EECS 230

ENGINEERING ELECTROMAGNETICS

Leland Pierce

Electrostatics 2

University of Michigan - Fall 2024

Chapter 4 Overview

Maxwell's Equations
- Electrostatics
- Magnetostatics

Charge density
Current density
Electric field from charges
Gauss's Law
Electric Scalar Potential Field
- Dipole Field
- Poisson's eqn

Conductors
- current
- resistance
- joule's law

Dielectrics
- polarization

Boundary Conditions
Capacitance
Potential Energy
Image method

$$\nabla \cdot \mathbf{D} = \rho_v,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

$$\mathbf{E} = \int_{v'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{v'} \hat{\mathbf{R}}' \frac{\rho_v \, dv'}{R'^2}.$$

(volume distribution)

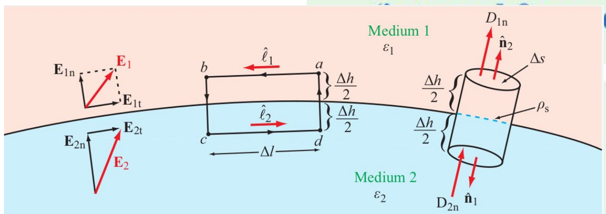
$$\nabla \cdot \mathbf{D} = \rho_v,$$

(differential form of Gauss's law)

$$\nabla \cdot \mathbf{D} = \rho_v,$$
$$\nabla \times \mathbf{E} = 0.$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q.$$

of Gauss's law)



$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l}$$

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J}.$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \mathbf{E} = -\nabla V.$$

$$Q = \int_{v} \rho_v \, dv \qquad \text{(C)}.$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (\text{A/m}^2) \quad \text{(Ohm's law)},$$

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2)$$

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}.$$

$$P = \int_v \mathbf{E} \cdot \mathbf{J} \, dv \qquad \text{(W)}$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.$$

Lecture Coverage

Today's lecture:

Review of Sections 4-1 through 4-4 of the book:

4-1: Maxwell's Equations
4-2: Charge and Current Distributions
4-3: Coulomb's Law
4-4: Gauss's Law

Section 4-5 of the book:

4-5: Voltage (Electric Scalar Potential)

Chapter 4 Review

Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho_V,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Empirically derived from many measurements

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}.$$

E: Electric Field
H: Magnetic Field
J: Current Density
ρ_v: Charge Density

Chapter 4 Review

Static Conditions:

$$\frac{\partial \rho_v}{\partial t} = 0$$

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}},$$
$$\nabla \times \mathbf{E} = 0.$$

$$\frac{\partial \mathbf{J}}{\partial t} = 0$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J}.$$

Electric and Magnetic Fields are decoupled.

Chapter 4 Review

Charge density and Total Charge

$$Q = \int_{\mathcal{V}} \rho_{\mathrm{v}} \, d\mathcal{V} \qquad \text{(C)}$$

Moving charges generate Current Density

$$\mathbf{J} = \rho_{\mathrm{v}} \mathbf{u} \qquad (\mathrm{A/m^2})$$

Current is the Current Density moving across some surface:

$$I = \int_S \mathbf{J} \cdot d\mathbf{s}$$

Chapter 4 Review

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \hat{\mathbf{R}}' \frac{\rho_\mathrm{v} \, d\upsilon'}{R'^2}$$ **(volume distribution)**

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_\mathrm{s} \, ds'}{R'^2}$$ **(surface distribution)**

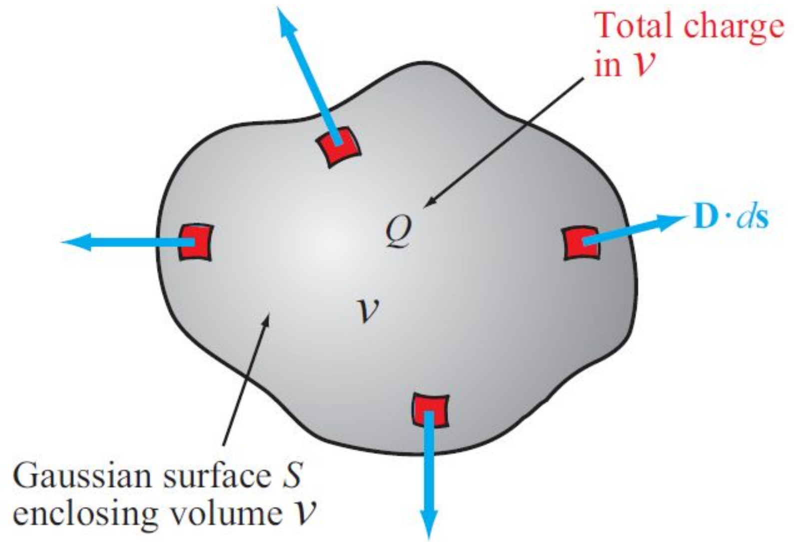
$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}$$ **(line distribution)**

Chapter 4 Review

Gauss's Law

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.2$$

(Integral form of Gauss's law).

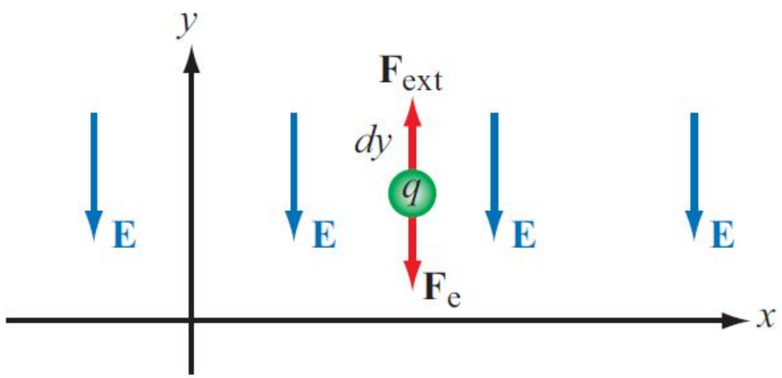


or:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\mathcal{V}} \rho_{\mathrm{v}} \, d\mathcal{V}$$

where the closed-surface S is the boundary of $\mathcal{V}$

4.5 Electric Scalar Potential



Minimum force needed to move charge against **E** field:

$$\mathbf{F}_{\text{ext}} = -\mathbf{F}_{\text{e}} = -q\mathbf{E}.$$

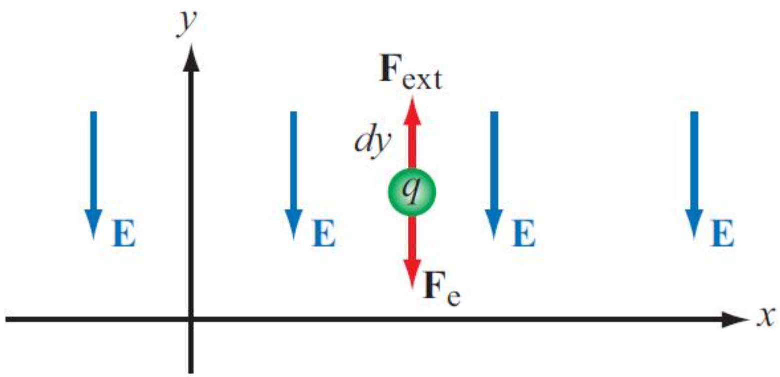
Work, or the energy expended, to move the charge some distance:

$$dW = \mathbf{F}_{\text{ext}} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J)}$$

Example: both **E** and motion oriented along y-direction:

$$dW = -q(-\hat{\mathbf{y}}E) \cdot \hat{\mathbf{y}}\, dy = qE\, dy.$$

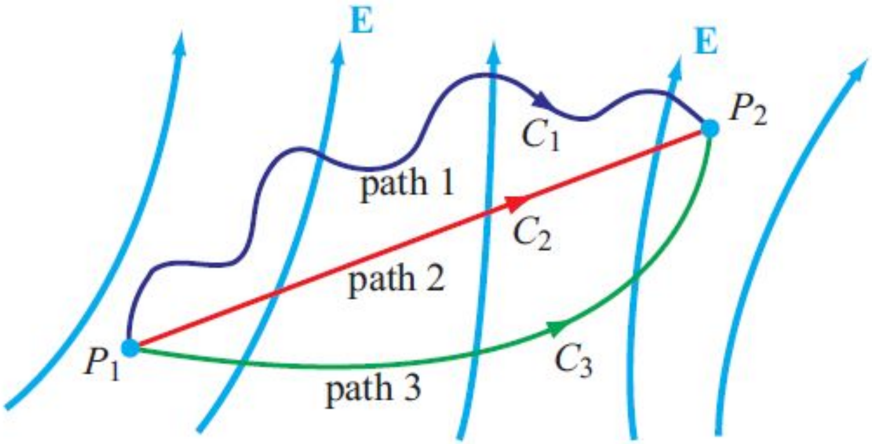
4-5 Electric Scalar Potential



Differential Electric Potential:

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J/C or V)}$$

4-5 Electric Scalar Potential



Potential difference when moving a point charge from point P_1 to point P_2:

Integrate along the path:

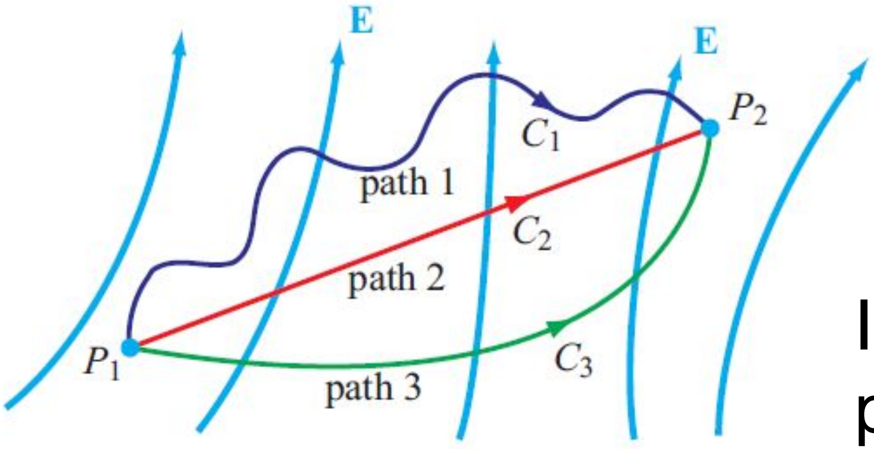
$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l},$$

Same result for ANY path:

Example: voltage difference in a circuit

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

4-5 Electric Scalar Potential

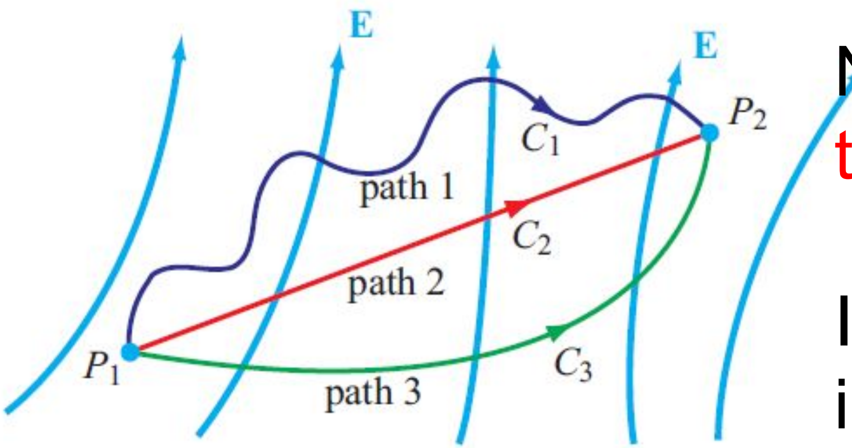


If integrate around a CLOSED path:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \qquad \textbf{(electrostatics)}$$

This is where KVL in circuits comes from.

