EECS 230
ENGINEERING ELECTROMAGNETICS
Leland Pierce

Electrostatics 1

Chapter 4 Overview

Maxwell's Equations
- Electrostatics
- Magnetostatics

Charge density
Current density
Electric field from charges
Gauss's Law
Electric Scalar Potential Field
- Dipole Field
- Poisson's eqn

Conductors
- current
- resistance
- joule's law

Dielectrics
- polarization

Boundary Conditions
Capacitance
Potential Energy
Image method

$$\nabla \cdot \mathbf{D} = \rho_v, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

$$\mathbf{E} = \int_{\upsilon'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \hat{\mathbf{R}}' \frac{\rho_v \, d\upsilon'}{R'^2}.$$

(volume distribution)

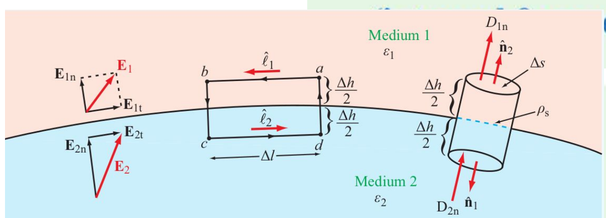
$$\nabla \cdot \mathbf{D} = \rho_v,$$

(differential form of Gauss's law)

$$\nabla \cdot \mathbf{D} = \rho_v, \quad \nabla \times \mathbf{E} = 0.$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q.$$

of Gauss's law)



$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}.$$

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \mathbf{E} = -\nabla V.$$

$$Q = \int_{\upsilon} \rho_v \, d\upsilon \qquad \text{(C)}.$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (\text{A/m}^2) \quad \text{(Ohm's law)},$$

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2)$$

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}.$$

$$P = \int_{\upsilon} \mathbf{E} \cdot \mathbf{J} \, d\upsilon \quad \text{(W)}$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.$$

Lecture Coverage

Today's lecture:

Sections 4-1 through 4-4 of the book:

4-1: Maxwell's Equations
4-2: Charge and Current Distributions
4-3: Coulomb's Law
4-4: Gauss's Law

4.1 Maxwell's Equations

$$\nabla \cdot \mathbf{D} = \rho_V,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Empirically derived from many measurements

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}.$$

E: Electric Field
H: Magnetic Field
J: Current Density
ρ_v: Charge Density

4.1 Maxwell's Equations

Empirically derived from many measurements

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{V}},$$

$$\nabla$$

$$\nabla$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Boldface letters, like **E**, **H**, are vectors and functions of space and time.

E: Electric Field
H: Magnetic Field
J: Current Density
ρ_v: Charge Density

4.1 Maxwell's Equations

Static Conditions:

Electrostatics

$$\frac{\partial \rho_v}{\partial t} = 0$$

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}},$$
$$\nabla \times \mathbf{E} = 0.$$

Magnetostatics

$$\frac{\partial \mathbf{J}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J}.$$

Electric and Magnetic Fields are decoupled.

4.2 Charge Distributions

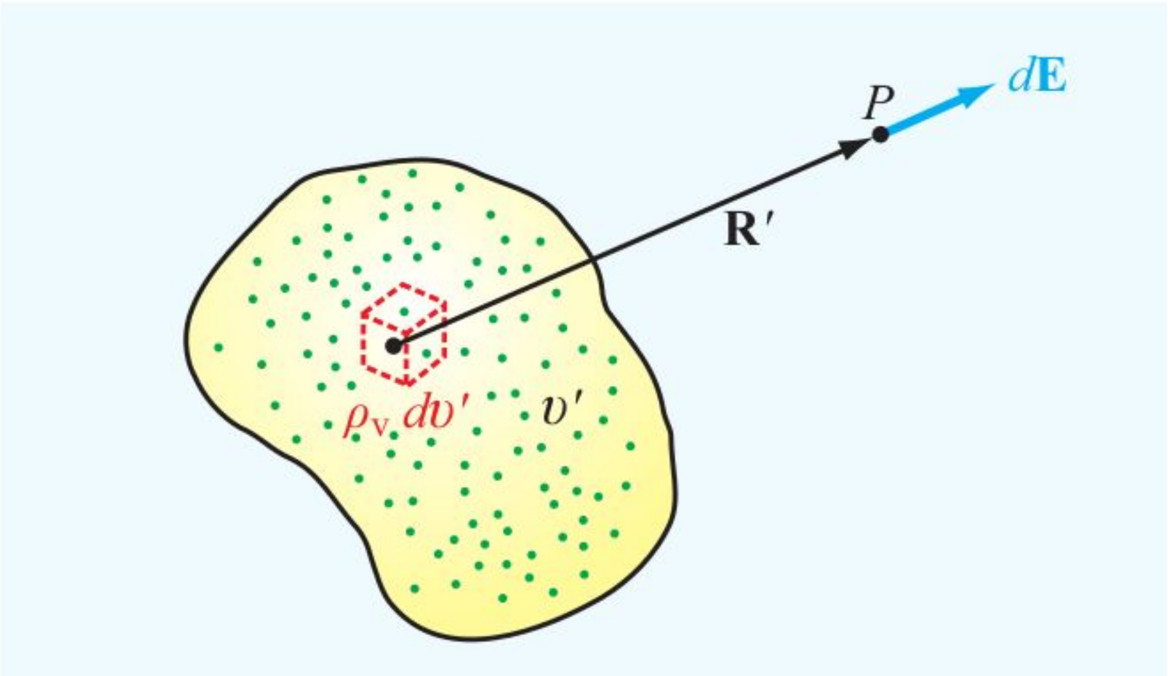
Volume charge density:

$$\rho_v = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV}$$

(C/m^3)

Total Charge in a Volume

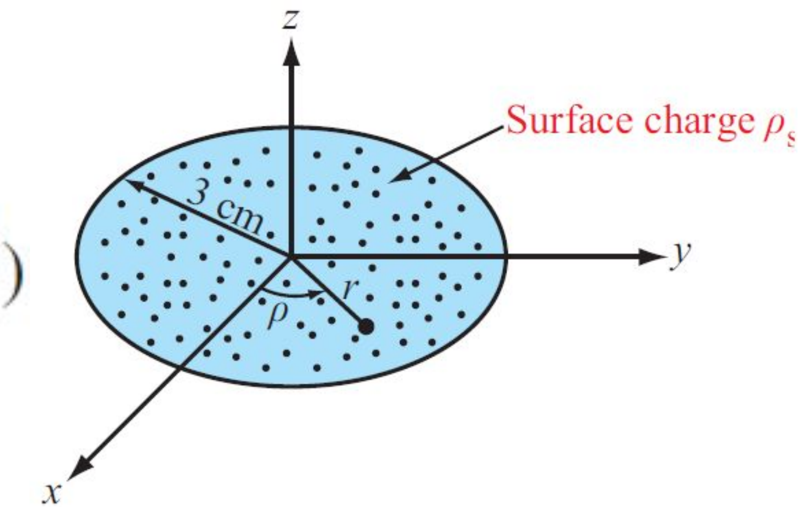
$$Q = \int_V \rho_v \, dV \qquad \text{(C)}$$



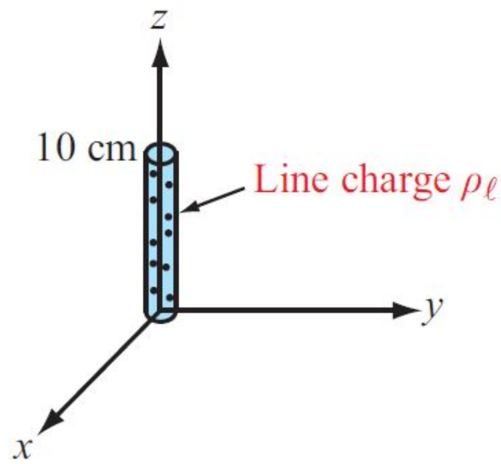
4.2 Charge Distributions

Surface and Line Charge Densities

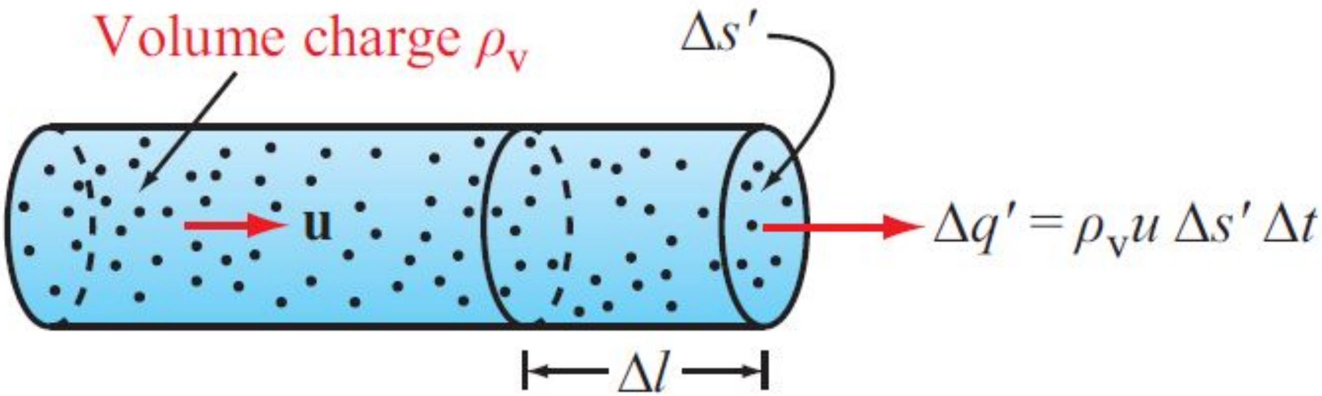
$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\text{C/m}^2)$$



$$\rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\text{C/m})$$



4.2 Current Density



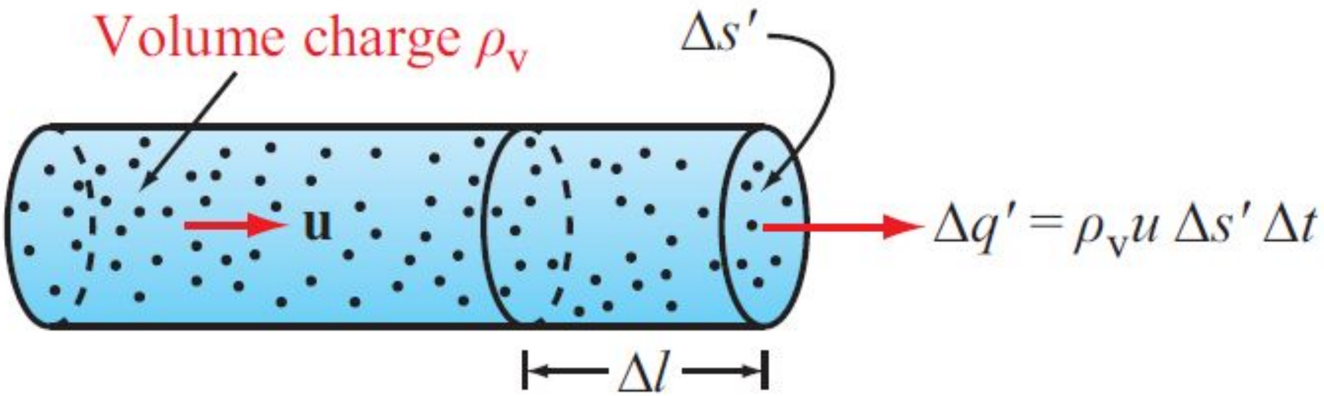
Wire with charge density ρ_v.
cross-sectional area Δs
charges moving with velocity u.

Amount of charge passing a spot during time interval Δt:

$$\Delta q = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} \, \Delta t \, ,$$

$(C/m^3)(m/sec)(m^2)(sec)$

4.2 Current Density



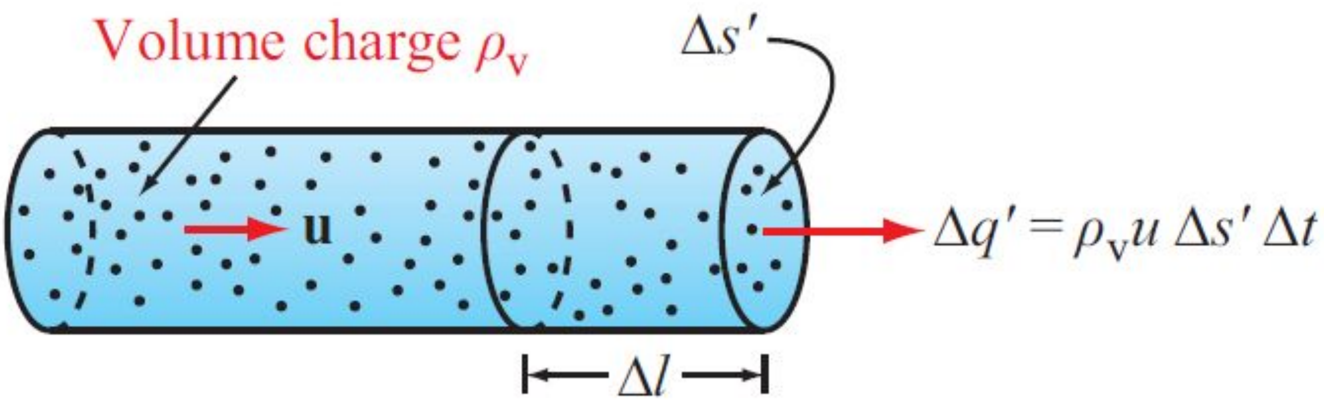
Amount of charge passing a spot during time interval Δt:

$$\Delta q = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} \, \Delta t,$$

Hence, the current is:

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}.$$

4.2 Current Density



$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}.$$

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2)$$

J is called the current density

4.2 Current Density

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}. \qquad (4.12)$$

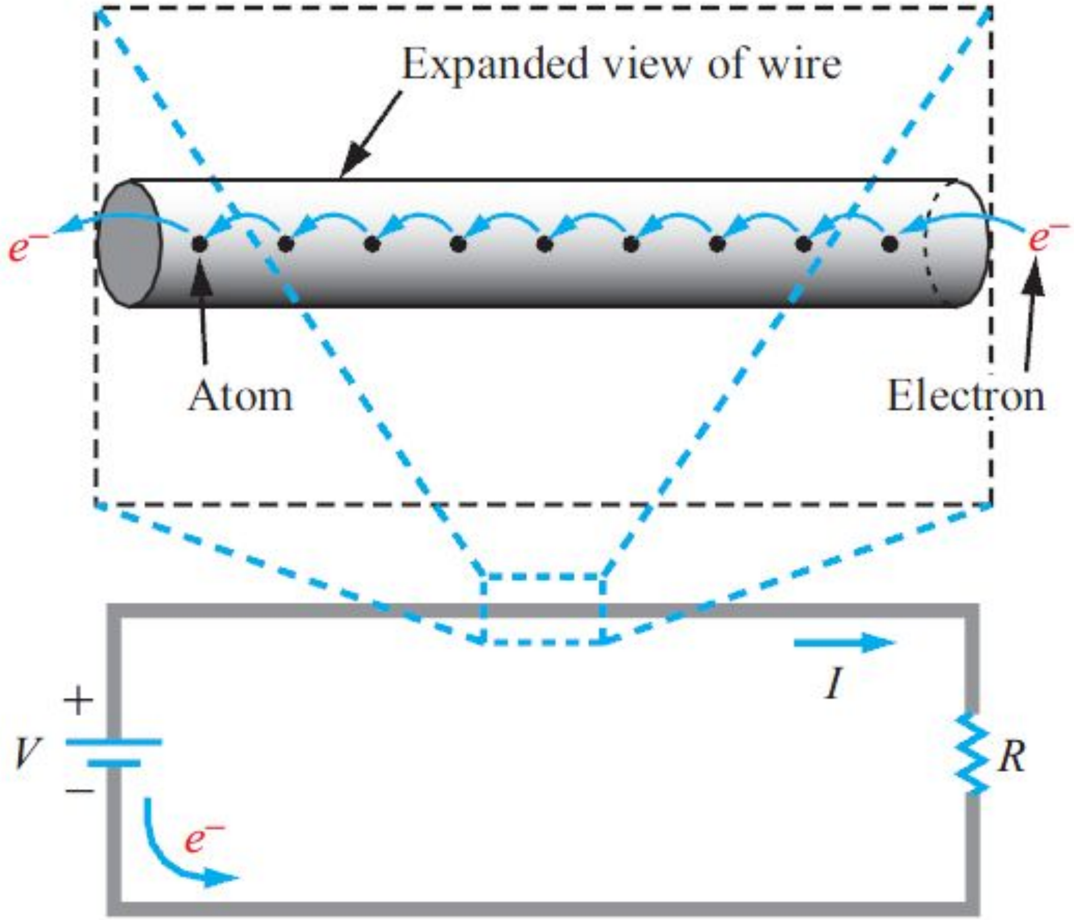
Two kinds of current:

1. Conduction current:
   charges moving relative to their host material.
   like moving electrons in a wire.

2. Convection current:
   movement of charged matter

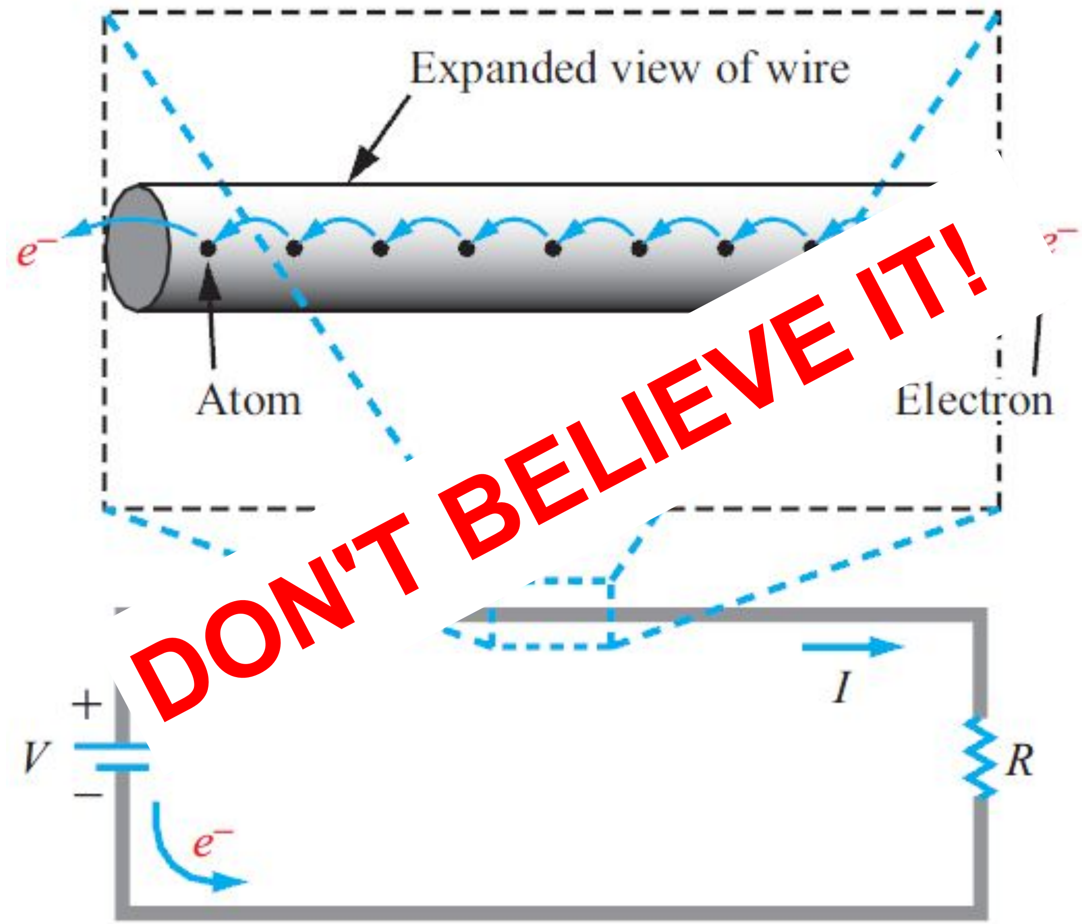
4.2 Convection vs. Conduction

Conduction current: movement of free electrons in a wire



4.2 Convection vs. Conduction

Conduction current: movement of free electrons in a wire



4.2 Quantum Mechanics

Understanding the movement of electrons in matter requires an understanding of quantum mechanics.

