EECS 230
ENGINEERING ELECTROMAGNETICS
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Electrostatics 5

University of Michigan - Fall 2024

Chapter 4 Overview

Maxwell's Equations
- Electrostatics
- Magnetostatics

Charge density
Current density
Electric field from charges
Gauss's Law
Electric Scalar Potential Field
- Dipole Field
- Poisson's eqn

Conductors
- current
- resistance
- joule's law

Dielectrics
- polarization

Boundary Conditions
Capacitance
Potential Energy
Image method

$$\nabla \cdot \mathbf{D} = \rho_v,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

$$\mathbf{E} = \int_{\upsilon'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \hat{\mathbf{R}}' \frac{\rho_v \, d\upsilon'}{R'^2}.$$

(volume distribution)

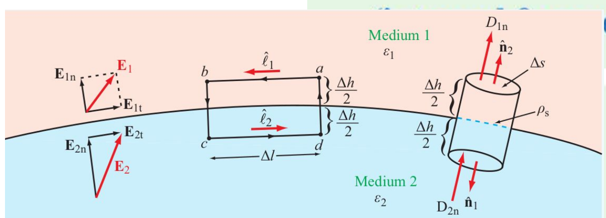
$$\nabla \cdot \mathbf{D} = \rho_v,$$

(differential form of Gauss's law)

$$\nabla \cdot \mathbf{D} = \rho_v,$$
$$\nabla \times \mathbf{E} = 0.$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q.$$

of Gauss's law)



$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l}$$

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J}.$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \mathbf{E} = -\nabla V.$$

$$Q = \int_{\upsilon} \rho_v \, d\upsilon \qquad \text{(C)}.$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (\text{A/m}^2) \quad \text{(Ohm's law)},$$

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2)$$

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}.$$

$$P = \int_{\upsilon} \mathbf{E} \cdot \mathbf{J} \, d\upsilon \qquad \text{(W)}$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.$$

Lecture Coverage

Today's lecture:

Review of Sections 4-1 through 4-8 of the book:

4-1: Maxwell's Equations

4-2: Charge and Current Distributions

4-3: Coulomb's Law

4-4: Gauss's Law

4-5: Voltage (Electric Scalar Potential)

4-6: Conductors

4-7: Dielectrics

4-8: Boundary Conditions

Sections 4-9, 4-10 of the book:

4-9: Capacitance

4-10: Electrostatic Potential Energy

Chapter 4 Review

Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{V}},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Empirically derived from many measurements

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}.$$

E: Electric Field
H: Magnetic Field
J: Current Density
ρ_v: Charge Density

Chapter 4 Review

Static Conditions:

Electrostatics

$$\frac{\partial \rho_v}{\partial t} = 0$$

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}},$$

$$\nabla \times \mathbf{E} = 0.$$

Magnetostatics

$$\frac{\partial \mathbf{J}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

Electric and Magnetic Fields are decoupled.

Chapter 4 Review

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \hat{\mathbf{R}}' \frac{\rho_{\mathrm{v}}\, d\upsilon'}{R'^2}$$ **(volume distribution)**

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_{\mathrm{s}}\, ds'}{R'^2}$$ **(surface distribution)**

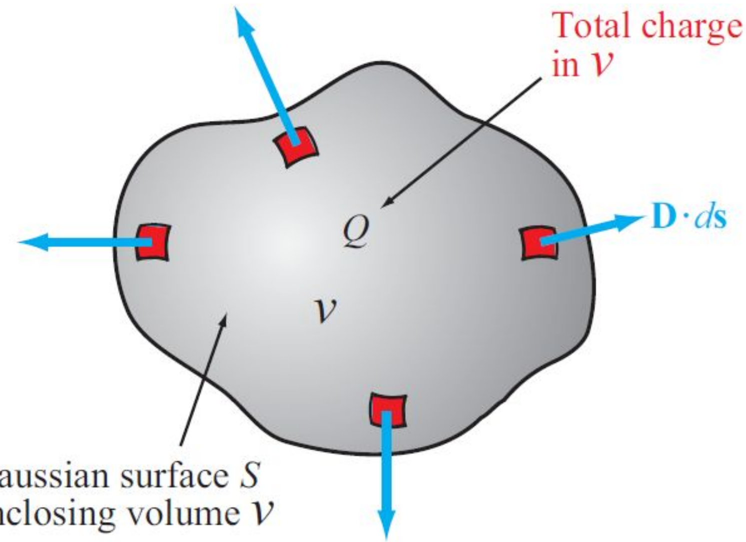
$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_{\ell}\, dl'}{R'^2}$$ **(line distribution)**

Chapter 4 Review

Gauss's Law

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.2$$

(Integral form of Gauss's law).



or:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\mathcal{V}} \rho_{\mathrm{v}} \, d\mathcal{V}$$

where the closed-surface S is the boundary of $\mathcal{V}$

Chapter 4 Review

Voltage:

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \qquad \text{(V)}. \qquad (4.43)$$

N Point Charges:

$$V = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|} \qquad \text{(V)}.$$

Chapter 4 Review

$$V = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \frac{\rho_{\mathrm{v}}}{R'} \, d\upsilon' \qquad \textbf{(volume distribution)},$$

$$V = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_{\mathrm{s}}}{R'} \, ds' \qquad \textbf{(surface distribution)},$$

$$V = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_{\ell}}{R'} \, dl' \qquad \textbf{(line distribution)}.$$

$$R' = |\mathbf{R} - \mathbf{R}_i|$$

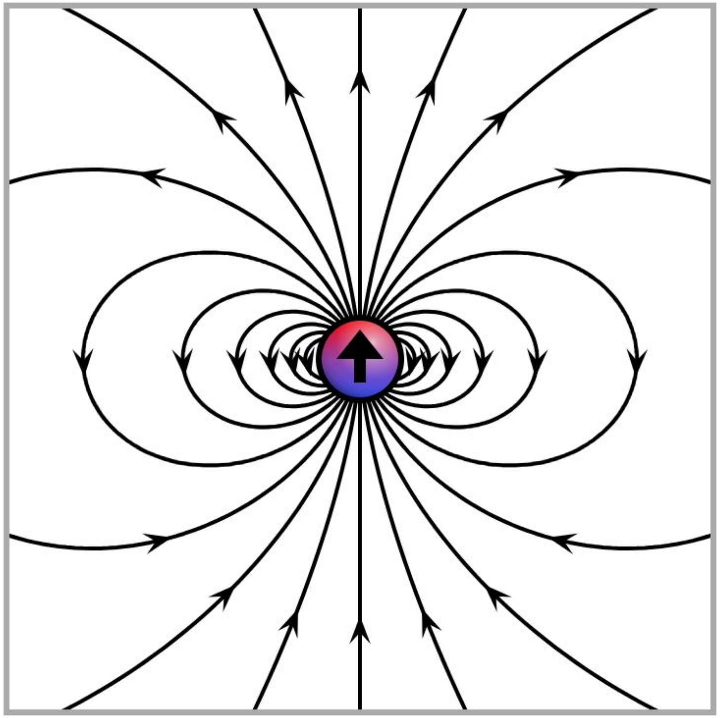
Chapter 4 Review

$$\mathbf{E} = -\nabla V.$$

Electric Dipole:

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbf{R}}\, 2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad \text{(V/m)}$$

(wikipedia.org)



Chapter 4 Review

Since:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \textbf{(Poisson's equation)}$$

if ρ_v=0:

$$\nabla^2 V = 0 \qquad \textbf{(Laplace's equation)}$$

Useful for problems where V is known on boundaries.

Chapter 4 Review

$$\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2) \quad (\text{Ohm's law}).$$

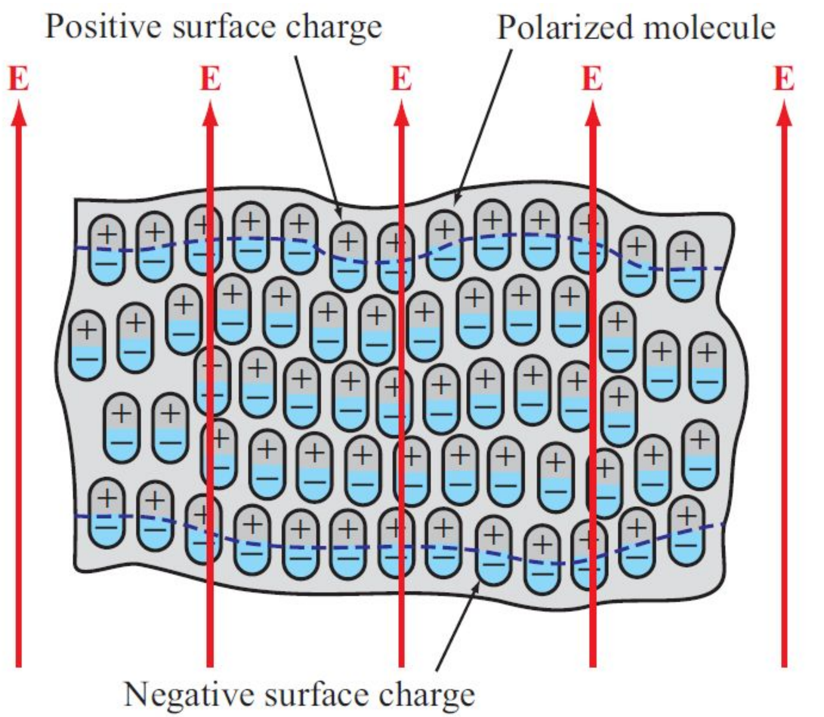
$$P = \int_V \mathbf{E} \cdot \mathbf{J} \, dV \qquad (\text{W}) \quad (\text{Joule's law})$$

Chapter 4 Review

In free space: $\mathbf{D} = \varepsilon_0 \mathbf{E}$

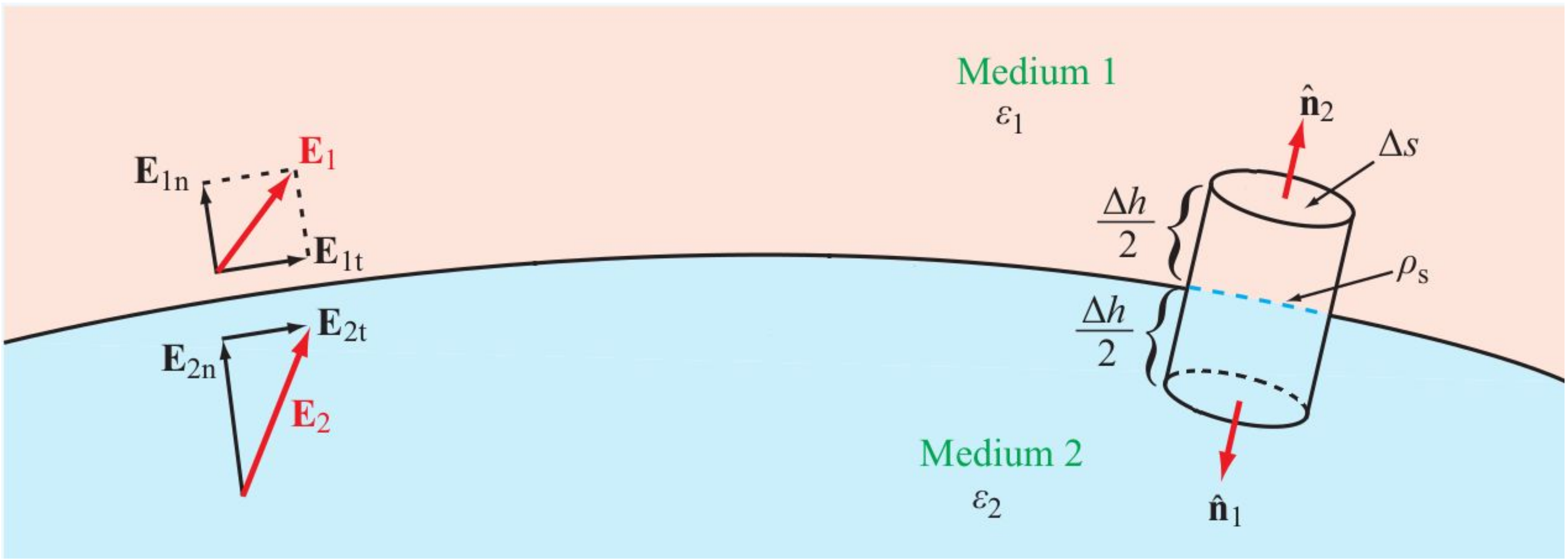
In a dielectric: $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E},$$