4-5 Electric Scalar Potential



Note that this is ONLY true in the electrostatic case.

If **E** is time-varying, this integral is not necessarily zero.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \qquad \text{(electrostatics)}$$

4-5 Electric Potential Due to Charges

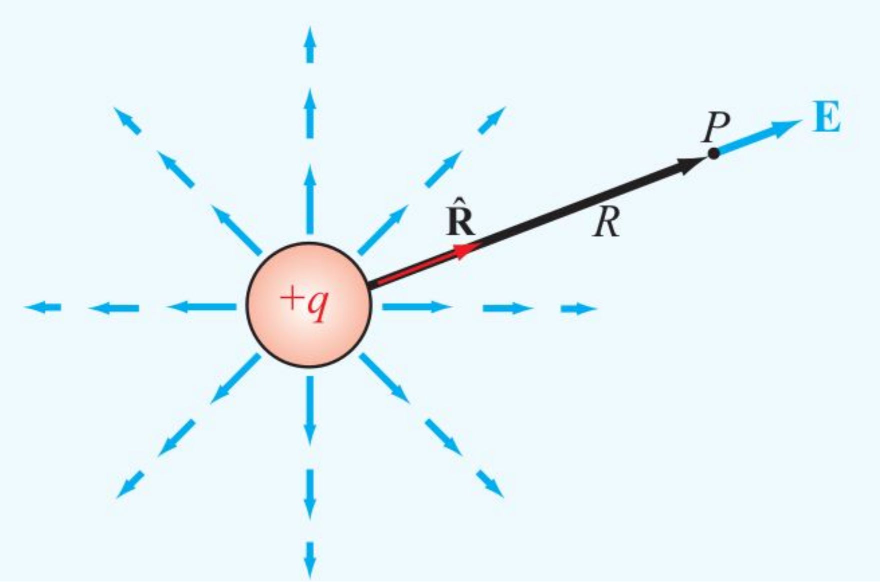
Need to choose a reference location and value.

In circuits choose a ground node and assign it 0V.

For non-circuit problems, choose infinity:

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \qquad \text{(V)}$$

4-5 Electric Potential Due to Charges



For a point charge, determine V at range R:

From Coulomb's law:

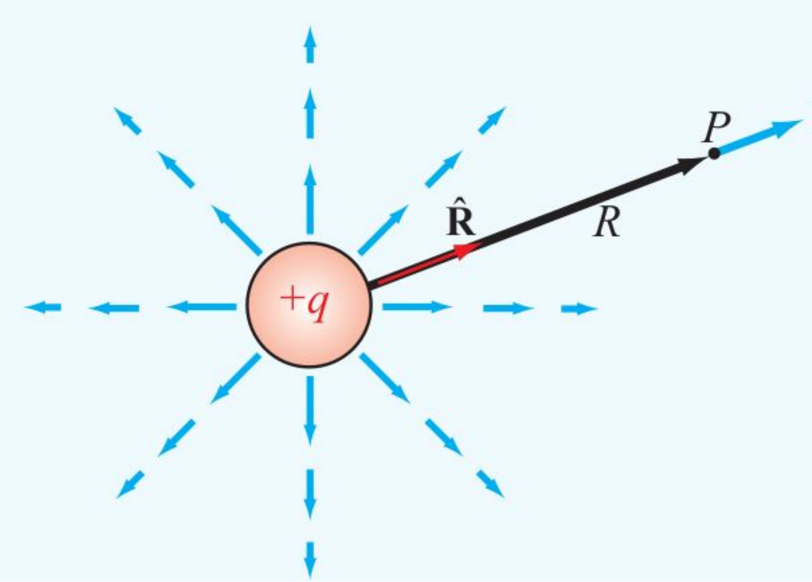
$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}$$

$$d\boldsymbol{l} = \hat{\mathbf{R}}\, dR$$

plug in:

$$V = -\int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \right) \cdot \hat{\mathbf{R}}\, dR$$

4-5 Electric Potential Due to Charges



$$V = -\int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \right) \cdot \hat{\mathbf{R}} dR$$

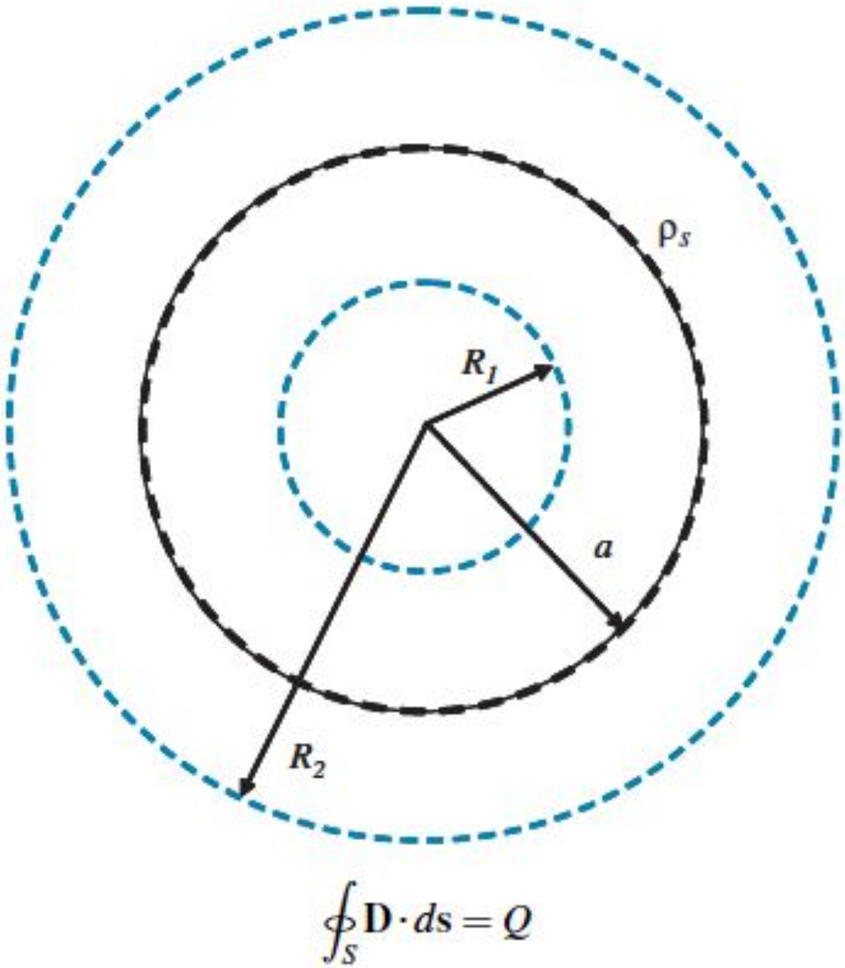
$$V = -\frac{q}{4\pi\varepsilon} \int_{\infty}^{R} R^{-2} dR$$

$$V = -\frac{q}{4\pi\varepsilon} \left[-R^{-1} \right]_{\infty}^{R}$$

$$V = \frac{q}{4\pi\varepsilon} \left[\frac{1}{R} - \frac{1}{\infty} \right]$$

$$V = \frac{q}{4\pi\varepsilon R}$$

Example 1

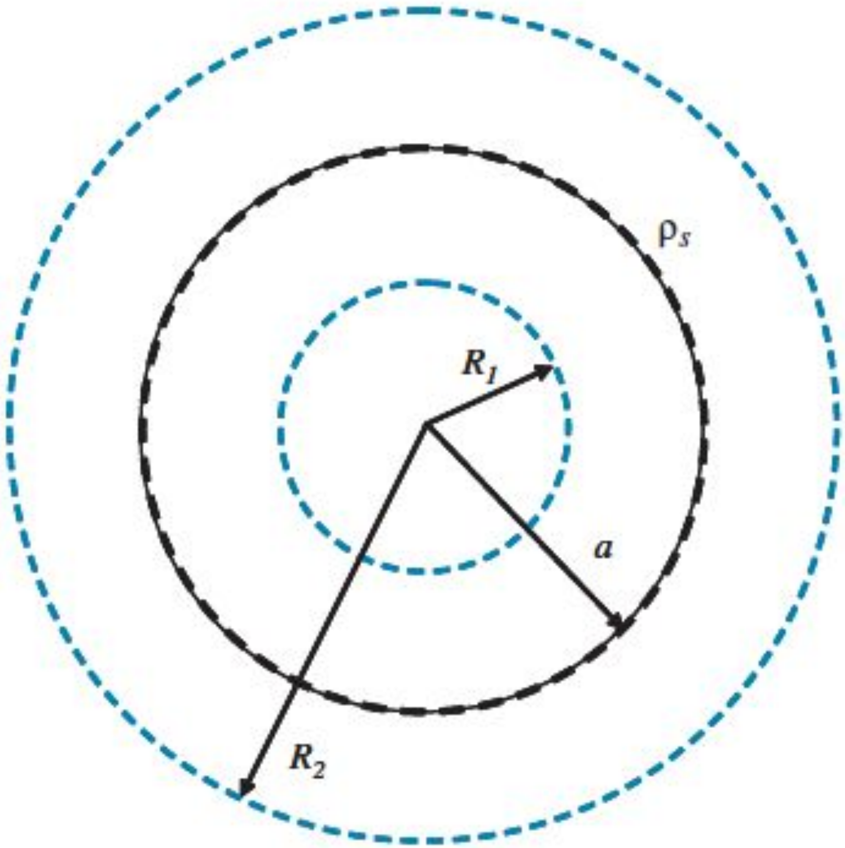


Given:
Spherical shell with a surface charge:
Solved for **E** already:

$$\mathbf{E} = \begin{cases} 0 & : R < a \\ \hat{\mathbf{R}}\,\dfrac{\rho_s a^2}{\epsilon R^2} & : R > a \end{cases}$$

Find: *V* everywhere

Example 1



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Solution:

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \qquad \text{(V)}.$$

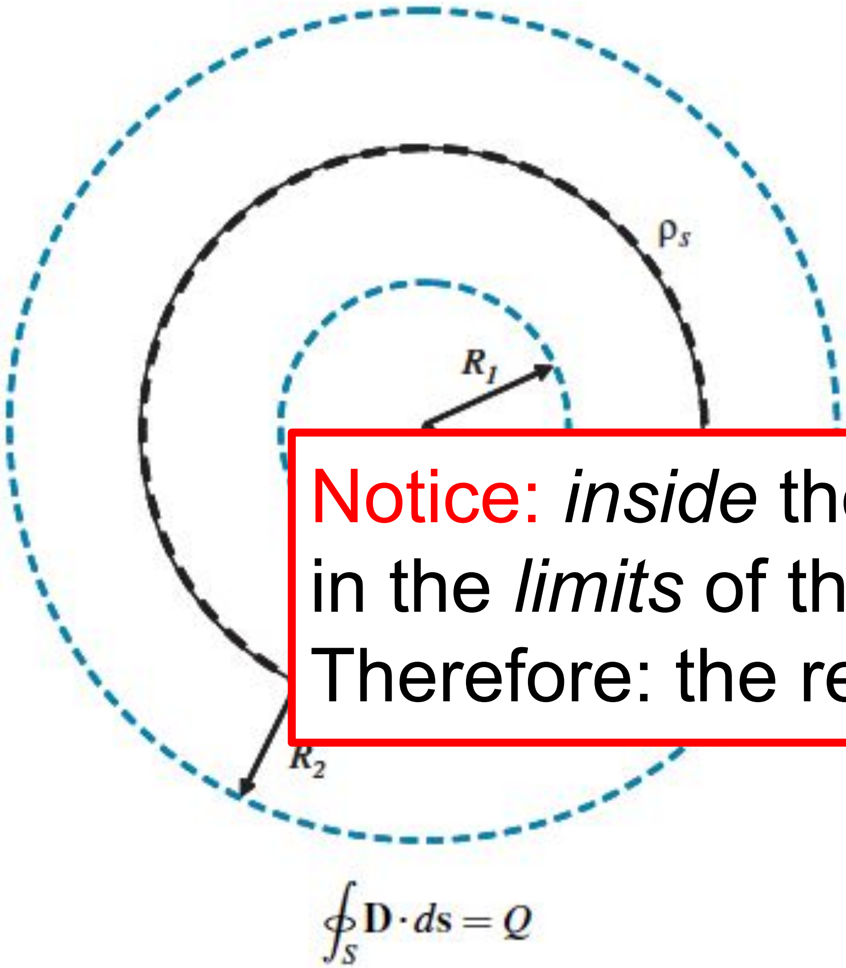
choose:

$d\mathbf{l}$ is a radial line segment:

$d\mathbf{l} = \hat{\mathbf{R}}\, dR'$

R' is the radial distance

Example 1

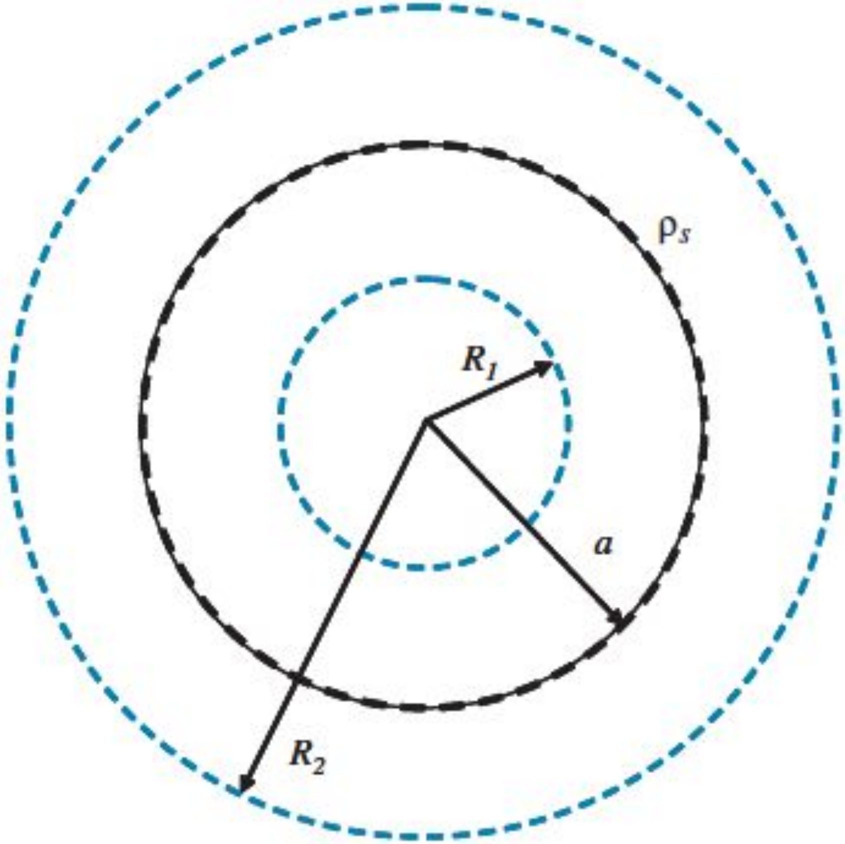


For *R*>*a*:

$$V(R) = -\int_{\infty}^{R} \hat{\mathbf{R}}\,\frac{\rho_s a^2}{\epsilon R'^2}\cdot \hat{\mathbf{R}}\,dR'$$

Notice: *inside* the integral we use *R*', and in the *limits* of the integral we use *R*. Therefore: the result is in terms of *R*

Example 1



$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$

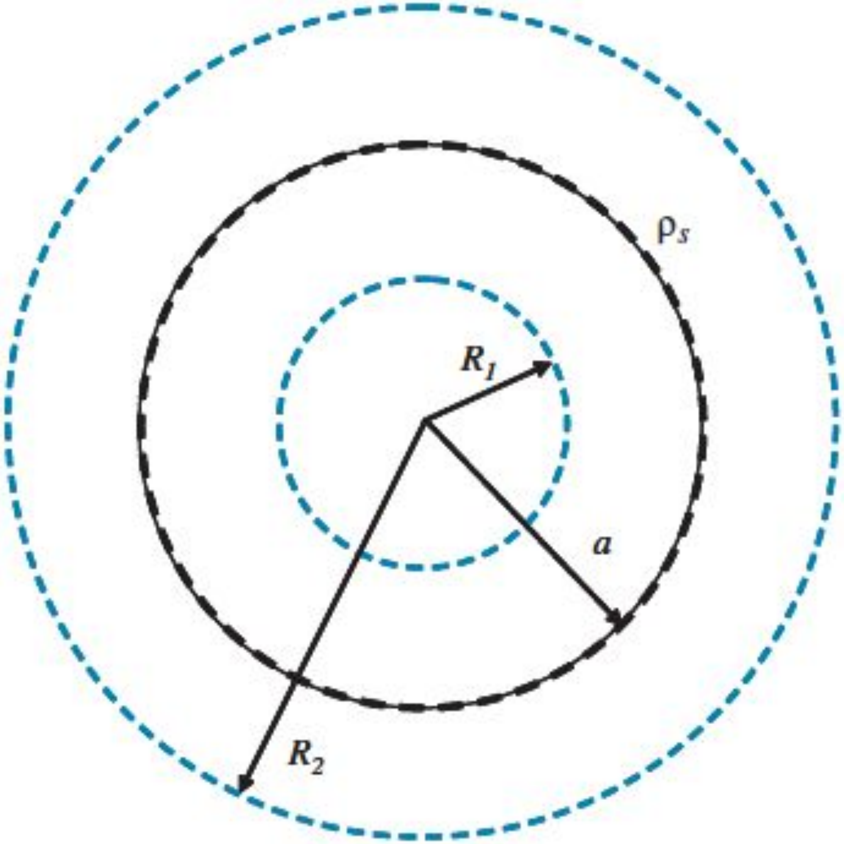
For *R*>*a*:

$$V(R) = -\int_{\infty}^{R} \hat{\mathbf{R}} \frac{\rho_s a^2}{\epsilon R'^2} \cdot \hat{\mathbf{R}} dR'$$

$$V(R) = -\frac{\rho_s a^2}{\epsilon} \int_{\infty}^{R} R'^{-2} dR'$$

$$V(R) = -\frac{\rho_s a^2}{\epsilon} \left(\frac{-1}{R'} \right) \Big|_{\infty}^{R}$$

Example 1



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

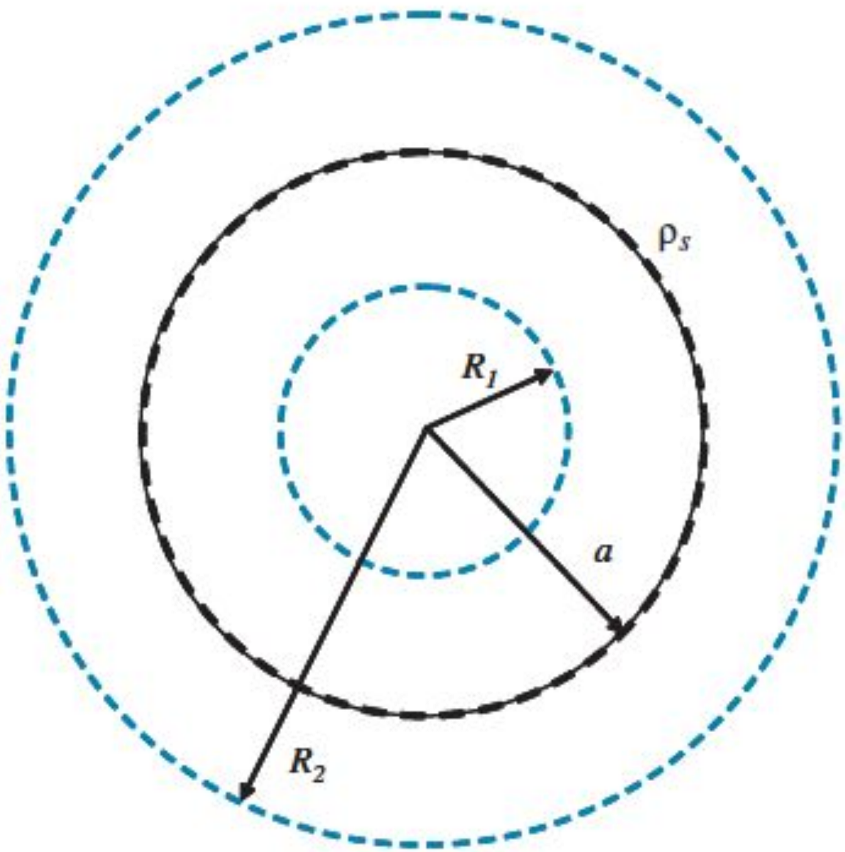
For $R>a$:

$$V(R) = -\frac{\rho_s a^2}{\epsilon}\left(\frac{-1}{R'}\right)\Big|_{\infty}^{R}$$

