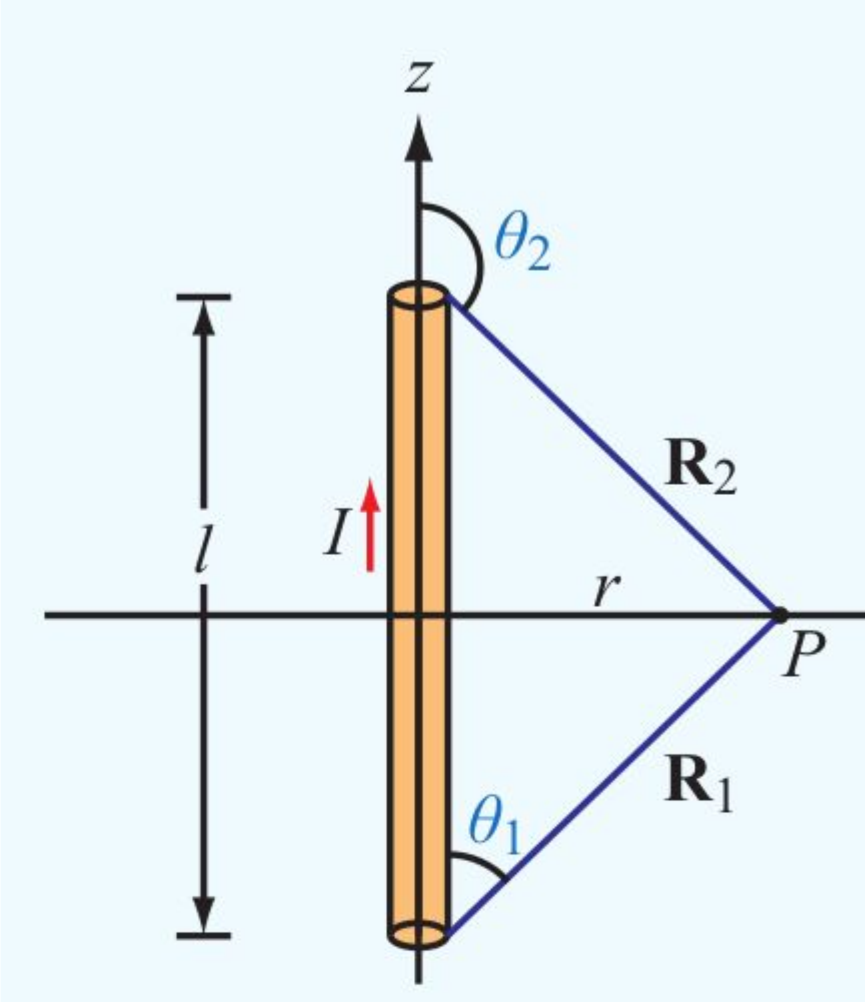
$$z = -r\frac{1}{\tan\theta}$$

$$\mathrm{d}z = -r(-1)\tan^{-2}\theta\sec^2\theta\mathrm{d}\theta$$

$$\mathrm{d}z = r\frac{\tan^{-2}\theta}{\cos^2\theta}\mathrm{d}\theta$$

$$\mathrm{d}z = r\frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\cos^2\theta}\mathrm{d}\theta$$

$$\mathrm{d}z = r\frac{1}{\sin^2\theta}\mathrm{d}\theta$$



Cont.

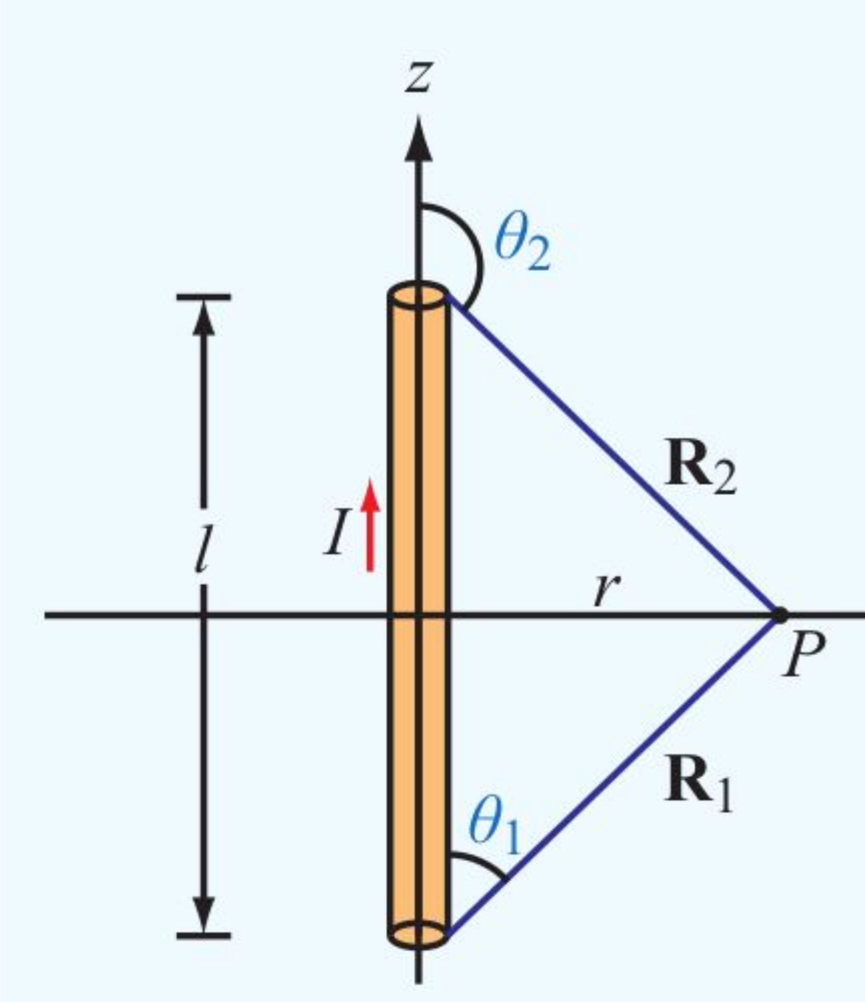
Example 5-2: Straight Current

$$\frac{\sin\theta}{R^2}\mathrm{d}z = \sin\theta\frac{\sin^2\theta}{r^2}r\frac{1}{\sin^2\theta}\mathrm{d}\theta$$

$$\frac{\sin\theta}{R^2}\mathrm{d}z = \frac{1}{r}\sin\theta\mathrm{d}\theta$$

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi r}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$$

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi r}(\cos\theta_1 - \cos\theta_2)$$



Cont.

Example 5-2: Straight Current

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi r}(\cos\theta_1 - \cos\theta_2)$$

Since:

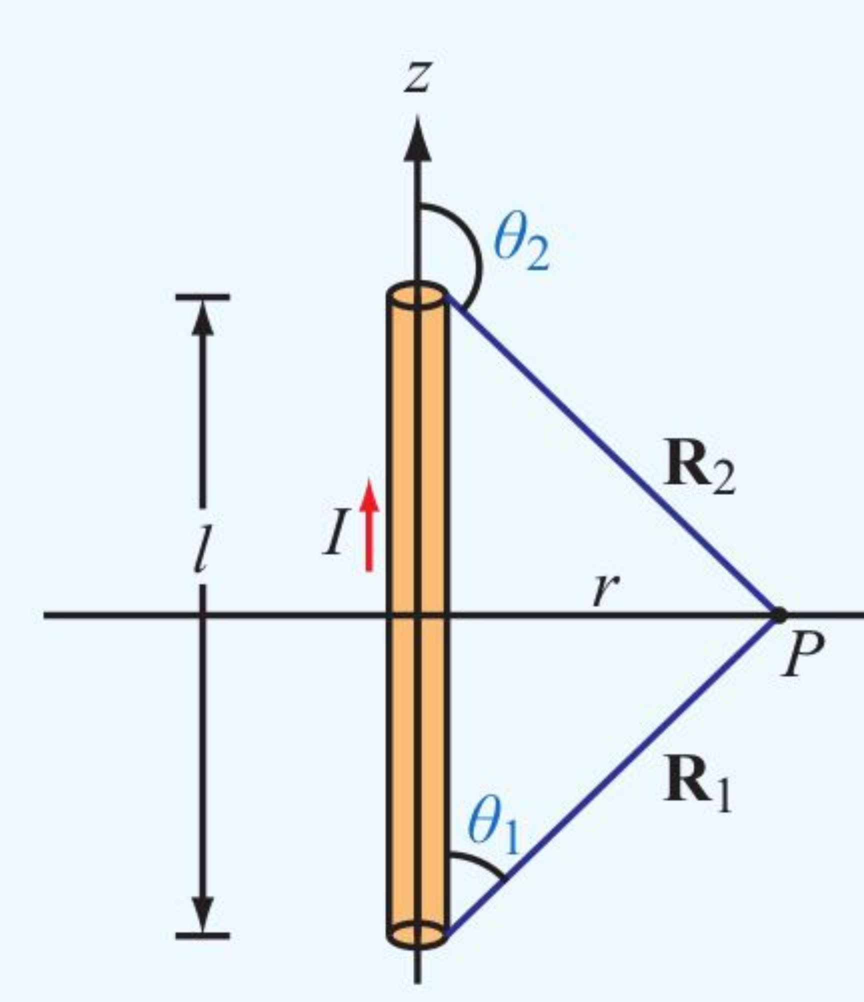
$$\theta_2 = 180^\circ - \theta_1$$

$$\cos\theta_2 = \cos(180^\circ - \theta_1)$$

$$\cos\theta_2 = \cos 180^\circ \cos\theta_1 + \sin 180^\circ \sin\theta_1$$

$$\cos\theta_2 = -\cos\theta_1$$

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r}\cos\theta_1$$



Cont.