This class does not require QM, nor does it use it.

The mathematical models used in this class are "classical", applying only to large-scale situations, and hence incorrect or misleading when it comes to the very small.

Use Quantum Electromagnetics concepts and tools when your application involves situations that are very small.

4.3 Coulomb's Law

Electric field at point P due to single charge

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2} \qquad \text{(V/m)}$$

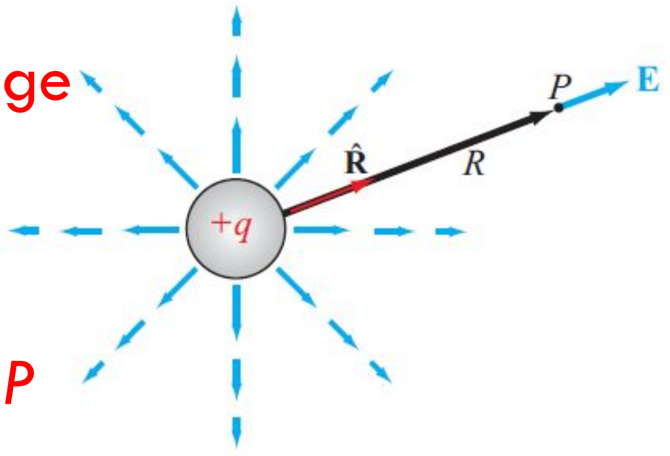


Figure 4-3: Electric-field lines due to a charge q.

Electric force on a test charge placed at *P*

$$\mathbf{F} = q'\mathbf{E} \qquad \text{(N)}$$

4.3 Coulomb's Law

Electric flux density **D**

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \approx (1/36\pi) \times 10^{-9} \qquad \text{(F/m)}$$

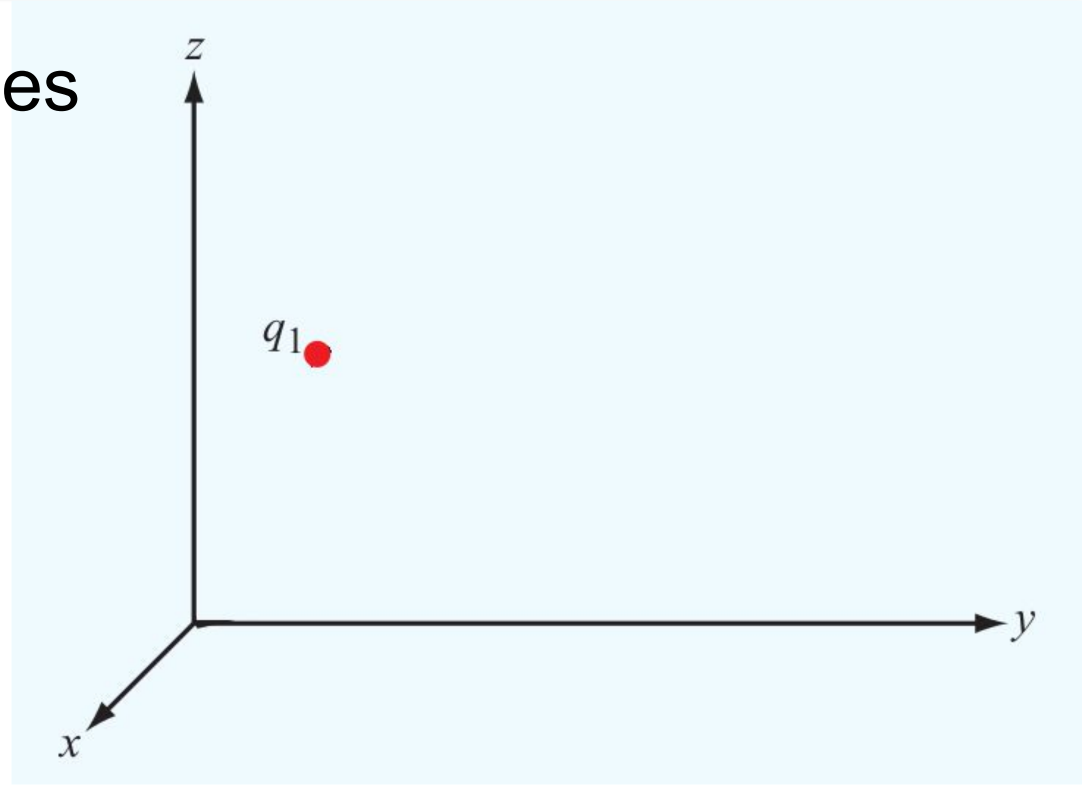
Material is:

Linear: $\varepsilon \neq f(|\mathbf{E}|)$

Isotropic: $\varepsilon \neq f(\text{direction of } \mathbf{E})$

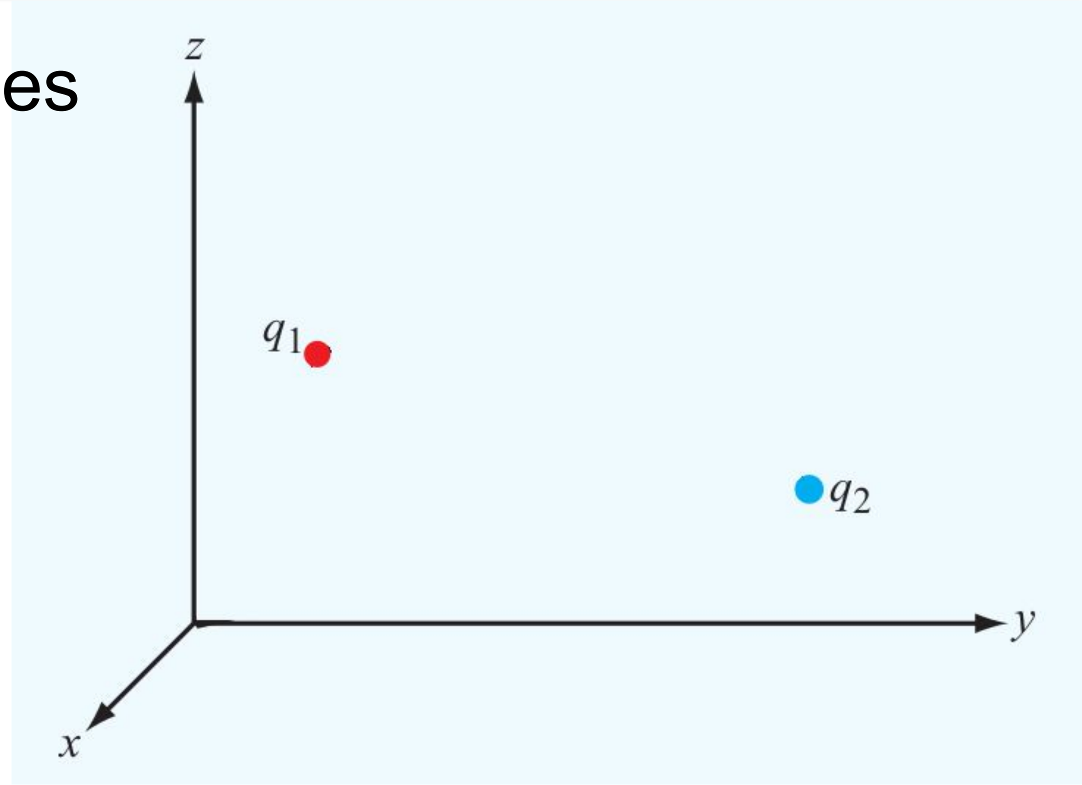
4.3 Electric Field of Multiple Charges

Start by placing 2 charges somewhere in space:



4.3 Electric Field of Multiple Charges

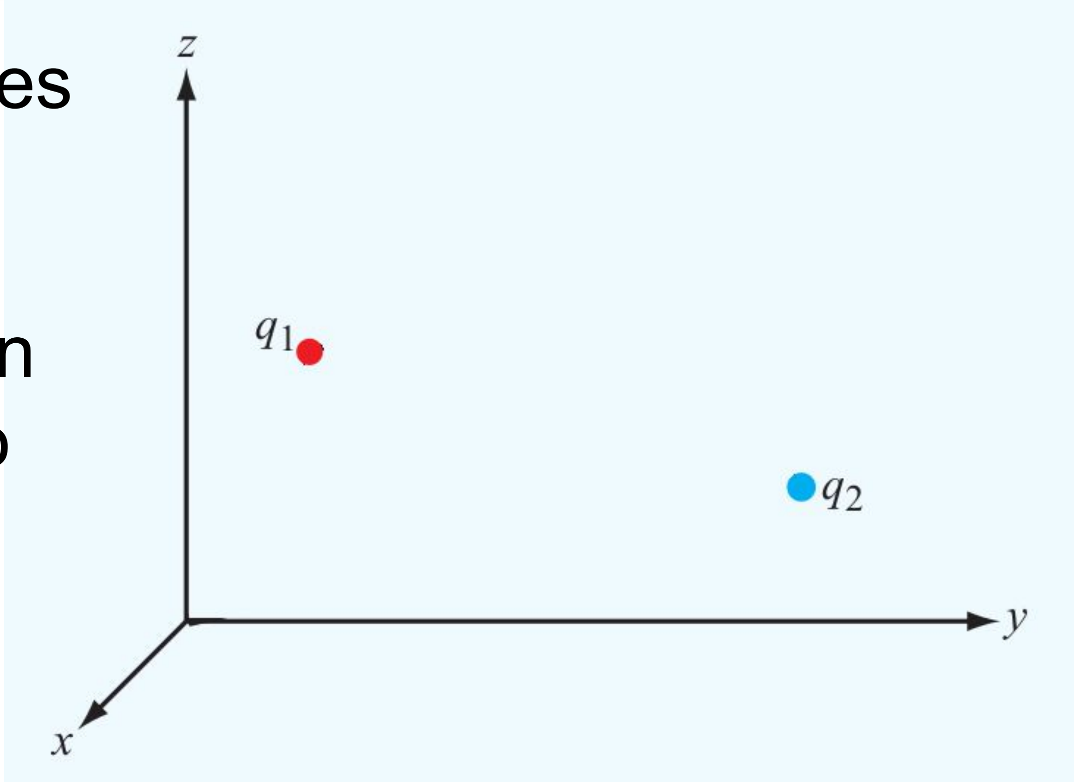
Start by placing 2 charges somewhere in space:



4.3 Electric Field of Multiple Charges

Start by placing 2 charges somewhere in space:

Next, identify a point *P* in space where we want to determine the **E** field.

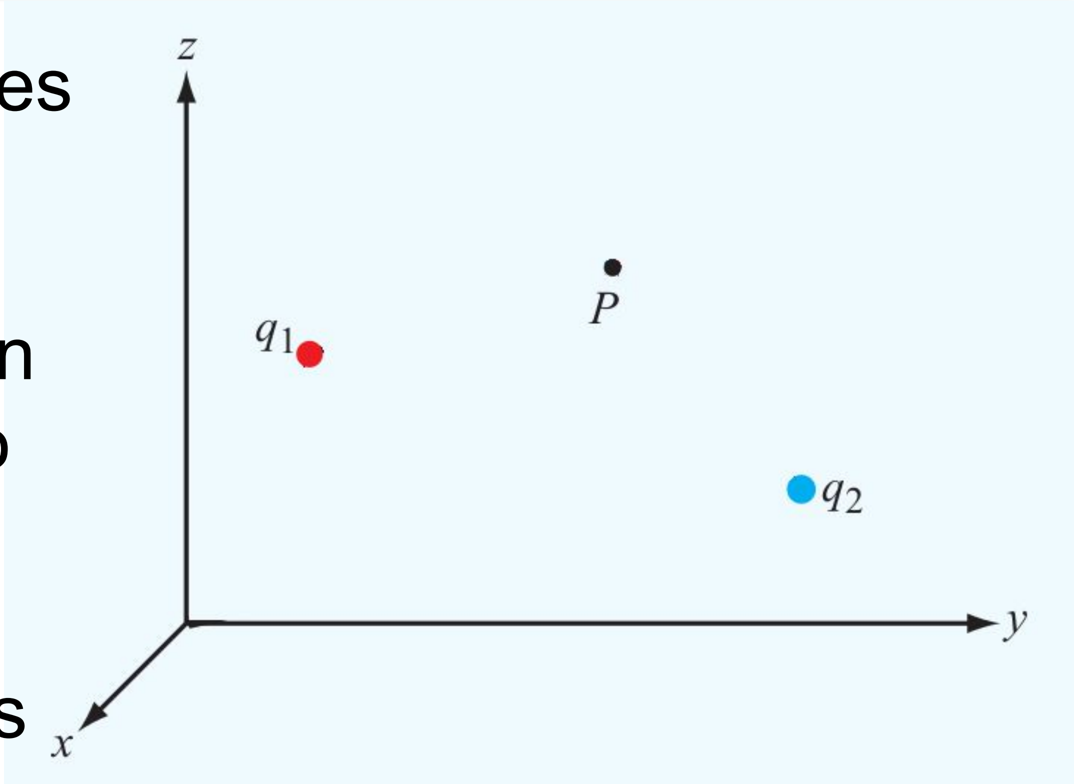


4.3 Electric Field of Multiple Charges

Start by placing 2 charges somewhere in space:

Next, identify a point *P* in space where we want to determine the **E** field.

The position of point *P* is given by a vector **R** from the origin

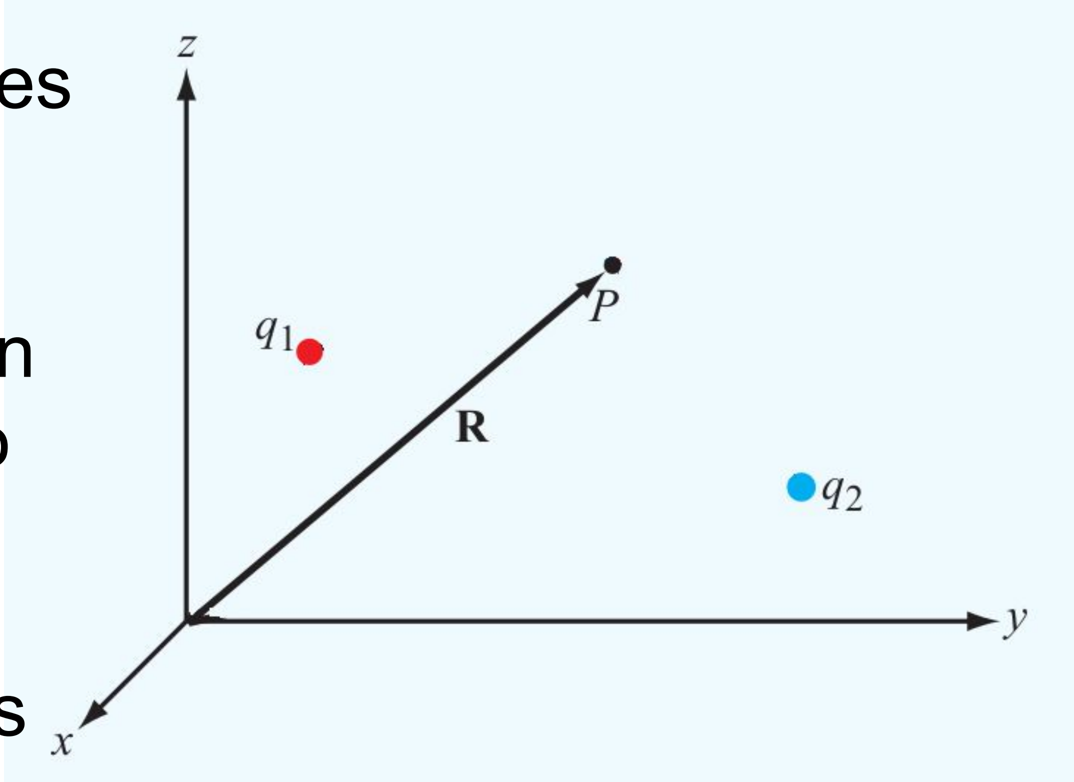


4.3 Electric Field of Multiple Charges

Start by placing 2 charges somewhere in space:

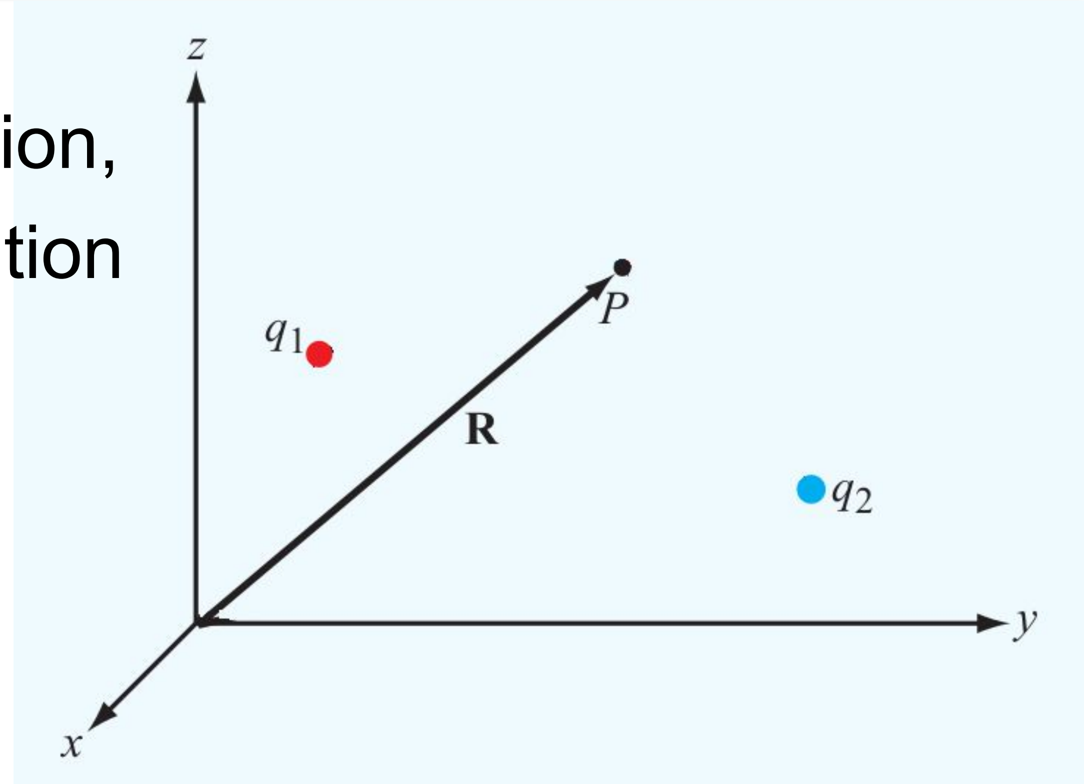
Next, identify a point *P* in space where we want to determine the **E** field.