$$V(R) = -\frac{\rho_s a^2}{\epsilon}\left(\frac{-1}{R} - 0\right)$$

$$V(R) = \frac{\rho_s a^2}{\epsilon R}$$

Example 1



$$\oint_S \mathbf{D} \cdot ds = Q$$

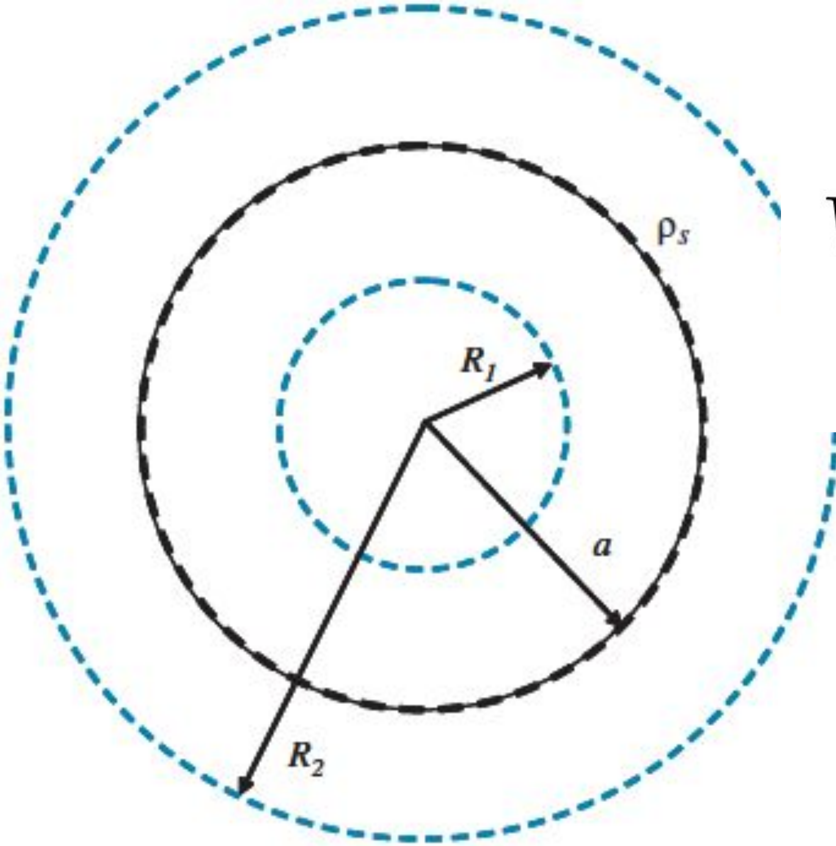
For *R*<*a*:

The integral when **E**=0, is 0.

So the Voltage does not change inside the sphere.

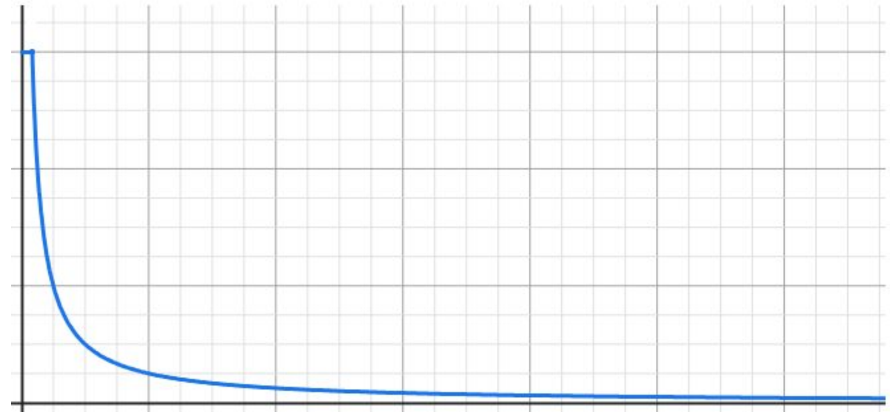
So the Voltage inside is the same as the voltage at *R*=*a*.

Example 1

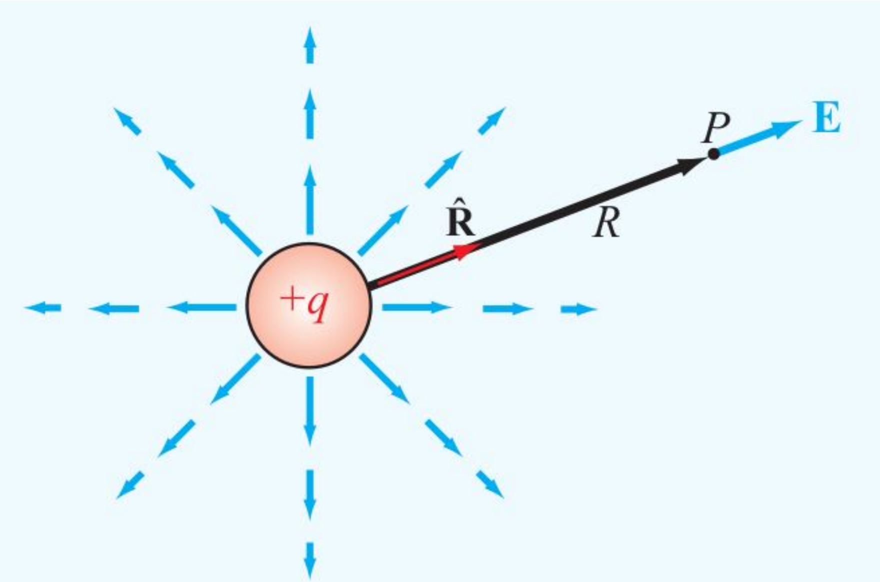


$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

$$V(R) = \begin{cases} \dfrac{\rho_s a}{\epsilon} & : R < a \\ \dfrac{\rho_s a^2}{\epsilon R} & : R > a \end{cases}$$



4-5 Electric Potential Due to Charges



For a point charge at the origin:

$$V = \frac{q}{4\pi\varepsilon R}$$

if NOT at the origin:

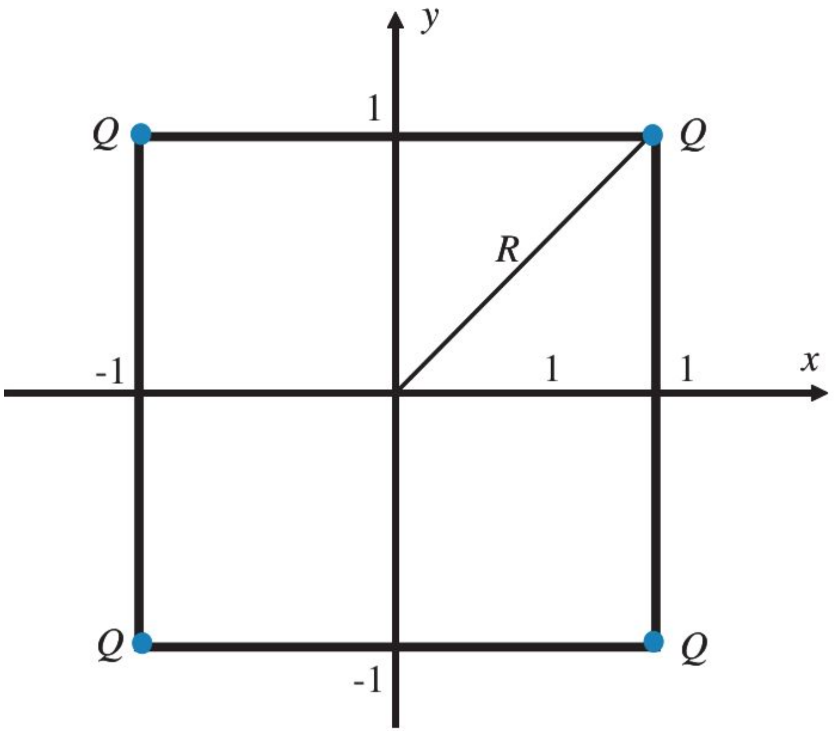
$$V = \frac{q}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_1|}$$

For N point charges:

$$V = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{R} - \mathbf{R}_i|} \quad \text{(V)}$$

Exercise 4-10 Electric Potential

Given: 4: 20 μC charges
at corners of 2m box:
Find: Potential, *V*, at origin.



Solution:

$$V = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{R}-\mathbf{R}_i|}$$

$$|\mathbf{R}-\mathbf{R}_i| = \sqrt{1^2+1^2} = \sqrt{2}\,\mathrm{m}$$

Exercise 4-10 Electric Potential

Solution:

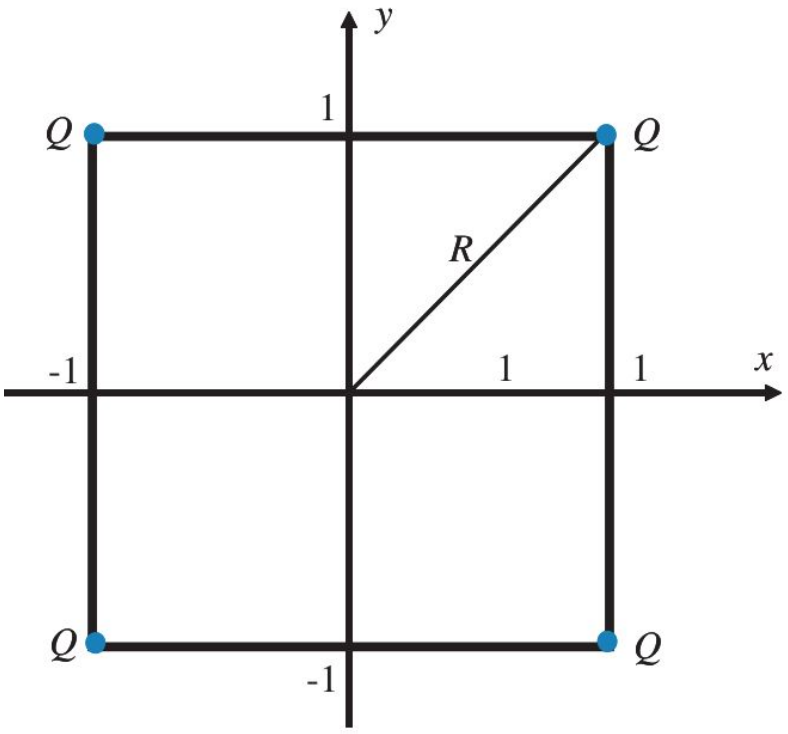
$$V = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{4} \frac{20\,\mu\text{C}}{\sqrt{2}\,\text{m}}$$

$$V = \frac{80\,\mu\text{C}}{(\sqrt{2}\,\text{m})(4\pi)\varepsilon_0}$$

$$V = \frac{80\,\mu\text{C}}{(\sqrt{2}\,\text{m})(4\pi)(8.85 \times 10^{-12}\,\text{F/m})}$$