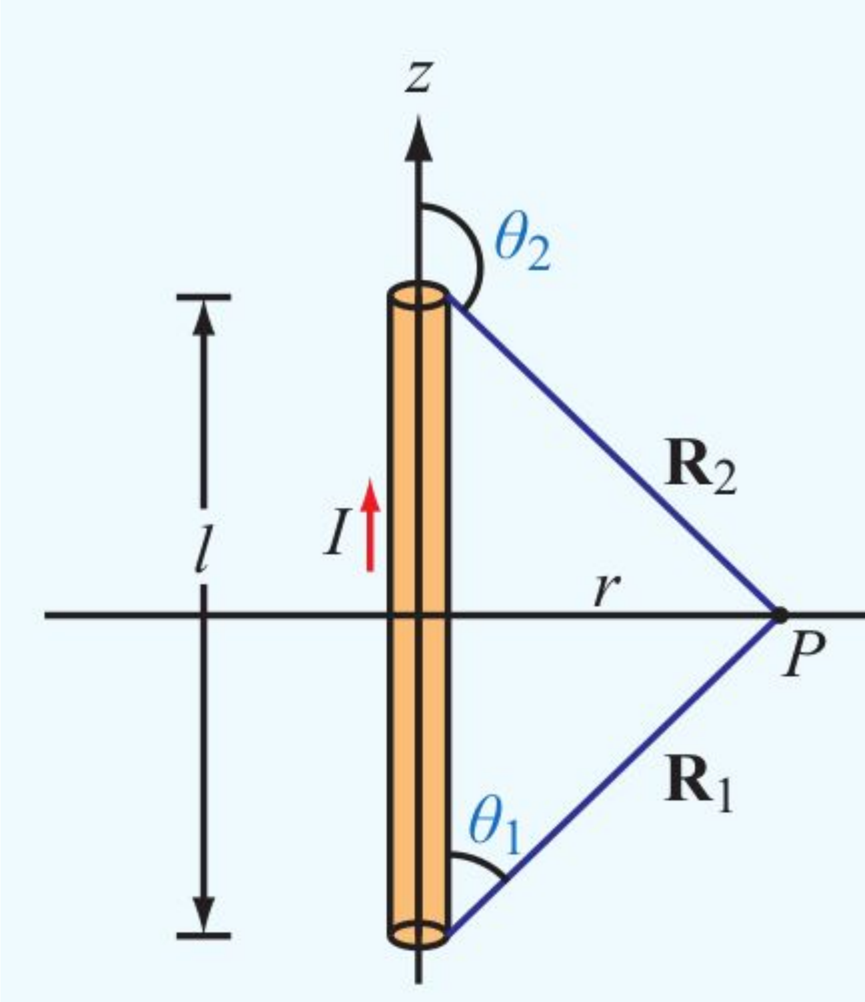
Example 5-2: Straight Current

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r} \cos\theta_1$$

$$\cos\theta_1 = \frac{l/2}{\sqrt{r^2 + (l/2)^2}}$$

$$\cos\theta_1 = \frac{l}{\sqrt{4r^2 + l^2}}$$

$$\mathbf{B} = \mu_0 \mathbf{H} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I l}{2\pi r\sqrt{4r^2 + l^2}}$$



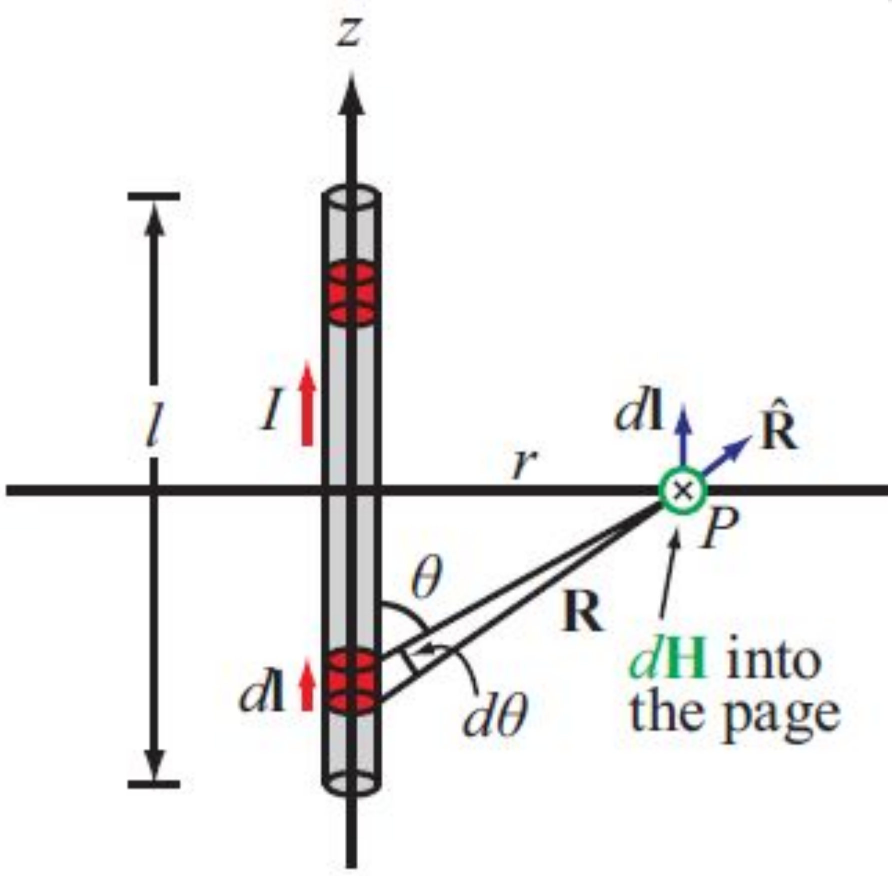
Cont.

Example 5-2: Straight Current

$$\mathbf{B} = \mu_0 \mathbf{H} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I l}{2\pi r \sqrt{4r^2 + l^2}}$$

For a very long wire:
In the limit as $l \to \infty$

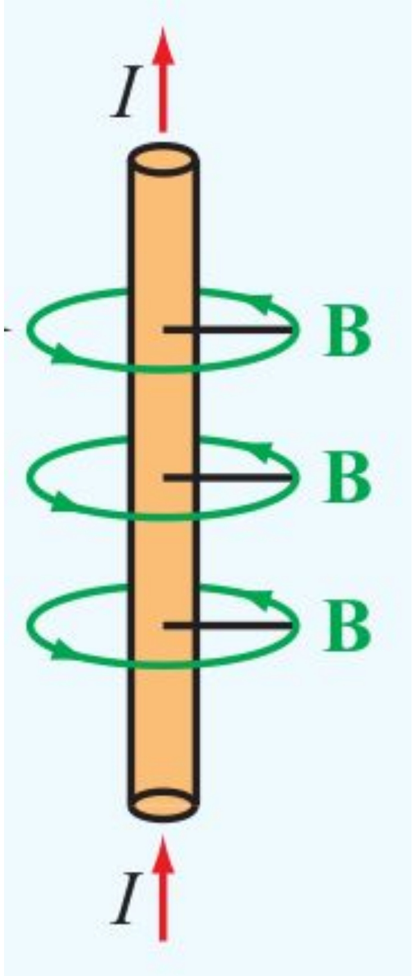
$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$