The position of point *P* is given by a vector **R** from the origin



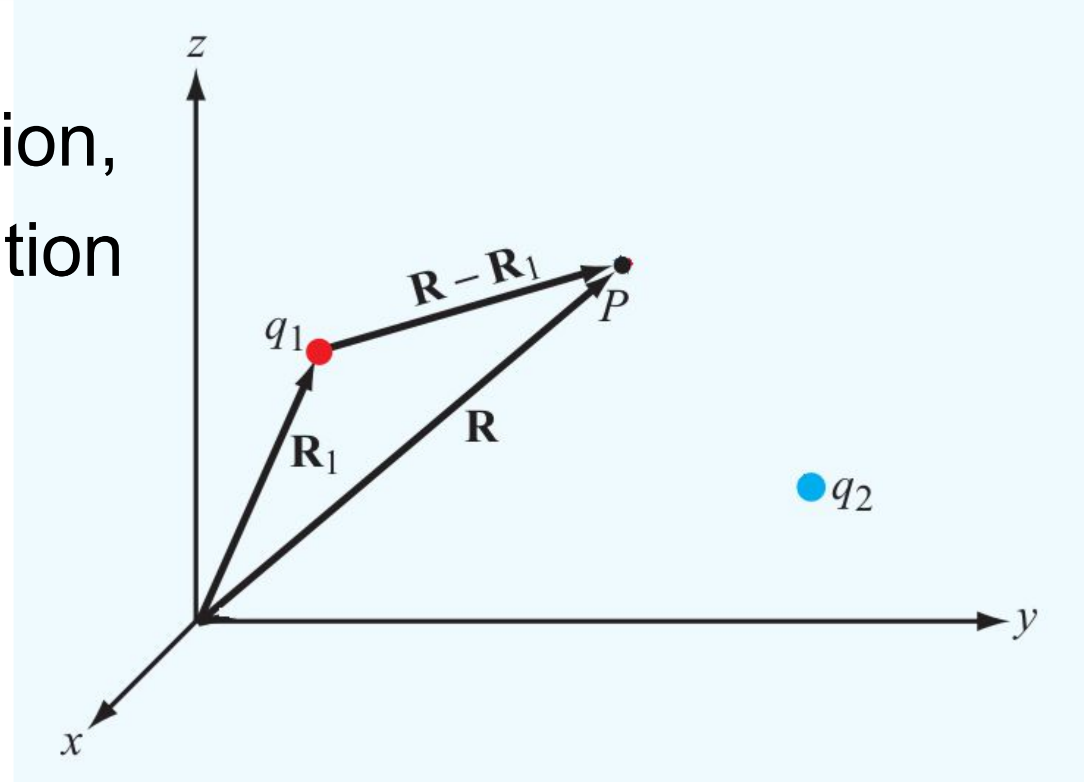
4.3 Electric Field of Multiple Charges

To get **E** due to *q1* we need to identify its position, $\mathbf{R}_1$, and it's *relative* position to point *P*: $\mathbf{R}\text{-}\mathbf{R}_1$



4.3 Electric Field of Multiple Charges

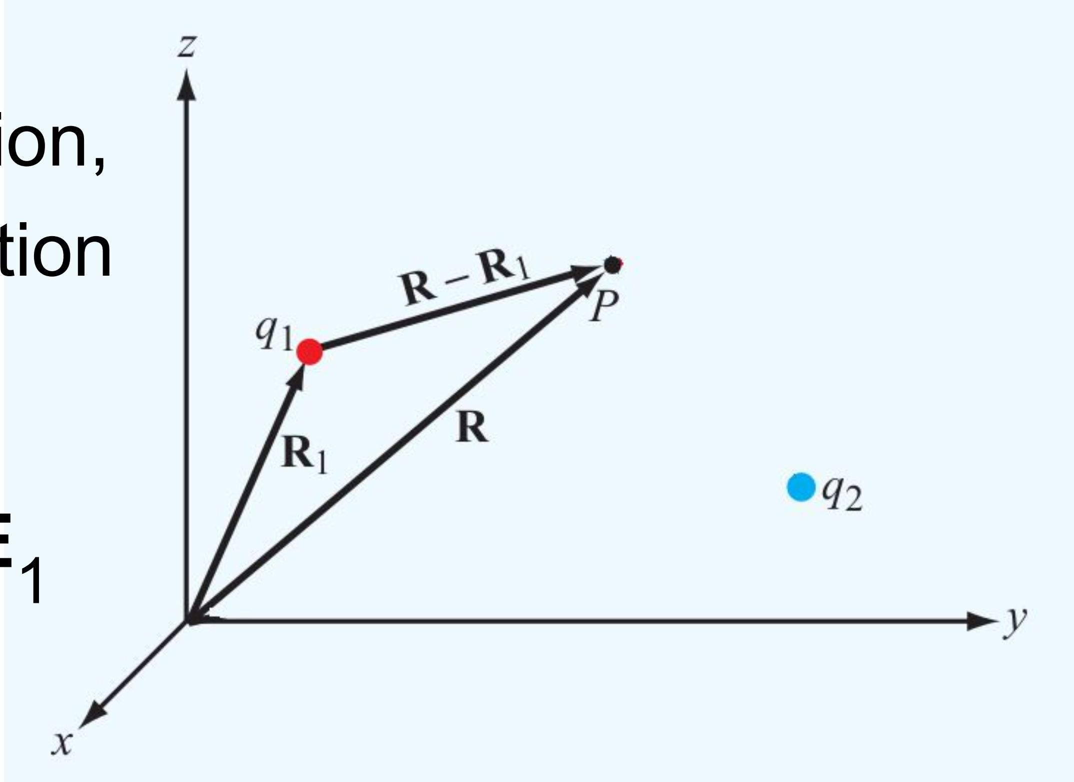
To get **E** due to *q1* we need to identify its position, $\mathbf{R}_1$, and it's *relative* position to point *P*: $\mathbf{R}-\mathbf{R}_1$



4.3 Electric Field of Multiple Charges

To get **E** due to *q1* we need to identify its position, $\mathbf{R}_1$, and it's *relative* position to point *P*: $\mathbf{R}-\mathbf{R}_1$

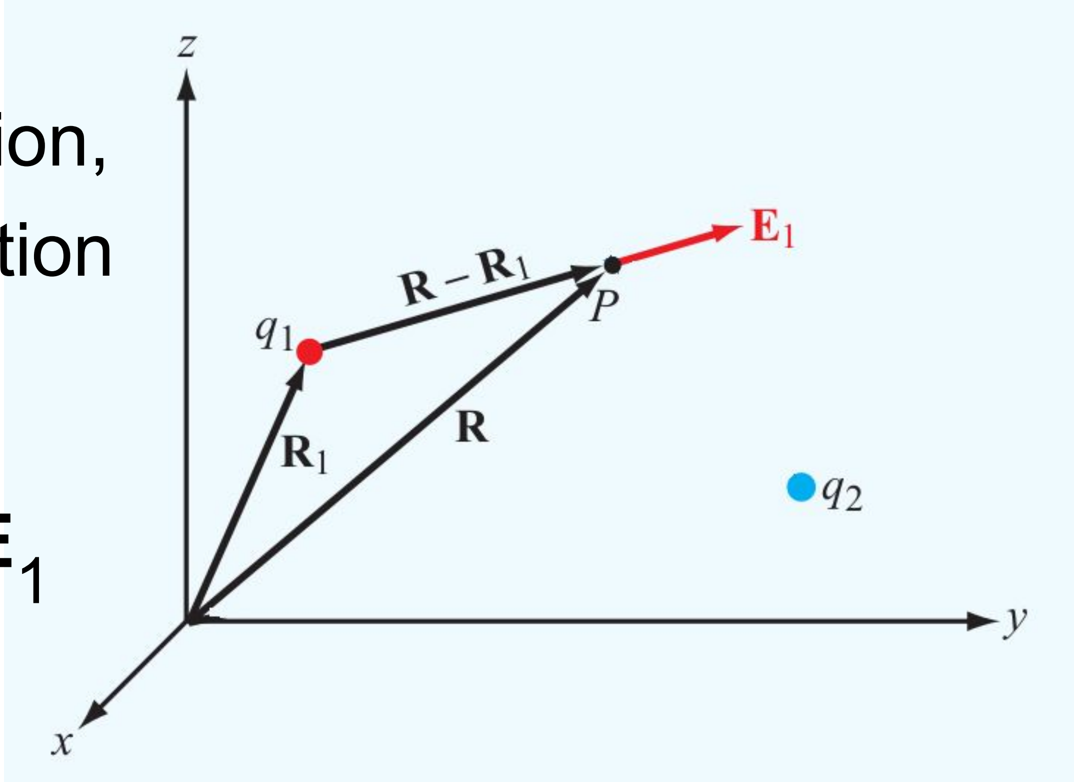
Calculate **E** due to *q1*: $\mathbf{E}_1$



4.3 Electric Field of Multiple Charges

To get **E** due to *q1* we need to identify its position, $\mathbf{R}_1$, and it's *relative* position to point *P*: $\mathbf{R}-\mathbf{R}_1$

Calculate **E** due to *q1*: $\mathbf{E}_1$

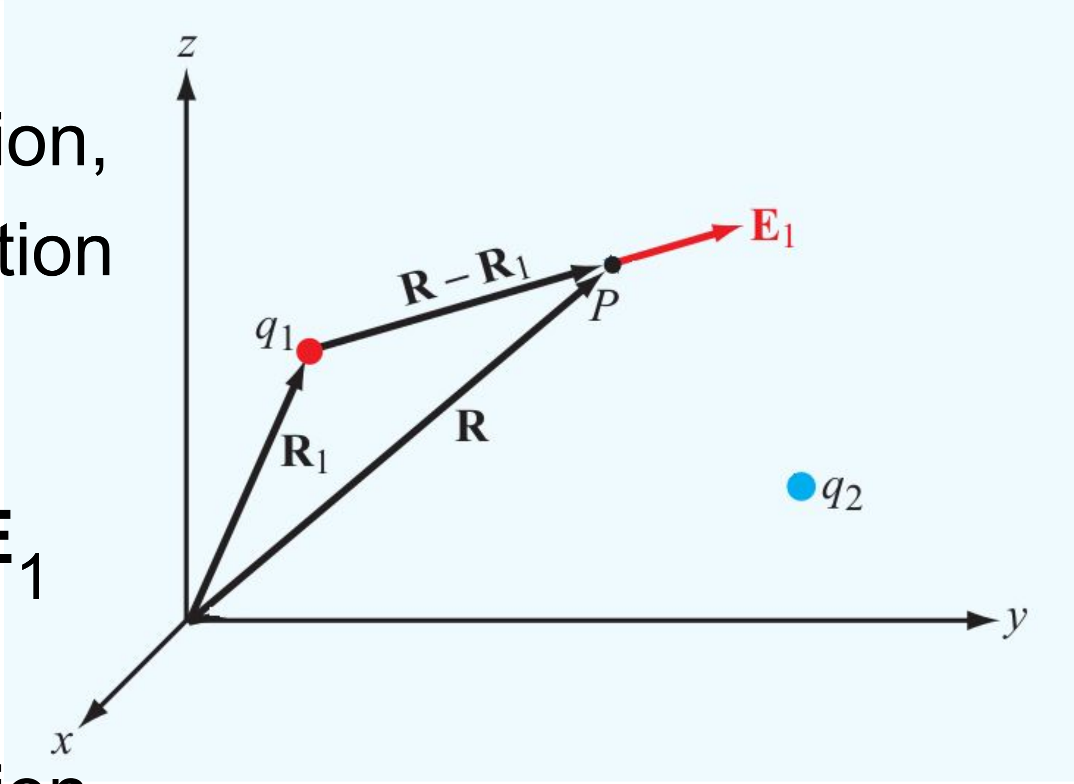


4.3 Electric Field of Multiple Charges

To get **E** due to *q1* we need to identify its position, $\mathbf{R}_1$, and it's *relative* position to point *P*: $\mathbf{R}-\mathbf{R}_1$

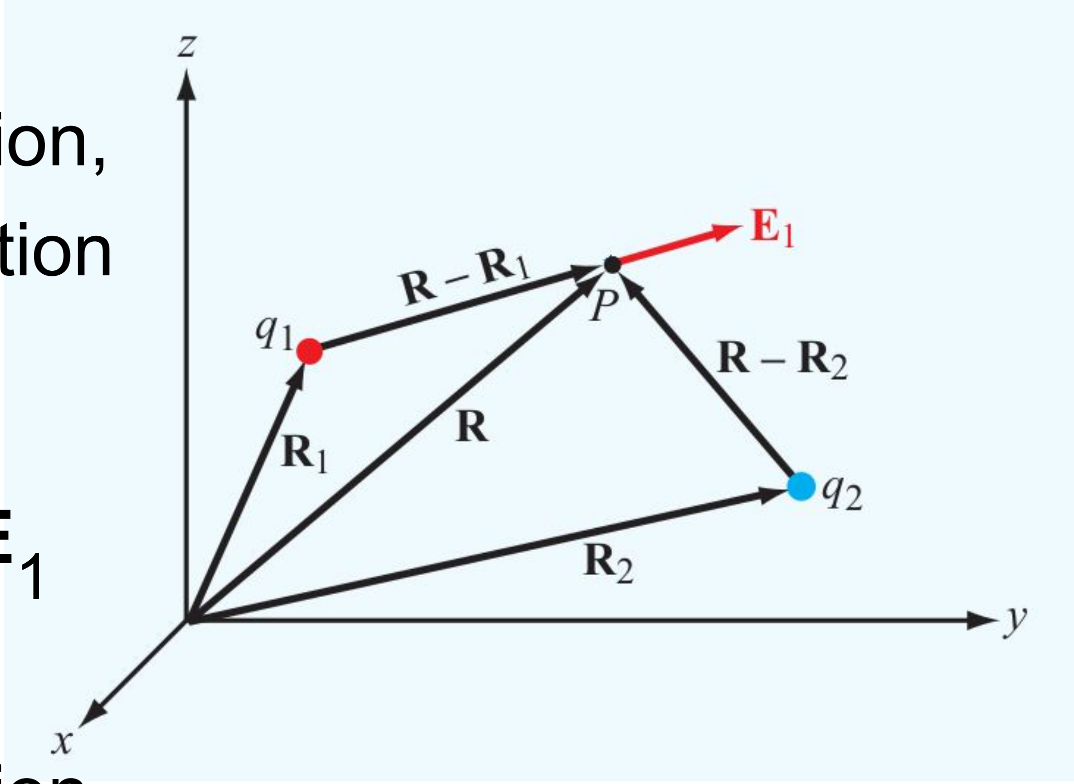
Calculate **E** due to *q1*: $\mathbf{E}_1$

To get **E** due to *q2* we need to identify its position, $\mathbf{R}_2$, and it's *relative* position to point *P*: $\mathbf{R}-\mathbf{R}_2$



4.3 Electric Field of Multiple Charges

To get **E** due to *q1* we need to identify its position, $\mathbf{R}_1$, and it's *relative* position to point *P*: $\mathbf{R}\text{-}\mathbf{R}_1$

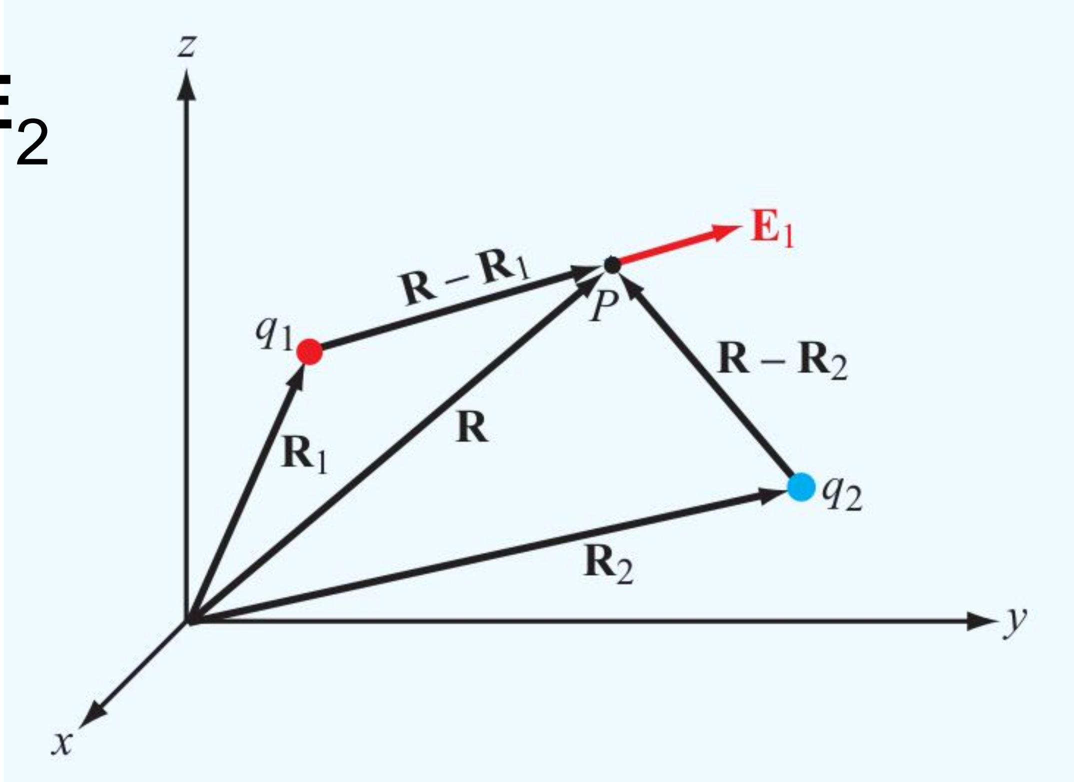


Calculate **E** due to *q1*: $\mathbf{E}_1$

To get **E** due to *q2* we need to identify its position, $\mathbf{R}_2$, and it's *relative* position to point *P*: $\mathbf{R}\text{-}\mathbf{R}_2$

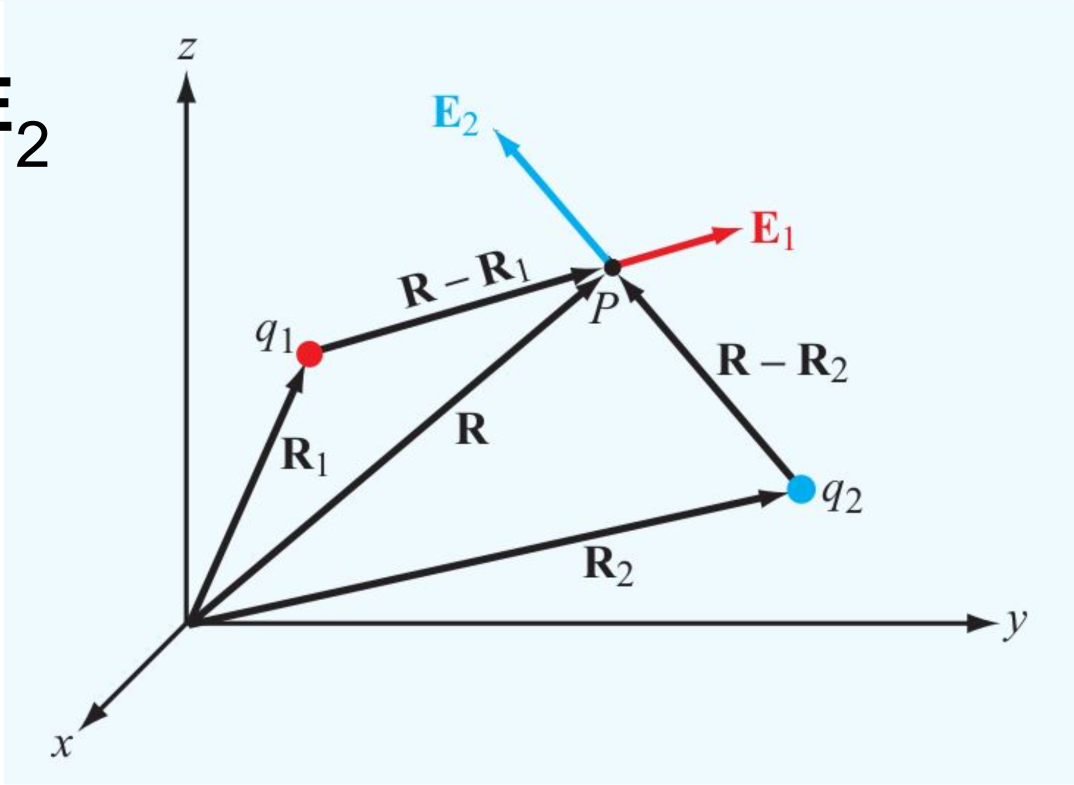
4.3 Electric Field of Multiple Charges

Calculate **E** due to *q2*: $\mathbf{E}_2$



4.3 Electric Field of Multiple Charges

Calculate **E** due to *q2*: $\mathbf{E}_2$



Total **E** is sum:

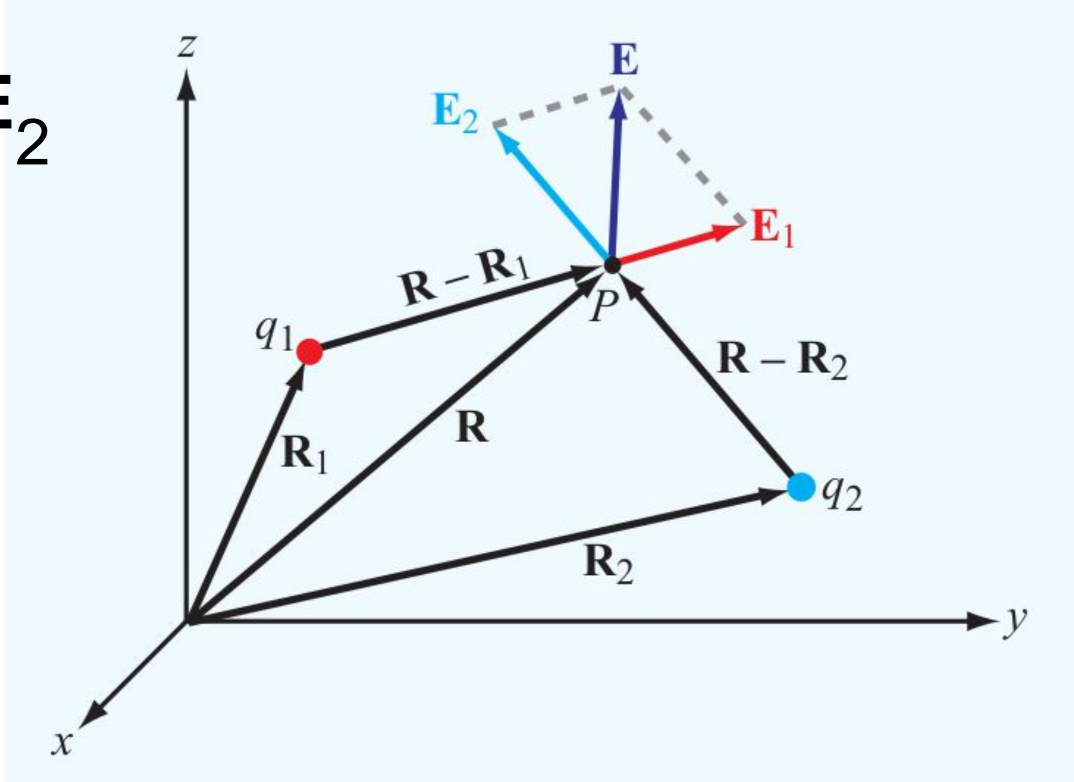
$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

4.3 Electric Field of Multiple Charges

Calculate **E** due to *q2*: $\mathbf{E}_2$

Total **E** is sum:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$



4.3 Electric Field of Multiple Charges

In general:

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2} \qquad \text{(V/m)}$$

so:

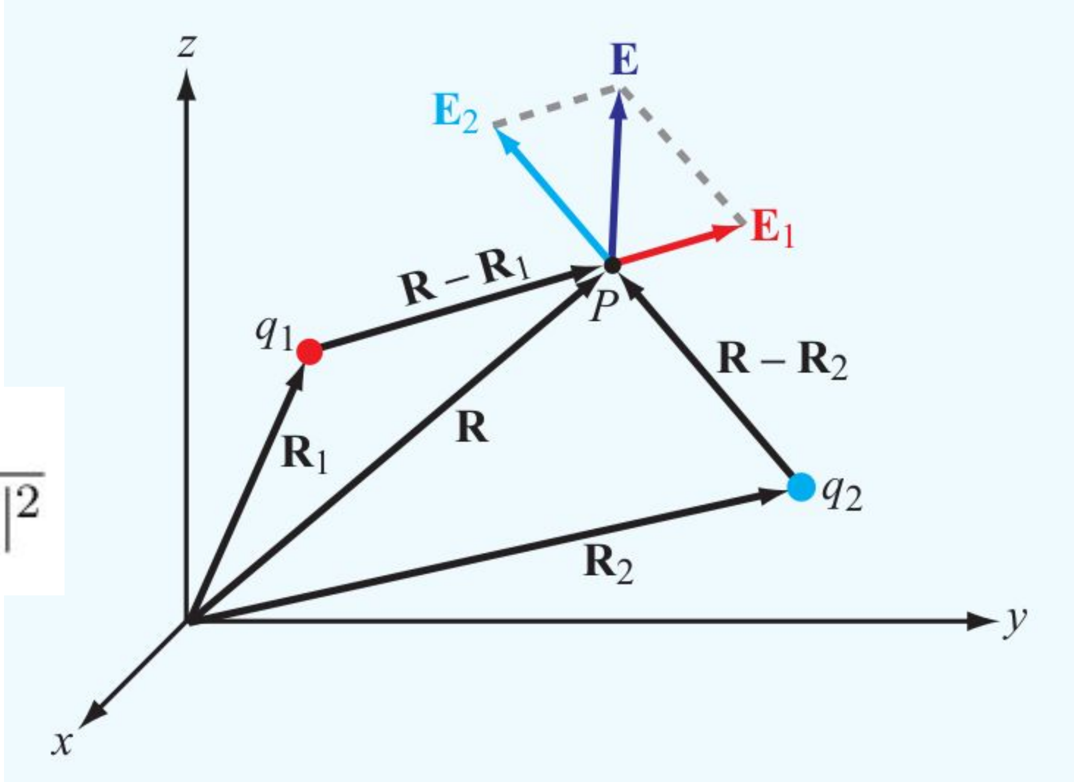
$$\mathbf{E}_1 = \widehat{\mathbf{R} - \mathbf{R}_1} \frac{q_1}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_1|^2}$$

where:

$$\widehat{\mathbf{R} - \mathbf{R}_1} = \frac{\mathbf{R} - \mathbf{R}_1}{|\mathbf{R} - \mathbf{R}_1|}$$

so:

$$\mathbf{E}_1 = (\mathbf{R} - \mathbf{R}_1) \frac{q_1}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_1|^3}$$



4.3 Electric Field of Multiple Charges

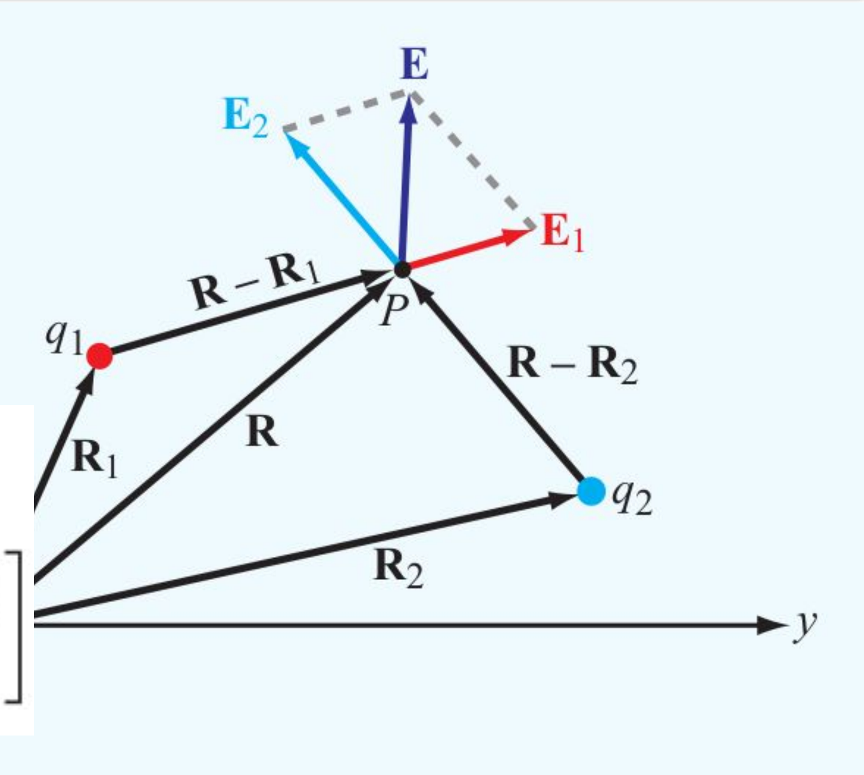
Similarly:

$$\mathbf{E}_2 = (\mathbf{R} - \mathbf{R}_2)\frac{q_2}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_2|^3}$$

Hence:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

$$= \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R} - \mathbf{R}_1)}{|\mathbf{R} - \mathbf{R}_1|^3} + \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{|\mathbf{R} - \mathbf{R}_2|^3}\right]$$



For N charges:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3}$$

Ex. 4-3 Electric Field of 2 Charges

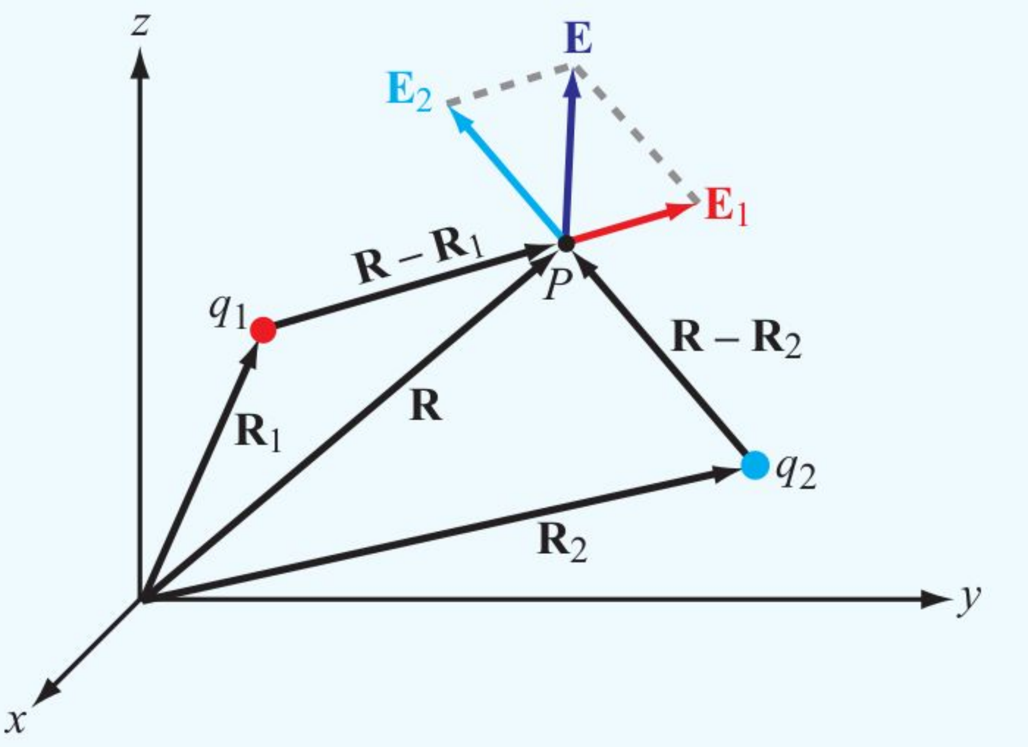
Given: in free space:

$q1 = 2 \times 10^{-5}$ C

$q2 = -4 \times 10^{-5}$ C

$P_1 = (1, 3, -1)$ meters

$P_2 = (-3, 1, -2)$ meters



Find: **E** at $P = (3, 1, -2)$ m

Solution:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3}$$

Ex. 4-3 Electric Field of 2 Charges

Solution:

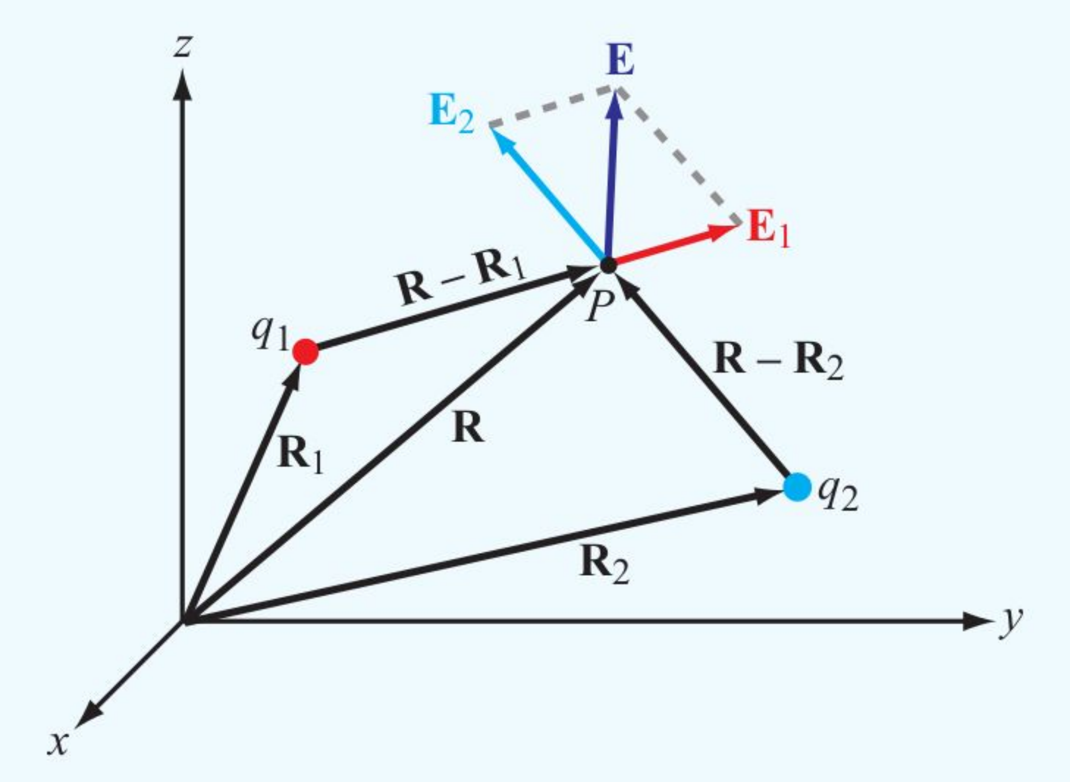
$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3}$$

so need:

$$\mathbf{R}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}},$$

$$\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2,$$

$$\mathbf{R} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2.$$



Ex. 4-3 Electric Field of 2 Charges

Solution:

$$\mathbf{R}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}},$$

$$\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2,$$

$$\mathbf{R} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2.$$

$$\mathbf{R} - \mathbf{R}_1 = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2 - (\hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}})$$

$$\mathbf{R} - \mathbf{R}_1 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}$$

$$|\mathbf{R} - \mathbf{R}_1| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$|\mathbf{R} - \mathbf{R}_1|^3 = 3^3 = 27$$

Ex. 4-3 Electric Field of 2 Charges

Solution:

$$\mathbf{R}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}},$$

$$\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2,$$

$$\mathbf{R} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2.$$

$$\mathbf{R} - \mathbf{R}_2 = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2 - (-\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2)$$

$$\mathbf{R} - \mathbf{R}_2 = \hat{\mathbf{x}}6 + \hat{\mathbf{y}}0 - \hat{\mathbf{z}}0$$

$$\mathbf{R} - \mathbf{R}_2 = \hat{\mathbf{x}}6$$

$$|\mathbf{R} - \mathbf{R}_2| = 6$$

$$|\mathbf{R} - \mathbf{R}_2|^3 = 216$$

Ex. 4-3 Electric Field of 2 Charges

Solution: plug in:

$$\mathbf{E} = \frac{1\times 10^{-5}\mathrm{C}}{4\pi\varepsilon_0}\left[\frac{2(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}})}{27} - \frac{4\hat{\mathbf{x}}6}{216}\right]$$

$$\mathbf{E} = \frac{1\times 10^{-5}\mathrm{C}}{4\pi\varepsilon_0}\left[\frac{2(216)(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}})}{27(216)} - \frac{4(27)\hat{\mathbf{x}}6}{27(216)}\right]$$

$$\mathbf{E} = \frac{1\times 10^{-5}\mathrm{C}}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}864 - \hat{\mathbf{y}}864 - \hat{\mathbf{z}}432}{5832} - \frac{\hat{\mathbf{x}}648}{5832}\right]$$

$$\mathbf{E} = \frac{1\times 10^{-5}\mathrm{C}}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}1 - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{27}\right]$$

Ex. 4-3 Electric Field of 2 Charges

Solution: plug in:

$$\mathbf{E} = \frac{1 \times 10^{-5}\mathrm{C}}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}1 - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{27}\right]$$

$$\mathbf{E} = \frac{1 \times 10^{-5}\mathrm{C}}{108\pi(8.85 \times 10^{-12}\,\mathrm{F/m})}(\hat{\mathbf{x}}1 - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2)$$

$$\boxed{\mathbf{E} = 3330(\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2)\ \mathrm{V/m}}$$

Ex. 4-3 Electric Field of 2 Charges

Solution: to calculate the force on $q3=8X10^{-5}$ C

$$\mathbf{F} = q\mathbf{E}$$

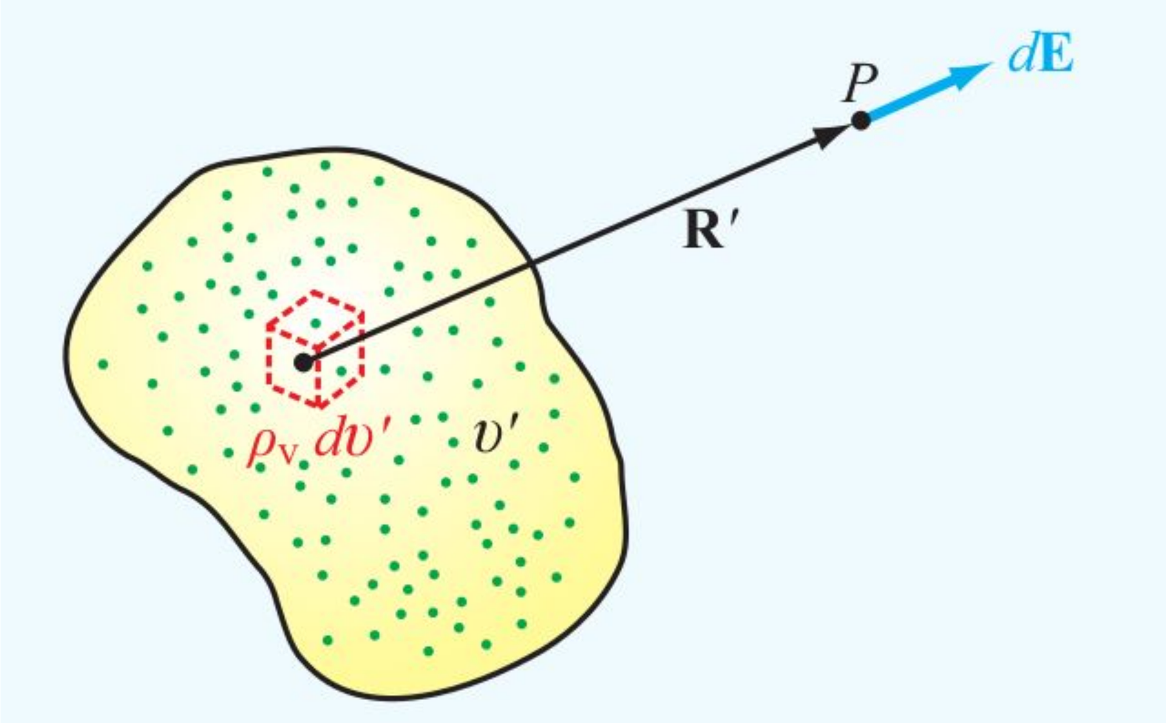
$$\mathbf{F} = q_3\mathbf{E}$$

$$\mathbf{F} = (8 \times 10^{-5}\text{C})3330(\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2)\ \text{V/m}$$

$$\boxed{\mathbf{F} = 266.4(\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2)\ \text{mN}}$$

4.3 Electric Field Due to Charge Distributions

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \hat{\mathbf{R}}' \frac{\rho_{\mathrm{v}}\, d\upsilon'}{R'^2} \qquad \textbf{(volume distribution)}$$