$$V = 0.51 \times 10^{6}\,\text{V}$$

$$\boxed{V = 0.51\,\text{MV}}$$



4-5 Electric Potential due to Charge Distributions

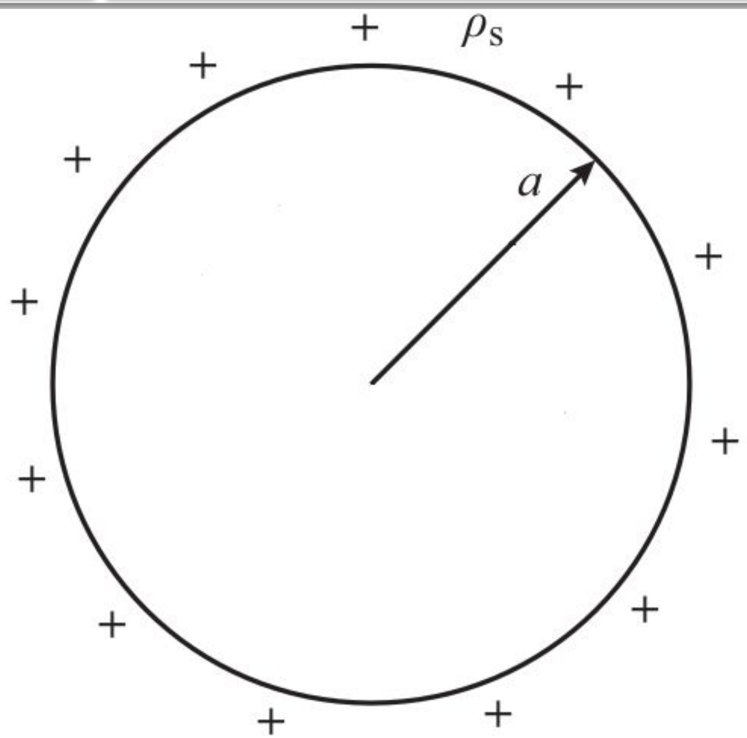
$$V = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \frac{\rho_{\mathrm{v}}}{R'}\, d\upsilon' \quad \textbf{(volume distribution)},$$

$$V = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_{\mathrm{s}}}{R'}\, ds' \quad \textbf{(surface distribution)},$$

$$V = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_{\ell}}{R'}\, dl' \quad \textbf{(line distribution)}.$$

$$R' = |\mathbf{R} - \mathbf{R}_i|$$

Exercise 4-11: Potential due to Charge Density



Given: Spherical shell, radius=a

surf charge density ρ_s

Find: at **R**=0: V and **E**

Solution:

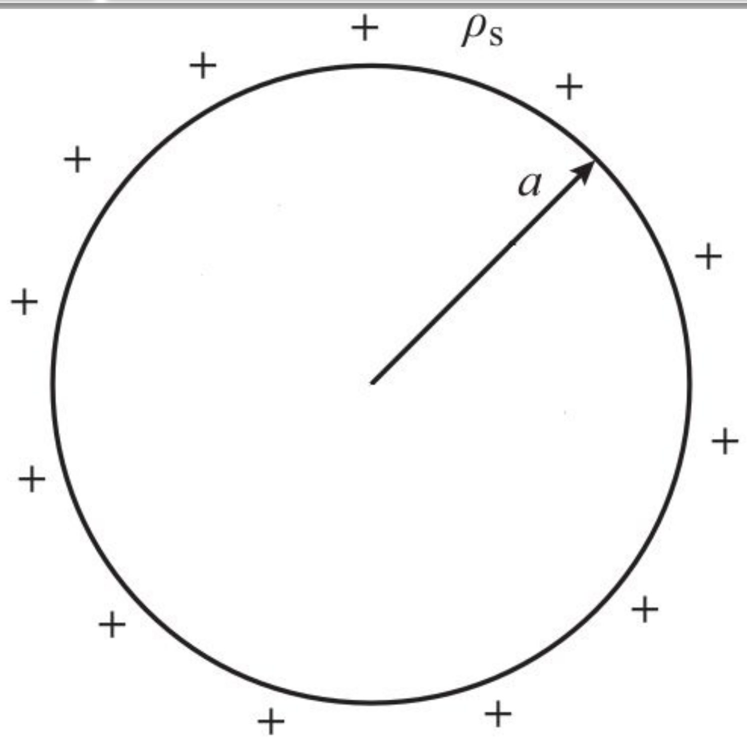
$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_s}{R'}\, ds'$$

where:

$R' = a$

$ds' = R'^2 \sin\theta \, d\theta \, d\phi$

Exercise 4-11: Potential due to Charge Density



Solution:

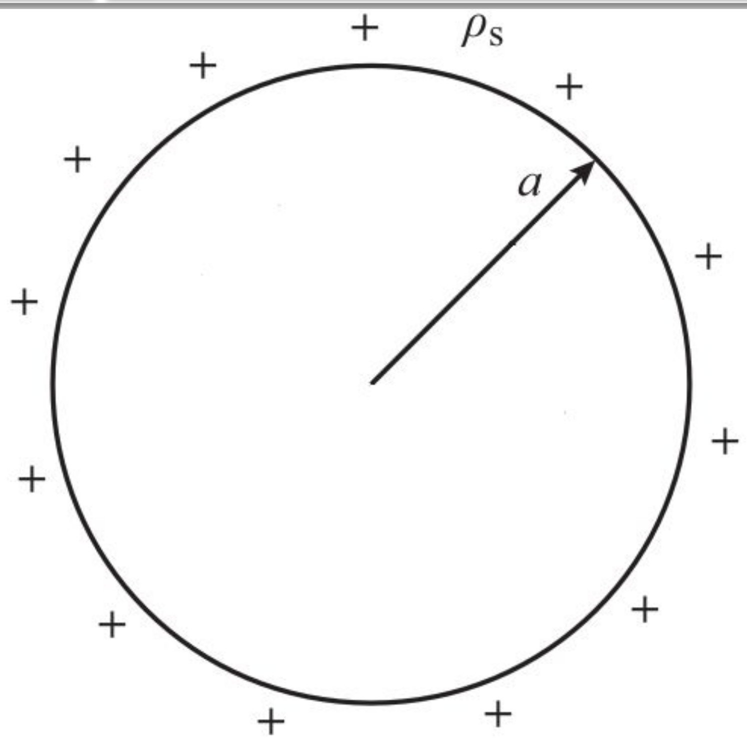
$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_s}{R'} ds'$$

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_s}{R'} R'^2 \sin\theta \, d\theta d\phi$$

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\rho_s}{R'} R'^2 \sin\theta \, d\theta d\phi$$

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho_s R' \sin\theta \, d\theta d\phi$$

Exercise 4-11: Potential due to Charge Density



Solution:

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\rho_s R' \sin\theta \, d\theta d\phi$$

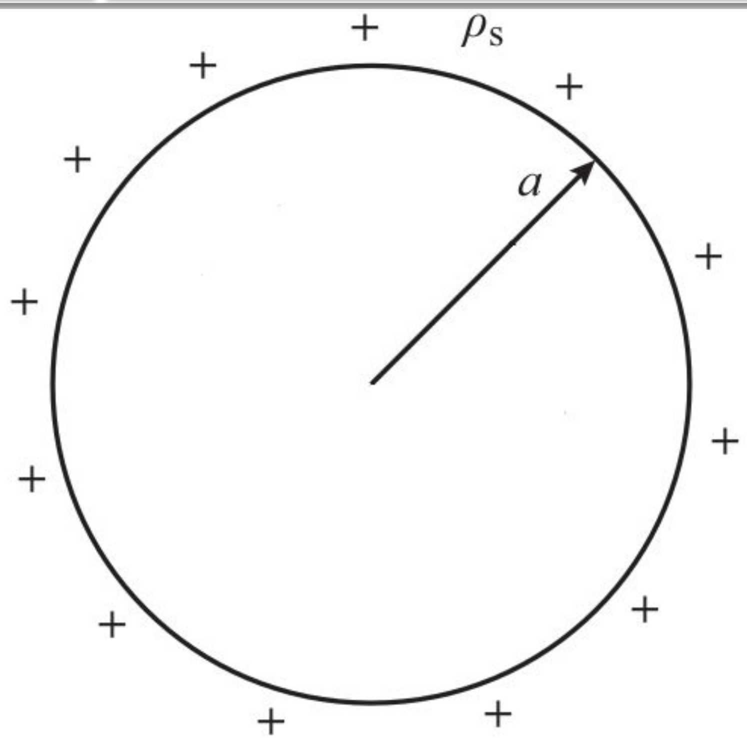
$$V(\mathbf{R}) = \frac{\rho_s R'}{4\pi\varepsilon}\int_{\theta=0}^{\pi}\sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$V(\mathbf{R}) = \frac{\rho_s R'}{4\pi\varepsilon}\left(-\cos\theta\right)\Big|_{\theta=0}^{\pi}(2\pi)$$

$$V(\mathbf{R}) = \frac{\rho_s R'}{4\pi\varepsilon}(2)(2\pi)$$

$$V(0) = \frac{\rho_s a}{\varepsilon}$$

Exercise 4-11: Potential due to Charge Density



Solution:

To solve for **E**, use Gauss' law:

$$\oint \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\varepsilon}$$

Since Q=0 inside the spherical shell, then **E**(0)=0

4-5 Relating **E** to *V*

Recall from vector calculus (ch 3) that for any well-behaved function *V*:

$$dV = \nabla V \cdot d\mathbf{l},$$

Since we know:

$$dV = -\mathbf{E} \cdot d\mathbf{l}.$$

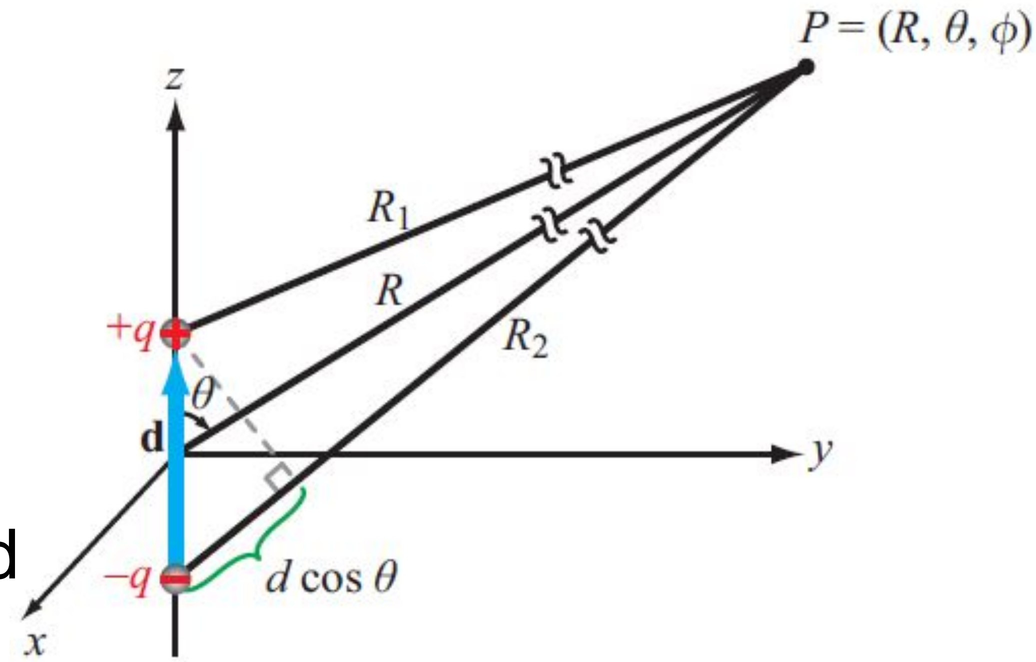
Then:

$$\mathbf{E} = -\nabla V.$$

Sometimes it's simpler to first calculate *V*, then **E**, instead of **E** directly.

