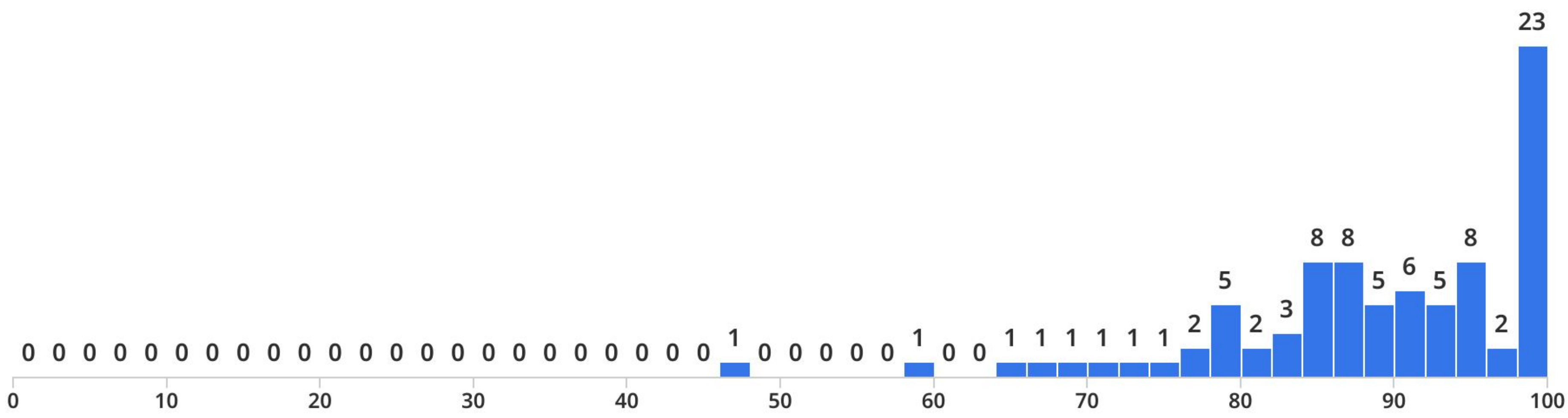
EECS 230
ENGINEERING ELECTROMAGNETICS
Leland Pierce

Electrostatics 3

University of Michigan - Fall 2024

Announcements



Minimum	Median	Maximum	Mean	Std Dev
46.0	90.0	100.0	88.78	10.76

Announcements

Homework 13, problem 3 is trickier than I intended.

If you separate the integral into 2 pieces:

one from -4 to 0 and another from 0 to 2

you end with infinity at R=0

To avoid this:

1. notice the radial symmetry in **E**, and just use the radial coordinate for the points, instead of z.

OR

2. Choose a path that avoids intersecting the origin.

Chapter 4 Overview

Maxwell's Equations

- Electrostatics
- Magnetostatics

Charge density

Current density

Electric field from charges

Gauss's Law

Electric Scalar Potential Field

- Dipole Field
- Poisson's eqn

Conductors

- current
- resistance
- joule's law

Dielectrics

- polarization

Boundary Conditions

Capacitance

Potential Energy

Image method

$$\nabla \cdot \mathbf{D} = \rho_v ,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} ,$$
$$\nabla \cdot \mathbf{B} = 0 ,$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} .$$

$$\mathbf{E} = \int_{v'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{v'} \hat{\mathbf{R}}' \frac{\rho_v \, dv'}{R'^2} .$$

(volume distribution)

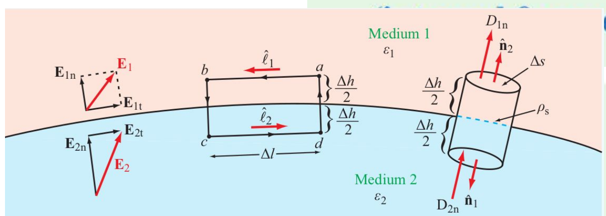
$$\nabla \cdot \mathbf{D} = \rho_v ,$$

(differential form of Gauss's law)

$$\nabla \cdot \mathbf{D} = \rho_v ,$$
$$\nabla \times \mathbf{E} = 0.$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q.$$

of Gauss's law)



$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l}$$

$$\nabla \cdot \mathbf{B} = 0 ,$$
$$\nabla \times \mathbf{H} = \mathbf{J} .$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \mathbf{E} = -\nabla V .$$

$$Q = \int_v \rho_v \, dv \qquad \text{(C)}.$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (\text{A/m}^2) \quad \text{(Ohm's law)},$$

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2)$$

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} .$$

$$P = \int_v \mathbf{E} \cdot \mathbf{J} \, dv \qquad \text{(W)}$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.$$

Lecture Coverage

Today's lecture:

Review of Sections 4-1 through 4-5 of the book:

4-1: Maxwell's Equations
4-2: Charge and Current Distributions
4-3: Coulomb's Law
4-4: Gauss's Law
4-5: Voltage (Electric Scalar Potential)

Section 4-6 of the book:

4-6: Conductors

Chapter 4 Review

Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho_V,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Empirically derived from many measurements

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}.$$

E: Electric Field
H: Magnetic Field
J: Current Density
ρ_v: Charge Density

Chapter 4 Review

Static Conditions:

$$\frac{\partial \rho_v}{\partial t} = 0$$

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_\mathrm{v},$$

$$\nabla \times \mathbf{E} = 0.$$

$$\frac{\partial \mathbf{J}}{\partial t} = 0$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

Electric and Magnetic Fields are decoupled.

Chapter 4 Review

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \hat{\mathbf{R}}' \frac{\rho_{\mathrm{v}}\, d\upsilon'}{R'^2}$$ (volume distribution)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_{\mathrm{s}}\, ds'}{R'^2}$$ (surface distribution)

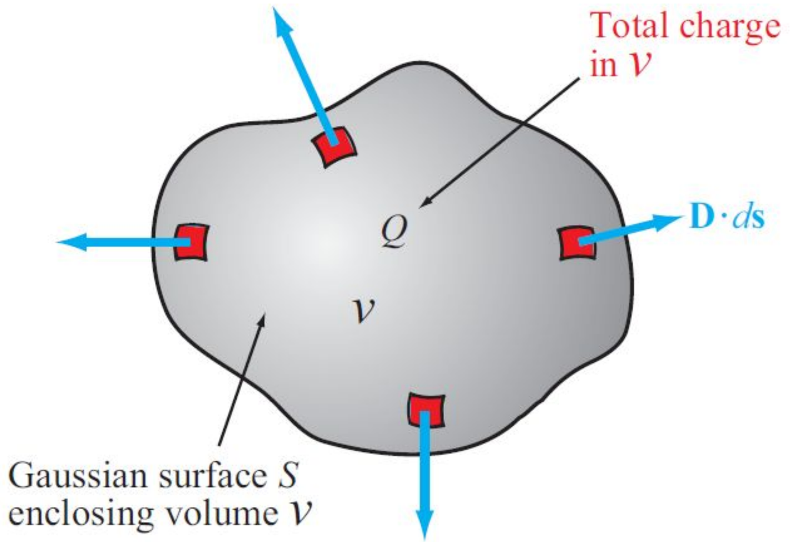
$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_{\ell}\, dl'}{R'^2}$$ (line distribution)

Chapter 4 Review

Gauss's Law

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.2$$

(Integral form of Gauss's law).



or:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho_V \, dV$$

where the closed-surface S is the boundary of $\mathcal{V}$

Chapter 4 Review

Voltage:

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \qquad \text{(V)}. \qquad (4.43)$$

N Point Charges:

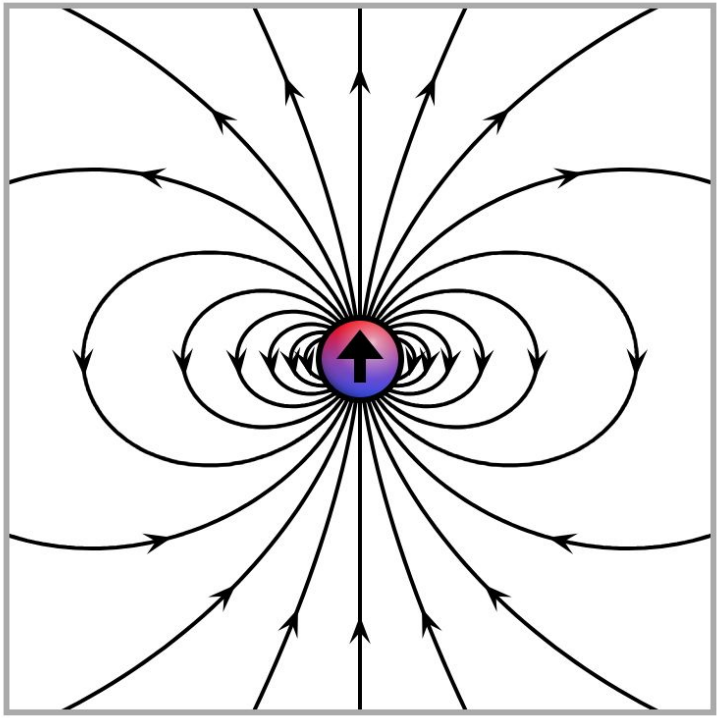
$$V = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|} \qquad \text{(V)}$$

Chapter 4 Review

$$\mathbf{E} = -\nabla V.$$

Electric Dipole:

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbf{R}}\, 2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad \text{(V/m)}$$



(wikipedia.org)

Chapter 4 Review

Since:

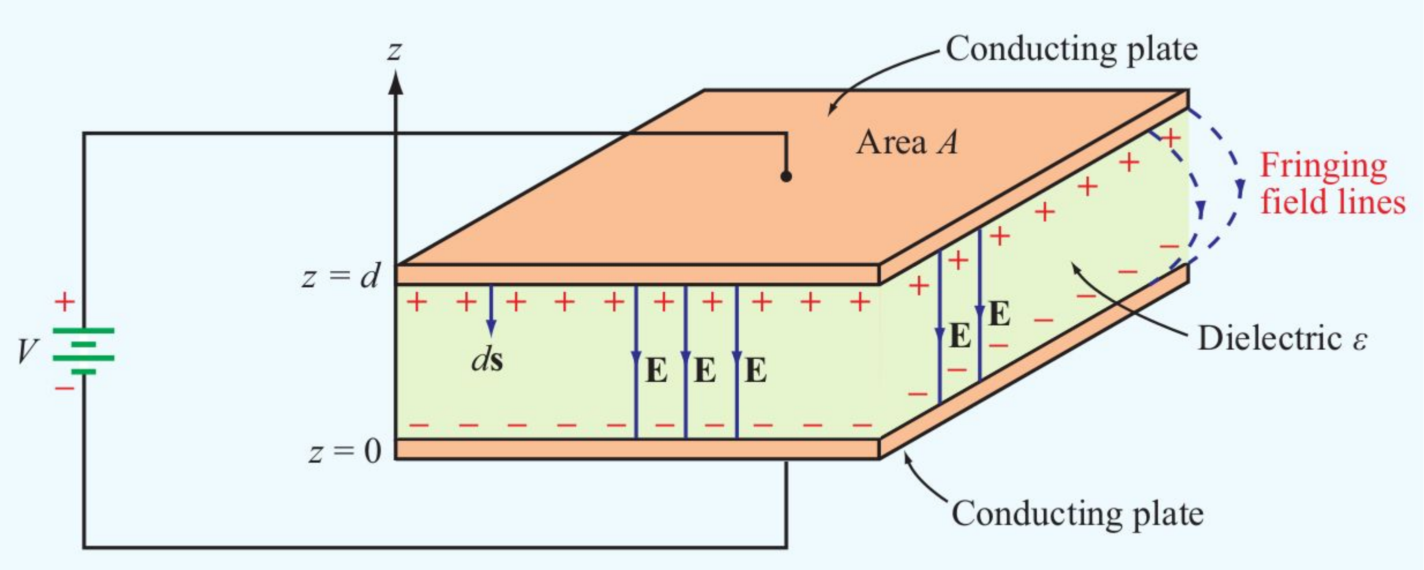
$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \textbf{(Poisson's equation)}$$

if $\rho_v = 0$:

$$\nabla^2 V = 0 \qquad \textbf{(Laplace's equation)}$$

Useful for problems where V is known on boundaries.

Chapter 4 Review



$$\mathbf{E} = -\hat{\mathbf{z}}\frac{V_0}{d}$$

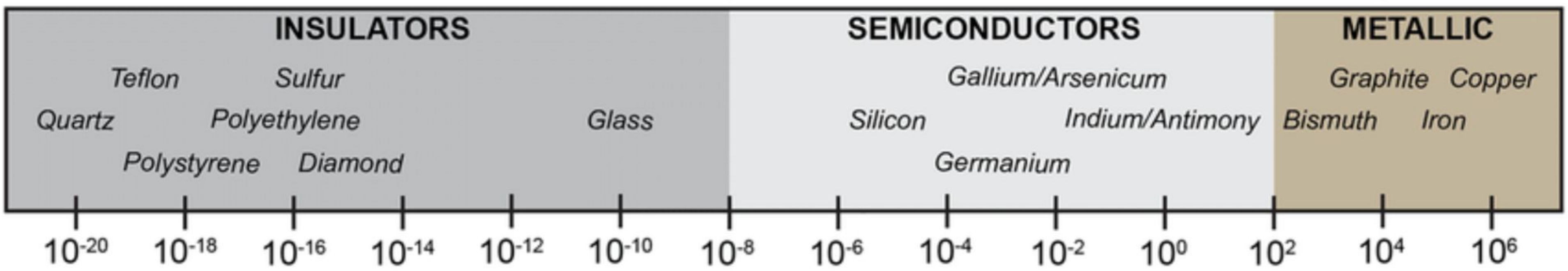
$$V(z) = \frac{V_0}{d}z$$

This approximation is OK for thin capacitors

Conductors and Conduction Current

Conduction Current Density:

$$\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2) \qquad (\text{Ohm's law})$$



Conductivity

Note how wide the range is:
over 27 orders of magnitude:
10^{-20} to 10^{7}

Conductivity: Conductors

Electrons move through material at a velocity known as the Drift Velocity:

$$\mathbf{u}_e = -\mu_e \mathbf{E} \qquad \text{(m/s)},$$

The electron mobility:

μ_e units: $m^2/(V\ sec)$

relates the induced velocity to the Electric field.