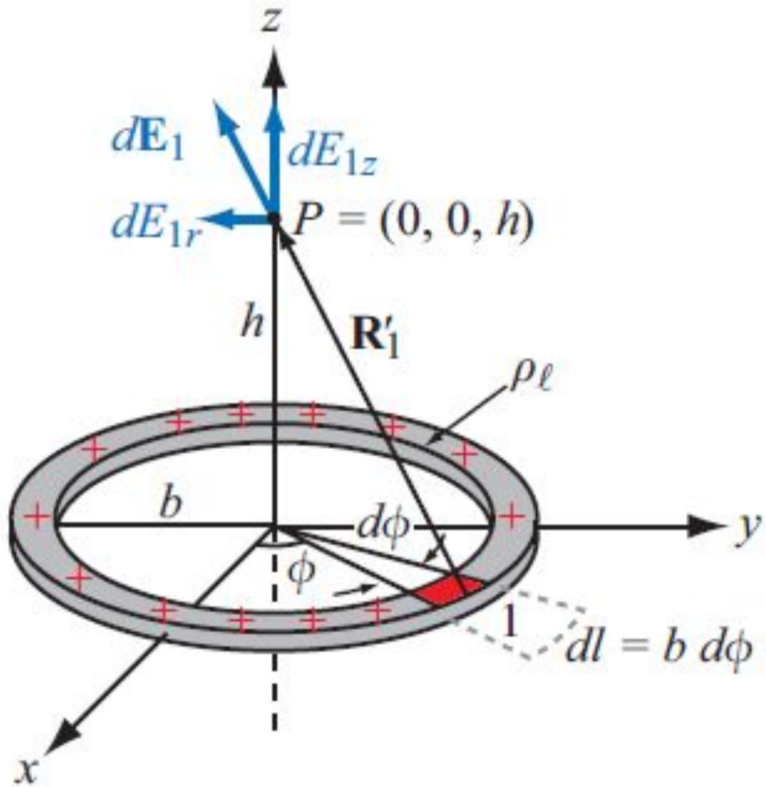
4.3 Electric Field Due to Charge Distributions

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\upsilon'} \hat{\mathbf{R}}' \, \frac{\rho_{\mathrm{v}} \, d\upsilon'}{R'^2}$$ **(volume distribution)**

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \, \frac{\rho_{\mathrm{s}} \, ds'}{R'^2}$$ **(surface distribution)**

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \, \frac{\rho_{\ell} \, dl'}{R'^2}$$ **(line distribution)**

Example 4-4



Given: thin ring: radius = b

line charge density = ρ_l

in x-y plane,

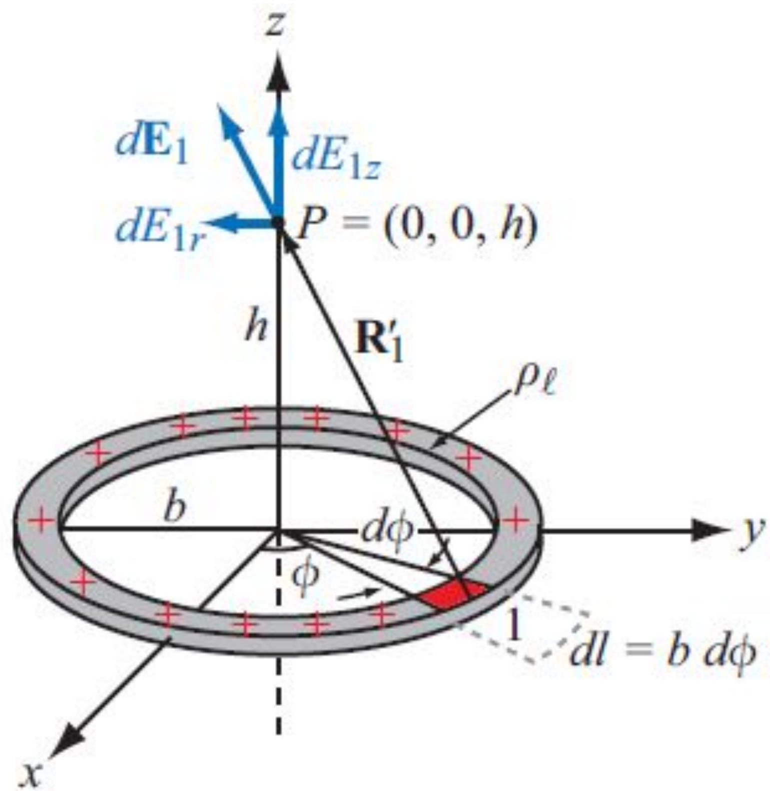
Find: $\mathbf{E}(0,0,h)$

(along the axis at distance h)

Cont.

Example 4-4

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}$$



Solution:

Find **E** due to small segment of charge:

center: $(b,\phi,0)$ (Cylindrical coords)

length: $dl = b\, d\phi$

charge in this segment: $dq = \rho_l\, dl = \rho_l\, b\, d\phi$

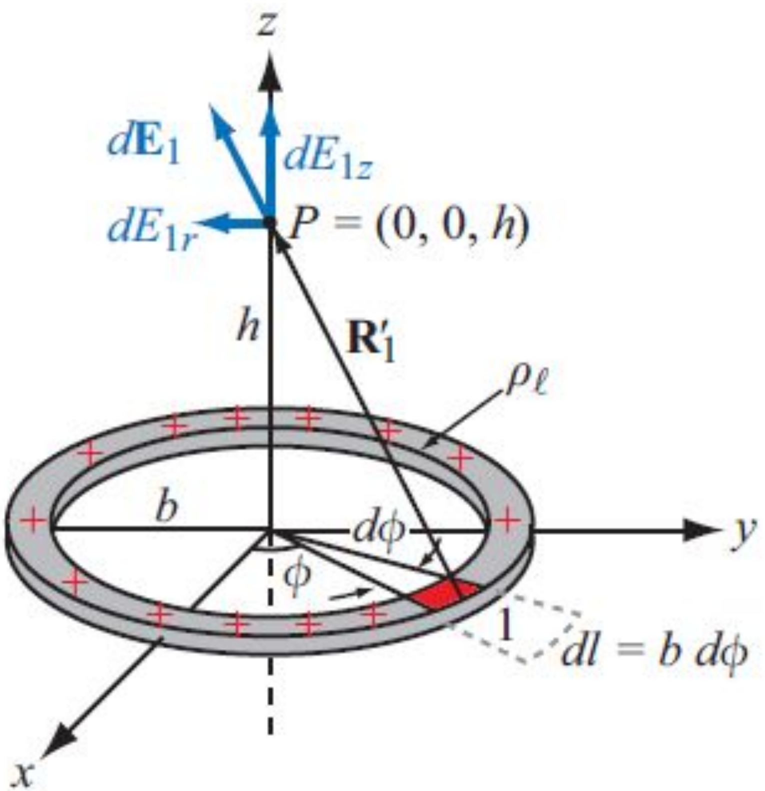
vector from this segment to $(0,0,h)$:

$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h$$

Cont.

Example 4-4

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}$$



$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h,$$

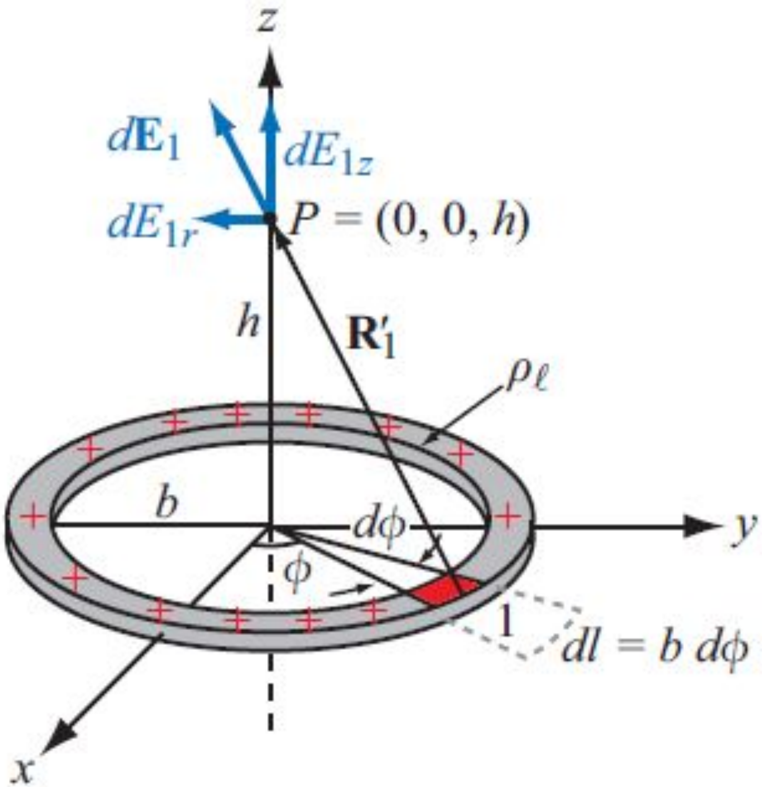
from which it follows that

$$R'_1 = |\mathbf{R}'_1| = \sqrt{b^2 + h^2}\,, \qquad \hat{\mathbf{R}}'_1 = \frac{\mathbf{R}'_1}{|\mathbf{R}'_1|} = \frac{-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h}{\sqrt{b^2 + h^2}}\,.$$

Cont.

Example 4-4

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}$$



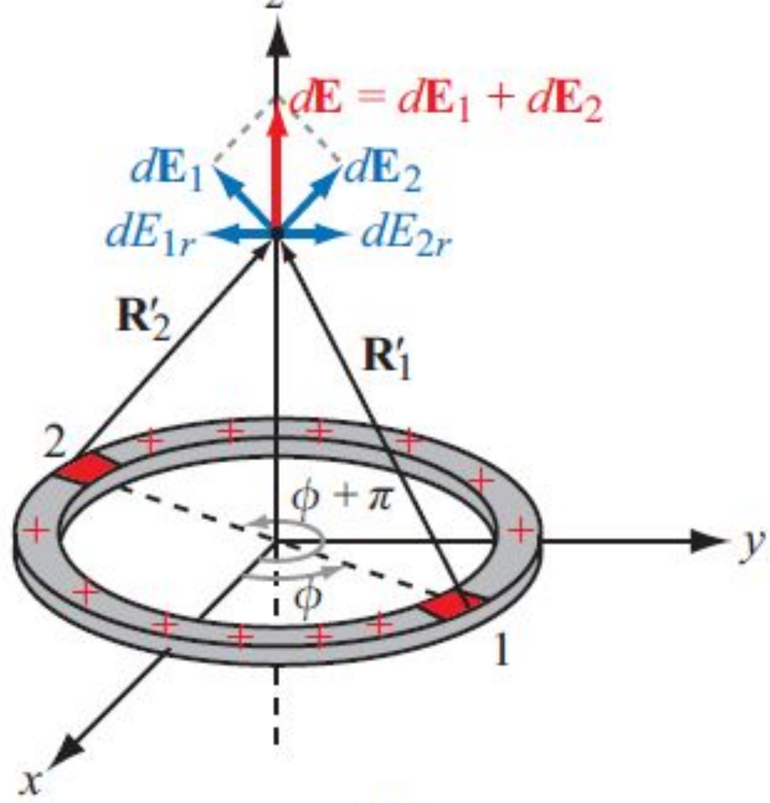
The electric field at $P = (0, 0, h)$ due to the charge in segment 1 therefore is

$$d\mathbf{E}_1 = \frac{1}{4\pi\varepsilon_0} \hat{\mathbf{R}}'_1 \frac{\rho_\ell \, dl}{{R'_1}^2} = \frac{\rho_\ell b}{4\pi\varepsilon_0} \frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2 + h^2)^{3/2}} \, d\phi .$$

Cont.

Example 4-4

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}$$



Segment 1 has an **E** field like: $-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h$

and from symmetry, for segment 2: $+\hat{\mathbf{r}}b + \hat{\mathbf{z}}h$

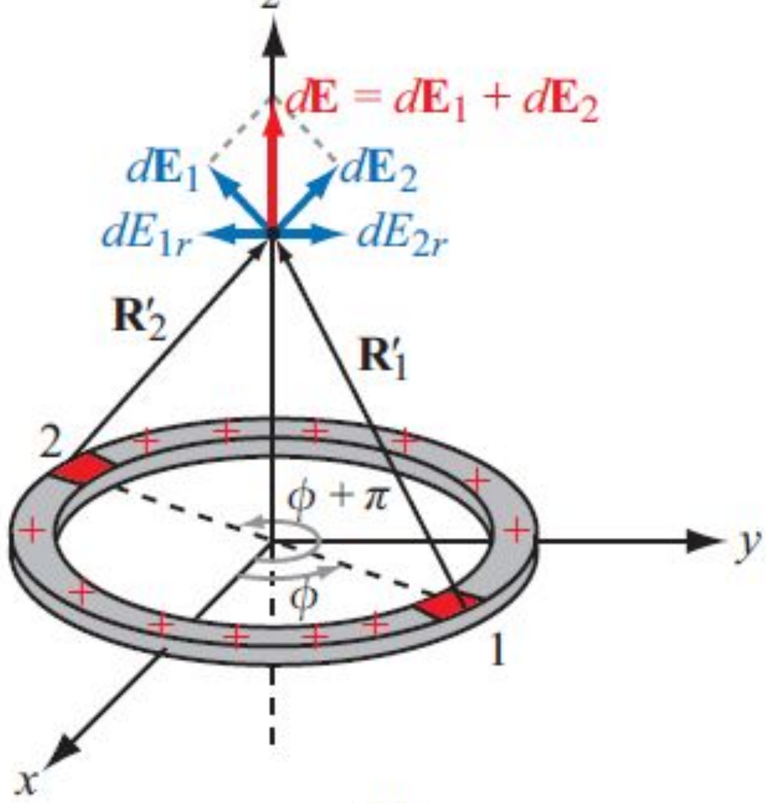
Therefore, their sum is:

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{z}} \frac{\rho_\ell bh}{2\pi\varepsilon_0} \frac{d\phi}{(b^2 + h^2)^{3/2}}$$

Cont.

Example 4-4

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}$$



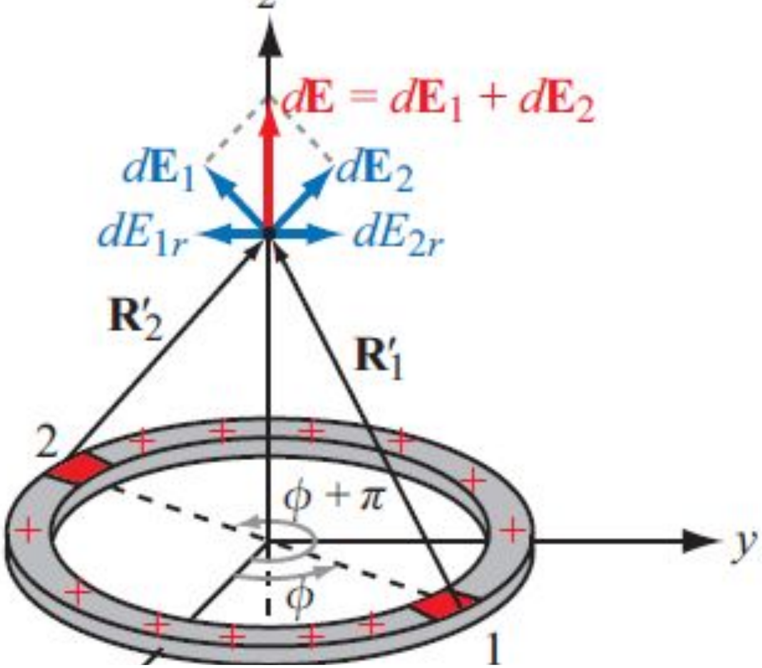
Since d**E** includes contributions from both halves of the ring,
We integrate d**E** around HALF the ring:

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_\ell b h}{2\pi\varepsilon_0 (b^2 + h^2)^{3/2}} \int_0^\pi d\phi$$

Cont.

Example 4-4

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}$$



$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_\ell bh}{2\pi\varepsilon_0 (b^2 + h^2)^{3/2}} \int_0^\pi d\phi$$

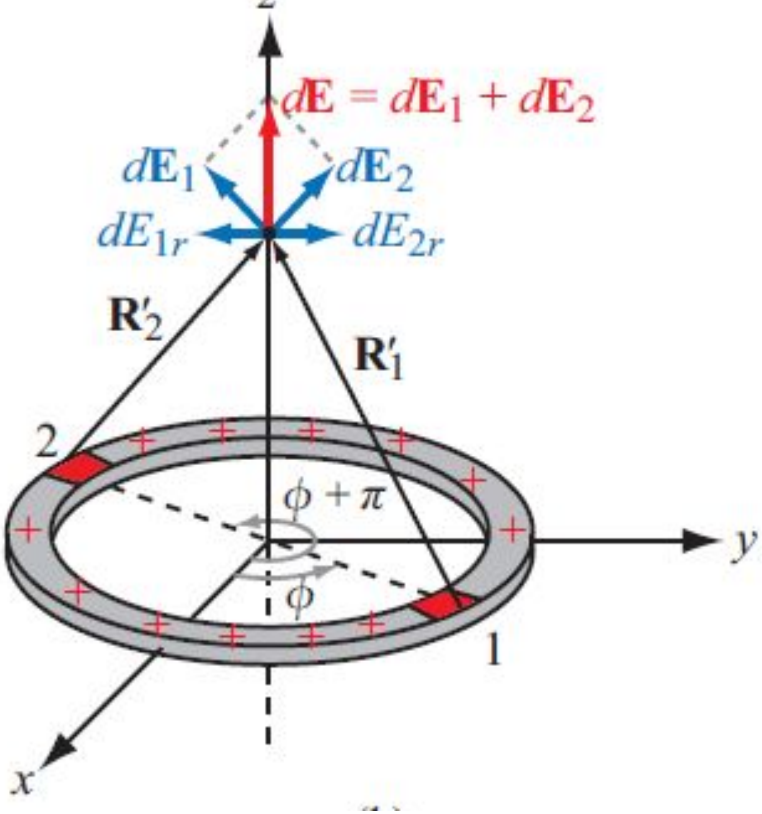
$$= \hat{\mathbf{z}} \frac{\rho_\ell bh}{2\varepsilon_0 (b^2 + h^2)^{3/2}}$$

$$= \hat{\mathbf{z}} \frac{h}{4\pi\varepsilon_0 (b^2 + h^2)^{3/2}} Q, \qquad (4.23)$$

where $Q = 2\pi b\rho_\ell$ is the total charge on the ring.

Cont.

Example 4-4



Special cases:

h=0 (ring center):
E = 0
the effect of the charges cancels out

h=∞:

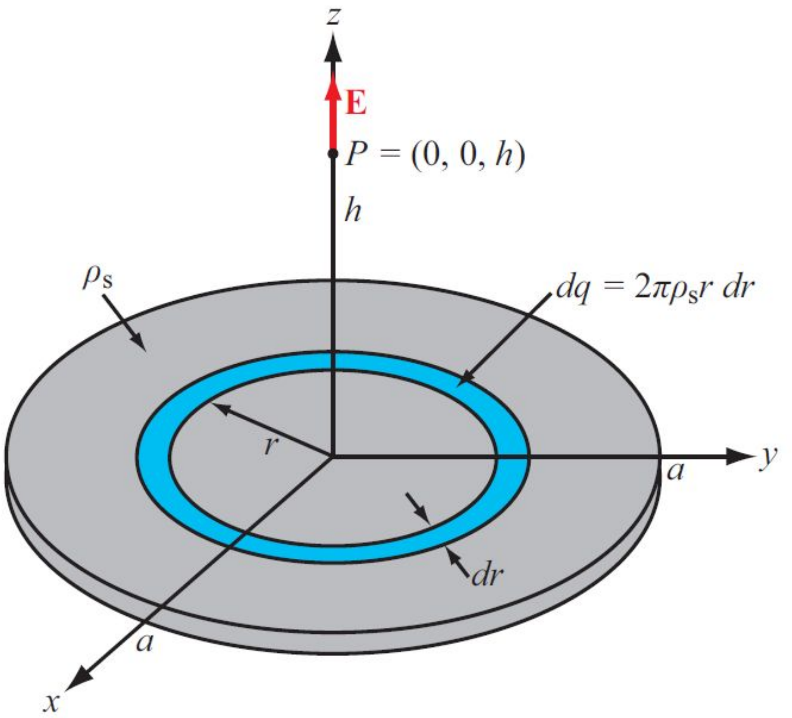
$$\mathbf{E} = \hat{\mathbf{z}} \frac{1}{4\pi\varepsilon_0 h^2} Q$$

like a point charge at the origin

Cont.