Example 4-9 Electric Dipole

Electric Dipole:
2 point charges
equal magnitudes
opposite polarities
separated by distance *d*



Want expressions for V and **E** when the observation point is "very far" from the dipole: *R*>>*d*

Cont.

Example 4-9 Electric Dipole

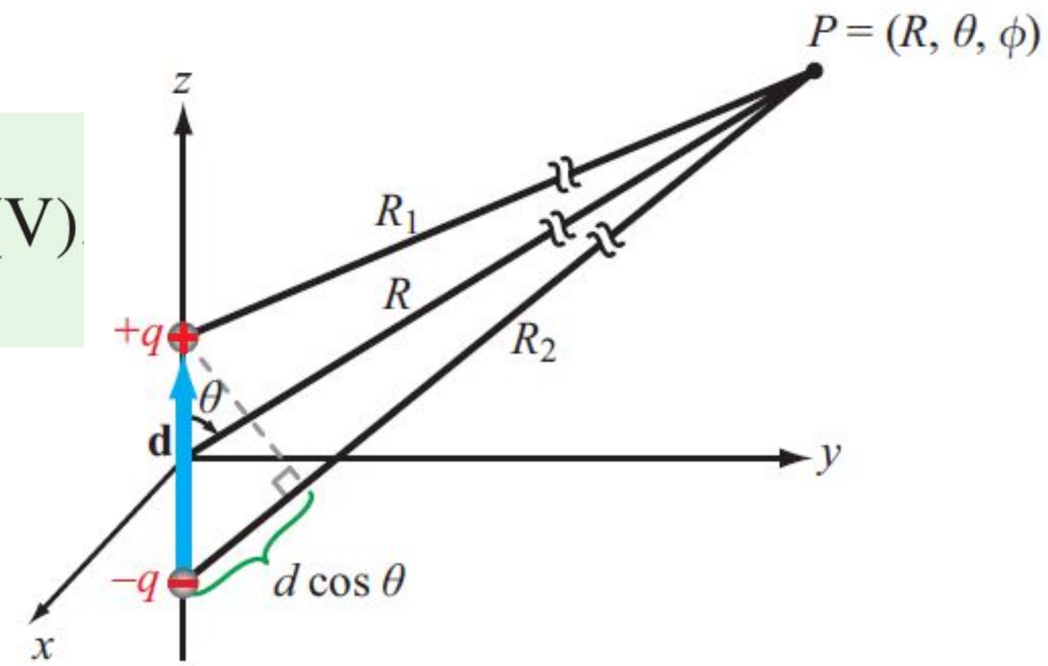
Since we know:

$$V = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{R}-\mathbf{R}_i|} \qquad \text{(V)}.$$

get:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1}+\frac{-q}{R_2}\right)$$

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{R_2-R_1}{R_1R_2}\right)$$



Cont.

Example 4-9 Electric Dipole

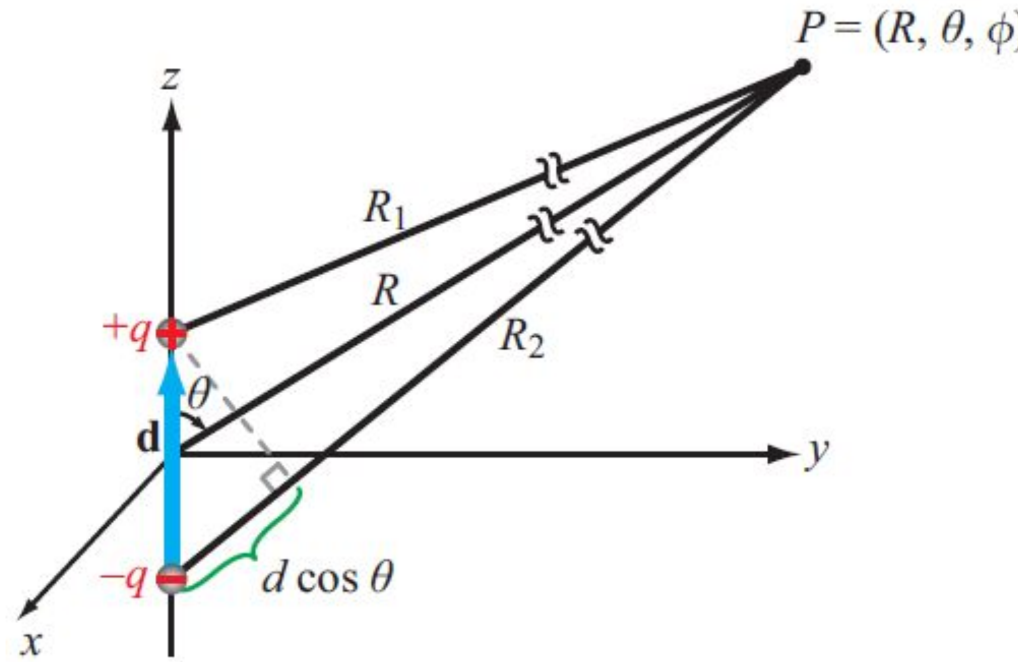
Since *d*<<*R*, we can make the approximations:

$$R_2 - R_1 \approx d\cos\theta,$$

$$R_1 R_2 \approx R^2$$

plug in:

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}$$



Cont.

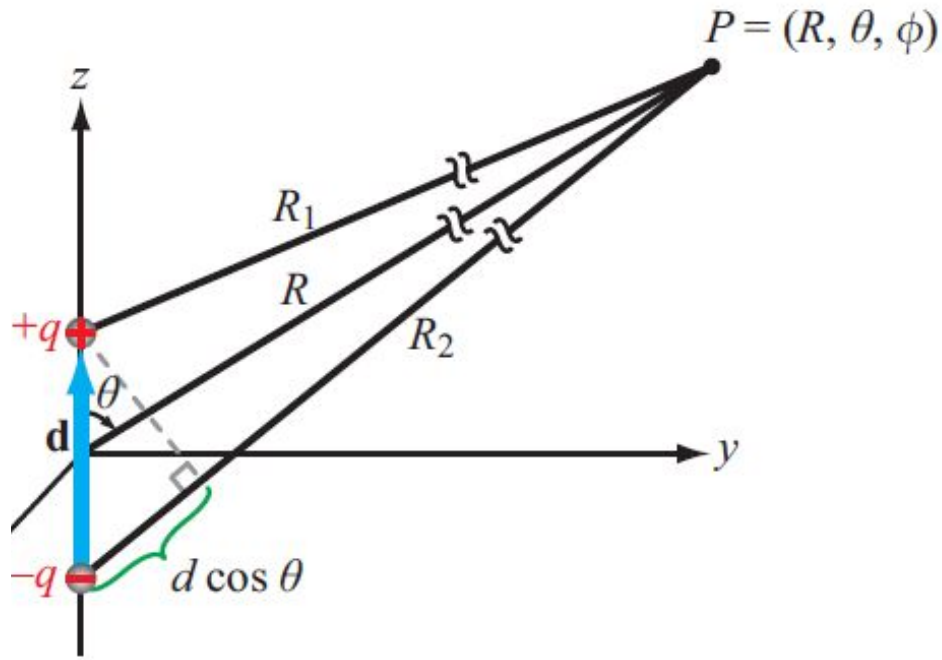
Example 4-9 Electric Dipole

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}$$

For a dipole that is arbitrarily-oriented:

$$qd\cos\theta = q\mathbf{d}\cdot\hat{\mathbf{R}} = \mathbf{p}\cdot\hat{\mathbf{R}}$$

where the dipole moment, **p**, is defined as: $\mathbf{p} = q\mathbf{d}$



so:

$$V = \frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{4\pi\varepsilon_0 R^2}$$

Cont.

Example 4-9 Electric Dipole

For a vertical dipole centered at the origin:

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}$$

$$\mathbf{E} = -\nabla V = -\left(\hat{\mathbf{R}}\,\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\,\frac{1}{R}\,\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{R\sin\theta}\,\frac{\partial V}{\partial \phi}\right)$$

Cont.

Example 4-9 Electric Dipole

Plug in V:

$$\mathbf{E} = -\frac{qd}{4\pi\varepsilon_0}\left(\hat{\mathbf{R}}\frac{\partial(R^{-2}\cos\theta)}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial(R^{-2}\cos\theta)}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial(R^{-2}\cos\theta)}{\partial\phi}\right)$$

$$\mathbf{E} = -\frac{qd}{4\pi\varepsilon_0}\left(\hat{\mathbf{R}}(-2R^{-3}\cos\theta) + \hat{\boldsymbol{\theta}}\frac{1}{R}(-R^{-2}\sin\theta) + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}(0)\right)$$

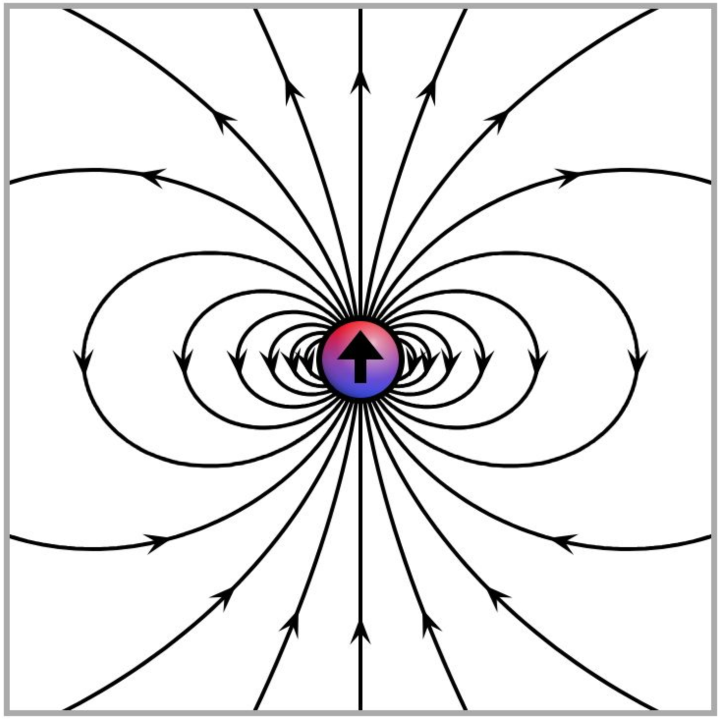
$$\mathbf{E} = -\frac{qd}{4\pi\varepsilon_0}\left(\hat{\mathbf{R}}(-2R^{-3}\cos\theta) + \hat{\boldsymbol{\theta}}(-R^{-3}\sin\theta)\right)$$

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbf{R}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right)$$

Cont.