Recall: $\mathbf{J} = \rho_v \mathbf{u}$

so: $\mathbf{J} = -\rho_v \mu_e \mathbf{E} = \sigma \mathbf{E}$

Conductivity: Conductors

For a *uniform* charge density:

Define: the Number Density of free electrons, N_e:

$$\rho_{\mathrm{ve}} = -N_{\mathrm{e}} e$$

N_e has units of: #electrons/m^3

Conductivity: Semiconductors

For semiconductor materials it is useful to model atoms in the material that have a "missing" electron as "holes". The holes are positively charged.

We then define a charge density for the holes: ρ_{vh}
And we can define the Number Density of holes, N_h:

$$\rho_{vh} = N_h e$$

N_h has units of: #holes/m^3

(EECS320 deals with semiconductors and holes in more detail)

Conductivity

Hence for semiconductors:

$$\sigma = -\rho_{\mathrm{ve}}\mu_{\mathrm{e}} + \rho_{\mathrm{vh}}\mu_{\mathrm{h}} = (N_{\mathrm{e}}\mu_{\mathrm{e}} + N_{\mathrm{h}}\mu_{\mathrm{h}})e \quad \text{(S/m)},$$

(semiconductor)

and for good conductors:

$$\sigma = -\rho_{\mathrm{ve}}\mu_{\mathrm{e}} = N_{\mathrm{e}}\mu_{\mathrm{e}}e \qquad \text{(S/m)}.$$

(good conductor)

where the contribution from the holes is negligible.

Conductivity

Special cases:

Perfect Dielectric: $\sigma = 0.$

Perfect Conductor: $\sigma = \infty$

$$\mathbf{J} = 0,$$

$$\mathbf{E} = 0.$$

Example 1

Given: copper wire

2mm diameter

σ = 5.8X10^7 S/m

μ_e = 0.0032 m^2/(V·s)

$|\mathbf{E}|$ = 20 mV/m

Find: (a) ρ_{ve}

(b) J

(c) I

(d) u_e

(e) N_e

Example 1

Solution:

(a)

$$\rho_{ve} = -\frac{\sigma}{\mu_e} = -\frac{5.8 \times 10^7}{0.0032} = -1.81 \times 10^{10} \text{ (C/m}^3\text{)}.$$

(b)

$$J = \sigma E = 5.8 \times 10^7 \times 20 \times 10^{-3} = 1.16 \times 10^6 \text{ (A/m}^2\text{)}.$$

Example 1

(c)

$$I = JA$$

$$= J\left(\frac{\pi d^2}{4}\right) = 1.16 \times 10^6 \left(\frac{\pi \times 4 \times 10^{-6}}{4}\right) = 3.64\ \text{A}.$$

Example 1

(d)

$$u_e = -\mu_e E = -0.0032 \times 20 \times 10^{-3} = -6.4 \times 10^{-5} \text{ m/s}.$$

The minus sign indicates that $\mathbf{u}_e$ is in the opposite direction of $\mathbf{E}$.

(e)

$$N_e = -\frac{\rho_{ve}}{e} = \frac{1.81 \times 10^{10}}{1.6 \times 10^{-19}} = 1.13 \times 10^{29} \text{ electrons/m}^3.$$

Example 2: Free Electron Density

Find: Compare the Free Electron Density in Copper to that in Aluminum.

Solution: $$N_e = \frac{\sigma}{\mu_e e}$$

For Aluminum: $\sigma = 3.5 \times 10^7$ S/m

$\mu_e = 0.0015\ m^2/(Vsec)$

For Copper: $\sigma = 5.8 \times 10^7$ S/m

$\mu_e = 0.005\ m^2/(Vsec)$

Example 2: Free Electron Density

Solution: Free Electron Density Ratio: Al/Cu:

$$\frac{3.5\mathrm{X}10^{7}\ \mathrm{S/m}\ /\ 0.0015\ \mathrm{m}^{2}/(\mathrm{Vsec})}{5.8\mathrm{X}10^{7}\ \mathrm{S/m}\ /\ 0.005\ \mathrm{m}^{2}/(\mathrm{Vsec})} = 2$$

So, Al has twice the free electron density of Cu, but that does not make up for it's much lower conductivity: most circuits use Cu.

However: because Al is lighter and lower cost, it is used in power lines.

Example 3:

Given: long wire: $L=100m$, $\sigma=2X10^7$ S/m

$J = 3X10^5$ A/m^2

Find: Voltage change across the length of the wire.

Solution:

Since $J=\sigma E$, $E= J/\sigma$

and so Voltage = $E L = J L / \sigma$

$= (3X10^5 A/m^2) (100m) / (2X10^7 S/m)$

$V = 1.5 V$

Example 4:

Given: Cylindrical bar of Silicon: r=4mm, L=8cm
Voltage difference=5V,
μ_e = 0.13 m^2/(Vsec), μ_h = 0.05 m^2/(Vsec)
N_e = 1.5X10^{16} electrons/m^3, $N_h = N_e$

Find: conductivity of Si, I, u_e, u_h, R, P

Example 4:

Solution:

$$\sigma = (N_e \mu_e + N_h \mu_h)e$$

$$= (1.5 \times 10^{16})(0.13 + 0.05)(1.6 \times 10^{-19})$$

$$= 4.32 \times 10^{-4} \quad \text{(S/m)}.$$

Example 4:

Solution:

$$I = JA = \sigma E A$$

$$= (4.32 \times 10^{-4}) \left(\frac{5\text{V}}{0.08} \right) (\pi (4 \times 10^{-3})^2)$$

$$I = 1.36 \qquad (\mu\text{A}).$$

Example 4:

Solution:

$$\mathbf{u}_e = -\mu_e \mathbf{E}$$

$$u_e = \mu_e E$$

$$u_e = \mu_e V/L$$

$$u_e = (0.13\mathrm{m}^2/(\mathrm{Vsec}))(5\,\mathrm{V})/0.08\,\mathrm{m}$$

$$\boxed{u_e = 8.125\ \mathrm{m/sec}}$$

Example 4:

Solution:

$$u_h = \mu_h E$$

$$u_h = \mu_h V/L$$

$$u_h = (0.05\text{m}^2/(\text{Vsec}))(5\,\text{V})/0.08\,\text{m}$$

$$\boxed{u_h = 3.125\ \text{m/sec}}$$

Example 4:

Solution:

$$R = V / I$$

$$R = 5\text{V} / 1.36\ \mu\text{A}$$

$$\boxed{R = 3.68\ \text{M}\Omega}$$

Example 4:

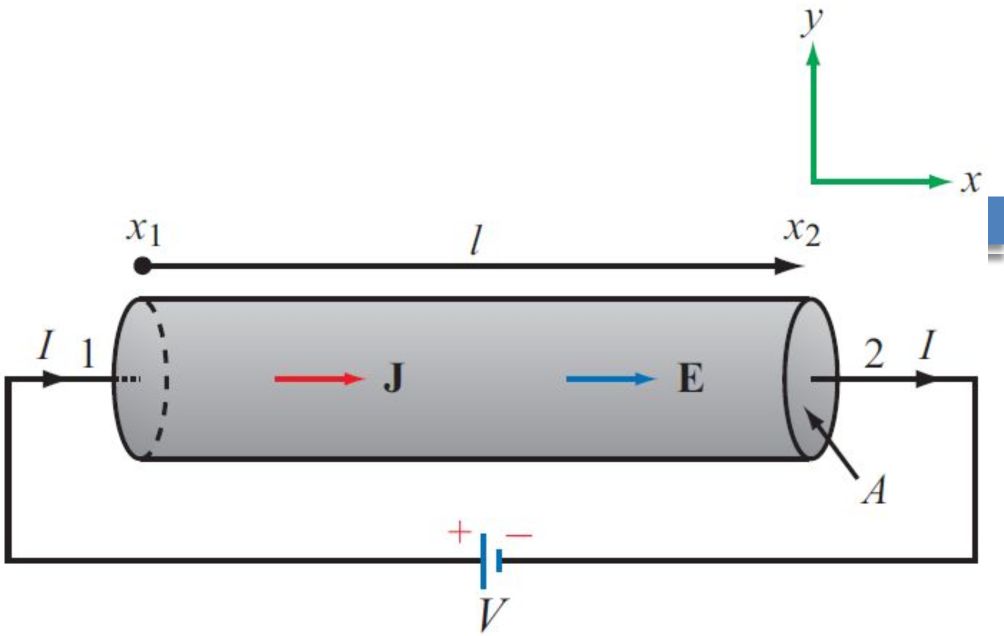
Solution:

$$P = I\,V$$

$$P = (1.36\ \mu\text{A})(\ 5\text{V})$$

$$\boxed{P = 6.8\ \mu\text{W}}$$

Resistance

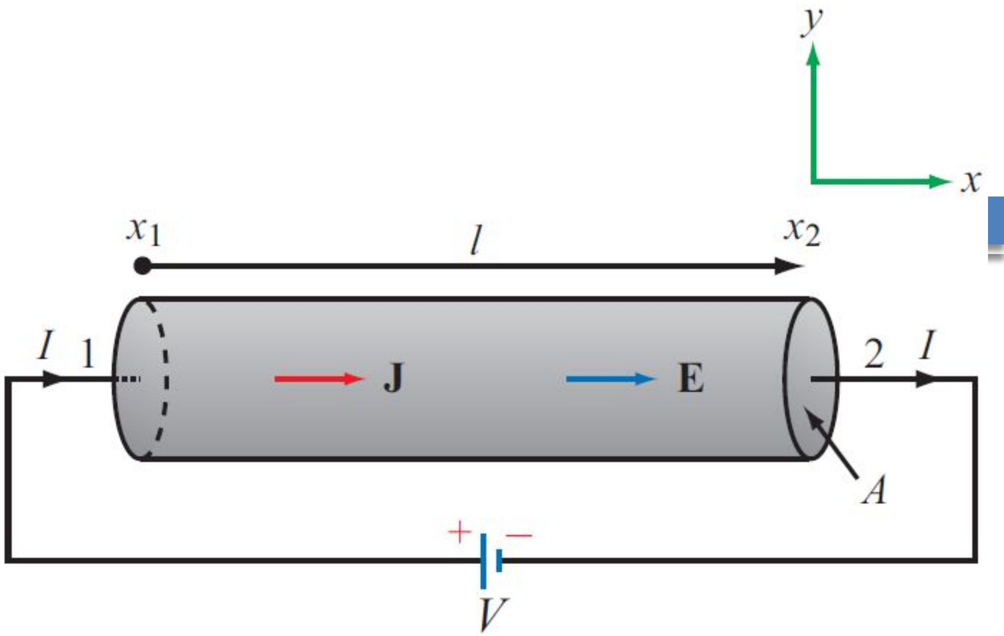


Consider the Longitudinal Resistor

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{x_2}^{x_1} \hat{\mathbf{x}} E_x \cdot \hat{\mathbf{x}}\, dl = E_x l \qquad \text{(V).} \qquad (4.68)$$

Resistance



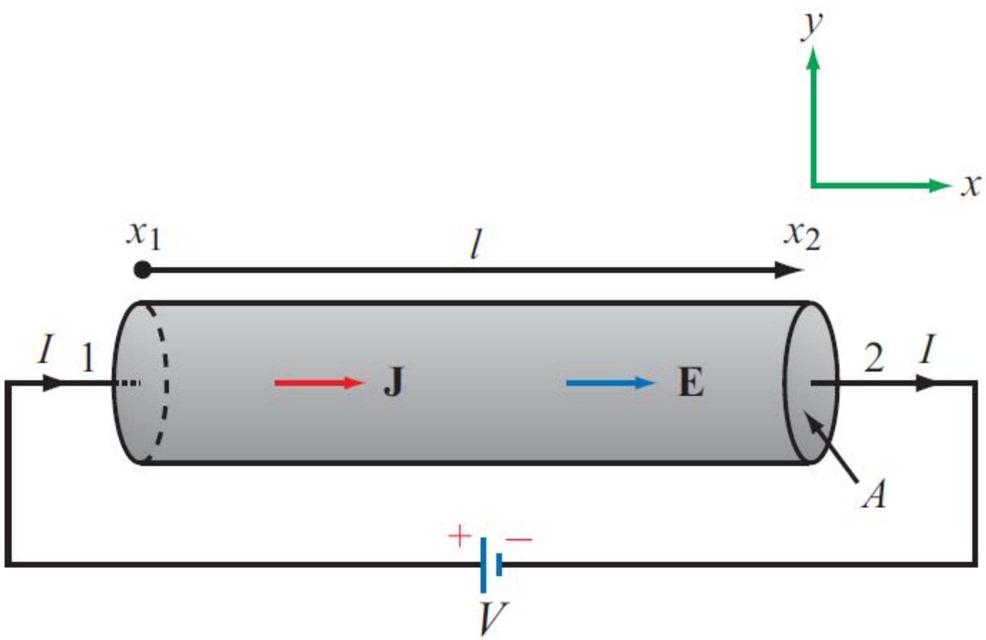
$V = E_x\, l$

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \qquad \text{(A)}.$$

Ohm's Law: $V = I\, R$: $R = V / I$

$$R = \frac{E_x l}{\sigma E_x A}$$ so: $$R = \frac{l}{\sigma A}$$

Resistance



For any conductor:

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} .$$

Example 5:

Given: copper wire: L=50m, σ=5.8X10^7 S/m

circular cross-section: r=2cm

voltage change along length = 1.5 mV

Find: Resistance, power dissipated.

Solution:

$$R = \frac{l}{\sigma A} = \frac{50}{5.8 \times 10^7 \times \pi (0.02)^2}$$

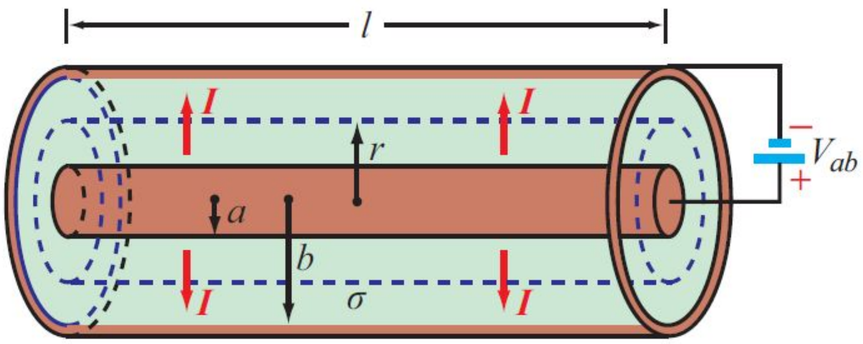
$$R = 6.9 \times 10^{-4}\ \Omega.$$

Example 5:

Solution:

$$P = \frac{V^2}{R} = \frac{(1.5 \times 10^{-3})^2}{6.9 \times 10^{-4}} = \boxed{3.3 \quad (\text{mW}).}$$