Example 4-5

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}$$



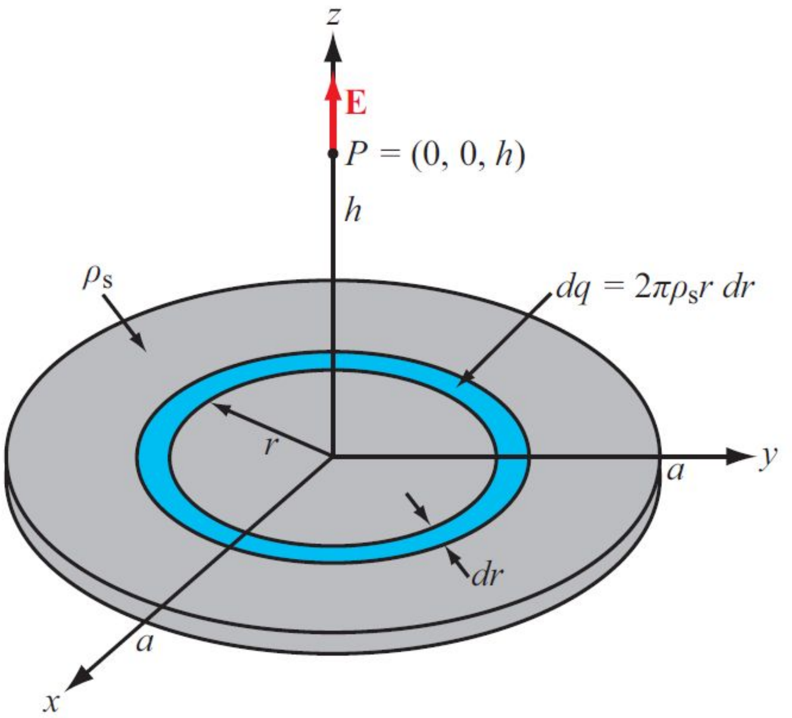
Given: Circular disk of radius: *a*
uniform charge density: ρ_s
in x-y plane

Find: $\mathbf{E}(0,0,h)$

Cont.

Example 4-5

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}$$



Solution:

Use the solution to Example 4-4.

The sheet is a summation of concentric rings.

each ring of radius r, width dr:

area: ds = 2π r dr

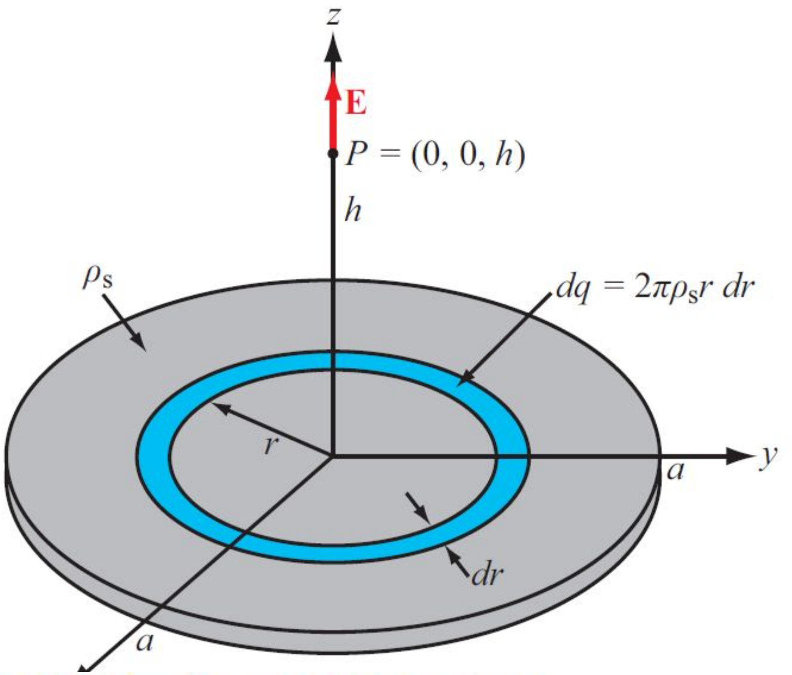
charge: dq = ρ_s ds = ρ_s 2π r dr

$$d\mathbf{E} = \hat{\mathbf{z}} \frac{h}{4\pi\varepsilon_0 (r^2 + h^2)^{3/2}} (2\pi\rho_s r \, dr)$$

Cont.

Example 4-5

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}$$



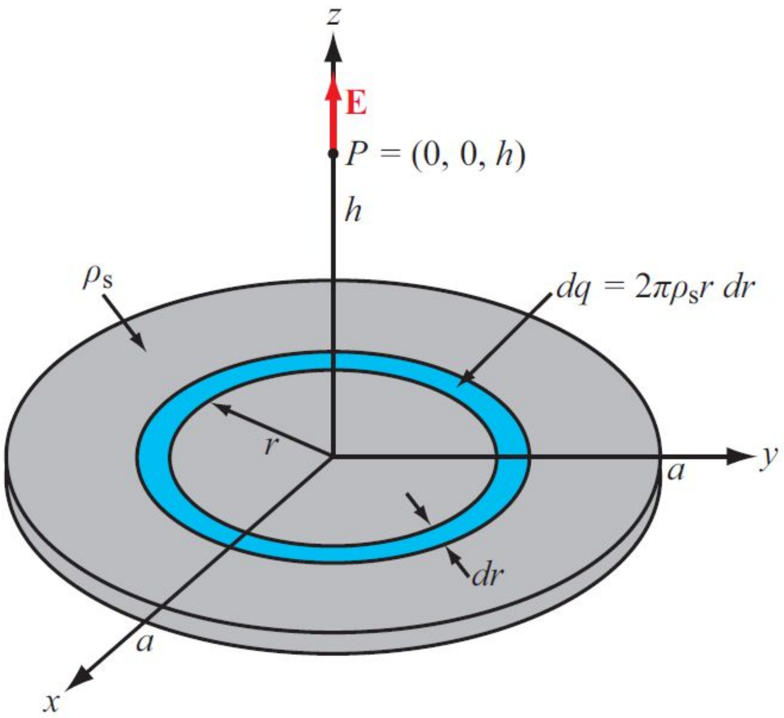
The total field at P is obtained by integrating the expression over the limits $r = 0$ to $r = a$:

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s h}{2\varepsilon_0} \int_0^a \frac{r \, dr}{(r^2 + h^2)^{3/2}}$$

Cont.

Example 4-5

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}$$



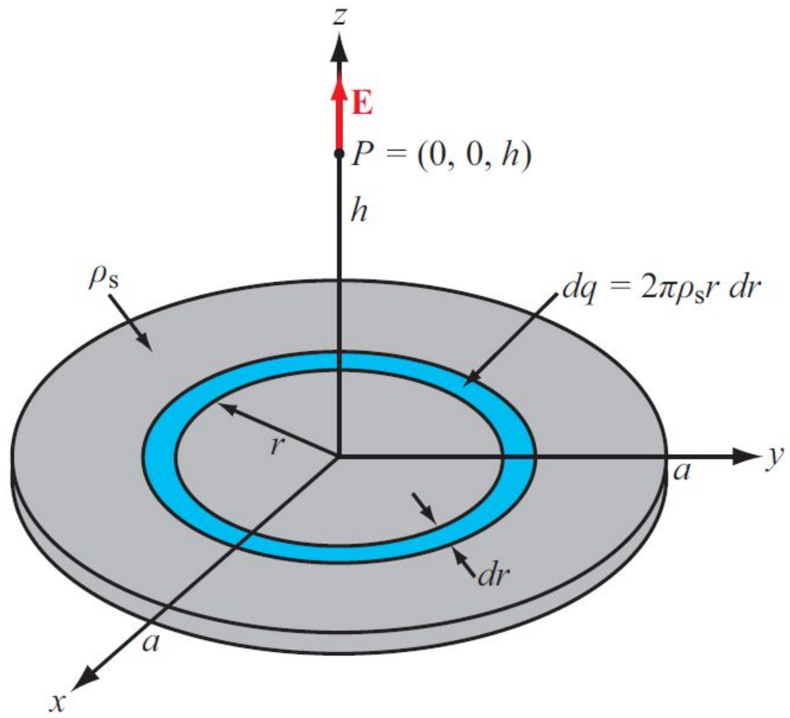
$$\int \frac{x \, dx}{(x^2 + c^2)^{3/2}} = -\frac{1}{(x^2 + c^2)^{1/2}}$$

$$\int_0^a \frac{r \, dr}{(r^2 + h^2)^{3/2}} = -\frac{1}{(r^2 + h^2)^{1/2}} \Bigg|_0^a$$

Cont.

Example 4-5

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}$$



Using the + square-root:

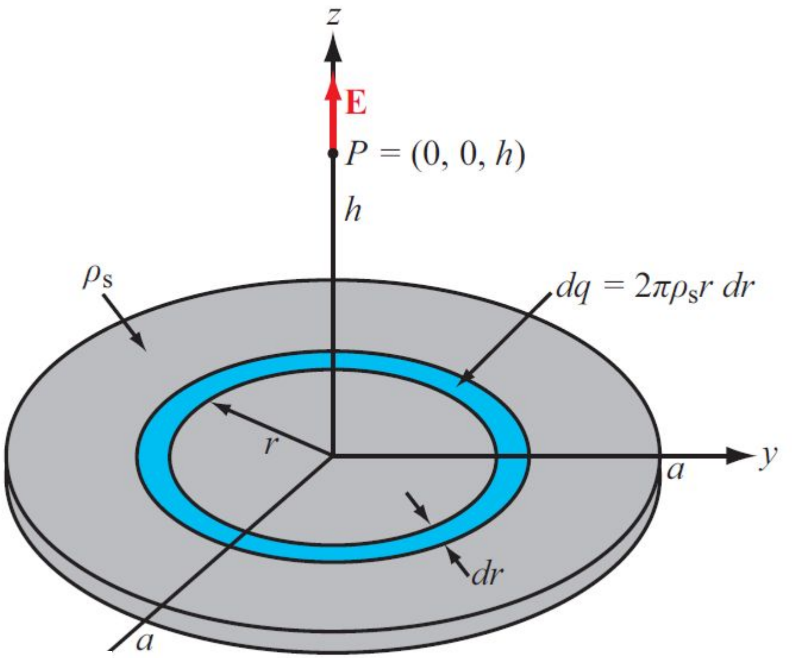
$$-\frac{1}{(r^2+h^2)^{1/2}}\bigg|_{r=0}^{a} = -\frac{1}{(a^2+h^2)^{1/2}} + \frac{1}{(h^2)^{1/2}}$$

$$-\frac{1}{(r^2+h^2)^{1/2}}\bigg|_{r=0}^{a} = -\frac{1}{(a^2+h^2)^{1/2}} + \frac{1}{|h|}$$

Cont.

Example 4-5

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}$$



$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s h}{2\varepsilon_0} \left[\frac{1}{|h|} - \frac{1}{(a^2 + h^2)^{1/2}} \right]$$

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[\frac{h}{|h|} - \frac{h}{(a^2 + h^2)^{1/2}} \right]$$

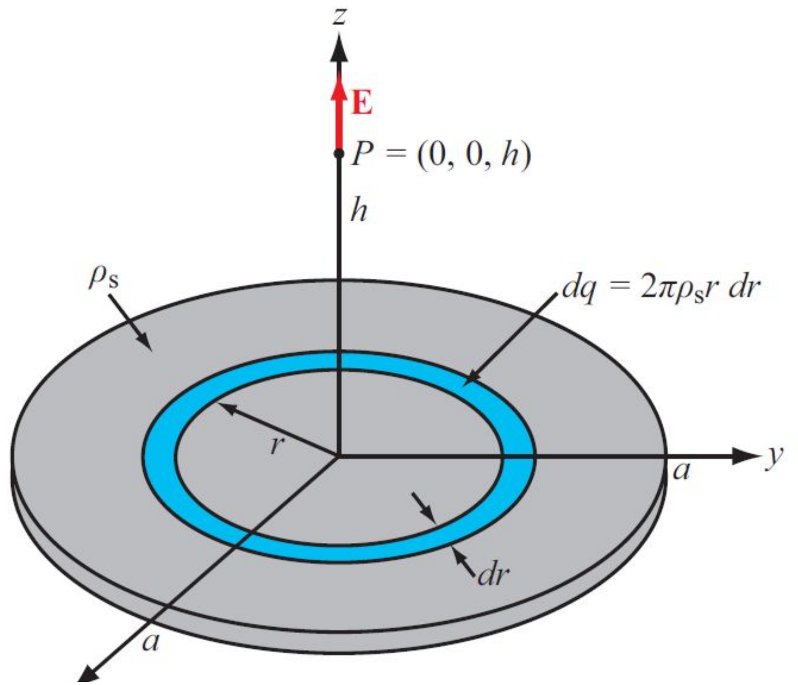
$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[1 - \frac{h}{(a^2 + h^2)^{1/2}} \right] \quad (h > 0)$$

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[-1 - \frac{h}{(a^2 + h^2)^{1/2}} \right] \quad (h < 0)$$

Cont.

Example 4-5

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}$$



$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[-1 - \frac{h}{(a^2 + h^2)^{1/2}} \right] \quad (h < 0)$$

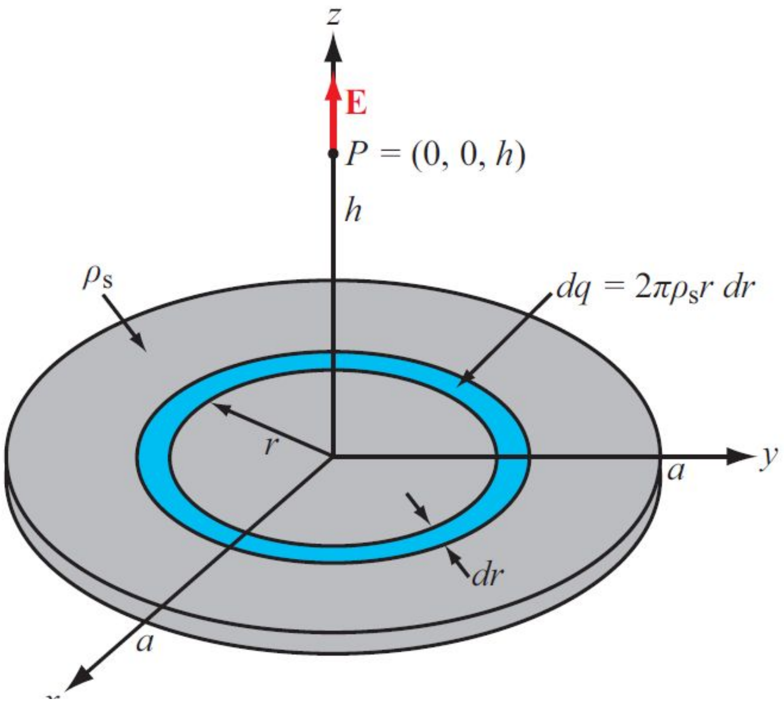
$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[-1 + \frac{|h|}{(a^2 + h^2)^{1/2}} \right] \quad (h < 0)$$

$$\mathbf{E} = -\hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[1 - \frac{|h|}{(a^2 + h^2)^{1/2}} \right] \quad (h < 0)$$

Cont.

Example 4-5

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_{S'} \hat{\mathbf{R}}' \frac{\rho_s\, ds'}{R'^2}$$



$$\mathbf{E} = -\hat{\mathbf{z}}\frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{|h|}{(a^2+h^2)^{1/2}}\right] \quad (h < 0)$$

$$\mathbf{E} = +\hat{\mathbf{z}}\frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{|h|}{(a^2+h^2)^{1/2}}\right] \quad (h > 0)$$

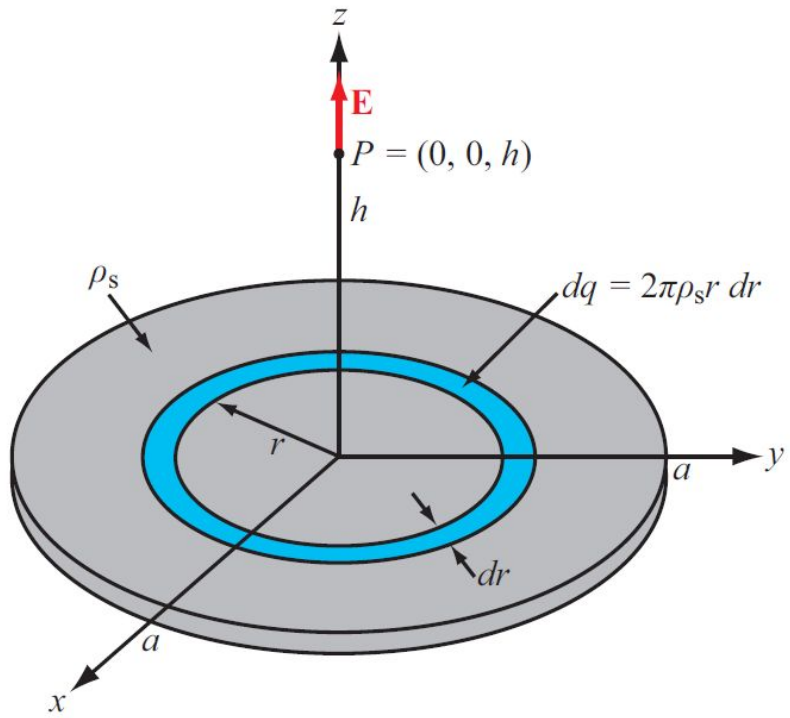
$$\mathbf{E} = \pm\hat{\mathbf{z}}\frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{|h|}{(a^2+h^2)^{1/2}}\right] \quad (+\,\text{if}\, h > 0, -\,\text{if}\, h < 0)$$

Cont.

Example 4-5

Special case:

$$\mathbf{E} = \pm\hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{|h|}{\sqrt{a^2+h^2}}\right]$$



a=∞ (infinite sheet)

$$\mathbf{E} = \pm\hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0} \qquad \text{(infinite sheet of charge).}$$

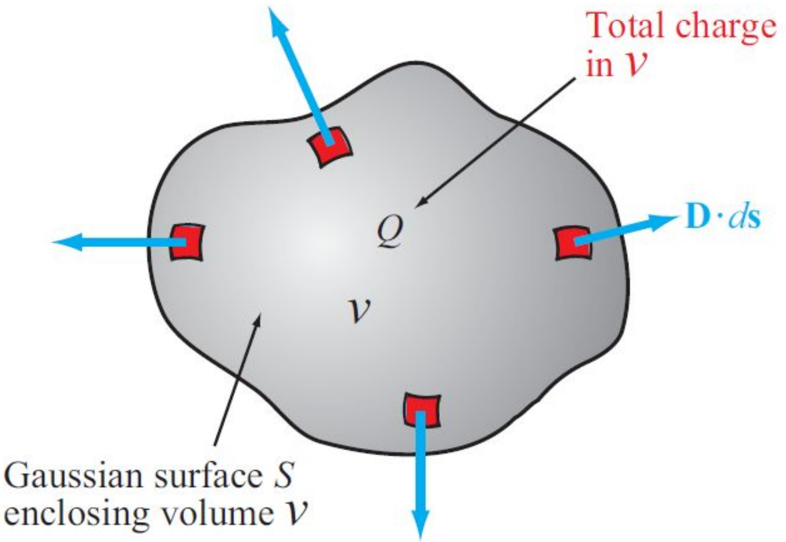
constant: does not depend on distance!

Cont.