Chapter 4 Review

Two dielectric materials:

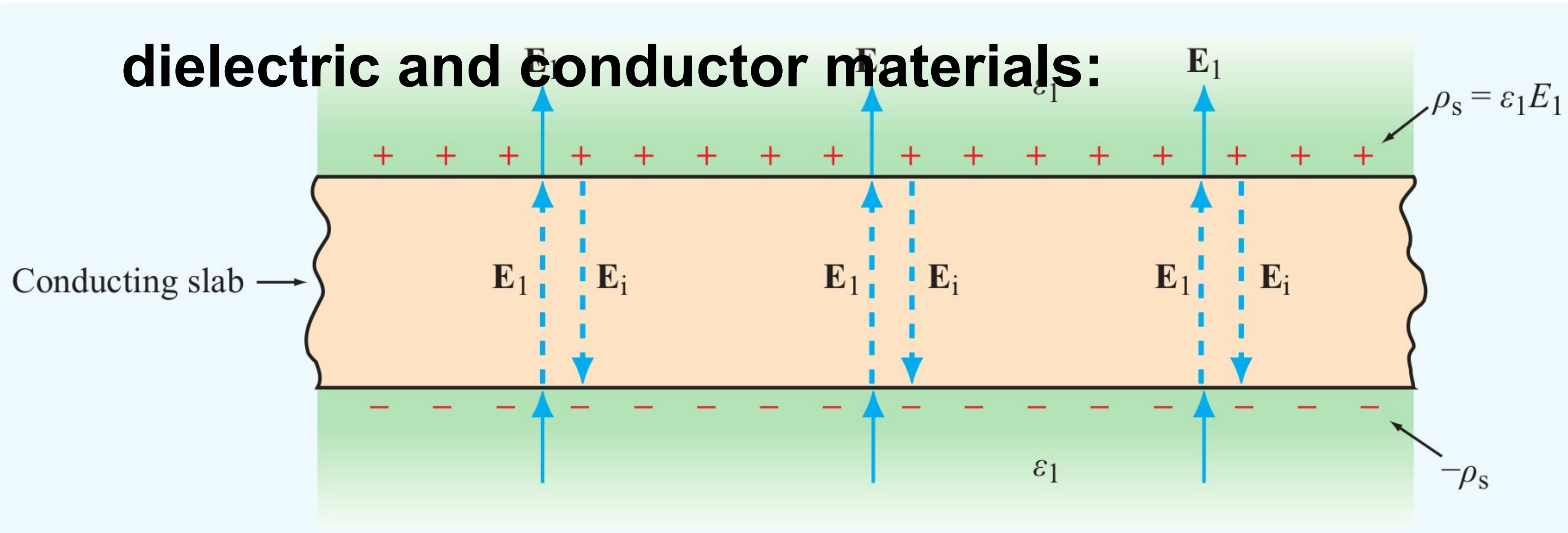


$$\mathbf{E}_{1t} = \mathbf{E}_{2t}$$

$$D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2).$$

Chapter 4 Review

dielectric and conductor materials:



$$E_{1t} = D_{1t} = 0, \qquad D_{1n} = \varepsilon_1 E_{1n} = \rho_s$$

$$\mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1 = \hat{\mathbf{n}}\rho_s,$$

(at conductor surface)

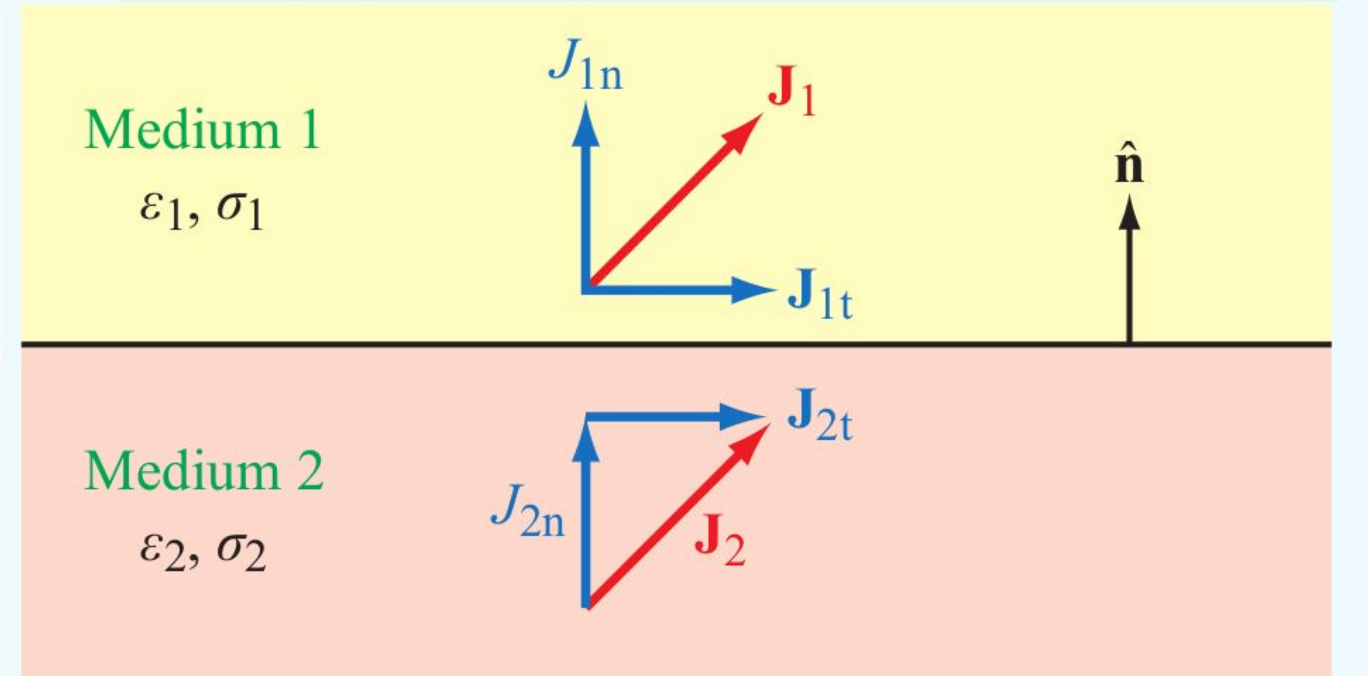
E is always normal to a conductor's surface

Chapter 4 Review

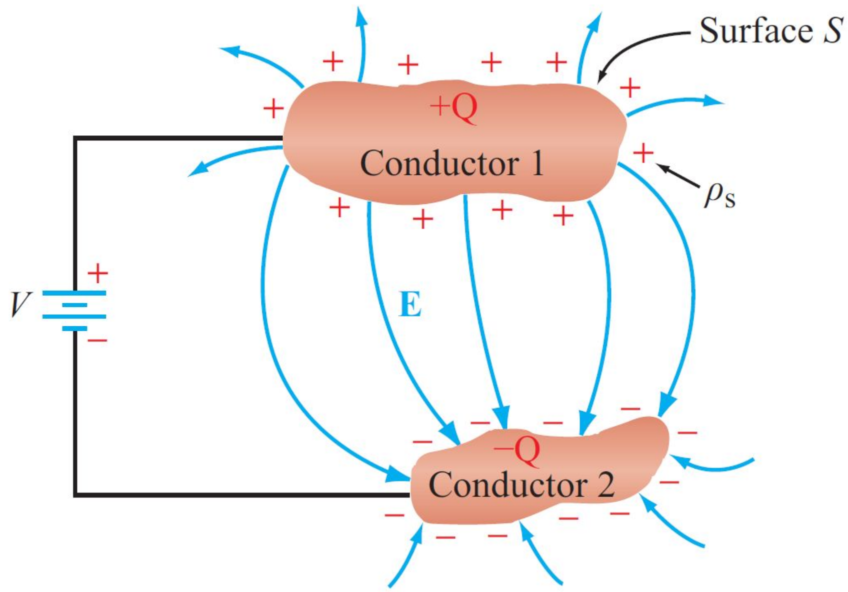
Two conducting materials:

$$\mathbf{E}_{1t} = \mathbf{E}_{2t}, \qquad \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$$

$$J_{1n}\left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right) = \rho_s \quad \textbf{(electrostatics)}.$$



4-9 Capacitance



The ***capacitance*** of a two-conductor configuration is defined as

$$C = \frac{Q}{V} \qquad \text{(C/V or F)}, \qquad (4.105)$$

Let's derive a formula for this using fields.

4-9 Capacitance

Recall from dielectric/conductor Boundary Conditions:

$$\mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1 = \hat{\mathbf{n}} \rho_s,$$

(at conductor surface)

Gauss' law:

$$Q = \int_S \rho_s \, ds = \int_S \varepsilon \hat{\mathbf{n}} \cdot \mathbf{E} \, ds = \int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}$$

$$V = V_{12} = -\int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l}$$

4-9 Capacitance

$$C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \qquad \text{(F)},$$

Recall:

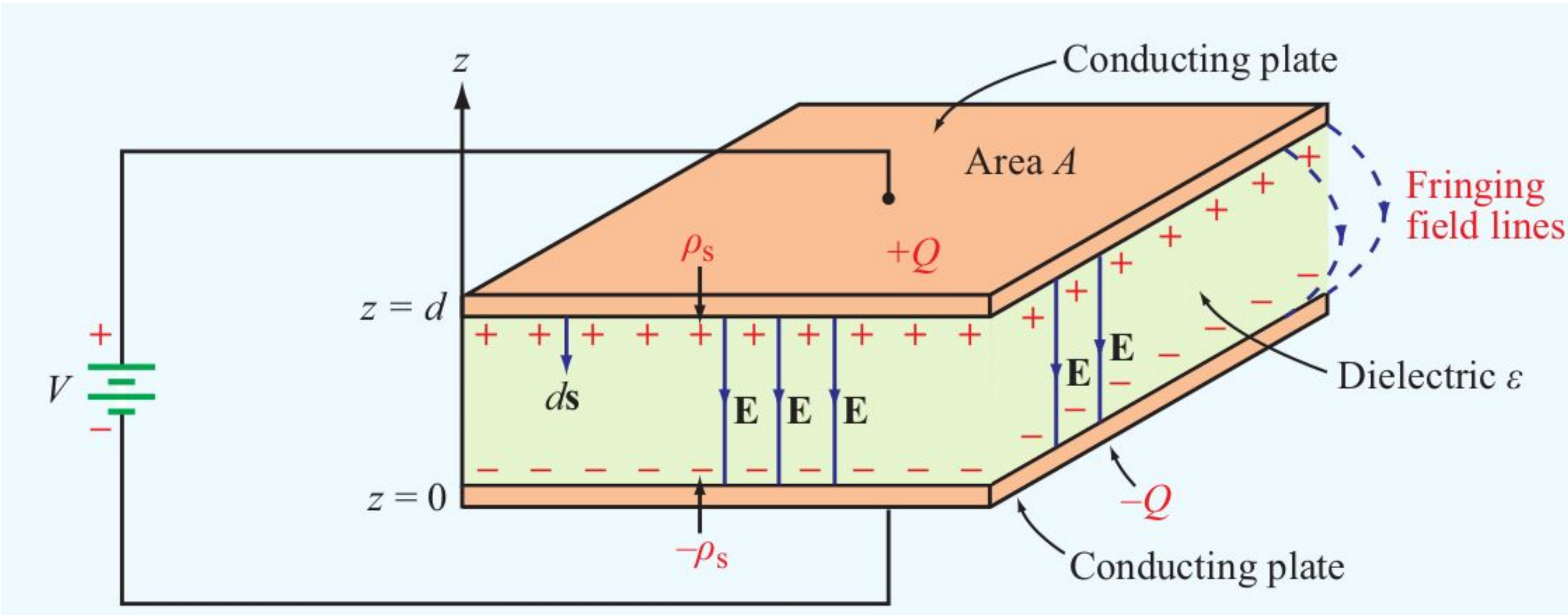
$$R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \qquad (\Omega)$$

So:

$$RC = \frac{\varepsilon}{\sigma}$$

So: easy to find C if know R.