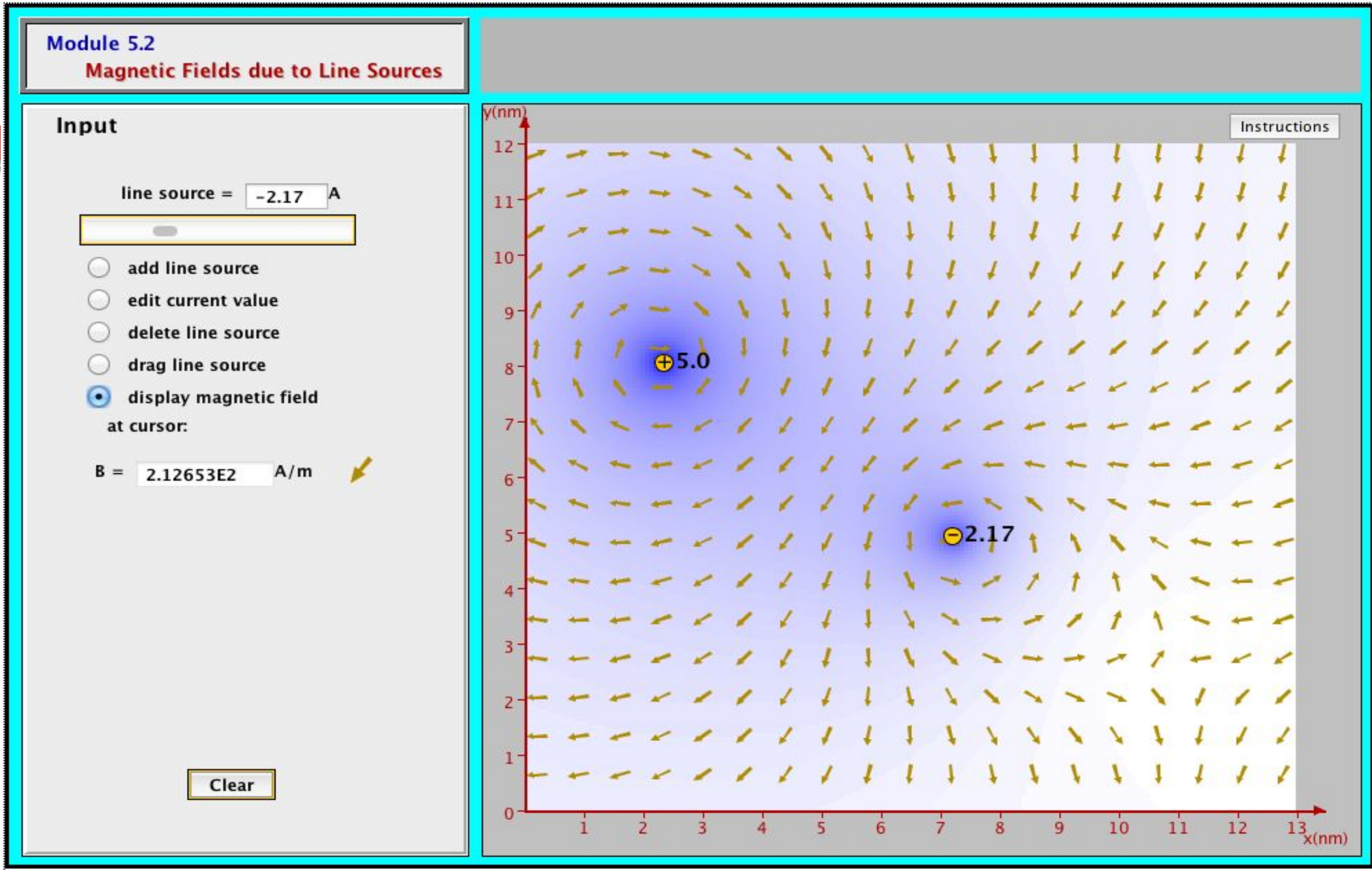


Example 5-2: Straight Current

Infinitely-long Wire

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$




Module 5.2
Magnetic Fields due to Line Sources
Input
line source = -2.17 A
add line source
edit current value
delete line source
drag line source
display magnetic field
at cursor:
B = 2.12653E2 A/m
Clear
Instructions
y(nm)
x(nm)
5.0
2.17

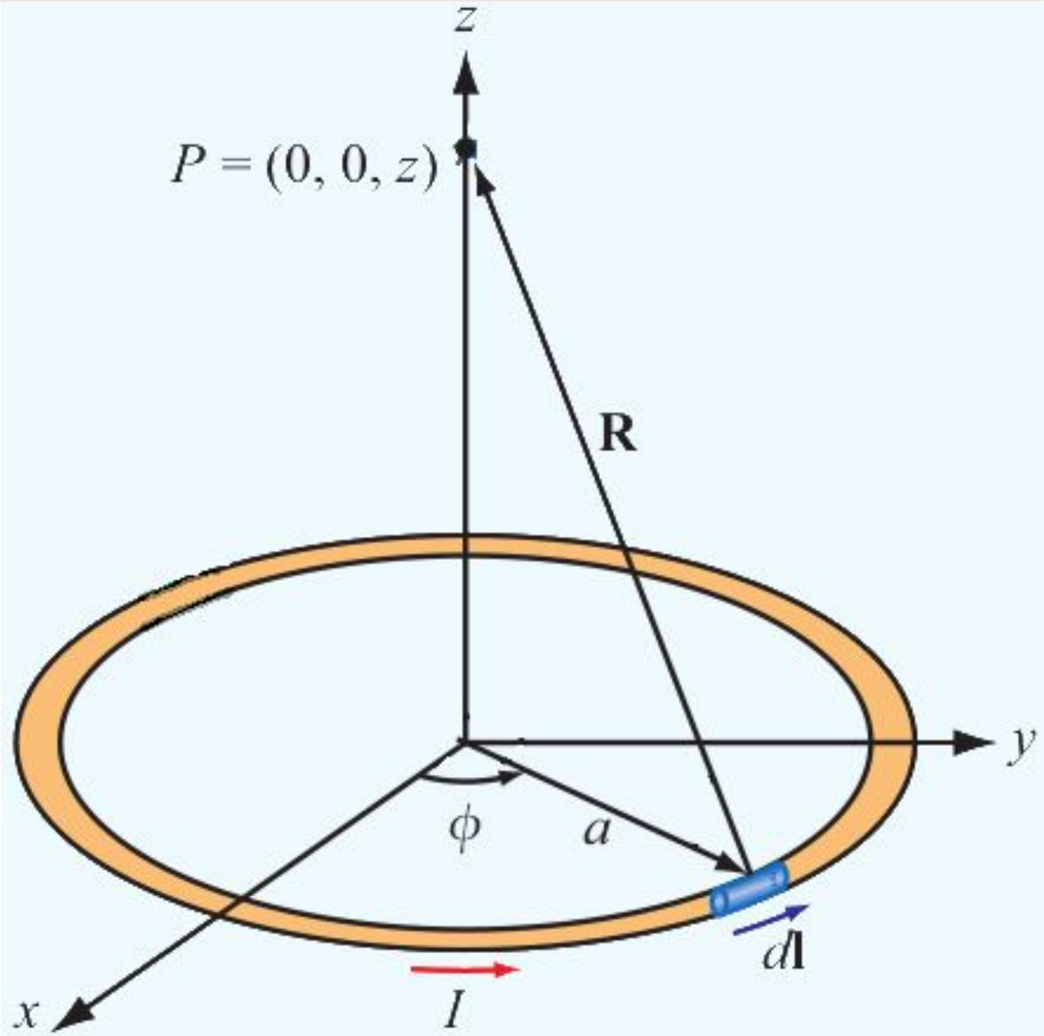
Example 5-3: Circular Current Loop

Given:

Current in a circular loop centered at origin in *x*-*y* plane radius *a*

Find:

H at point along *z*-axis



Cont.