Coax Conductance



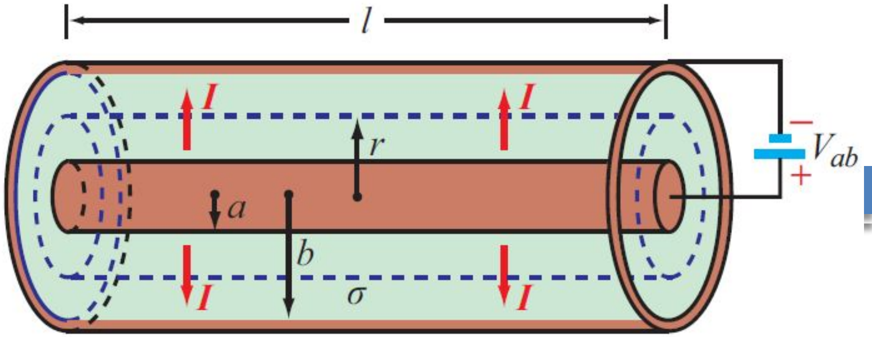
Given: a coaxial cable:
length = l
inner radius = *a*
outer radius = *b*
insulation conductivity = σ

Find: the conductance per unit length: *G*'

Radial current, I, flows through a cylindrical surface at distance *r*, with area

$$A = 2\pi r l$$

Coax Conductance



Hence:

$$\mathbf{J} = \hat{\mathbf{r}}\,\frac{I}{A} = \hat{\mathbf{r}}\,\frac{I}{2\pi r l}$$

and since: $\mathbf{J} = \sigma\mathbf{E}$

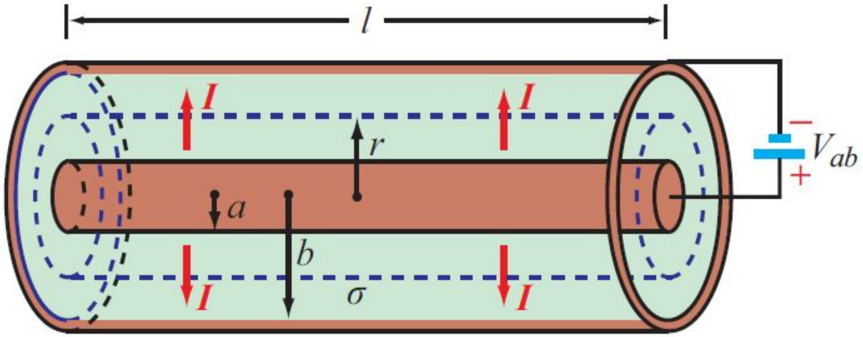
get:

$$\mathbf{E} = \hat{\mathbf{r}}\,\frac{I}{2\pi\sigma r l}$$

and for the voltage between inner and outer conductors:

$$d\mathbf{l} = \hat{\mathbf{r}}\,dr$$

Coax Conductance



Hence:

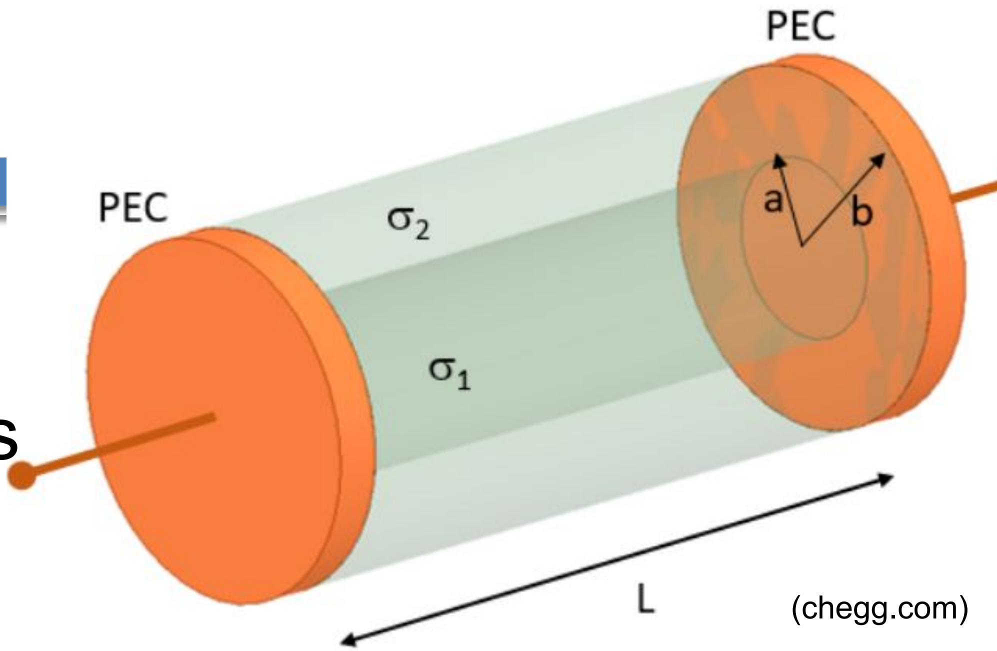
$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \frac{I}{2\pi\sigma l} \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, dr}{r} = \frac{I}{2\pi\sigma l} \ln\left(\frac{b}{a}\right)$$

the conductance per unit length is then:

$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab}l} = \boxed{\frac{2\pi\sigma}{\ln(b/a)}} \qquad \text{(S/m)}.$$

as given in Chapter 2.

Coaxial Resistor



Given: cylindrical matls
difft conductivities
PEC: conducting plates

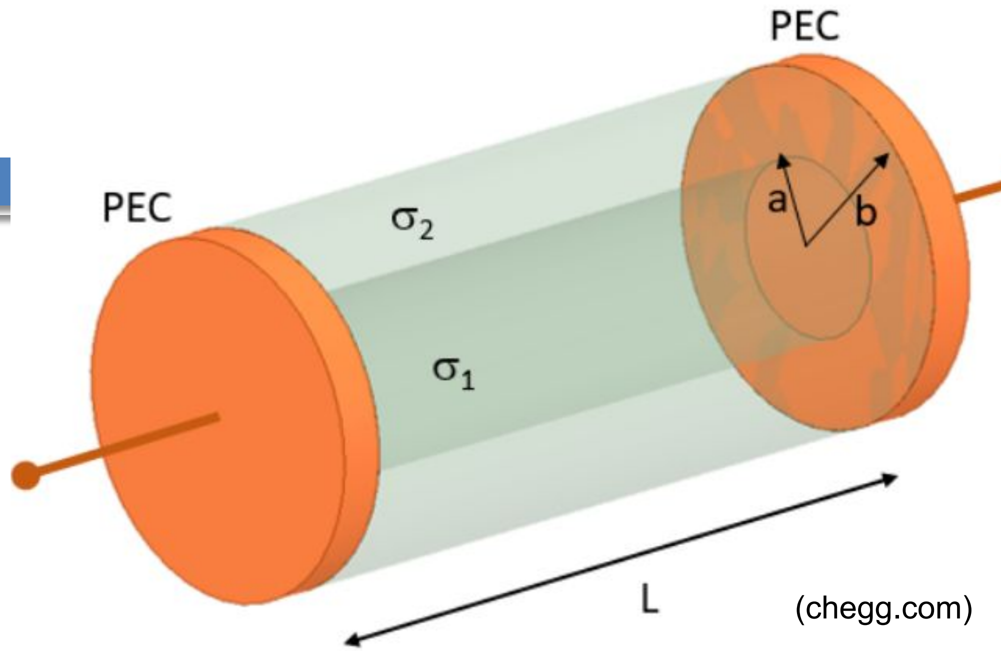
Find: expression for resistance

Solution: voltage at each end is same across plate
hence: **E** is along the length of the resistor
Like 2 resistors in parallel:

$$\frac{1}{R} = \frac{1}{R_{\text{inner}}} + \frac{1}{R_{\text{outer}}}:$$

Coaxial Resistor

Solution: $R = \frac{l}{\sigma A}$



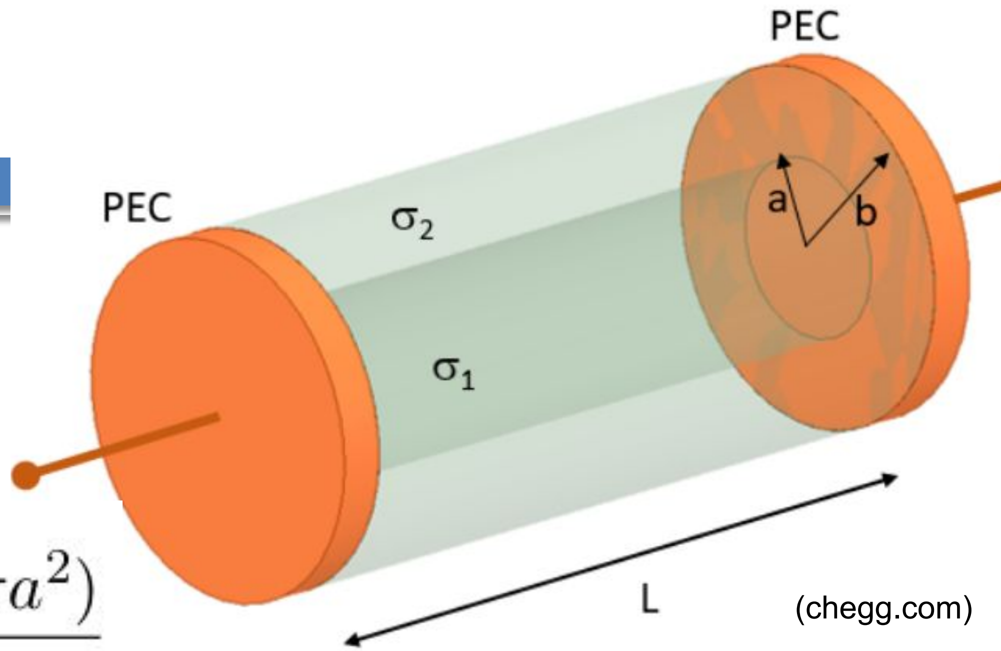
(chegg.com)

$$R_{\text{inner}} = \frac{L}{\sigma_1 \pi a^2}$$

$$R_{\text{outer}} = \frac{L}{\sigma_2(\pi b^2 - \pi a^2)}$$

$$\frac{1}{R} = \frac{\sigma_1 \pi a^2}{L} + \frac{\sigma_2(\pi b^2 - \pi a^2)}{L}$$

Coaxial Resistor



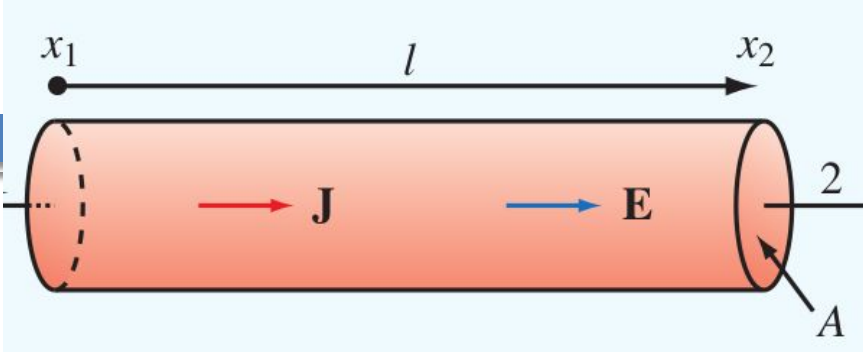
Solution: $$R = \frac{l}{\sigma A}$$

$$\frac{1}{R} = \frac{\sigma_1 \pi a^2}{L} + \frac{\sigma_2(\pi b^2 - \pi a^2)}{L}$$

$$\frac{1}{R} = \frac{\sigma_1 \pi a^2 + \sigma_2 \pi (b^2 - a^2)}{L}$$

$$R = \frac{L}{\sigma_1 \pi a^2 + \sigma_2 \pi (b^2 - a^2)}$$

Cylindrical Resistor



Given: carbon resistor

L=8cm, diameter=1mm

Find: Resistance

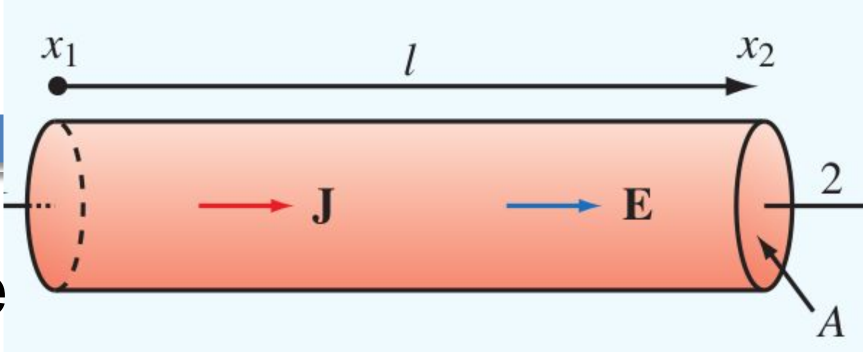
Solution: $$R = \frac{l}{\sigma A}$$

σ for carbon, from appendix B: 3X10^4 S/m

$$R = \frac{0.08\,\text{m}}{(3 \times 10^4\ \text{S/m})\pi(0.5 \times 10^{-3}\ \text{m})^2}$$

$$\boxed{R = 3.4\,\Omega}$$

Cylindrical Resistor



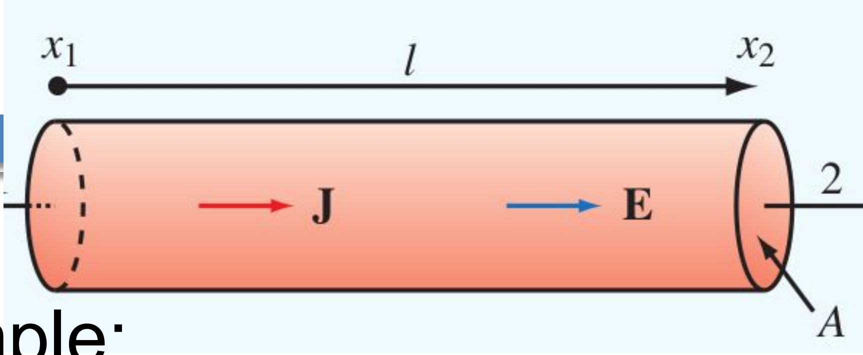
Find: Reduce the resistance by 40% by coating with copper: what is the thickness needed?