4-9 Parallel-Plate Capacitor



Idealized: lossless dielectric

E exists only in dielectric and is uniform

uniform surface charge density

4-9 Parallel-Plate Capacitor

Charge density on upper plate:

$$\rho_s = Q/A$$

Electric field points from +charges to -charges:

$$\mathbf{E} = -\hat{\mathbf{z}}E,$$

from boundary conditions:

$$\varepsilon \mathbf{E}_n = \rho_s$$

so:

$$\mathrm{E} = \rho_s/\varepsilon = \frac{Q}{\varepsilon A}$$

4-9 Parallel-Plate Capacitor

Use the line integral from the top plate to the bottom to get voltage:

$$V = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = -\int_0^d (-\hat{\mathbf{z}}E) \cdot \hat{\mathbf{z}}\, dz = Ed.$$

Substitute into the Capacitance eqn:

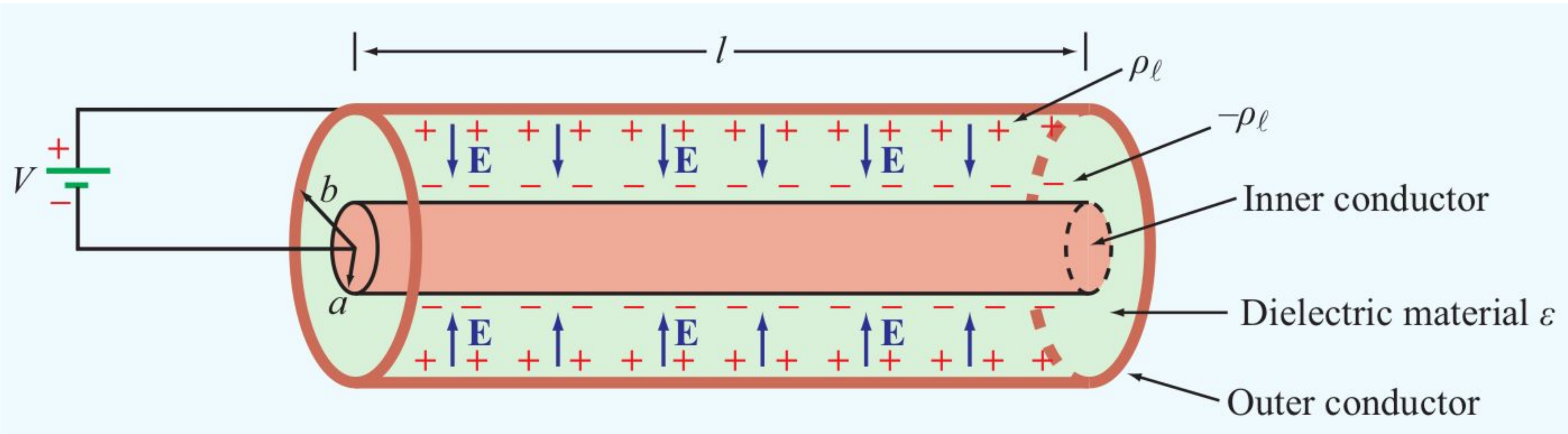
$$C = \frac{Q}{V} = \frac{Q}{Ed}$$

$$C = \frac{\varepsilon A}{d}$$

because:

$$\mathrm{E} = \frac{Q}{\varepsilon A}$$

4-9 Capacitance of Coaxial Line



Idealized: lossless dielectric

E exists only in dielectric and is uniform

uniform surface charge density

oriented along the *z*-axis

4-9 Capacitance of Coaxial Line

Outer conductor has +*Q*
Inner conductor has -*Q*

l=length of coax

Surface area of outer conductor: 2π*b* l
Surface area of inner conductor: 2π*a* l

Apply Gauss's Law: $$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Surface is cylindrical: surrounding the inner conductor, at some radius *r*:

$$d\mathbf{s} = \hat{\mathbf{r}} 2\pi r \, dz$$

4-9 Capacitance of Coaxial Line

Electric field is radial, and

directed from +charges to -charges:

$$\mathbf{E} = -\hat{\mathbf{r}} E$$

and is *not* a function of ϕ.

Charge enclosed is charge on inner conductor: -*Q*

Plug in:

$$\int_{z=0}^{l} -\varepsilon \hat{\mathbf{r}} E \cdot \hat{\mathbf{r}} 2\pi r \, dz = -Q$$

4-9 Capacitance of Coaxial Line

$$\int_{z=0}^{l} -\varepsilon \hat{\mathbf{r}} E \cdot \hat{\mathbf{r}} 2\pi r \, dz = -Q$$

$$2\pi\varepsilon r E \int_{z=0}^{l} dz = Q$$

$$2\pi\varepsilon r E l = Q$$

$$E = \frac{Q}{2\pi\varepsilon r l}$$

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{Q}{2\pi\varepsilon r l}$$

4-9 Capacitance of Coaxial Line

Then use the voltage eqn: (*d***l** is radial)

$$V = -\int_{\text{inner}}^{\text{outer}} \mathbf{E} \cdot d\mathbf{l}$$

$$V = -\int_{r=a}^{b} \left(-\hat{\mathbf{r}} \frac{Q}{2\pi \varepsilon r l} \right) \cdot (\hat{\mathbf{r}} dr)$$

$$V = \frac{Q}{2\pi \varepsilon l} \int_{r=a}^{b} \frac{1}{r} dr$$

$$V = \frac{Q}{2\pi \varepsilon l} \Big[\ln r \Big]_{r=a}^{b}$$

$$V = \frac{Q}{2\pi \varepsilon l} \ln \left(\frac{b}{a} \right)$$

4-9 Capacitance of Coaxial Line

Plug into equation for Capacitance:

$$C = \frac{Q}{V}$$

$$C = \frac{Q}{\frac{Q}{2\pi\varepsilon l}\ln(b/a)}$$

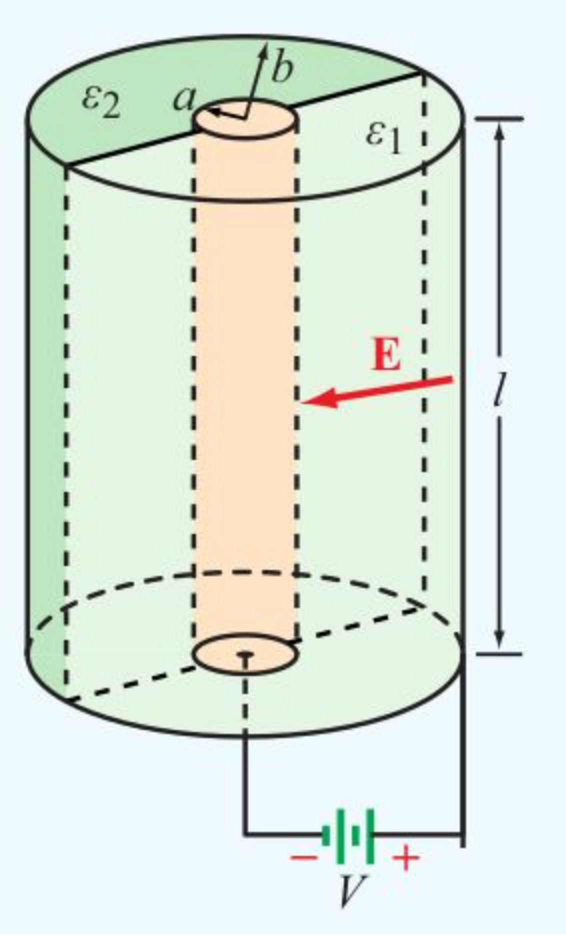
$$C = \frac{2\pi\varepsilon l}{\ln(b/a)}$$

$$C' = \frac{C}{l}$$

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)}$$

(as in Chapter 2)

Example: Divided Coax Capacitor



Given: Coax half-filled with
2 different insulators

Find: $\mathbf{E}_1$, $\mathbf{E}_2$, charge densities,
Capacitance

Solution:

The Electric field is radial inward.

$\mathbf{E}_1$ is in region 1: $\mathbf{E}_1 = -\hat{\mathbf{r}} E_1$

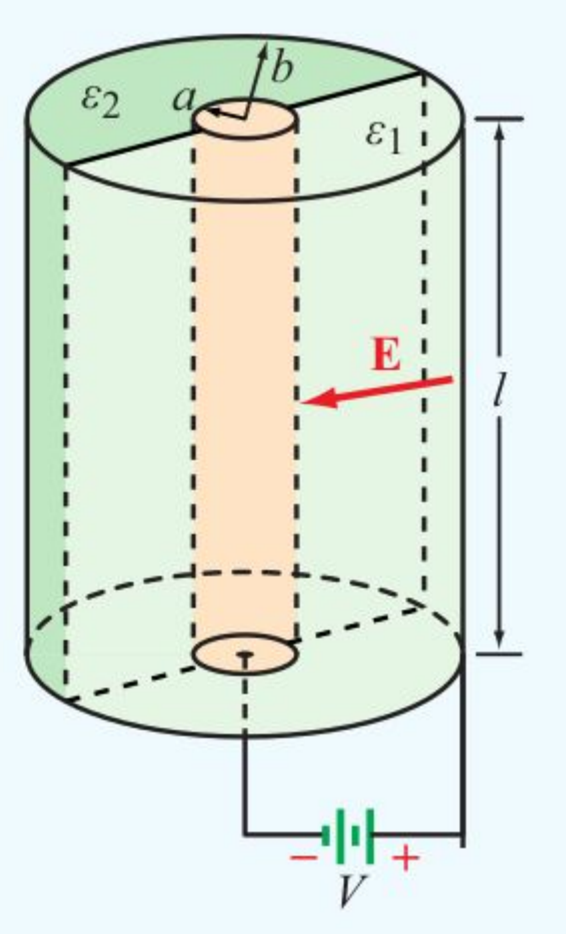
$\mathbf{E}_2$ is in region 2: $\mathbf{E}_2 = -\hat{\mathbf{r}} E_2$