Example 5-3: Circular Current Loop

Solution:

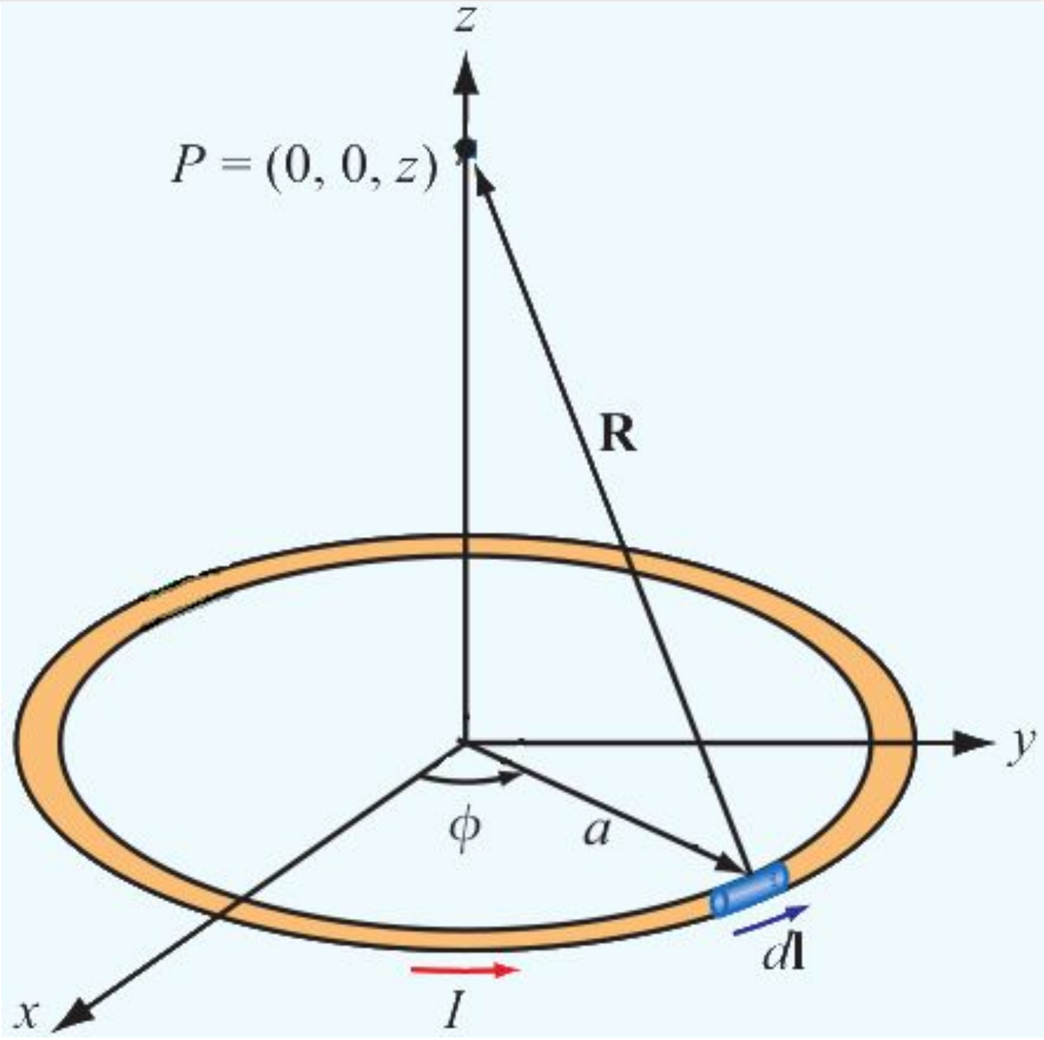
Use Biot-Savart formula:

$$\mathbf{H} = \frac{I}{4\pi} \int_{l} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2}$$

In cylindrical coords:

$$\mathrm{d}\mathbf{l} = a\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{R}} = \frac{1}{R}(-a\hat{\mathbf{r}} + z\hat{\mathbf{z}})$$



Cont.

Example 5-3: Circular Current Loop

$$\mathrm{d}\mathbf{l} = a\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} \qquad \hat{\mathbf{R}} = \frac{1}{R}(-a\hat{\mathbf{r}} + z\hat{\mathbf{z}})$$

$$\begin{aligned}
\mathrm{d}\mathbf{l} \times \hat{\mathbf{R}} &= \frac{a\mathrm{d}\phi}{R}\hat{\boldsymbol{\phi}} \times (-a\hat{\mathbf{r}} + z\hat{\mathbf{z}}) \\
&= \frac{a\mathrm{d}\phi}{R}(-a\,\hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} + z\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) \\
&= \frac{a\mathrm{d}\phi}{R}(-a(-\hat{\mathbf{z}}) + z(\hat{\mathbf{r}}))
\end{aligned}$$

$$\mathrm{d}\mathbf{l} \times \hat{\mathbf{R}} = \frac{a\mathrm{d}\phi}{R}(a\hat{\mathbf{z}} + z\hat{\mathbf{r}})$$

Cont.

Example 5-3: Circular Current Loop

$$\mathrm{d}\mathbf{l} \times \hat{\mathbf{R}} = \frac{a\,\mathrm{d}\phi}{R}(a\hat{\mathbf{z}} + z\hat{\mathbf{r}})$$

$$\mathbf{H} = \frac{I}{4\pi} \int_{l} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2}$$

$$\mathbf{H} = \frac{I}{4\pi} \int_{\phi=0}^{2\pi} \frac{1}{R^2} \frac{a}{R}(a\hat{\mathbf{z}} + z\hat{\mathbf{r}})\mathrm{d}\phi$$

Cont.

Example 5-3: Circular Current Loop

$$\mathbf{H} = \frac{I}{4\pi} \int_{\phi=0}^{2\pi} \frac{1}{R^2} \frac{a}{R} (a\hat{\mathbf{z}} + z\hat{\mathbf{r}}) \mathrm{d}\phi$$

Since $\hat{\mathbf{r}}$ changes with ϕ, express it as an explicit function of ϕ:

$$\hat{\mathbf{r}} = \cos\phi \hat{\mathbf{x}} + \sin\phi \hat{\mathbf{y}}$$

$$\mathbf{H} = \frac{Ia}{4\pi R^3} \int_{\phi=0}^{2\pi} (a\hat{\mathbf{z}} + z\cos\phi \hat{\mathbf{x}} + z\sin\phi \hat{\mathbf{y}}) \mathrm{d}\phi$$

Cont.

Example 5-3: Circular Current Loop

$$\mathbf{H} = \frac{Ia}{4\pi R^3}\int_{\phi=0}^{2\pi}(a\hat{\mathbf{z}} + z\cos\phi\hat{\mathbf{x}} + z\sin\phi\hat{\mathbf{y}})\mathrm{d}\phi$$

$$\mathbf{H} = \frac{Ia}{4\pi R^3}\left[2\pi a\hat{\mathbf{z}} + z\sin\phi\hat{\mathbf{x}}\Big|_0^{2\pi} - z\cos\phi\hat{\mathbf{y}}\Big|_0^{2\pi}\right]$$

$$\mathbf{H} = \frac{I2\pi a^2}{4\pi R^3}\hat{\mathbf{z}} = \frac{Ia^2}{2R^3}\hat{\mathbf{z}} \qquad R = \sqrt{a^2 + z^2}$$

Cont.

Example 5-3: Circular Current Loop

Since the magnetic moment of a loop in the *x*-*y* plane is:

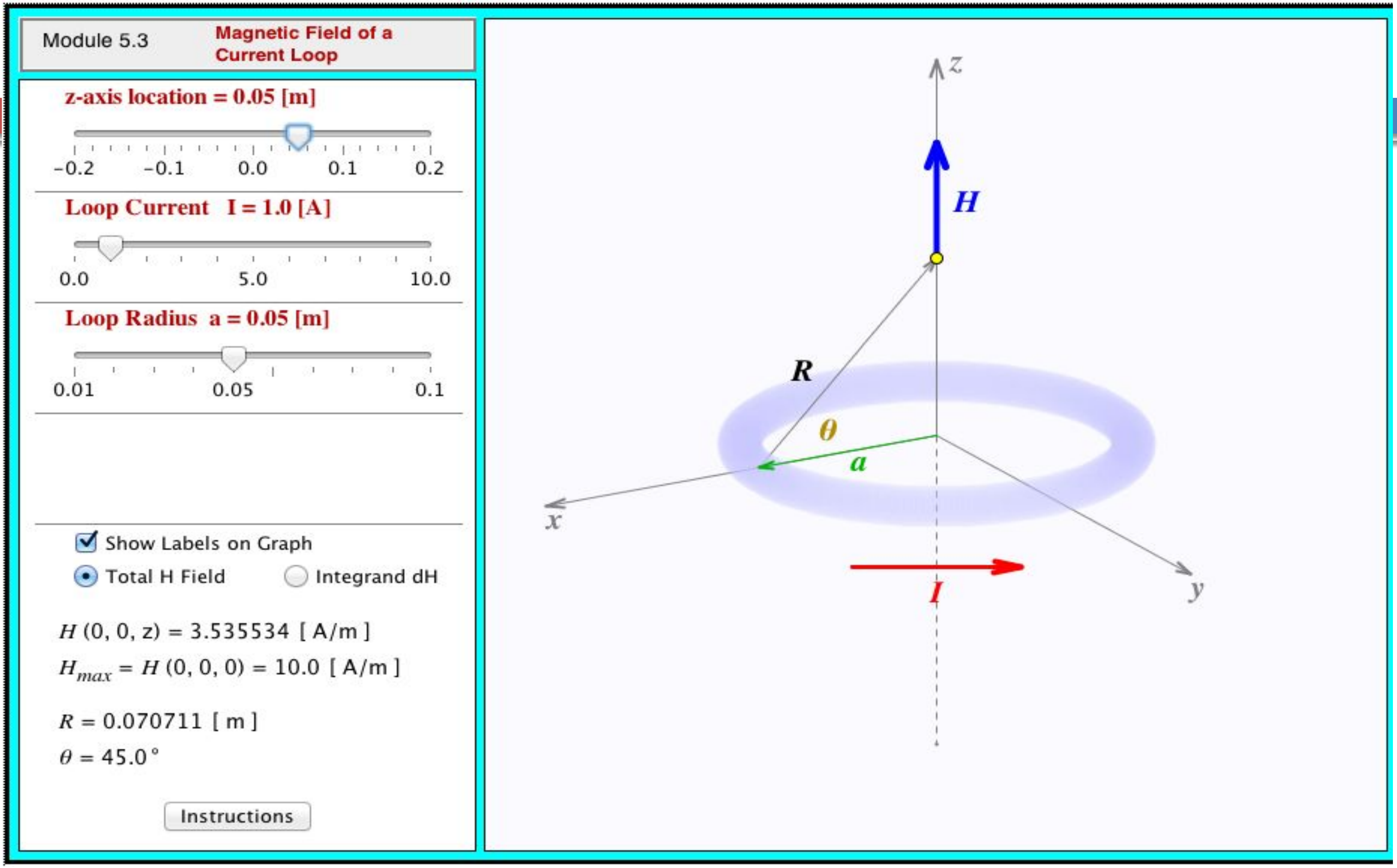
$$\mathbf{m} = \hat{\mathbf{z}} I A$$

the magnetic moment of this loop is:

$$\mathbf{m} = \hat{\mathbf{z}} I \pi a^2$$

so field along *z*-axis:

$$\mathbf{H} = \frac{I a^2}{2R^3} \hat{\mathbf{z}} = \frac{\mathbf{m}}{2\pi R^3}$$

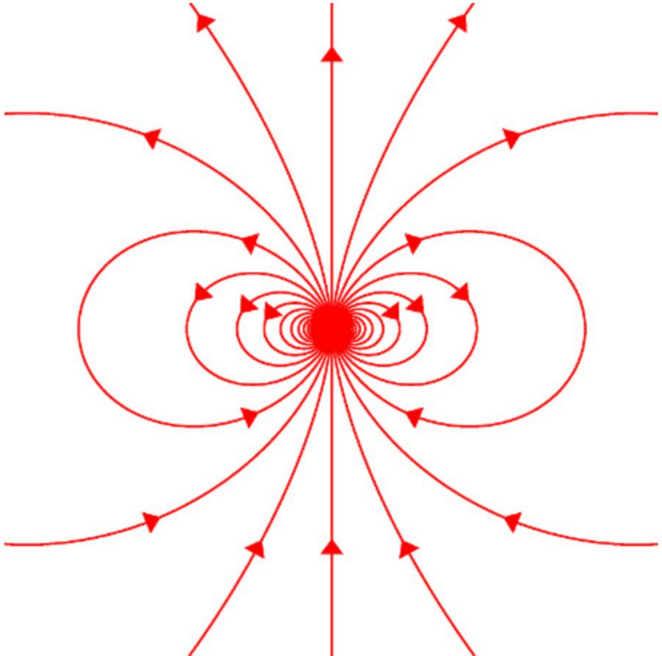


Module 5.3
Magnetic Field of a Current Loop
z-axis location = 0.05 [m]
−0.2 −0.1 0.0 0.1 0.2
Loop Current I = 1.0 [A]
0.0 5.0 10.0
Loop Radius a = 0.05 [m]
0.01 0.05 0.1
Show Labels on Graph
Total H Field
Integrand dH
H (0, 0, z) = 3.535534 [A/m]
Hmax = H (0, 0, 0) = 10.0 [A/m]
R = 0.070711 [m]
θ = 45.0°
Instructions
z
H
R
θ
a
x
y
I

5-2 Magnetic Dipole

Solving for the fields **everywhere** far from a current loop results in:

$$\mathbf{H} = \frac{m}{4\pi R^3}\left(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \qquad (\text{for } R \gg a).$$

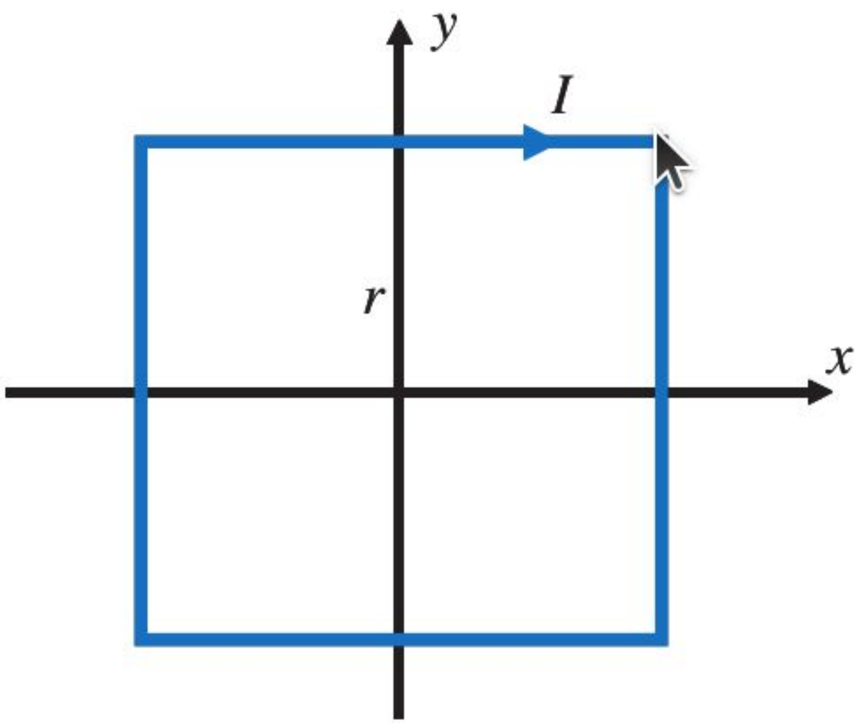


(physics.stackexchange.com)

Exercise 5.8 Current Loop

Given: square loop:
side lengths: 40cm
current I = 5A

Find: **H** at the origin



Solution:

For one finite wire:

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{Il}{2\pi r \sqrt{4r^2 + l^2}}$$

in this case:

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{z}}$$

$$r = l/2$$

Exercise 5.8 Current Loop

Solution:

For one finite wire:

$$\mathbf{H} = -\hat{\mathbf{z}}\frac{Il}{2\pi(l/2)\sqrt{l^2 + l^2}}$$

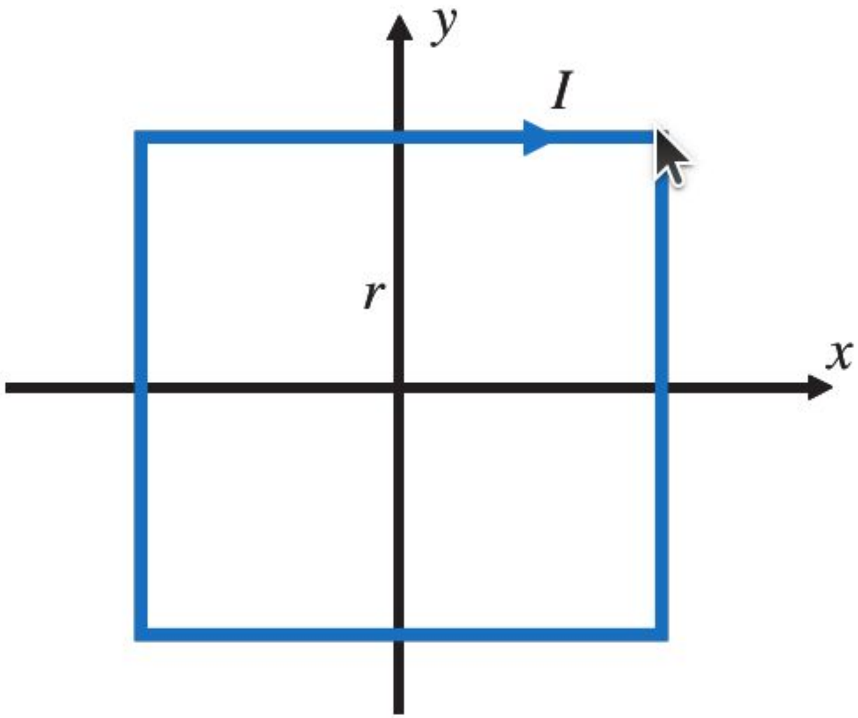
$$\mathbf{H} = -\hat{\mathbf{z}}\frac{I}{\pi l\sqrt{2}}$$

for 4 wires:

$$\mathbf{H} = -\hat{\mathbf{z}}\frac{4I}{\pi l\sqrt{2}}$$

$$\mathbf{H} = -\hat{\mathbf{z}}\frac{4(5\,\text{A})}{\pi(0.4\,\text{m})\sqrt{2}}$$