Solution: like previous example:

$$R = \frac{L}{\sigma_1 \pi a^2 + \sigma_2 \pi (b^2 - a^2)}$$

want $R_{new} = 0.6\,R = 0.6\,(3.4\ \Omega) = 2.04\ \Omega$

Cylindrical Resistor



Solution: like previous example:

$$R = \frac{L}{\sigma_1 \pi a^2 + \sigma_2 \pi (b^2 - a^2)} = 2.04\ \Omega$$

with: $L = 0.08$m, $a = 0.5 \times 10^{-3}$m,

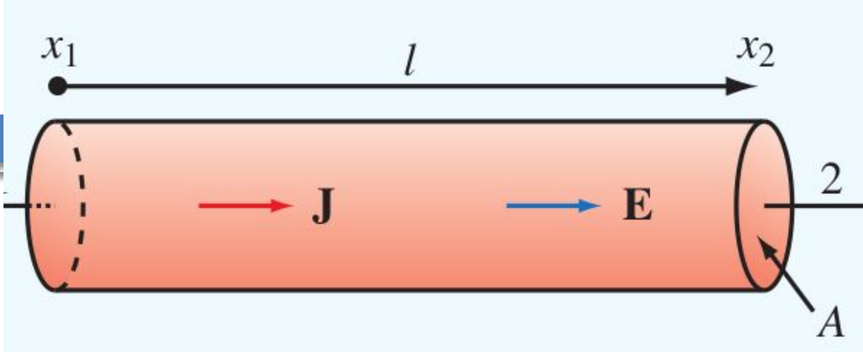
$\sigma_1 = 3 \times 10^4$ S/m (carbon)

$\sigma_2 = 5.8 \times 10^7$ S/m (copper)

Solve for b:

$b = 5.00086 \times 10^{-4}$ m

Cylindrical Resistor



Solution:

$$b = 5.00086 \text{ X}10^{-4} \text{ m}$$

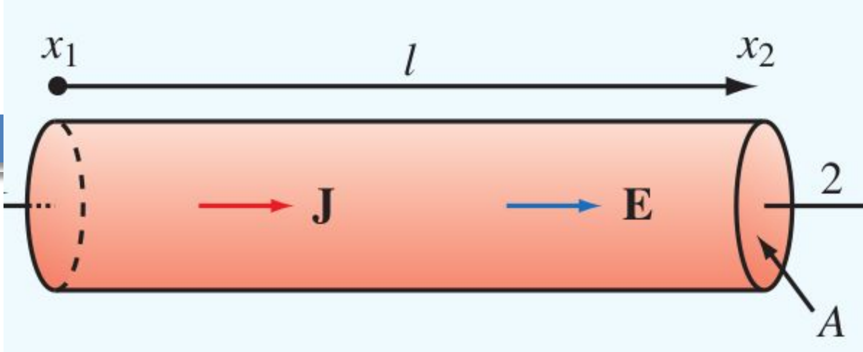
$$\text{Thickness} = b\text{-}a = 5.00086 \text{ X}10^{-4} \text{ m} - 5.0 \text{ X}10^{-4} \text{ m}$$

$$= 0.00086 \text{ X}10^{-4} \text{ m}$$

Thickness = 0.086 μm

(a very thin copper coating)

Joule's Law



Assume: a conductor:

Charge density: ρ_v

Charge in a differential volume: $dq = \rho_v d\mathcal{V}$

Force due to applied Electric field:

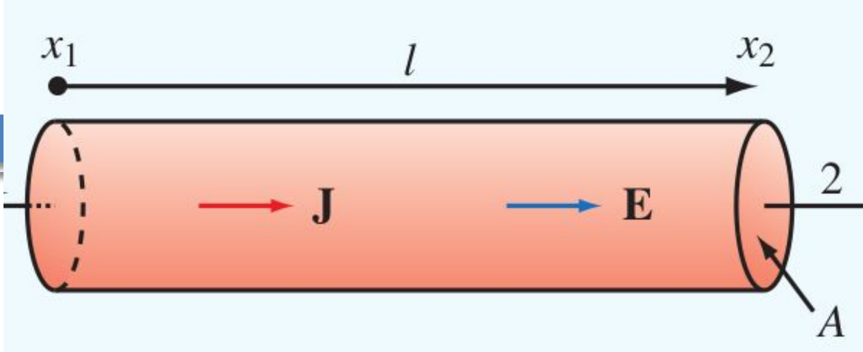
$$\mathbf{F} = dq\mathbf{E}$$

$$\mathbf{F} = (\rho_v d\mathcal{V})\,\mathbf{E}$$

Differential Work (energy) to move dq a length $d\boldsymbol{l}$:

$$dW = \mathbf{F} \cdot d\boldsymbol{l}$$

Joule's Law



Differential Work (energy) to move *dq* a length *d**l***:

$$dW = \mathbf{F} \cdot d\boldsymbol{l}$$

Differential Power to move it:

$$dW / dt = \mathbf{F} \cdot d\boldsymbol{l} / dt$$

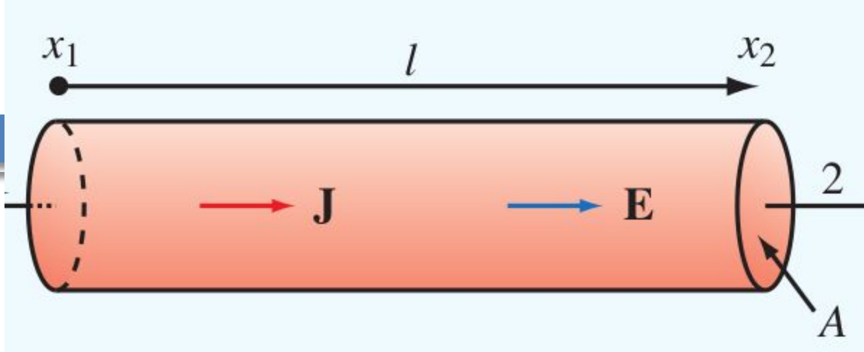
$$dW / dt = \mathbf{F} \cdot \mathbf{u}_e$$

$$dW / dt = (\rho_v d\mathcal{V})\, \mathbf{E} \cdot \mathbf{u}_e$$

$$dW / dt = \mathbf{E} \cdot \rho_v \mathbf{u}_e d\mathcal{V}$$

$$dP = \mathbf{E} \cdot \mathbf{J} d\mathcal{V}$$

Joule's Law



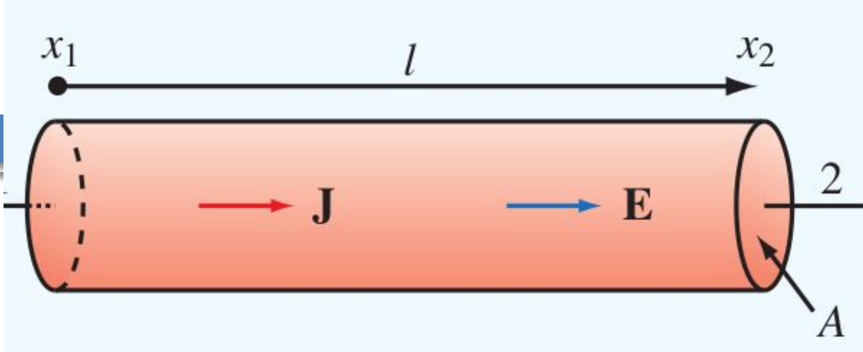
Differential Power to move it:

$$dP = \mathbf{E} \cdot \mathbf{J} d\mathcal{V}$$

Integrate over the volume:

$$P = \int_{\upsilon} \mathbf{E} \cdot \mathbf{J} \, d\upsilon \quad \text{(W)} \quad \textbf{(Joule's law)},$$