Example: Divided Coax Capacitor

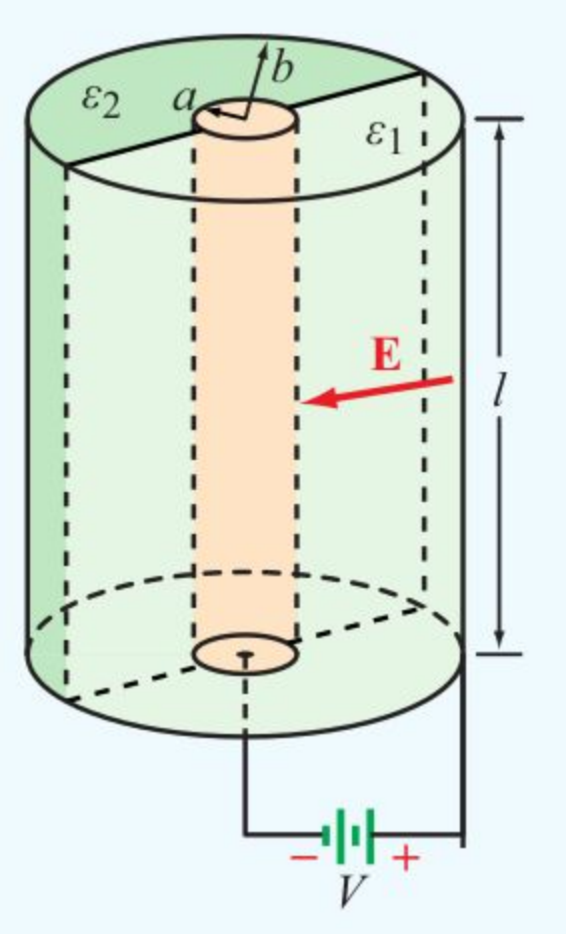
Solution:
Apply Boundary Conditions:

1. Along boundary between two insulators:
   Tangential **E** is continuous:

$$-\hat{\mathbf{r}}E_1 = -\hat{\mathbf{r}}E_2 = -\hat{\mathbf{r}}E$$



Example: Divided Coax Capacitor



Solution:

Apply Boundary Conditions:

2. Along outer edge of inner conductor: $r=a$

Normal **D** is discontinuous:

region 1: $\mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1 = \hat{\mathbf{n}} \rho_{s1}$

$$-\hat{\mathbf{r}}\,\varepsilon_1 E = \hat{\mathbf{r}}\,\rho_{s1}$$

$$\rho_{s1} = -\varepsilon_1 E.$$

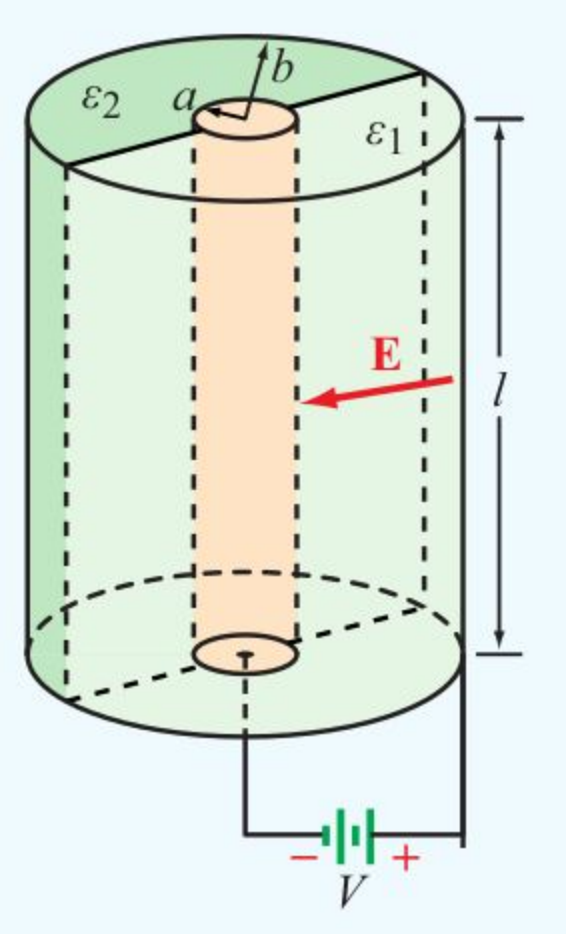
Example: Divided Coax Capacitor

Solution:
Apply Boundary Conditions:

2. Along outer edge of
inner conductor: $r=a$
Normal **D** is discontinuous:

region 2: $\mathbf{D}_2 = \varepsilon_2 \mathbf{E}_2 = \hat{\mathbf{n}}\, \rho_{s2}$

$$\rho_{s2} = -\varepsilon_2 E.$$



Example: Divided Coax Capacitor

Solution:

Apply Boundary Conditions:

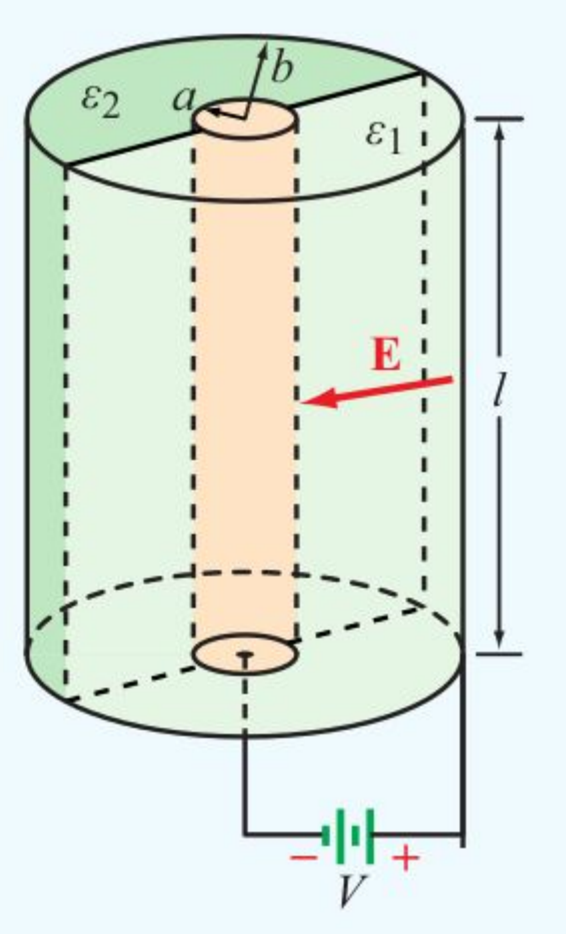
Electric fields are the same:

$$\mathbf{E}_1 = \mathbf{E}_2 = -\hat{\mathbf{r}}E$$

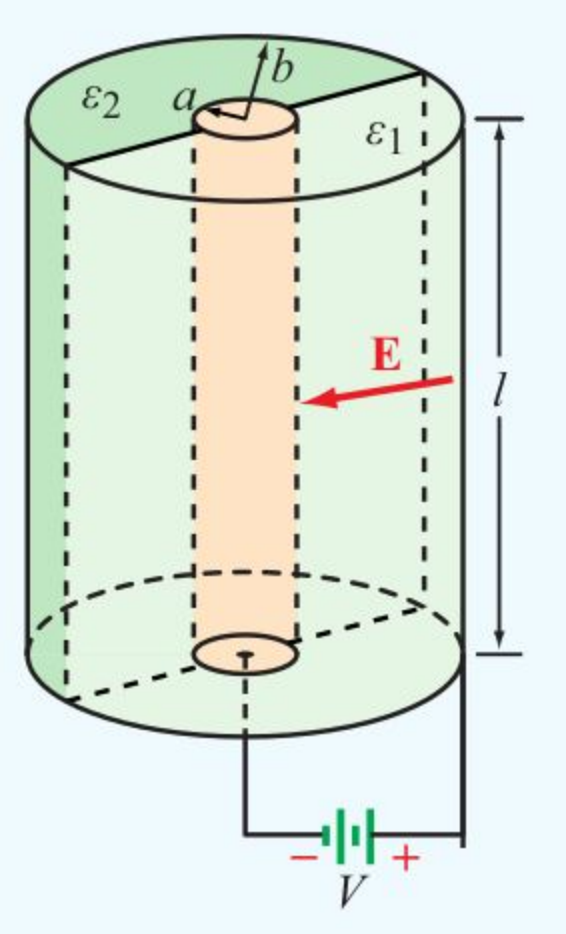
Charge densities different:

$$\rho_{s1} = -\varepsilon_1 E.$$

$$\rho_{s2} = -\varepsilon_2 E.$$



Example: Divided Coax Capacitor



Solution:

Apply Boundary Conditions:

Find the expression for *E*:

we know:

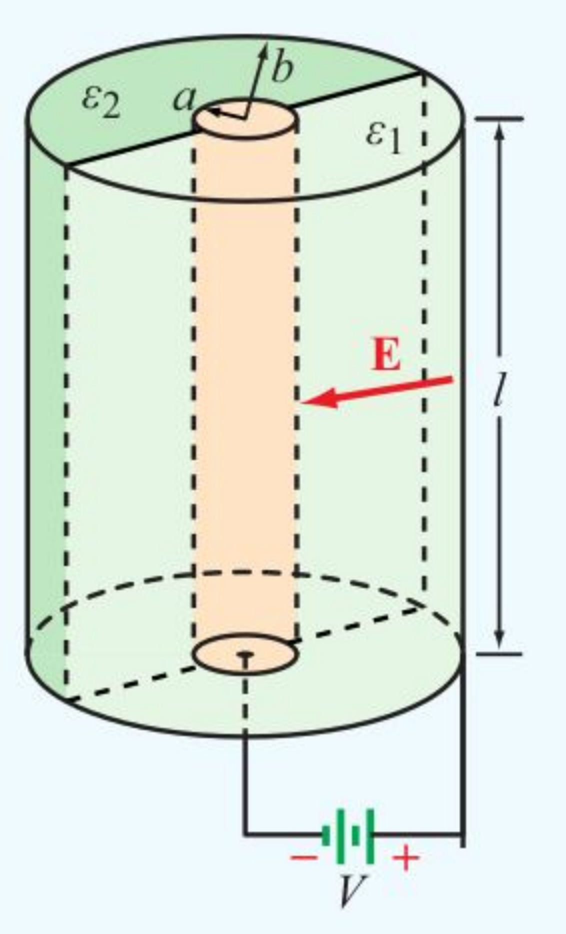
$$E = \frac{Q}{2\pi\varepsilon r l}$$

$$V = \frac{Q}{2\pi\varepsilon l} \ln\left(\frac{b}{a}\right)$$

so:

$$\frac{Q}{2\pi\varepsilon l} = \frac{V}{\ln(b/a)}$$

Example: Divided Coax Capacitor



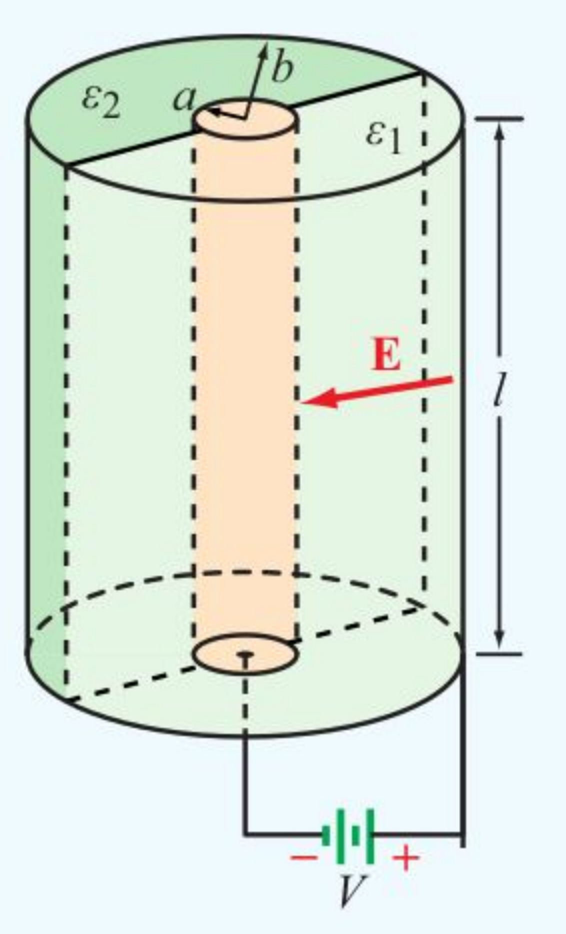
Solution:

Find the expression for *E*:

$$E = \frac{V}{r \ln(b/a)}$$

Example: Divided Coax Capacitor

Solution:
V is the same across both pieces, so like 2 capacitors in parallel.