4-4 Gauss's Law

$$\nabla \cdot \mathbf{D} = \rho_v$$

(Differential form of Gauss's law),



4-4 Gauss's Law

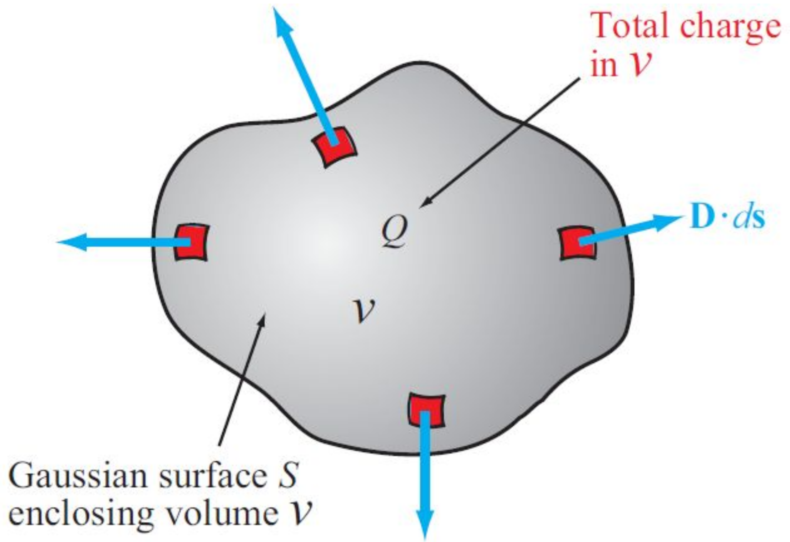
Application of the divergence theorem gives:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \oint_{S} \mathbf{D} \cdot d\mathbf{s}.$$

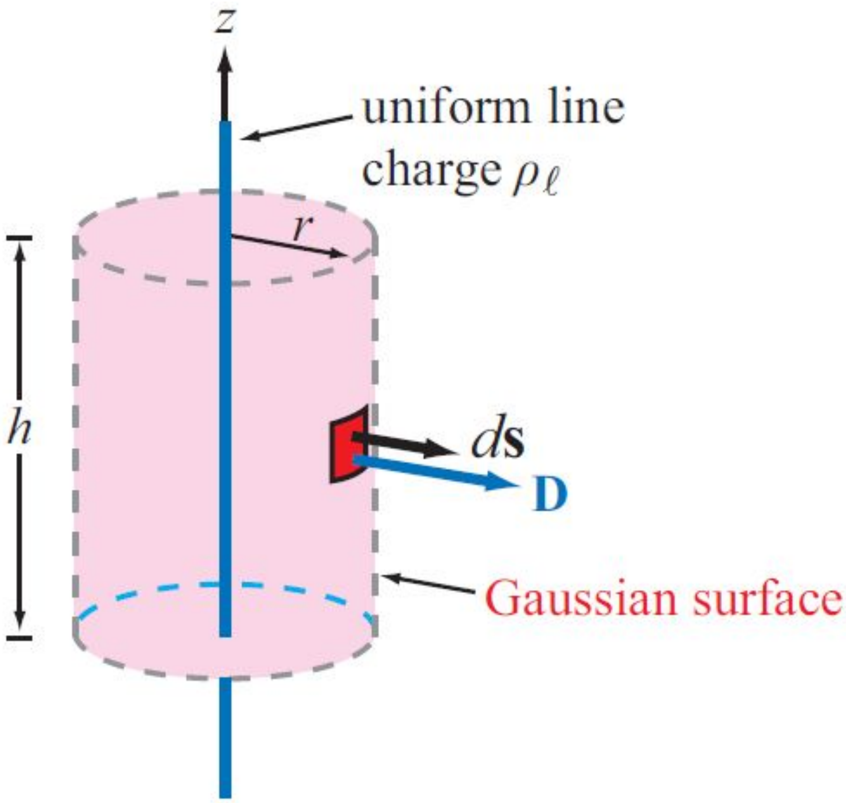
$$\int_{\mathcal{V}} \rho_v \, d\mathcal{V} = \oint_{S} \mathbf{D} \cdot d\mathbf{s}$$

$$\oint_{S} \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).



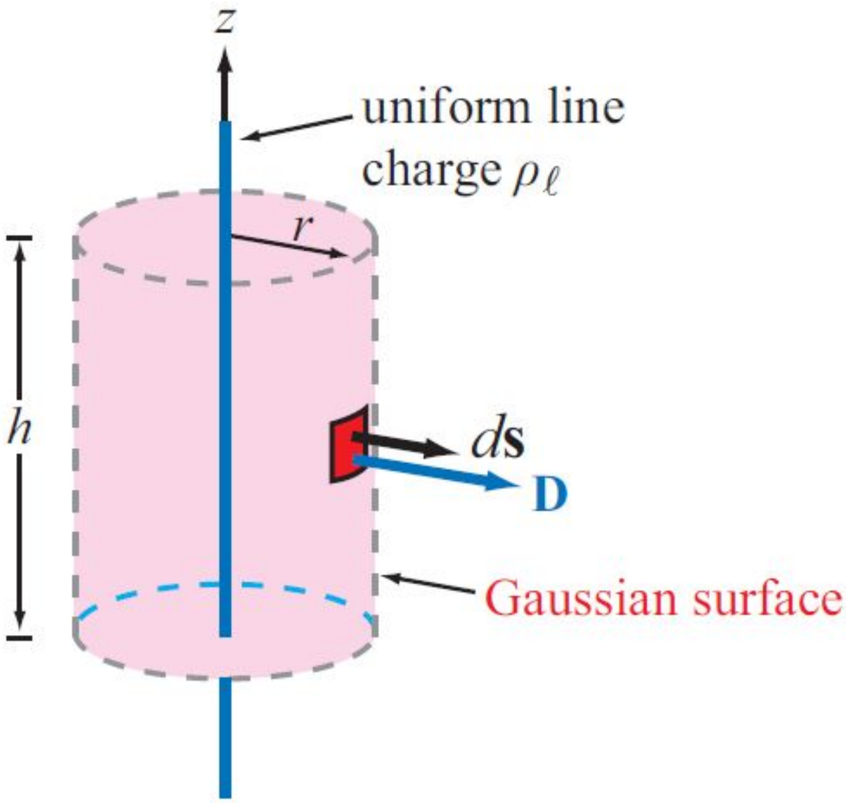
Example 4-6



Given: line charge density: ρ_l
infinite
along z-axis
in free space

Find: Using Gauss's Law:
E everywhere

Example 4-6



(Don't make it infinite, since that often causes problems...)

Solution:

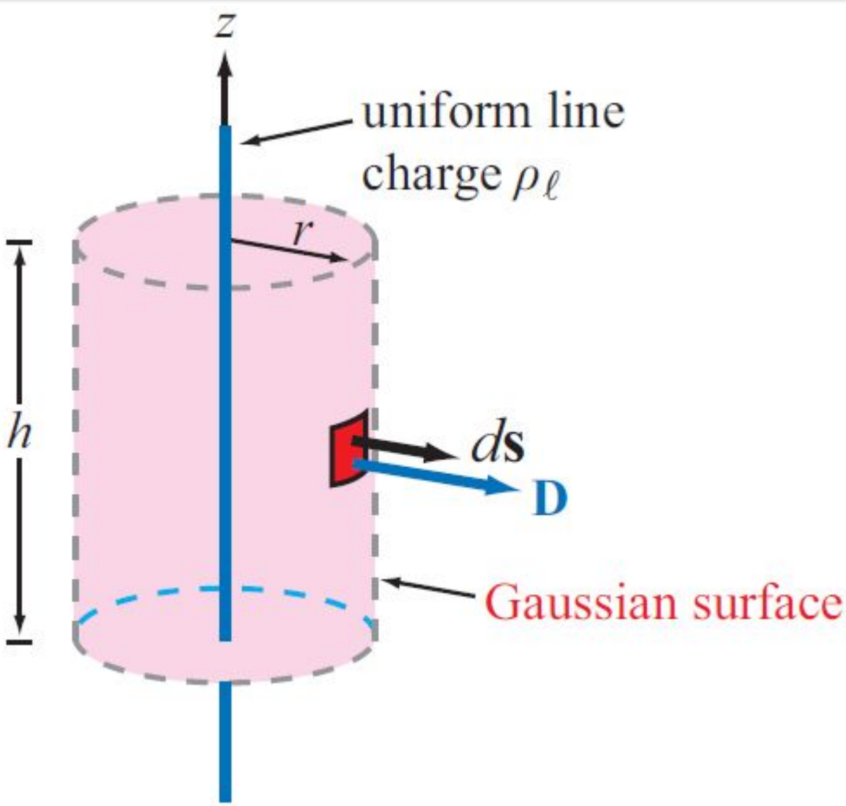
Since we are using Gauss's law:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q.$$

we need to identify an appropriate surface.

Construct an imaginary cylinder of radius *r* and height *h*

Example 4-6

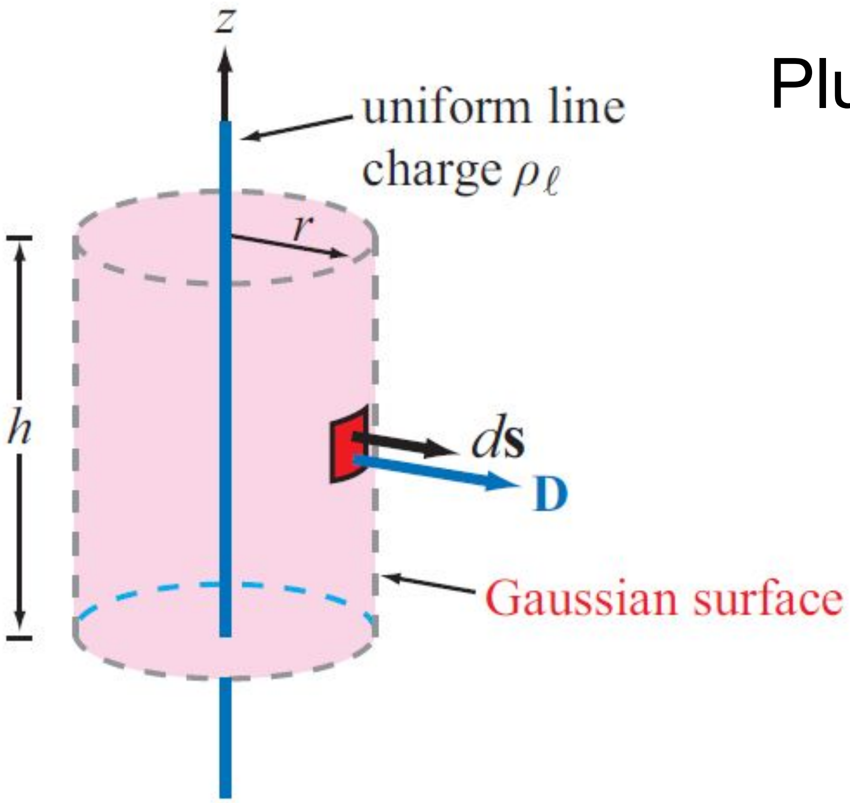


Note that we will integrate over the curved surface of the cylinder, and not include the 2 flat ends.

ds is surface patch on this curved surface:
direction: $\hat{\mathbf{r}}$
magnitude: $r\,d\phi\,dz$

D is radial: $\hat{\mathbf{r}} D_r$

Example 4-6



Plug in:

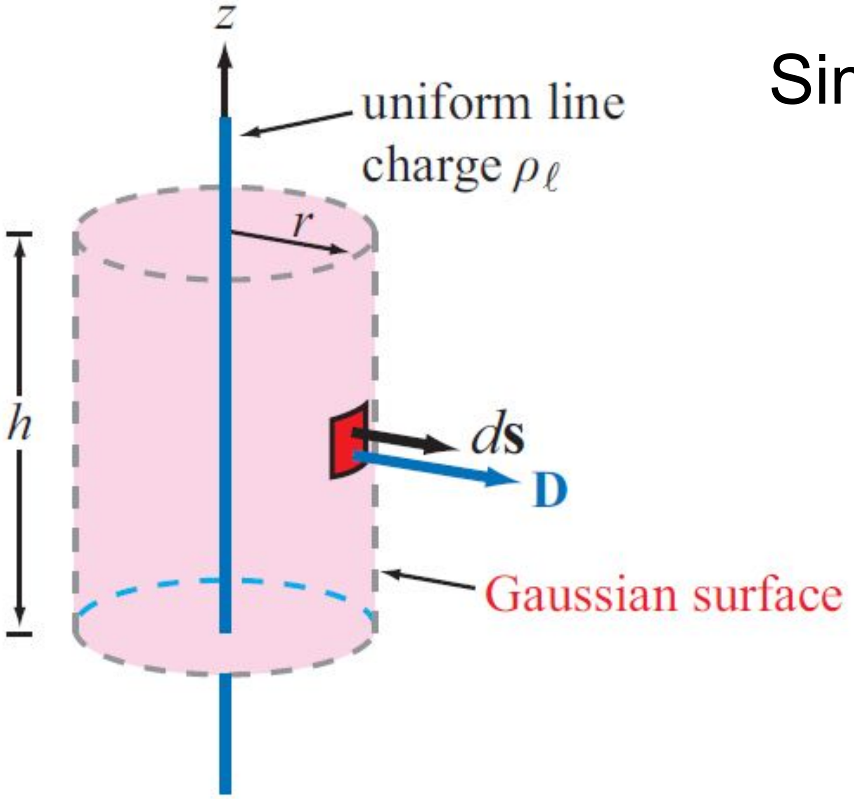
$$\int_{z=0}^{h} \int_{\phi=0}^{2\pi} \mathbf{D} \cdot \hat{\mathbf{r}} r \, d\phi \, dz = \rho_l \, h$$

$$\int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}} r \, d\phi \, dz = \rho_l \, h$$

$$D_r r \int_{z=0}^{h} \int_{\phi=0}^{2\pi} d\phi \, dz = \rho_l \, h$$

$$D_r r \int_{z=0}^{h} dz \int_{\phi=0}^{2\pi} d\phi = \rho_l \, h$$

Example 4-6



Simplify:

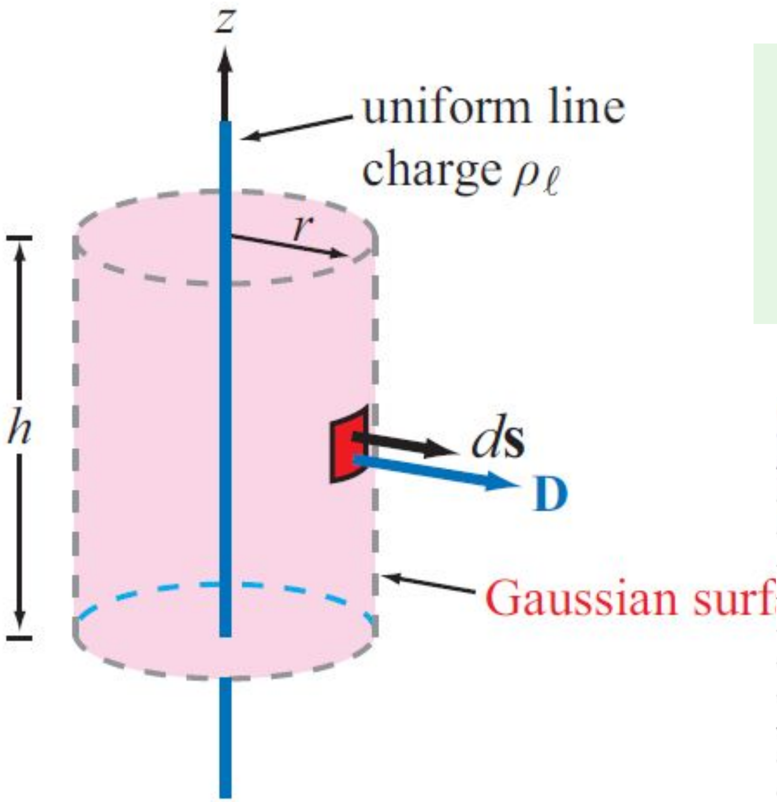
$$D_r r \int_{z=0}^{h} \int_{\phi=0}^{2\pi} d\phi\, dz = \rho_l\, h$$

$$D_r r \int_{z=0}^{h} dz \int_{\phi=0}^{2\pi} d\phi = \rho_l\, h$$

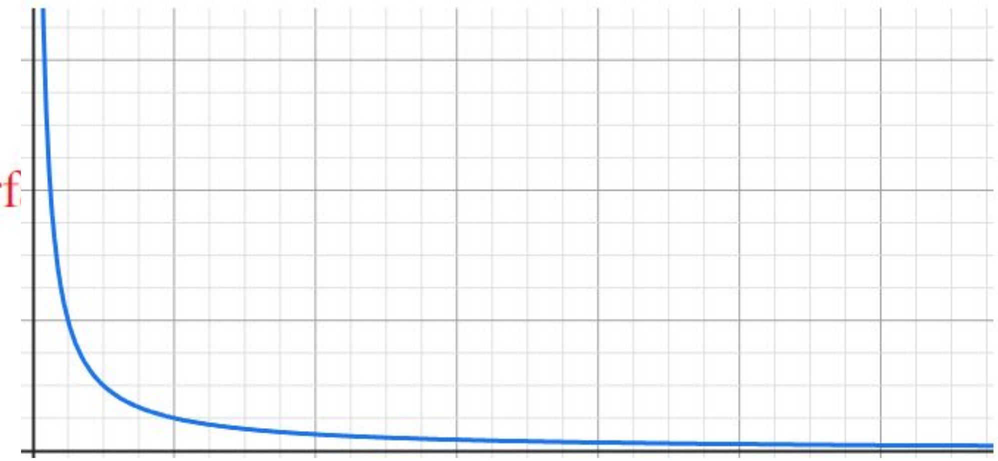
$$D_r r \left[z \right]_{z=0}^{h} \left[\phi \right]_{\phi=0}^{2\pi} = \rho_l\, h$$

$$D_r r\, h 2\pi = \rho_l\, h$$

Example 4-6

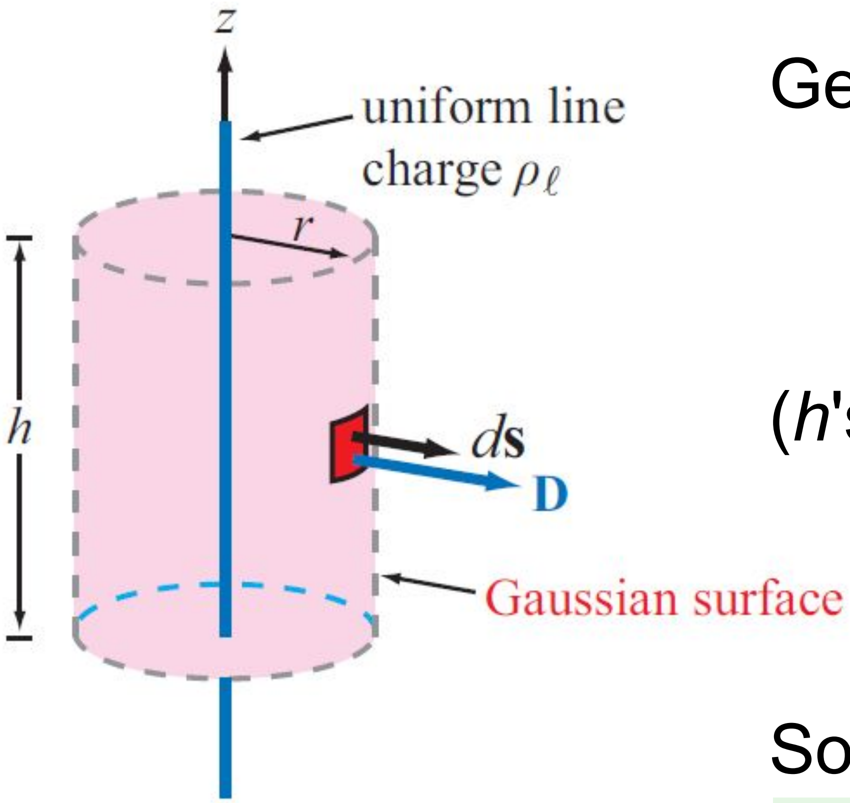


$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}}\,\frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}}\,\frac{\rho_\ell}{2\pi\varepsilon_0 r}$$



Note: invalid for r too small.