Joule's Law



$$P = \int_{\upsilon} \mathbf{E} \cdot \mathbf{J} \, d\upsilon \quad \text{(W)} \quad \text{(Joule's law)},$$

Since we know that: **J** = σ **E**:

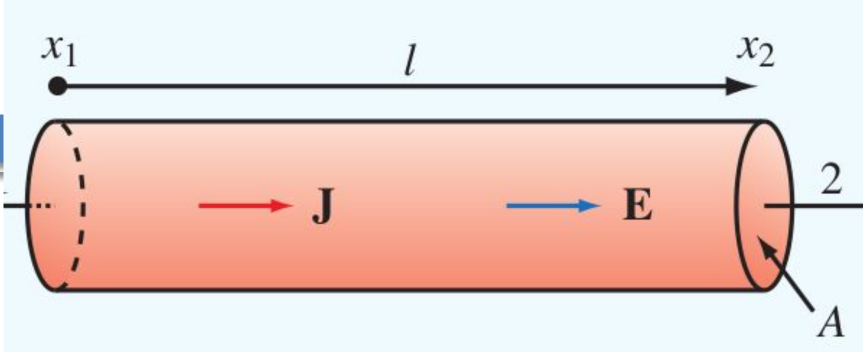
$$P = \int_{\mathcal{V}} \mathbf{E} \cdot \sigma \mathbf{E} \, d\mathcal{V}$$

$$P = \int_{\mathcal{V}} \sigma \mathbf{E} \cdot \mathbf{E} \, d\mathcal{V}$$

$$P = \int_{\mathcal{V}} \sigma |\mathbf{E}|^2 \, d\mathcal{V}$$

Joule's Law

$$P = \int_{\upsilon} \mathbf{E} \cdot \mathbf{J} \, d\upsilon \quad \text{(W)} \quad \textbf{(Joule's law)},$$



For a uniform resistor:

$$P = E\,J\,\mathcal{V}$$

$$P = \text{(V/m) (A/m}^2\text{) (m}^3\text{)}$$

$$P = \text{(V) (A)}$$

$$P = I\,V \quad \text{(Watts)}$$

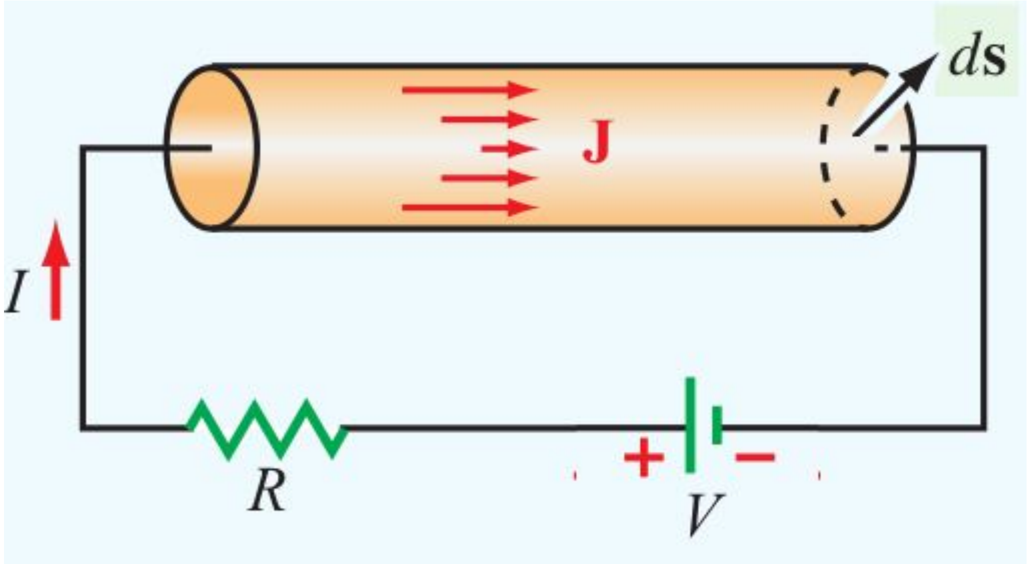
Just like in EECS215.

Joule's Law

Given: non-uniform wire: circular cross-section,

$r = r_0$, $L = L_0$, $\mathbf{J} = \hat{\mathbf{z}} \cdot J_0\, r^2$, $r < r_0$,

voltage across wire: V_0



Find: Power, P

Solution:

$$P = \int_{\upsilon} \mathbf{E} \cdot \mathbf{J}\, d\upsilon$$

cylindrical coords: $d\mathcal{V} = r\, dr\, d\phi\, dz$

uniform field along length: $\mathbf{E} = \hat{\mathbf{z}} \cdot V_0 / L_0$

Joule's Law

Solution:

$$P = \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J} \, d\mathcal{V}$$

cylindrical coords: $d\mathcal{V} = r \, dr \, d\phi \, dz$

$\mathbf{E} = \hat{\mathbf{z}} V_0 / L_0$

Plug in:

$$P = \int_{r=0}^{r_0} \int_{\phi=0}^{2\pi} \int_{z=0}^{L_0} \hat{\mathbf{z}} \frac{V_0}{L_0} \cdot \hat{\mathbf{z}} J_0 r^2 \, r \, dr \, d\phi \, dz$$

Joule's Law

Solution:

$$P = \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J} \, d\mathcal{V}$$

cylindrical coords: $d\mathcal{V} = r \; dr \; d\phi \; dz$

$\mathbf{E} = \hat{\mathbf{z}} \cdot V_0 / L_0$

Plug in:

$$P = \int_{r=0}^{r_0} \int_{\phi=0}^{2\pi} \int_{z=0}^{L_0} \hat{\mathbf{z}} \frac{V_0}{L_0} \cdot \hat{\mathbf{z}} J_0 r^2 \, r \, dr \, d\phi \, dz$$

$$P = J_0 \frac{V_0}{L_0} \int_{r=0}^{r_0} r^3 dr \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{L_0} dz$$

Joule's Law

Solution:

$$P = J_0 \frac{V_0}{L_0} \int_{r=0}^{r_0} r^3 dr \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{L_0} dz$$

$$P = J_0 \frac{V_0}{L_0} L_0 \int_{r=0}^{r_0} r^3 dr \int_{\phi=0}^{2\pi} d\phi$$

$$P = J_0 \frac{V_0}{L_0} L_0 (2\pi) \int_{r=0}^{r_0} r^3 dr$$

$$P = J_0 \frac{V_0}{L_0} L_0 (2\pi) \left[\frac{r^4}{4} \right]_{r=0}^{r_0}$$

Joule's Law

Solution:

$$P = J_0 \frac{V_0}{L_0} L_0 (2\pi) \left[\frac{r^4}{4} \right]_{r=0}^{r_0}$$

$$P = J_0 \frac{V_0}{L_0} L_0 (2\pi) \frac{r_0^4}{4}$$

$$\boxed{P = \pi J_0 V_0 \frac{r_0^4}{2}}$$

Homework

Homework 14 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Sections 4-7, 4-8:

Dielectrics
Boundary Conditions