Capacitance for a uniform coax:

$$C = \frac{2\pi\varepsilon l}{\ln(b/a)}$$

so for each half:

$$C = \frac{\pi\varepsilon l}{\ln(b/a)}$$

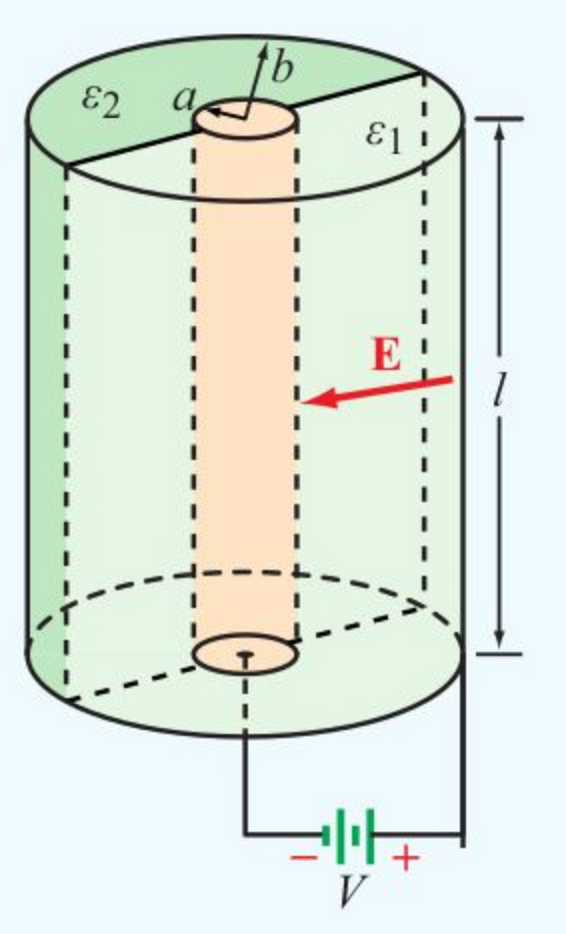
Example: Divided Coax Capacitor

Solution:

Capacitance for each half:

$$C_1 = \frac{\pi \varepsilon_1 l}{\ln(b/a)}$$

$$C_2 = \frac{\pi \varepsilon_2 l}{\ln(b/a)}$$



Example: Divided Coax Capacitor

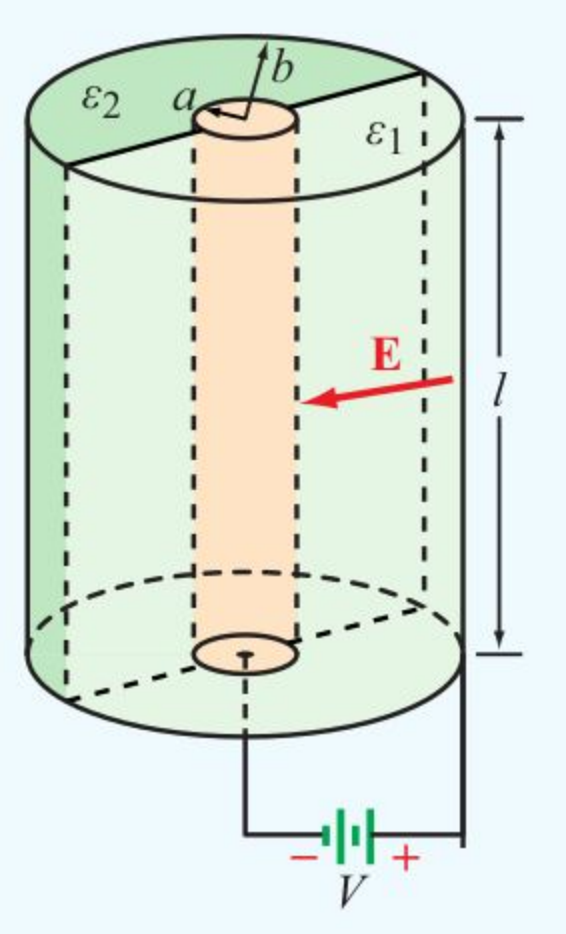
Solution:

Total capacitance:

$$C = C_1 + C_2$$

$$C = \frac{\pi\varepsilon_1 l}{\ln(b/a)} + \frac{\pi\varepsilon_2 l}{\ln(b/a)}$$

$$\boxed{C = \frac{\pi(\varepsilon_1 + \varepsilon_2) l}{\ln(b/a)}}$$



4-10 Electrostatic Potential Energy

While charging a capacitor from $v=0$ to $v=V$, we transfer $+q$ onto one conductor, $-q$ onto the other.

Since:

$$C = \frac{Q}{V}$$

we can say that during charging: $v = q/C$

Since voltage is defined as dW_e / dq:

$$dW_e = v\, dq = (q/C)\, dq$$

Total work when charging the capacitor:

$$W_e = \int_0^Q \frac{q}{C}\, dq$$

4-10 Electrostatic Potential Energy

Total work when charging the capacitor:

$$W_e = \int_{q=0}^{Q} \frac{q}{C} dq$$

$$W_e = \frac{1}{C} \int_{q=0}^{Q} q dq$$

$$W_e = \frac{1}{C} \left[\frac{q^2}{2} \right]_{q=0}^{Q}$$

$$\boxed{W_e = \frac{1}{2} \frac{Q^2}{C}}$$

4-10 Electrostatic Potential Energy

Substitute $Q = CV$ into this eqn:

$$W_e = \frac{1}{2} C V^2$$

which should be familiar from EECS 215.

For a parallel-plate capacitor with:

$$C = \varepsilon A/d \qquad V = Ed$$

this becomes:

$$W_e = \frac{1}{2} \frac{\varepsilon A}{d} (Ed)^2 = \frac{1}{2} \varepsilon E^2 (Ad)$$

4-10 Electrostatic Potential Energy

Since:

$$W_e = \frac{1}{2}\,\varepsilon E^2 (Ad)$$

the energy expended in charging up the capacitor is proportional to the volume (*Ad*) of the capacitor.

This energy is being stored in the electric field in the dielectric.

4-10 Electrostatic Potential Energy

Electrostatic energy density: W_e/volume:

$$w_{\mathrm{e}} = \frac{1}{2}\,\varepsilon E^2$$

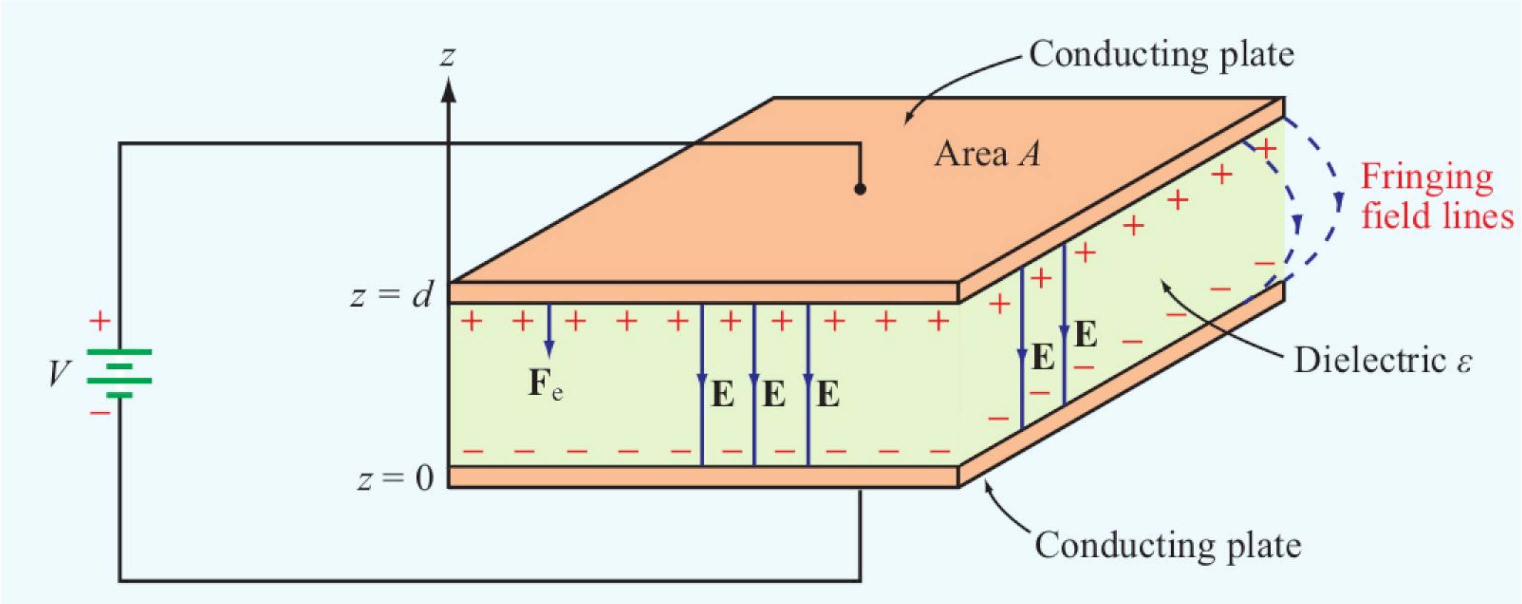
We can integrate over the volume to get the total:

$$W_{\mathrm{e}} = \frac{1}{2}\int_{\upsilon} \varepsilon E^2 \, d\upsilon$$

4-10 Capacitor Force

For a parallel-plate capacitor, the oppositely-charged plates attract each other.
The force on the upper plate is of the form:

$$\mathbf{F}_e = -\hat{\mathbf{z}}\, F_e$$



4-10 Capacitor Force

We can determine this force using W_e.