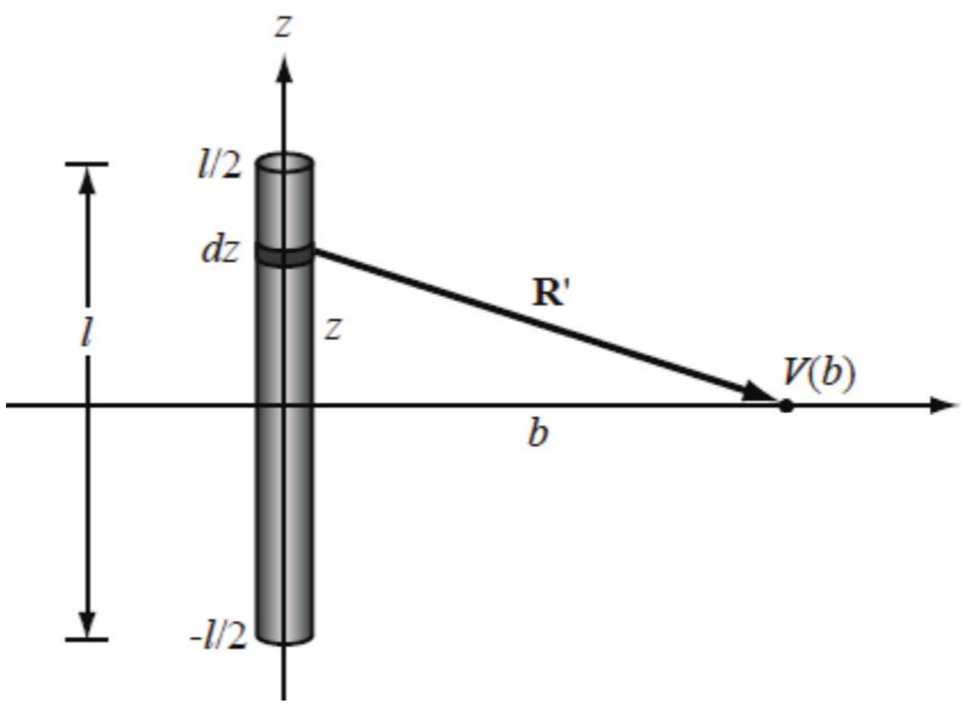
Example 4-9 Electric Dipole

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad \text{(V/m)}$$



(wikipedia.org)

Example: Line Charge



Given: line charge ρ_l along z-axis from -l/2 to l/2

Find: $V(r$=b)

Solution:

$$V(b) = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_l}{R'}\, dl'$$

where:

$$R' = \sqrt{z^2 + b^2}$$

Example: Line Charge

Solution:

and: dl' = dz

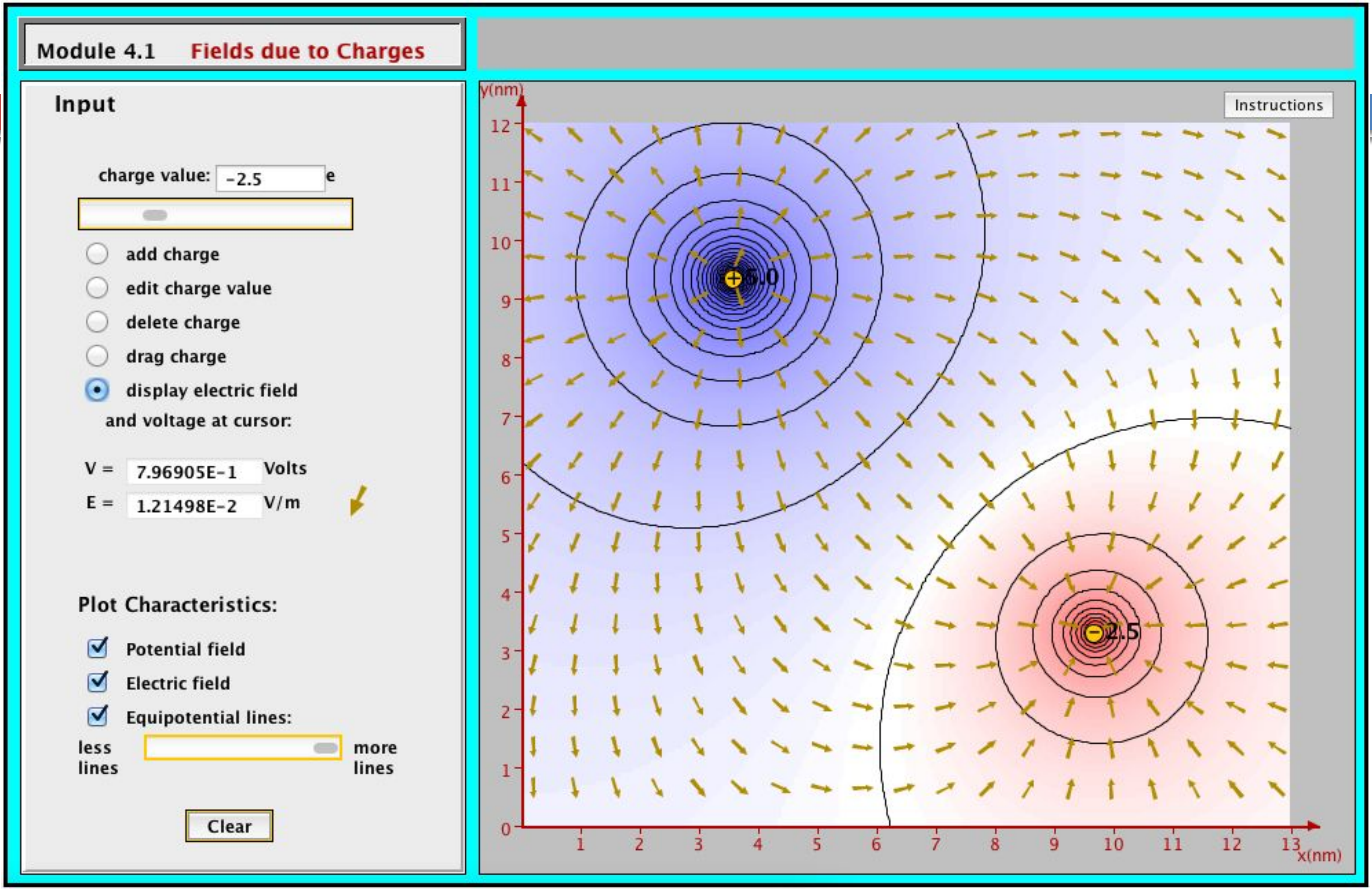
$$V(b) = \frac{\rho_l}{4\pi\varepsilon}\int_{z=-l/2}^{l/2}\frac{dz}{\sqrt{z^2+b^2}}$$

$$V(b) = \frac{\rho_l}{4\pi\varepsilon}\left(\ln(z+\sqrt{z^2+b^2})\right)_{z=-l/2}^{l/2}$$

$$V(b) = \frac{\rho_l}{4\pi\varepsilon}\left(\ln(l/2+\sqrt{(l/2)^2+b^2}) - \ln(-l/2+\sqrt{(-l/2)^2+b^2})\right)$$

$$V(b) = \frac{\rho_l}{4\pi\varepsilon}\ln\left(\frac{l/2+\sqrt{(l/2)^2+b^2}}{-l/2+\sqrt{(-l/2)^2+b^2}}\right)$$

$$V(b) = \frac{\rho_l}{4\pi\varepsilon}\ln\left(\frac{l+\sqrt{l^2+4b^2}}{-l+\sqrt{l^2+4b^2}}\right)$$



Module 4.1 Fields due to Charges
Input
charge value: -2.5 e
add charge
edit charge value
delete charge
drag charge
display electric field
and voltage at cursor:
V = 7.96905E-1 Volts
E = 1.21498E-2 V/m
Plot Characteristics:
Potential field
Electric field
Equipotential lines:
less lines
more lines
Clear
Instructions
y(nm)
x(nm)

4-5 Poisson's Equation

We know: $\nabla \cdot \mathbf{D} = \rho_V \, ,$ and: $\mathbf{D} = \varepsilon \mathbf{E}$

so: $\nabla \cdot \mathbf{E} = \dfrac{\rho_V}{\varepsilon}$

and since: $\mathbf{E} = -\nabla V \, .$

we get:

$$\nabla \cdot (\nabla V) = -\frac{\rho_V}{\varepsilon}$$

4-5 Poisson's Equation

Since:

$$\nabla \cdot (\nabla V) = -\frac{\rho_{\mathrm{v}}}{\varepsilon}$$

and from Ch3, we know:

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

get:

$$\nabla^2 V = -\frac{\rho_{\mathrm{v}}}{\varepsilon}$$ **(Poisson's equation)**

4-5 Laplace's Equation

Since:

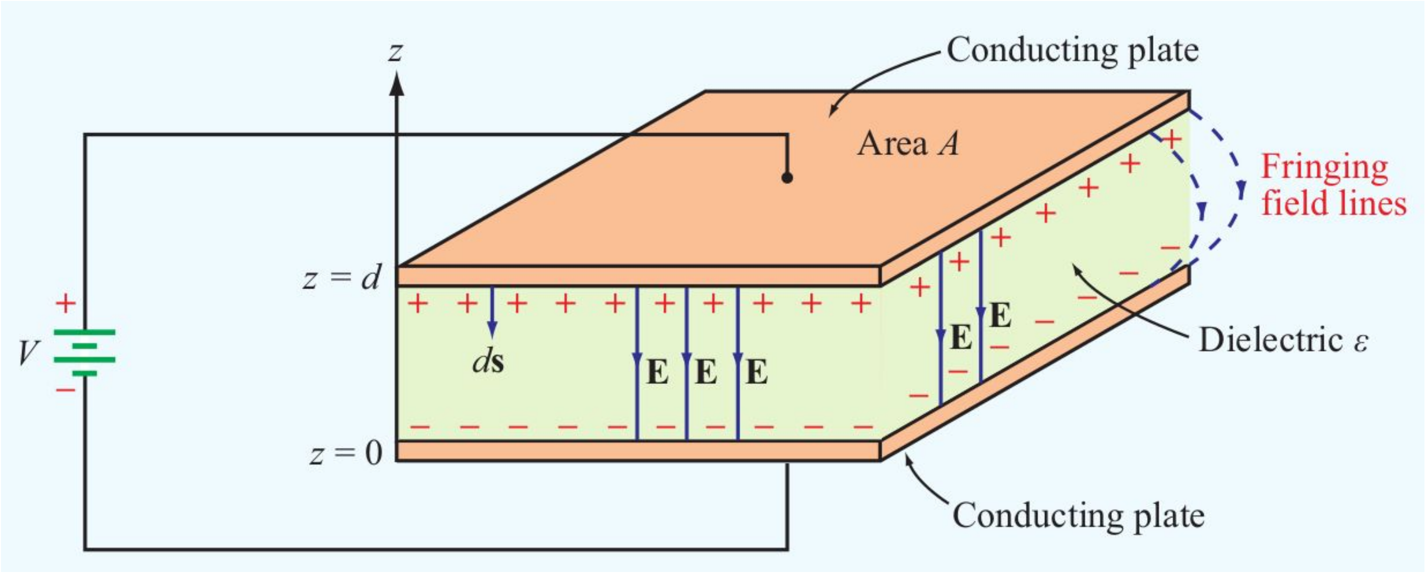
$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \textbf{(Poisson's equation)}$$

if ρ_v=0:

$$\nabla^2 V = 0 \qquad \textbf{(Laplace's equation)}$$

Useful for problems where *V* is known on boundaries.