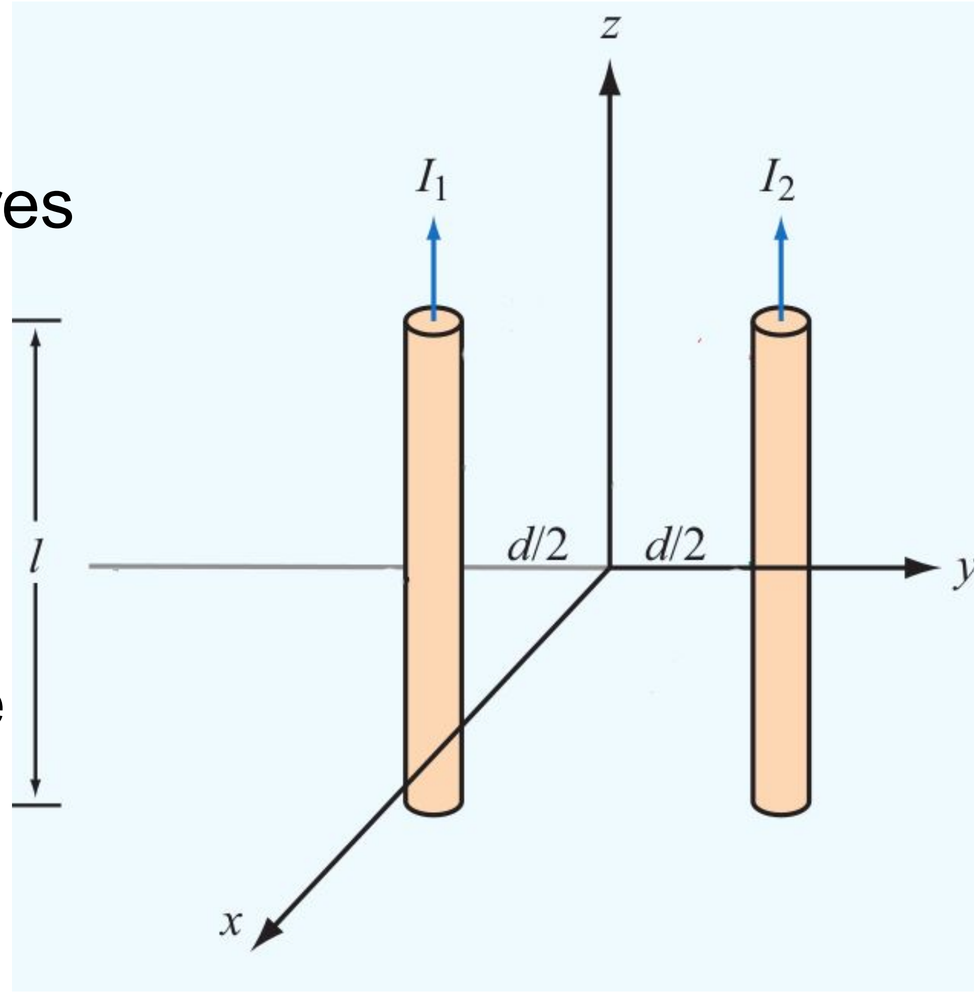
$$\boxed{\mathbf{H} = -\hat{\mathbf{z}}11.25\,\text{A/m}}$$



5-2 Forces on Parallel Conductors

Given:
2 parallel infinitely-long wires
current along +*z*-axis
separation: *d*

Find:
Force per unit length
exerted by one wire on the other



5-2 Forces on Parallel Conductors

Solution:

Force is:

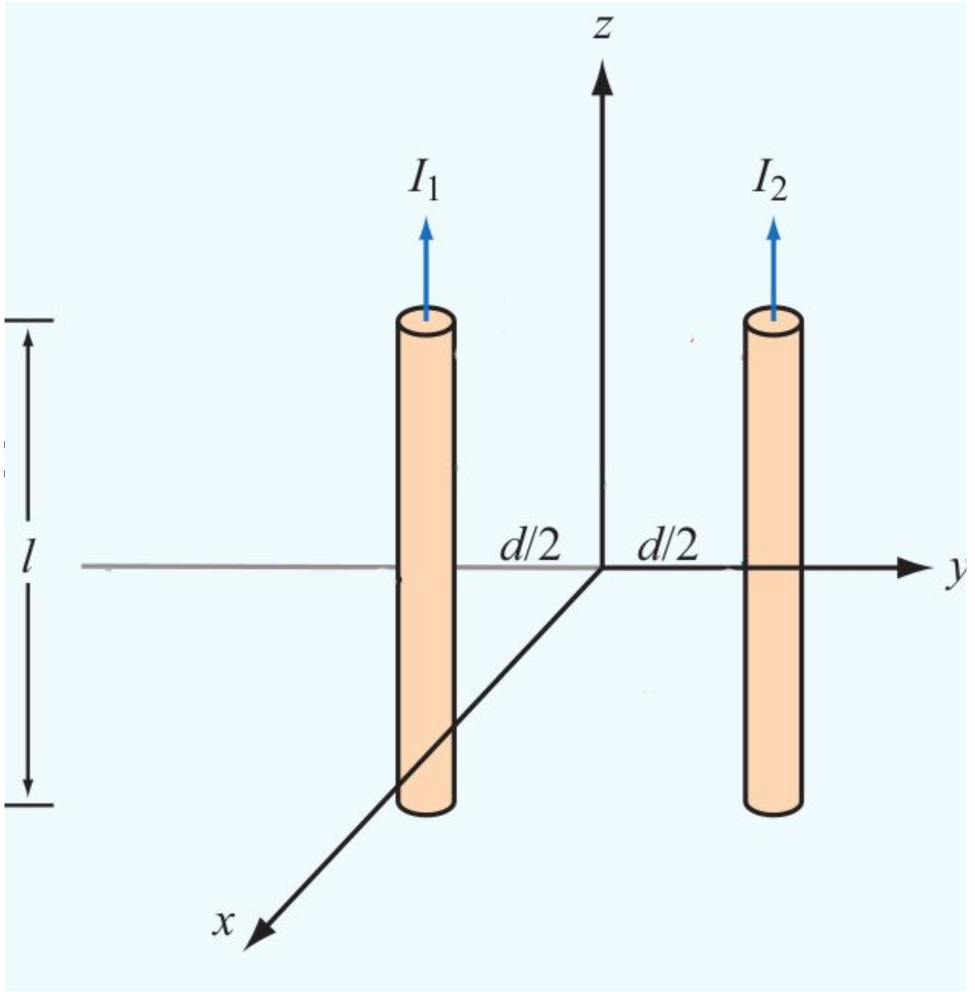
$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}$$

Field for infinitely long wire:

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$

Field from wire1 at wire2:

$$\mathbf{B}_1 = -\hat{\mathbf{x}} \frac{\mu_0 I_1}{2\pi d}$$



5-2 Forces on Parallel Conductors

Solution:

Force is:

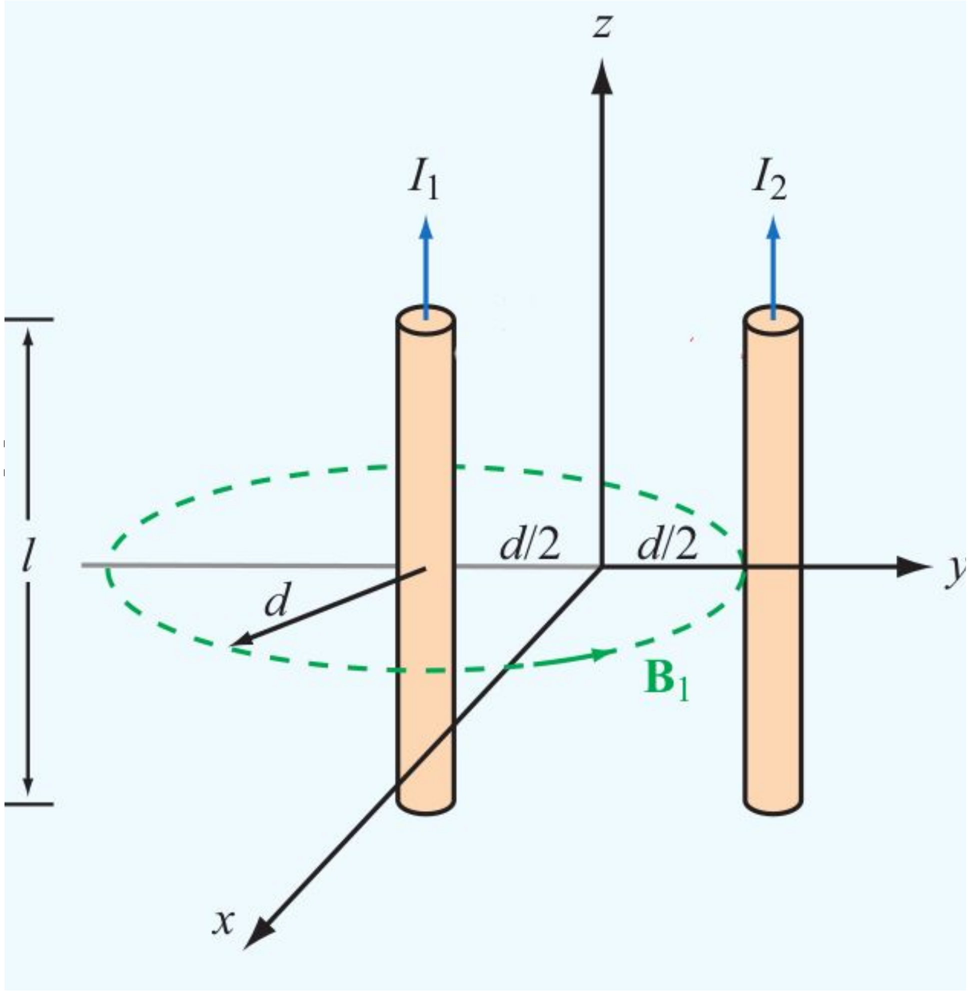
$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}$$

Field for infinitely long wire:

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$

Field from wire1 at wire2:

$$\mathbf{B}_1 = -\hat{\mathbf{x}} \frac{\mu_0 I_1}{2\pi d}$$



5-2 Forces on Parallel Conductors

For a length-l portion of wire2:

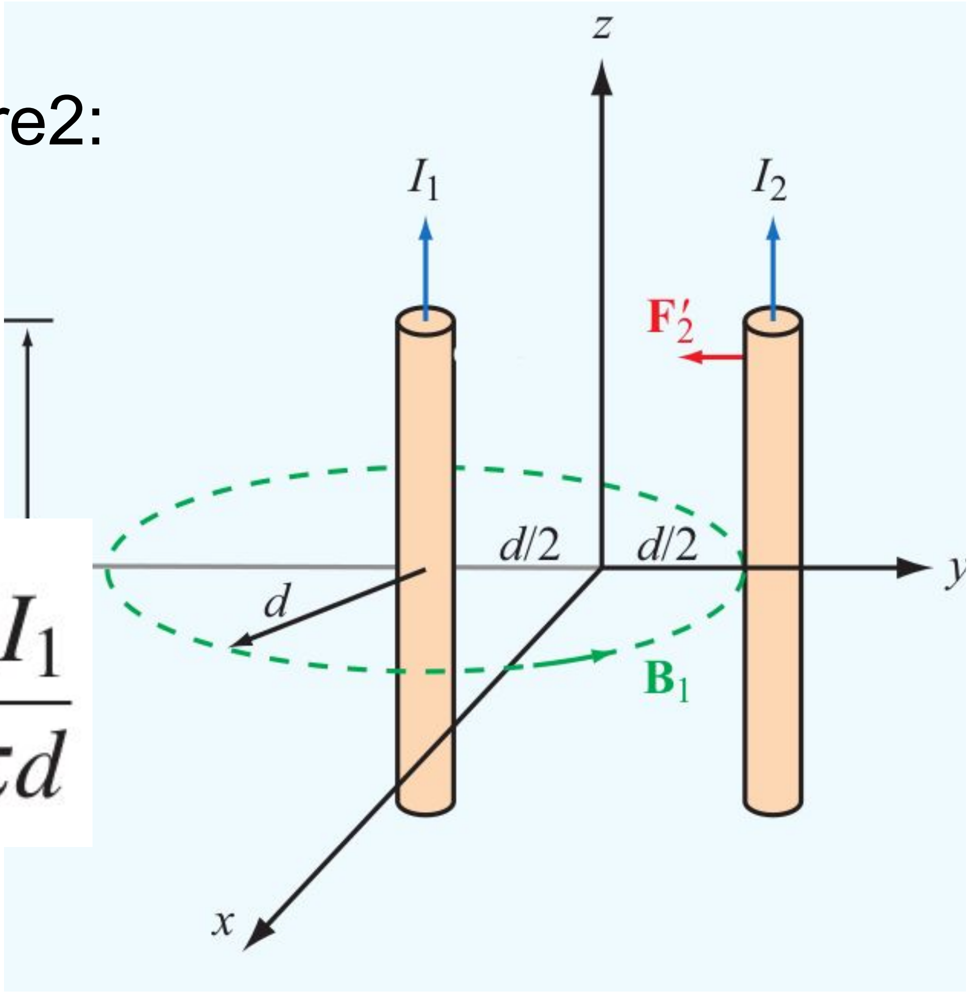
$$q\mathbf{u} = I_2 l \hat{\mathbf{z}}$$

so force on wire2:

$$\mathbf{F}_2 = I_2 l \hat{\mathbf{z}} \times \mathbf{B}_1$$

$$\mathbf{F}_2 = I_2 l \hat{\mathbf{z}} \times (-\hat{\mathbf{x}}) \frac{\mu_0 I_1}{2\pi d}$$

$$\mathbf{F}_2 = -\hat{\mathbf{y}} \frac{\mu_0 I_1 I_2 l}{2\pi d}$$



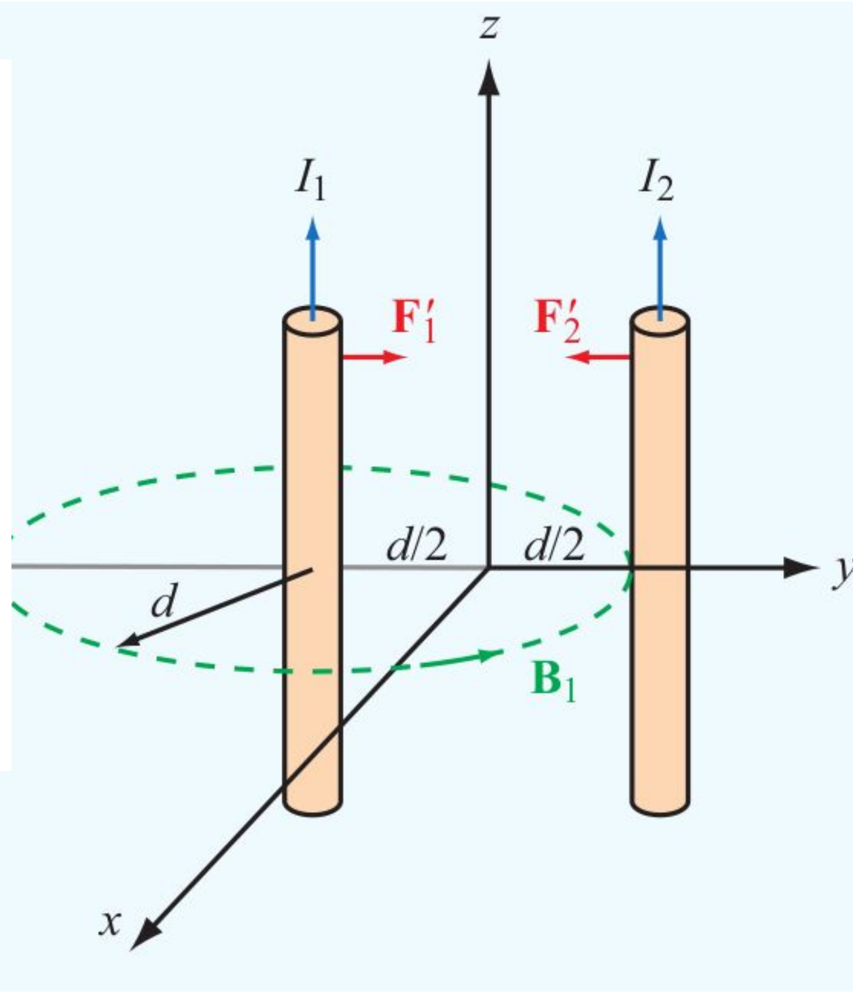
5-2 Forces on Parallel Conductors

So the force per unit length from wire1 on wire2 is:

$$\mathbf{F}_2' = -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}$$

similarly, from wire2 on wire1:

$$\mathbf{F}_1' = \hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}$$

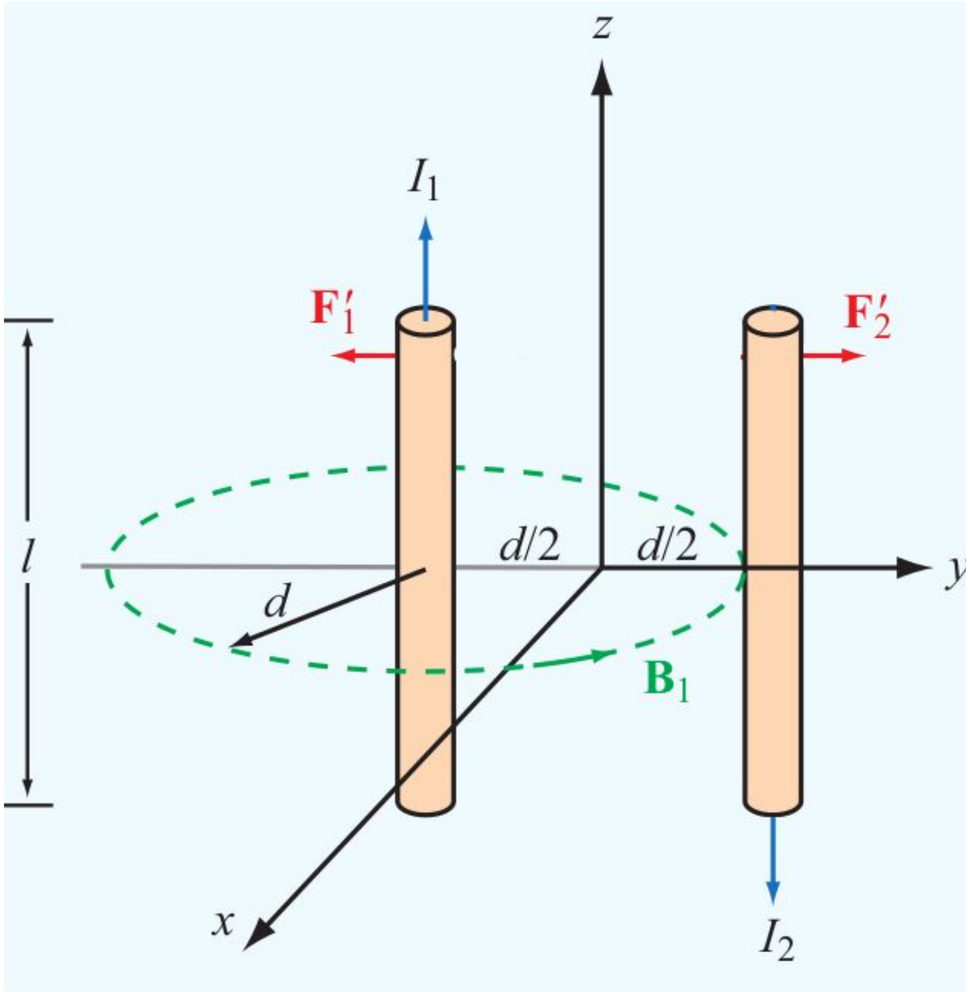


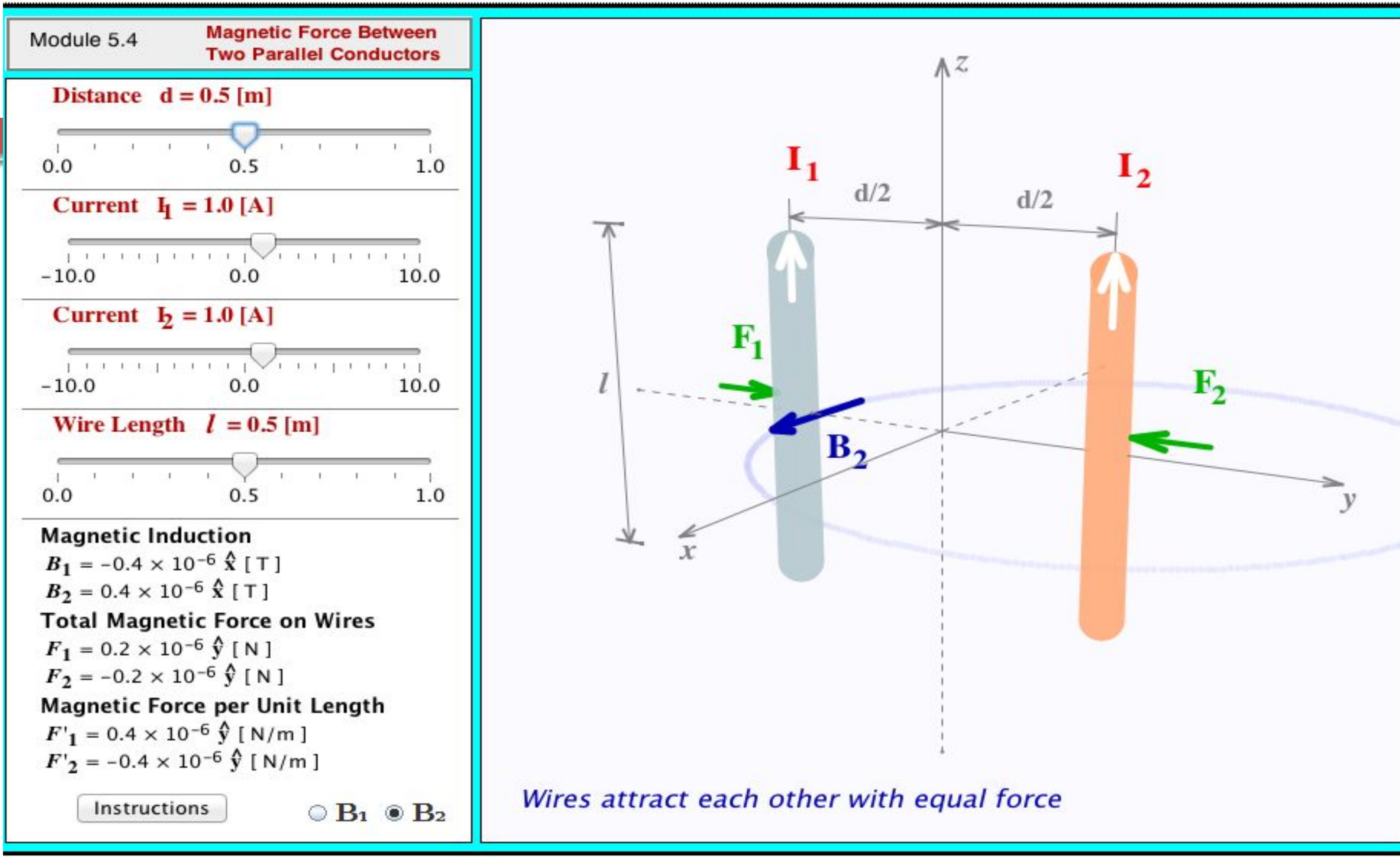
These forces pull the two wires together.

5-2 Forces on Parallel Conductors

If the currents went in opposite directions:

These forces would push the wires apart.




Module 5.4
Magnetic Force Between Two Parallel Conductors
Distance d = 0.5 [m]
0.0 0.5 1.0
Current I1 = 1.0 [A]
-10.0 0.0 10.0
Current I2 = 1.0 [A]
-10.0 0.0 10.0
Wire Length l = 0.5 [m]
0.0 0.5 1.0
Magnetic Induction
B1 = -0.4 × 10^-6 x̂ [T]
B2 = 0.4 × 10^-6 x̂ [T]
Total Magnetic Force on Wires
F1 = 0.2 × 10^-6 ŷ [N]
F2 = -0.2 × 10^-6 ŷ [N]
Magnetic Force per Unit Length
F'1 = 0.4 × 10^-6 ŷ [N/m]
F'2 = -0.4 × 10^-6 ŷ [N/m]
Instructions
B1 B2
z
I1
I2
d/2
d/2
l
F1
F2
B2
x
y
Wires attract each other with equal force

Homework

Homework 18 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Sections 5-3 through 5-4:

Magnetic Field from Currents
Ampere's Law
Magnetic Vector Potential Field
- Poisson's eqn
- Magnetic Flux