Change the plate spacing from constant *d* to variable *z*. so:

$$C = \varepsilon A / z,$$

then:

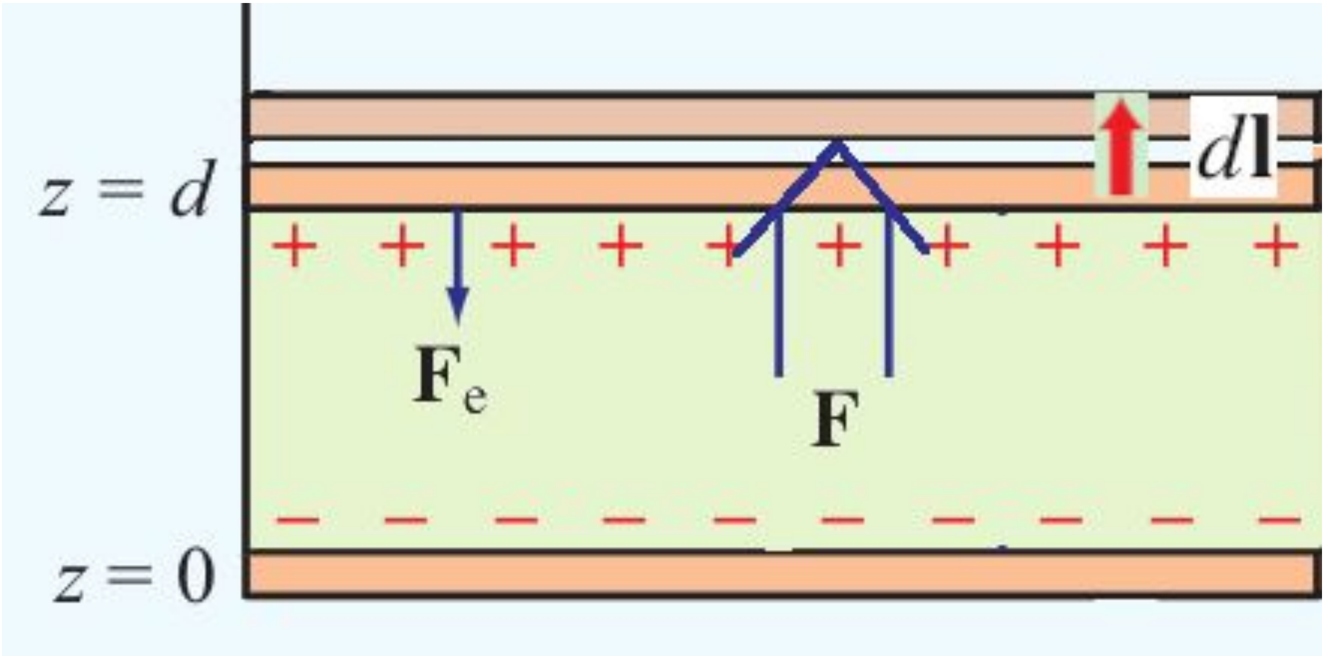
$$W_e = \frac{1}{2} CV^2 = \frac{1}{2} \frac{\varepsilon A V^2}{z}$$

4-10 Capacitor Force

apply an external force, **F**, to move the upper plate up:

Since: dWork = **F** · $d\mathbf{l}$

then: dWork = **F** · $\hat{\mathbf{z}}\, dz$



4-10 Capacitor Force

This causes the electrostatic energy in the capacitor to decrease, since: $W_e \propto 1/z$, so:

dWork = -dW_e

$$dW_e = -dW = -\mathbf{F} \cdot \hat{\mathbf{z}} dz$$

Since: $\mathbf{F}_e = -\mathbf{F}$

$$dW_e = \mathbf{F}_e \cdot \hat{\mathbf{z}} dz$$

$$dW_e = (-\hat{\mathbf{z}} F_e) \cdot \hat{\mathbf{z}} dz$$

$$dW_e = -F_e dz$$

4-10 Capacitor Force

From our expression for W_e:

$$\frac{dW_e}{dz} = \frac{d}{dz}\left\{\frac{1}{2}\frac{\varepsilon A V^2}{z}\right\}$$

$$\frac{dW_e}{dz} = -\frac{1}{2}\frac{\varepsilon A V^2}{z^2}$$

$$dW_e = -\frac{1}{2}\frac{\varepsilon A V^2}{z^2}dz$$

$$dW_e = -F_e dz$$

$$F_e = \frac{1}{2}\frac{\varepsilon A V^2}{z^2}$$

4-10 Capacitor Force

Since:

$$F_e = \frac{1}{2} \frac{\varepsilon A V^2}{z^2}$$

at *z*=*d*, get:

$$\boxed{F_e = \frac{1}{2}\, \varepsilon\, \frac{A V^2}{d^2}}$$

4-10 Capacitor Force

So, we get the electrostatic force acting on the upper plate:

$$\mathbf{F}_e = -\hat{\mathbf{z}}\,\frac{1}{2}\,\varepsilon A\,\frac{V^2}{d^2} \qquad \text{(N)}.$$

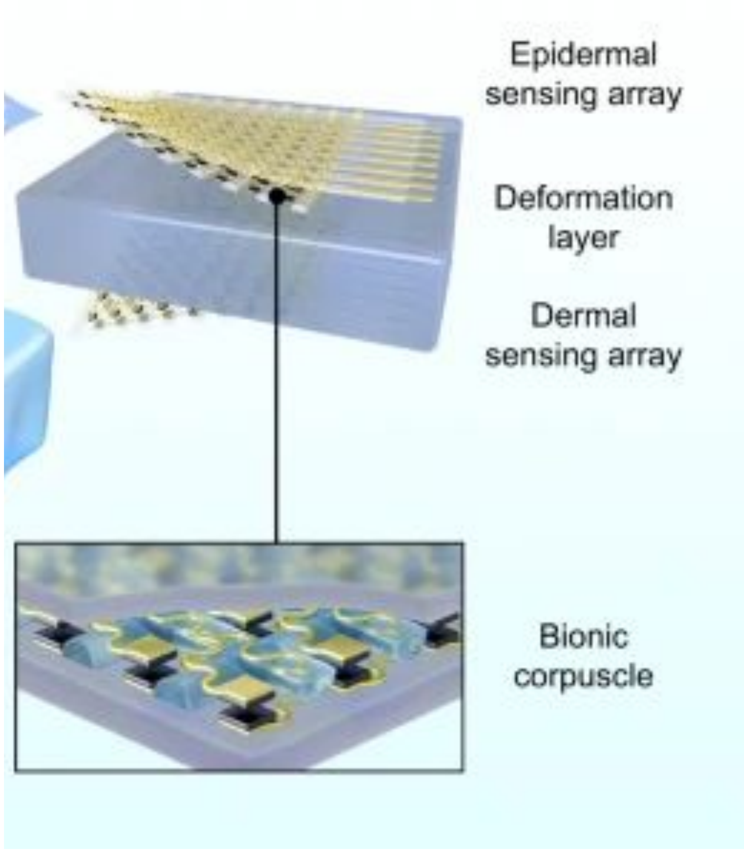
(parallel-plate capacitor)

4-10 Capacitor Force

Applications:

One can imagine using a "squishy" dielectric.
And making an array of really small capacitors.

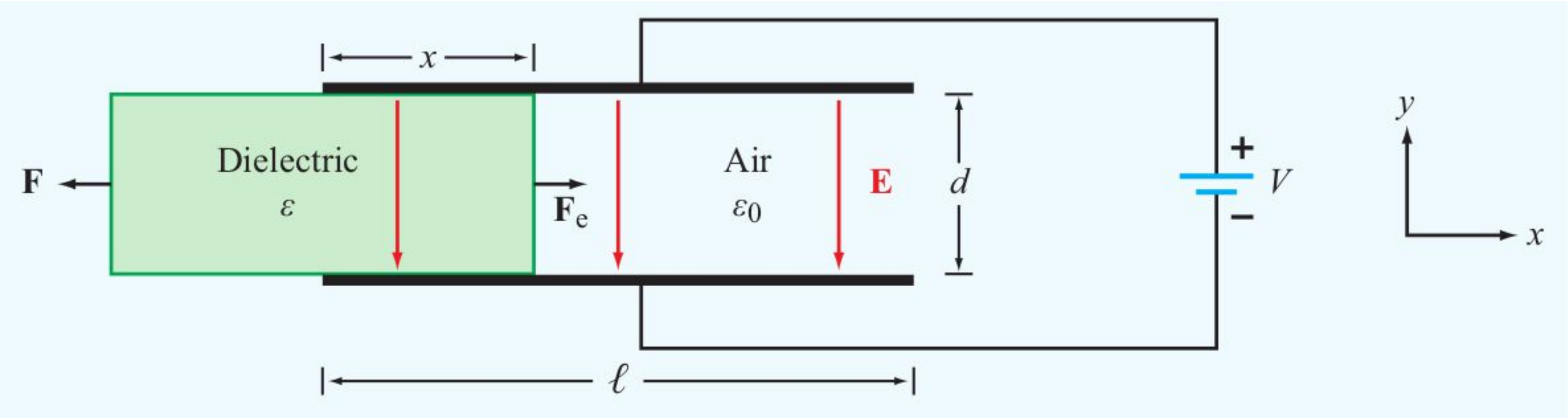
Then you could use it for a capacitive touch sensor.
Maybe even image fingerprints.
Or put on a robot hand for force feedback.



(nature.com)

Or a biologically-inspired robot actuator...

4-10 Capacitor Actuator

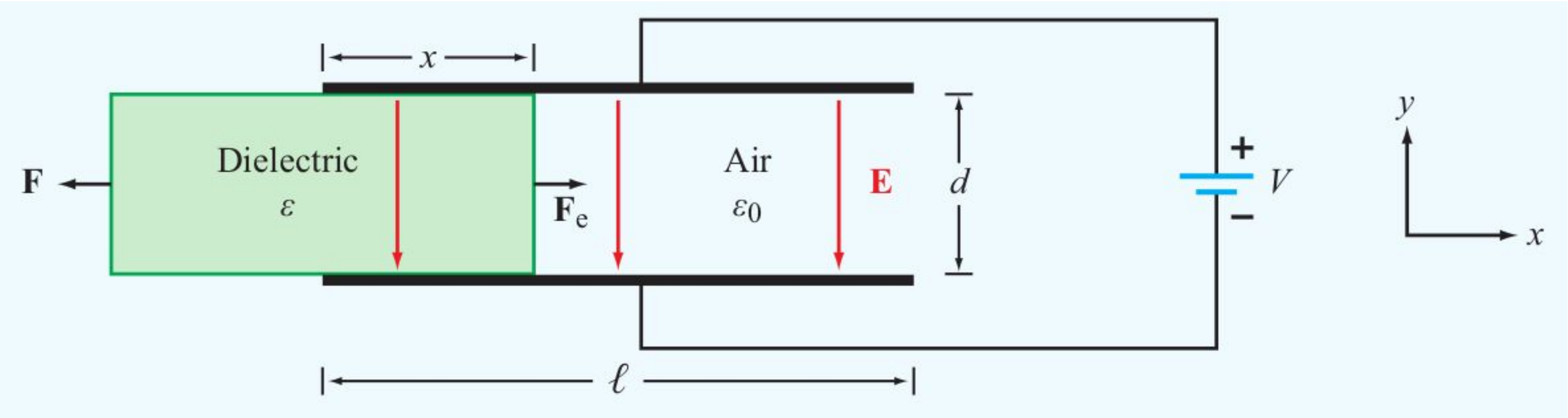


Parallel-plate Capacitor:

Dielectric can **move:** slides along the *x*-direction
Dielectric dimensions: $\ell \times w \times d$

Calculate: Electrostatic Energy, W_e, and the Force, F_e.