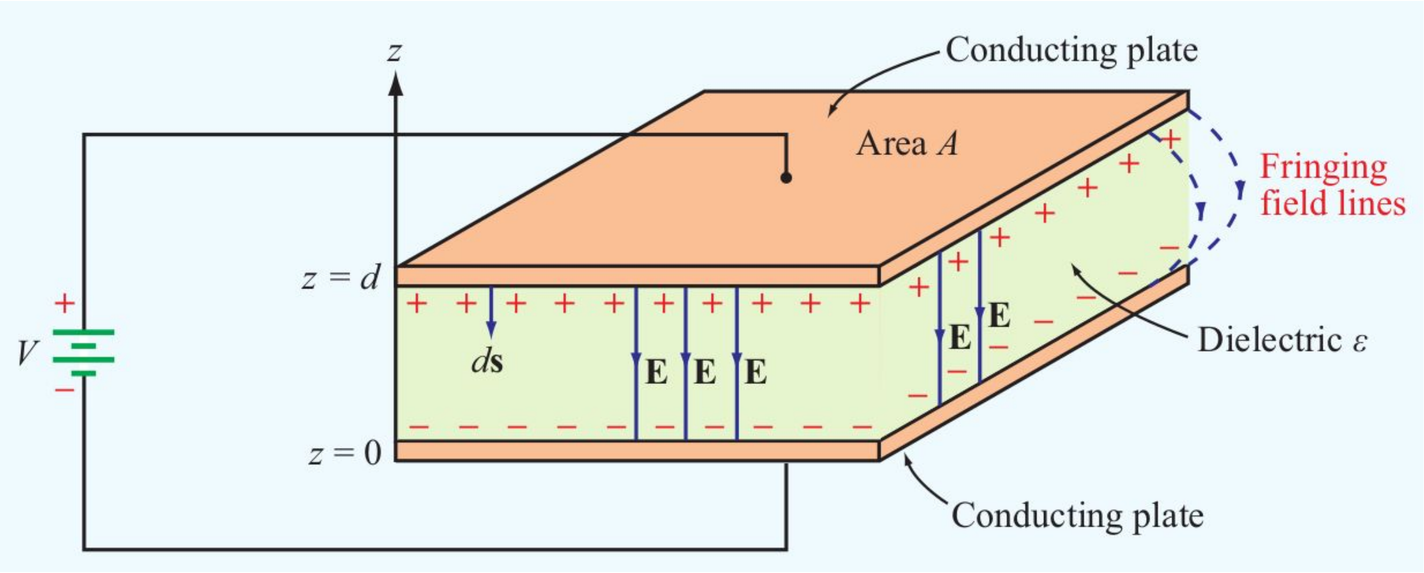
Example: Capacitor



Given: two parallel rectangular sheets of metal, with insulation of thickness *d* in the *z*-direction, with a voltage V_0 across them. assume: $\rho_v=0$

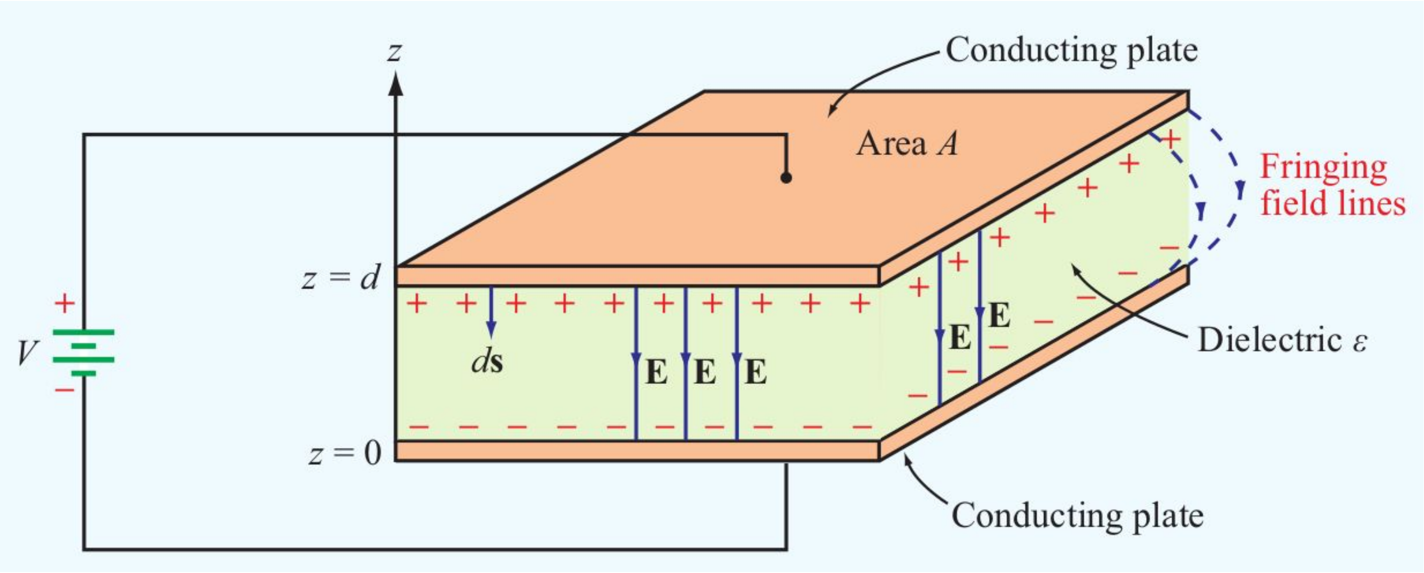
Find: *V*, **E**

Example: Capacitor



Solution: Approximate with sheets that are infinite.
so: $V = V(z)$

Example: Capacitor



Solution: $V = V(z)$: Plug into Laplace's Equation:

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\boxed{\frac{\partial^2 V}{\partial z^2} = 0}$$

Example: Capacitor

Solution: Solve this differential equation:

$$\frac{d}{dz}\left(\frac{dV}{dz}\right) = 0$$

$$\int d\left(\frac{dV}{dz}\right) = \int 0\, dz$$

$$\frac{dV}{dz} = A$$

(*A* some constant)

$$dV = A\, dz$$

$$\int dV = \int A\, dz$$

$$V(z) = Az + B$$

(*A*, *B* some constants)

Example: Capacitor

Solution: Apply boundary conditions:

$$V(z) = Az + B$$

$$V(z = 0) = 0$$

$$V(z = d) = V_0$$

Solve for unknown constants:

$$V(0) = 0 = A(0) + B$$

$$B = 0$$

$$V(d) = Ad = V_0$$

$$A = \frac{V_0}{d}$$

Example: Capacitor

Solution: So for our case:

$$V(z) = \frac{V_0}{d} z$$

and so:

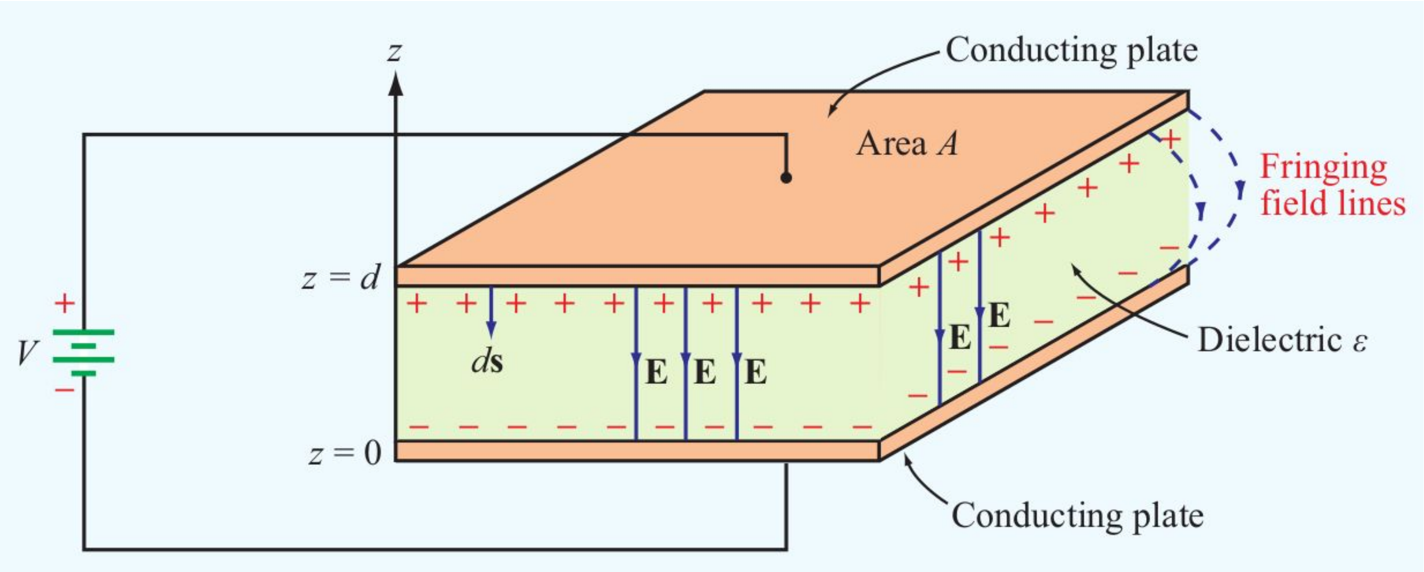
$$\mathbf{E} = -\nabla V$$

$$\mathbf{E} = -\left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) V$$

$$\mathbf{E} = -\hat{\mathbf{z}} \frac{dV}{dz}$$

$$\mathbf{E} = -\hat{\mathbf{z}} \frac{V_0}{d}$$

Example: Capacitor



$$\mathbf{E} = -\hat{\mathbf{z}}\frac{V_0}{d}$$

$$V(z) = \frac{V_0}{d}z$$

This approximation is OK for thin capacitors

Homework

Homework 13 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Section 4-6:

Conductors