Example 4-6



Get:

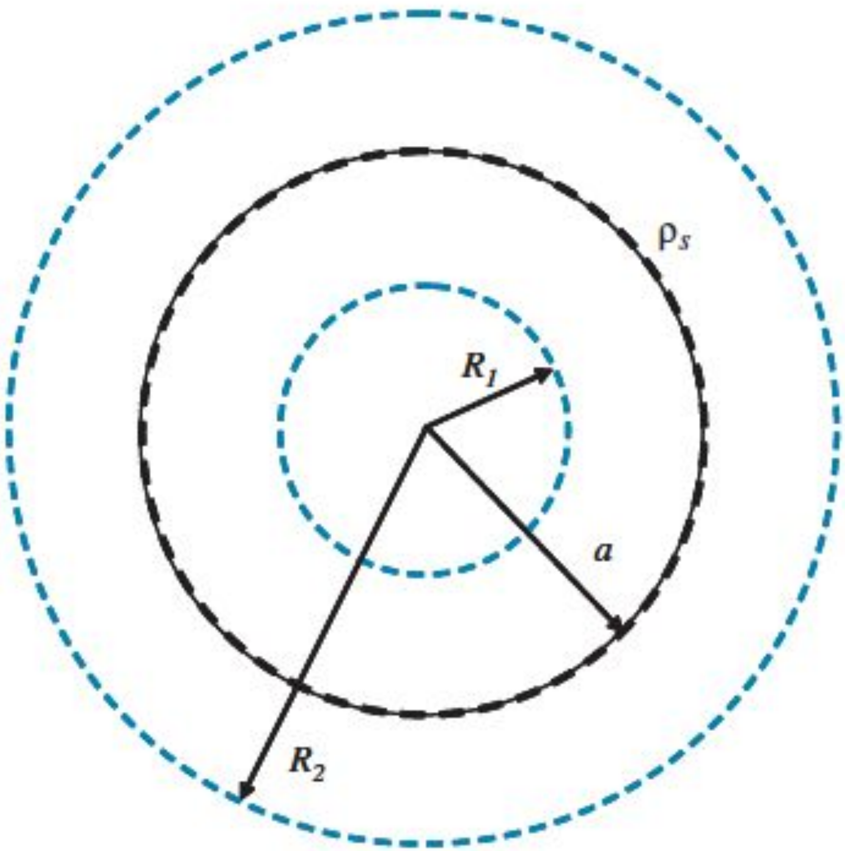
$$D_r = \frac{\rho_\ell}{2\pi r}$$

(*h*'s cancel)

Solving for **E**:

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}}\,\frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}}\,\frac{\rho_\ell}{2\pi\varepsilon_0 r}$$

Exercise 4-8

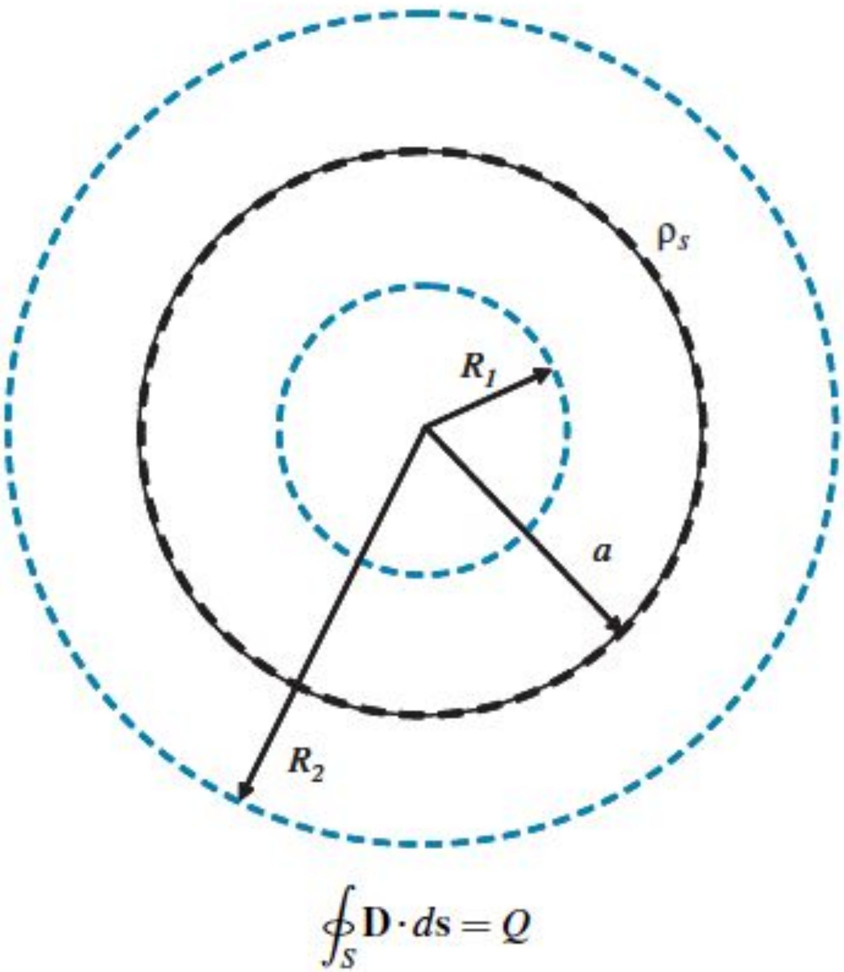


$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Given: surface charge density: ρ_s on sphere, radius a in some medium with ε

Find: Using Gauss's Law: **E** everywhere (both inside and outside the spherical shell)

Exercise 4-8



Solution:

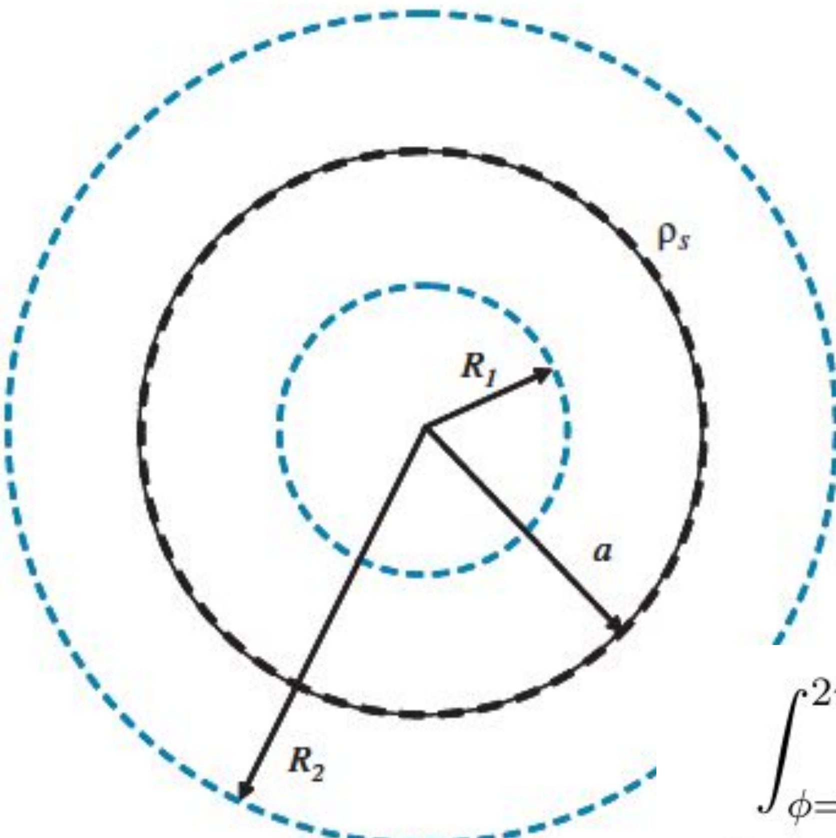
Since we are using Gauss's law:

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = Q.$$

we need to identify an appropriate surface.

Choose a spherical surface of radius $R_1 < a$ for fields inside the sphere, and $R_2 > a$ for fields outside the sphere.

Exercise 4-8



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

d**s** is surface patch on this spherical surface:

direction: $\hat{\mathbf{R}}$

magnitude: $R^2 \sin\theta \; d\theta \; d\phi$

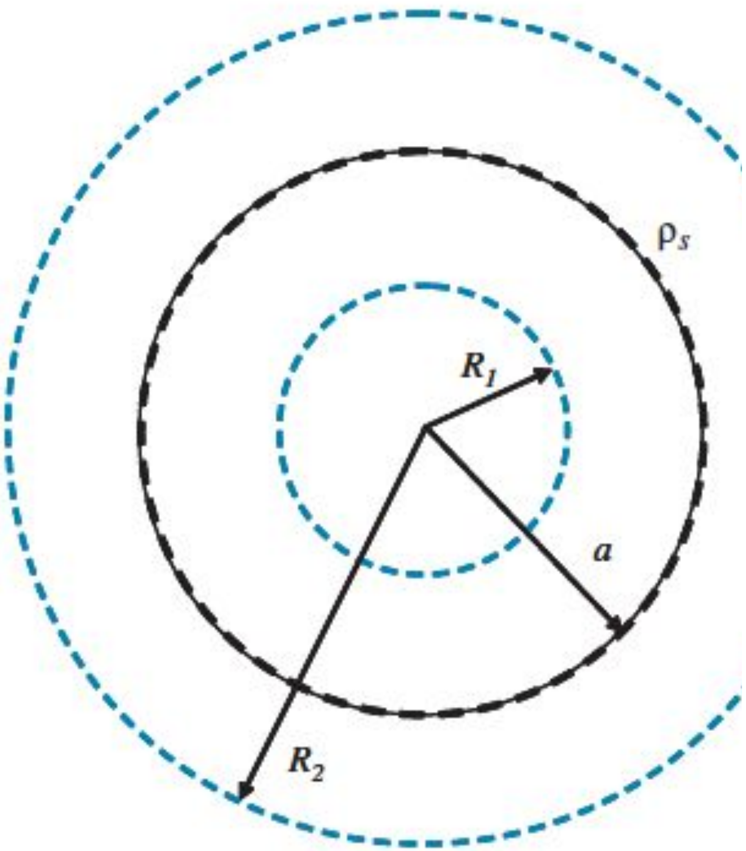
D is radial

Plug in:

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \mathbf{D} \cdot \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi = Q_{\text{enclosed}}$$

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \hat{\mathbf{R}} D_R \cdot \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi = Q_{\text{enclosed}}$$

Exercise 4-8



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Simplify:

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \hat{\mathbf{R}} D_R \cdot \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi = Q_{\text{enclosed}}$$

$$D_R R^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi = Q_{\text{enclosed}}$$

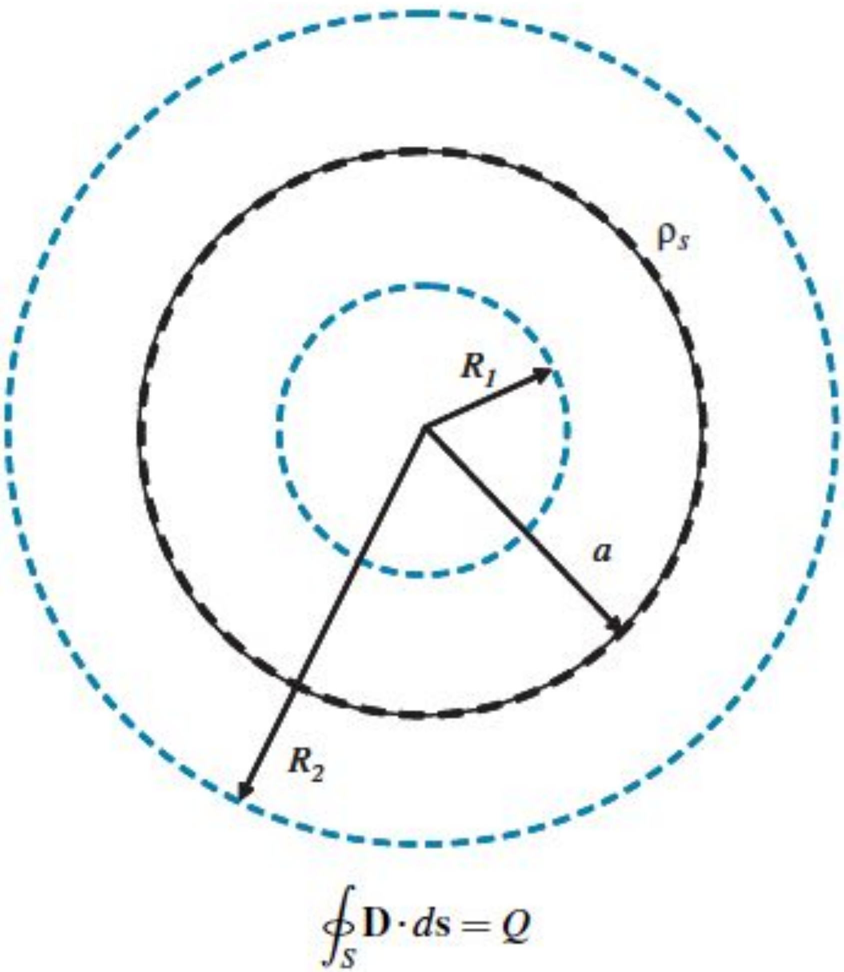
$$D_R R^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta = Q_{\text{enclosed}}$$

$$D_R R^2 2\pi \Big[-\cos\theta \Big]_{\theta=0}^{\pi} = Q_{\text{enclosed}}$$

$$D_R R^2 2\pi 2 = Q_{\text{enclosed}}$$

$$\boxed{D_R = \frac{Q_{\text{enclosed}}}{4\pi R^2}}$$

Exercise 4-8



For the surface inside the spherical shell, $Q_{enclosed}$=0, so **D**=0, and **E**=0.

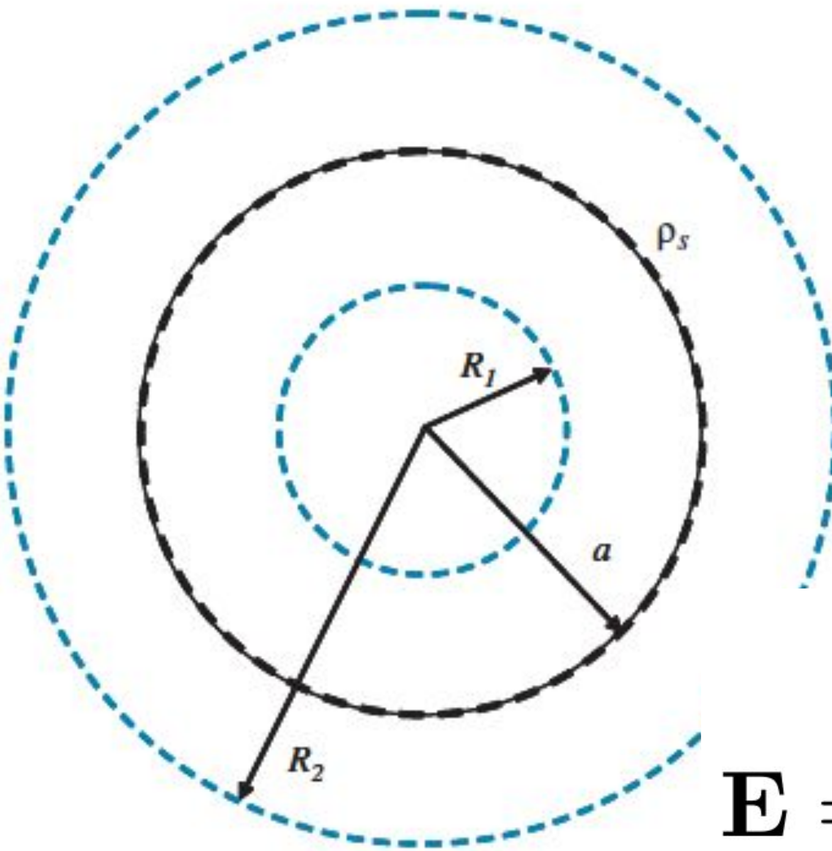
For the surface outside the spherical shell:

$Q_{enclosed} = \rho_s A = \rho_s \, 4\pi a^2$

So:

$$D_R = \frac{\rho_s 4\pi a^2}{4\pi R^2} = \frac{\rho_s a^2}{R^2}$$

Exercise 4-8



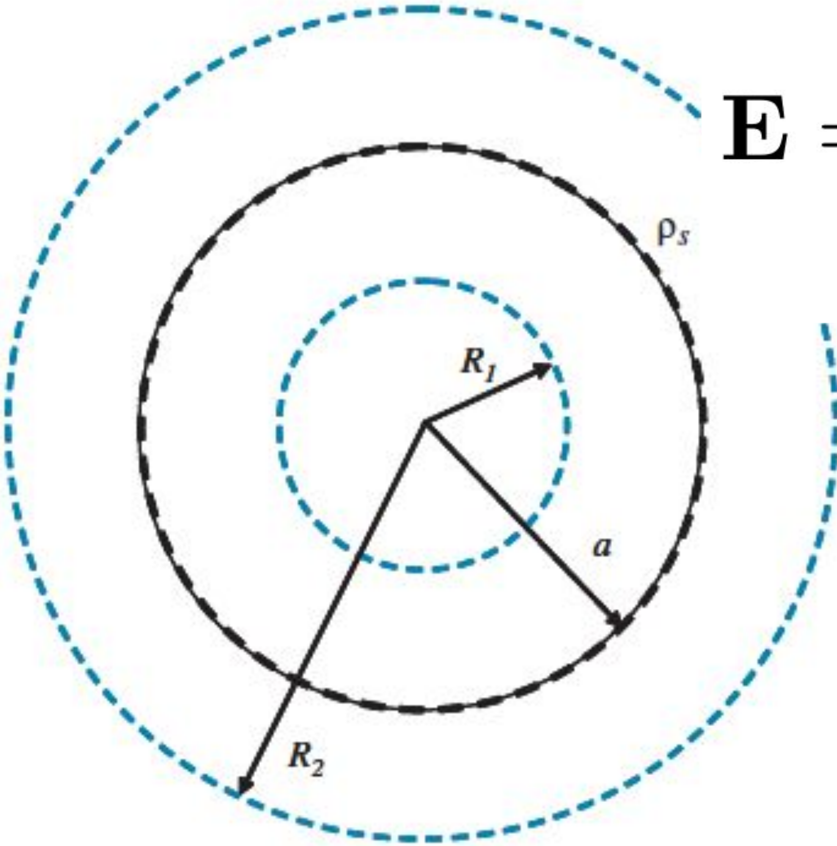
For outside the spherical shell:

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\hat{\mathbf{R}}}{\varepsilon} D_r = \hat{\mathbf{R}} \frac{\rho_s a^2}{\varepsilon R^2}$$

The complete solution:

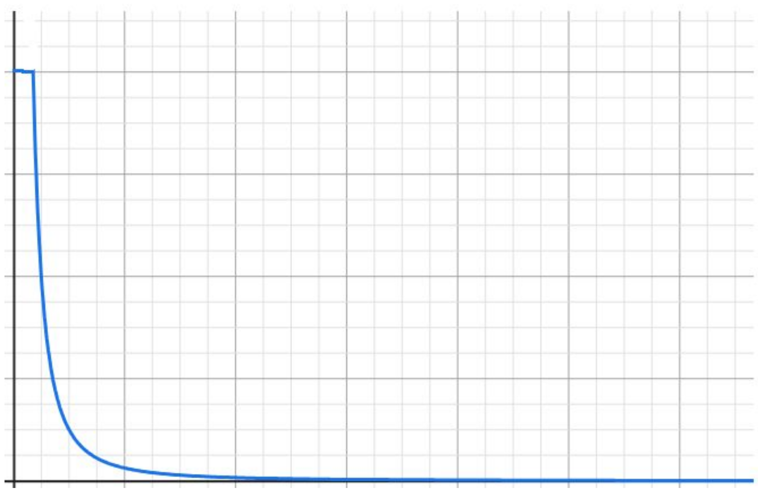
$$\mathbf{E} = \begin{cases} 0 & : R < a \\ \hat{\mathbf{R}} \dfrac{\rho_s a^2}{\epsilon R^2} & : R > a \end{cases}$$

Exercise 4-8



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

$$\mathbf{E} = \begin{cases} 0 & : R < a \\ \hat{\mathbf{R}}\,\dfrac{\rho_s a^2}{\epsilon R^2} & : R > a \end{cases}$$



Homework

Homework 12 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Section 4-5:

Electric Potential