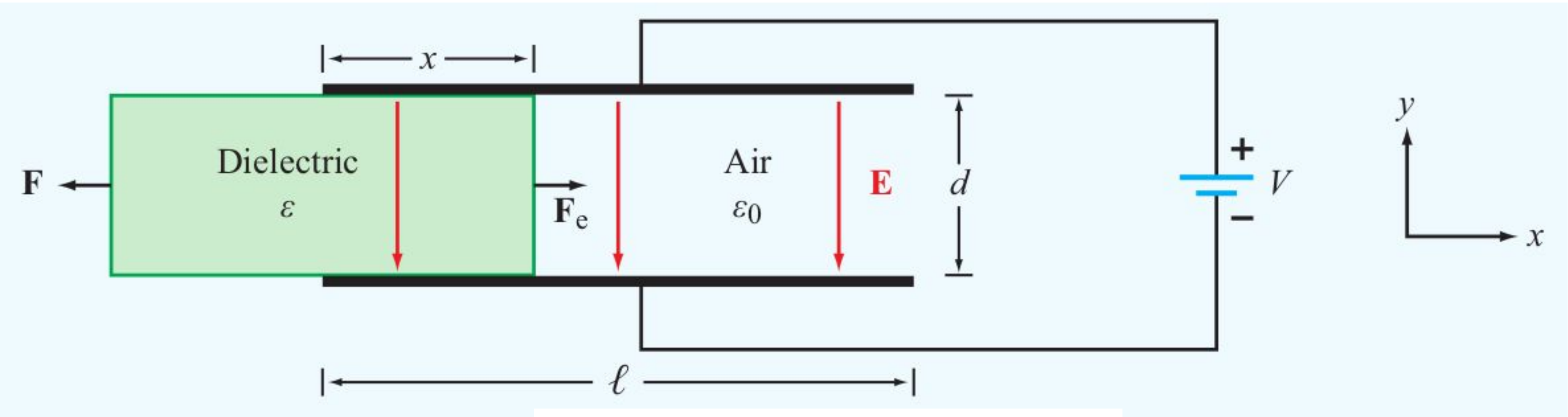
4-10 Capacitor Actuator



W_e has 2 components:

1. volume containing the dielectric block
2. volume containing air

4-10 Capacitor Actuator



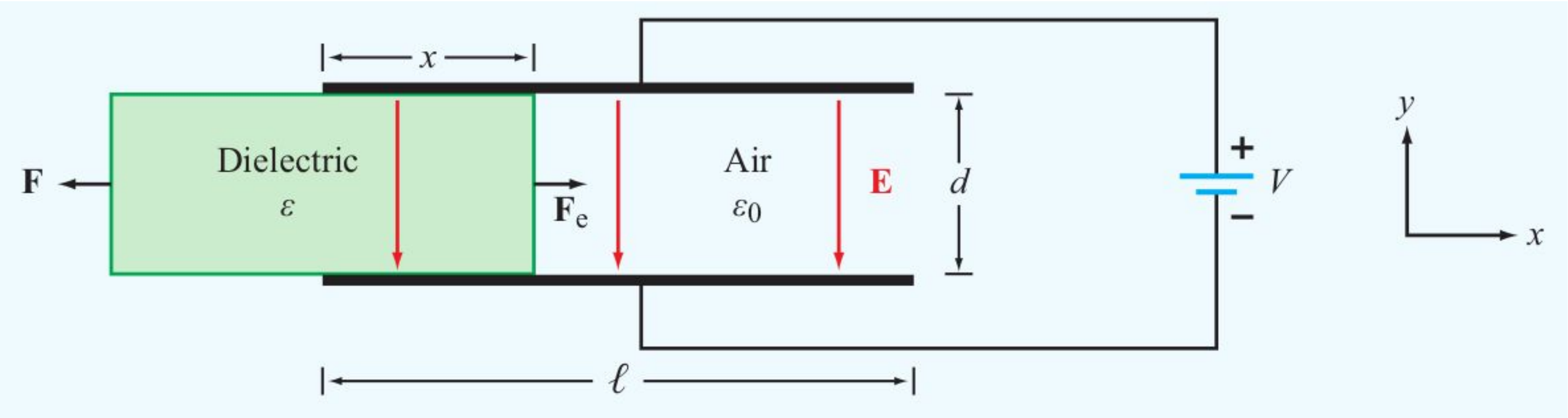
In general:

$$W_e = \frac{1}{2} \int_{\upsilon} \varepsilon E^2 \, d\upsilon$$

can divide this into 2 uniform pieces (diel & air):

$$W_e = \frac{1}{2} \, \varepsilon E^2 \upsilon_1 + \frac{1}{2} \, \varepsilon_0 E^2 \upsilon_2$$

4-10 Capacitor Actuator



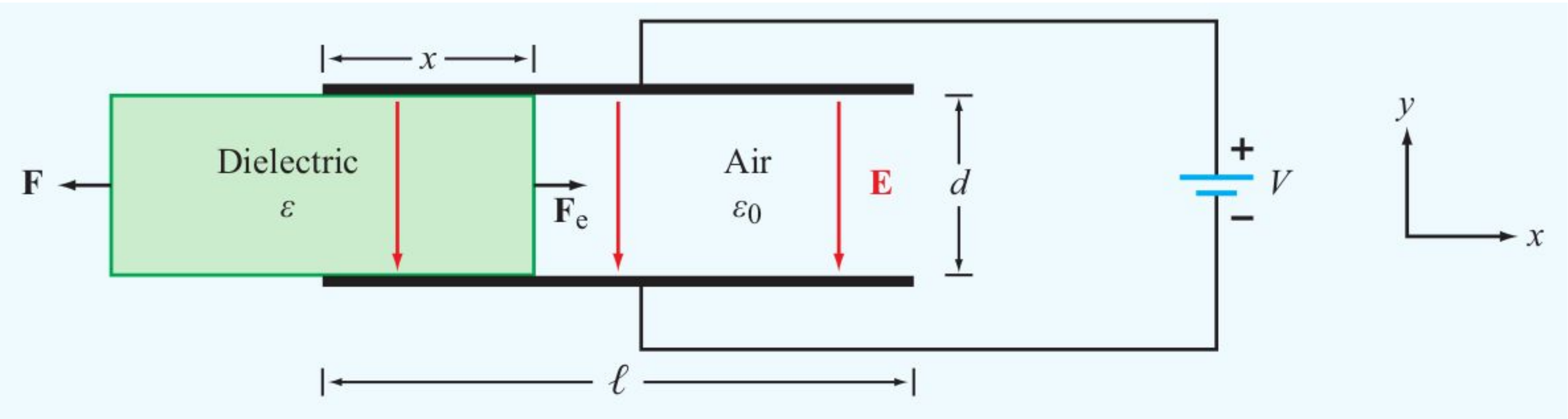
$$W_e = \frac{1}{2}\,\varepsilon E^2 \mathcal{V}_1 + \frac{1}{2}\,\varepsilon_0 E^2 \mathcal{V}_2$$

The electric field: $E = V/d$

Approximate E=0 outside the capacitor.

Dielectric volume within the capacitor: $\mathcal{V}_1 = x\,w\,d$

4-10 Capacitor Actuator

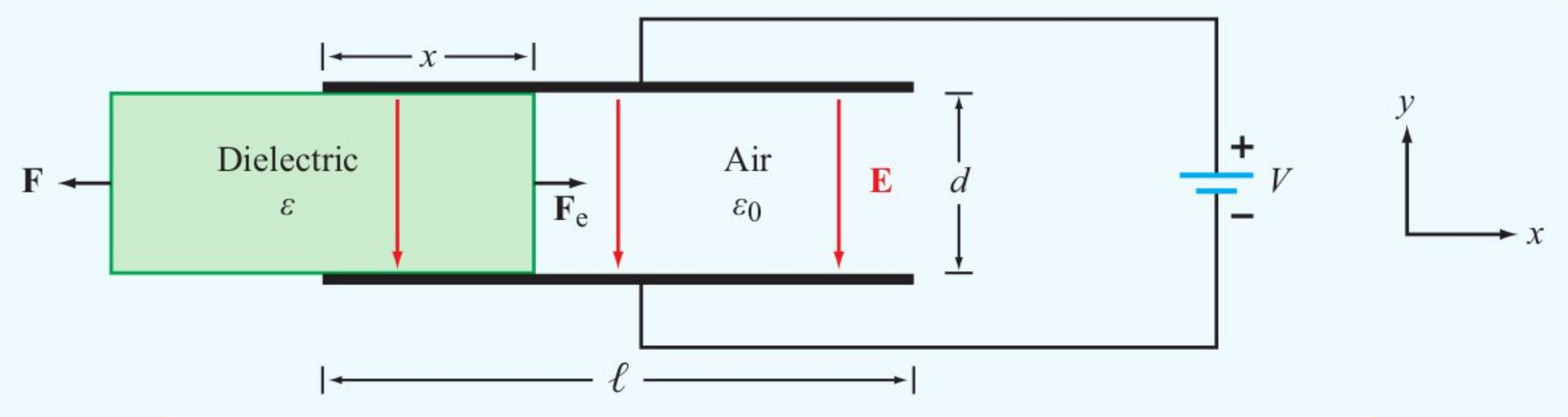


$$W_e = \frac{1}{2}\,\varepsilon E^2 \upsilon_1 + \frac{1}{2}\,\varepsilon_0 E^2 \upsilon_2$$

The electric field: $E = V/d$

Air volume within the capacitor: $\mathcal{V}_2 = (\ell - x)\, w\, d$

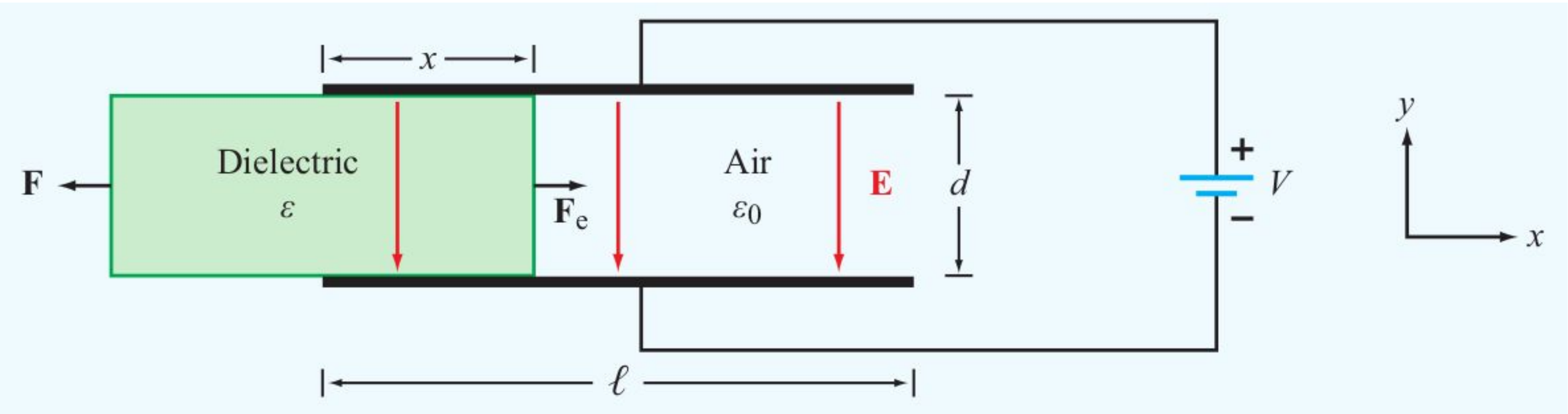
4-10 Capacitor Actuator



$$W_e = \frac{1}{2}\,\varepsilon\left(\frac{V}{d}\right)^2 xwd + \frac{1}{2}\,\varepsilon_0\left(\frac{V}{d}\right)^2 (\ell - x)wd$$

$$W_e = \frac{1}{2}\,\frac{V^2}{d}\,w[\varepsilon x + \varepsilon_0(\ell - x)].$$

4-10 Capacitor Actuator

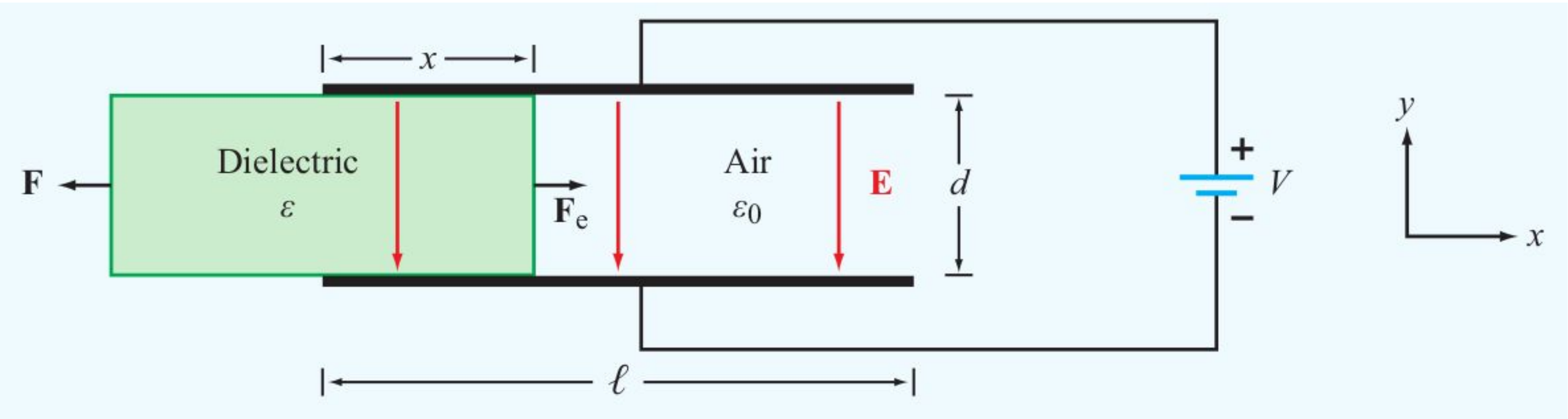


$$W_e = \frac{1}{2} \frac{V^2}{d} w[\varepsilon x + \varepsilon_0(\ell - x)].$$

Since $\varepsilon > \varepsilon_0$: Maximum W_e when $x=\ell$

This is when the dielectric is completely inside the capacitor

4-10 Capacitor Actuator



If apply an external force to move dielectric to left:

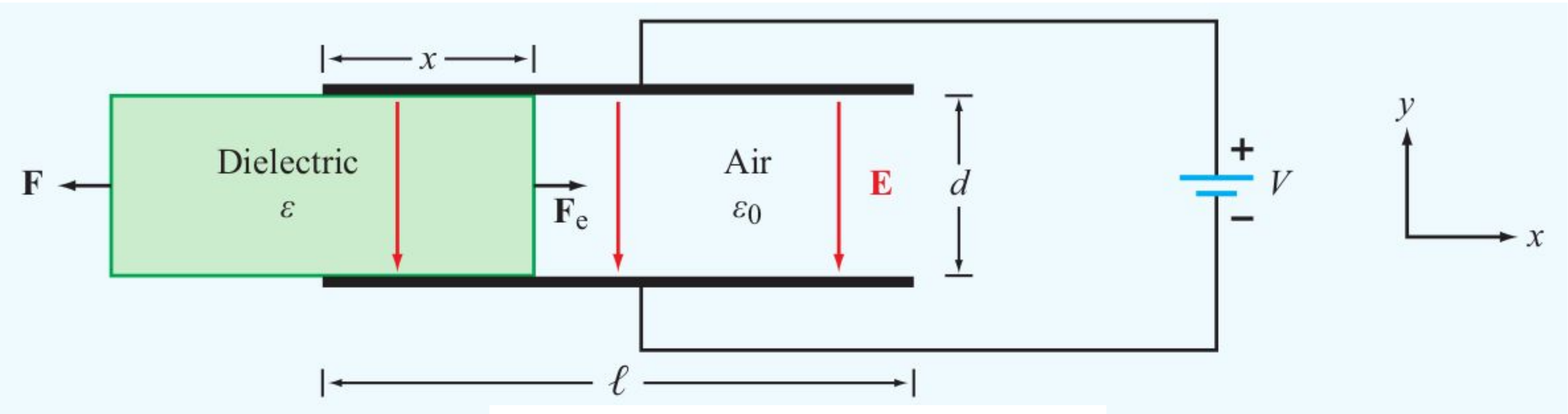
$$\text{Work} = \mathbf{F} \cdot d\mathbf{l}$$

$$d\text{Work} = \mathbf{F} \cdot \hat{\mathbf{x}}(-dx)$$

This causes electrostatic energy of capacitor to decrease:

$$d\text{Work} = -dW_e$$

4-10 Capacitor Actuator



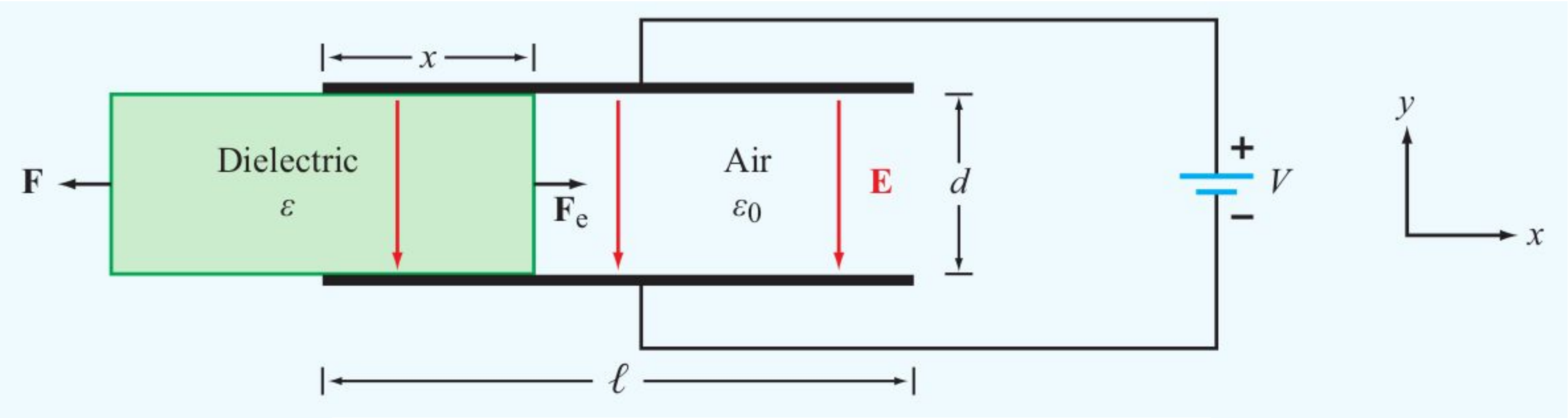
Since $\mathbf{F}_e$ = $-\mathbf{F}$:

$$dW_e = -\mathbf{F}_e \cdot \hat{x}(-dx)$$

$$dW_e = -\hat{x}F_e \cdot \hat{x}(-dx)$$

$$dW_e = F_e dx$$

4-10 Capacitor Actuator

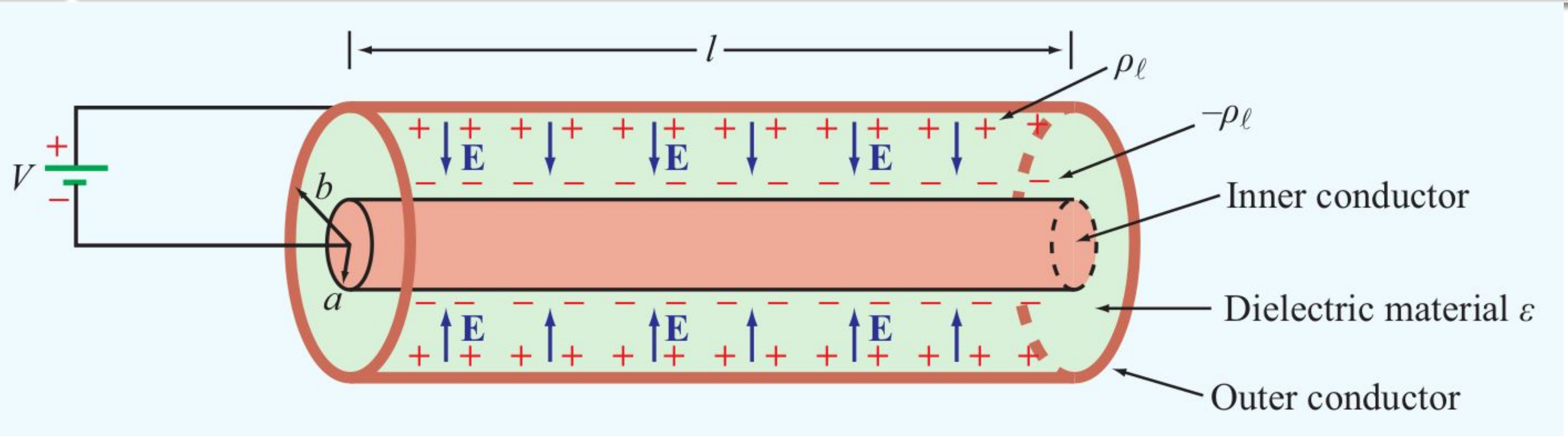


$$\mathbf{F}_e = \hat{\mathbf{x}} \frac{dW_e}{dx}$$

$$\mathbf{F}_e = \hat{\mathbf{x}} \frac{d}{dx} \left[\frac{1}{2} \frac{V^2}{d} w[\varepsilon x + \varepsilon_0(\ell - x)] \right]$$

$$\mathbf{F}_e = \hat{\mathbf{x}} \frac{1}{2} \frac{V^2}{d} w(\varepsilon - \varepsilon_0).$$

Exercise 4-18: Energy in Coax



Given: a=2cm, b=5cm, ε_r=4, ρ_l=1X10^{-4} C/m

$$E = \frac{\rho_l}{2\pi\varepsilon r}$$

Find: Energy stored in 20cm-length of coax

Exercise 4-18: Energy in Coax

Solution: $E = \dfrac{\rho_l}{2\pi\varepsilon r}$

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \varepsilon E^2 d\mathcal{V}$$

$$W_e = \frac{1}{2}\varepsilon \int_{\mathcal{V}} \left(\frac{\rho_l}{2\pi\varepsilon r}\right)^2 d\mathcal{V}$$

$$W_e = \frac{1}{2}\varepsilon \int_{\mathcal{V}} \left(\frac{\rho_l}{2\pi\varepsilon r}\right)^2 r dr d\phi dz$$

$$W_e = \frac{1}{2}\left(\frac{\rho_l}{2\pi\varepsilon}\right)^2 \varepsilon \int_{r=a}^{b} \int_{\phi=0}^{2\pi} \int_{z=0}^{l} \left(\frac{1}{r}\right)^2 r dr d\phi dz$$

Exercise 4-18: Energy in Coax

Solution:

$$W_e = \frac{1}{2}\left(\frac{\rho_l}{2\pi\varepsilon}\right)^2 \varepsilon \int_{r=a}^{b}\int_{\phi=0}^{2\pi}\int_{z=0}^{l}\left(\frac{1}{r}\right)^2 r\,dr\,d\phi\,dz$$

$$W_e = \frac{1}{2}\left(\frac{\rho_l}{2\pi\varepsilon}\right)^2 \varepsilon 2\pi l \int_{r=a}^{b}\frac{1}{r}dr$$

$$W_e = \frac{1}{2}\left(\frac{\rho_l}{2\pi\varepsilon}\right)^2 \varepsilon 2\pi l \Big[\ln r\Big]_{r=a}^{b}$$

$$\boxed{W_e = \frac{\rho_l^2 l}{4\pi\varepsilon}\ln(b/a)}$$

Exercise 4-18: Energy in Coax

Solution:

$$W_e = \frac{\rho_l^2 l}{4\pi\varepsilon}\ln(b/a)$$

$$W_e = \frac{(1\times 10^{-4}\,\mathrm{C/m})^2(0.2\,\mathrm{m})}{4\pi(4)(8.85\times 10^{-12}\,\mathrm{F/m})}\ln(5/2)$$

$$\boxed{W_e = 4.1\,\mathbf{J}}$$

Homework

Homework 16 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Section 4-11:

Image method for solving field problems