EECS 230
ENGINEERING ELECTROMAGNETICS
Leland Pierce

Vector Calculus

University of Michigan - Fall 2024

Announcements:

Exam1:

next monday, oct 7, 5:30-7PM, CHRYS220
or
next tuesday, oct 8, 5:30-7PM, EWRE185

bring:

calculator
ruler
compass
1-sheet handwritten

provided: test, paper, smith chart, 3-pg eqns

Prerequisite: MATH 215

This class assumes that you have had MATH 215 or equivalent.

This means:

Textbook:
Multivariable Calculus by James Stewart

Coverage:
Chapters 12 through 16

Chapter 3 Overview

Vector Algebra:
- **dot product**
- cross product
- triple product

Coordinate System
- Cartesian
- Cylindrical
- Spherical

Coordinate Transforms
- Cartesian to Cylindrical
- Cartesian to Spherical

Gradient

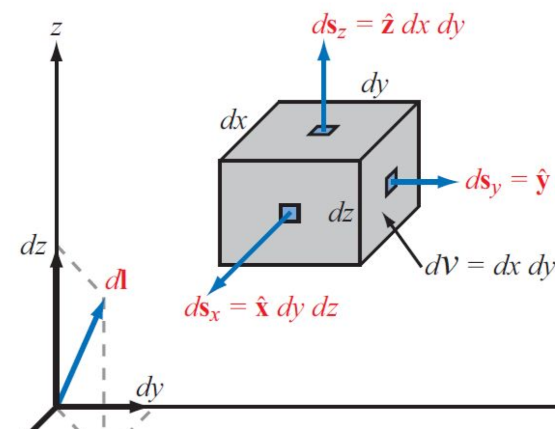
Divergence

Curl
- Stokes' Theorem

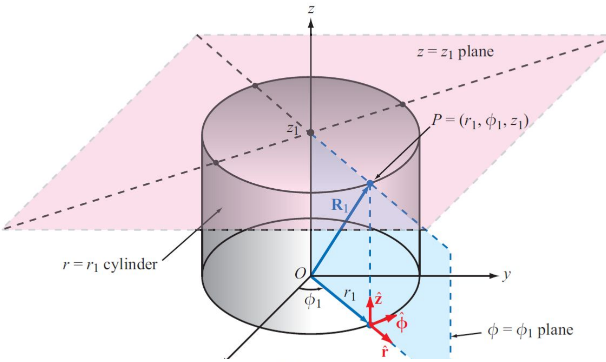
Laplacian

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta_{AB}$$

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{n}}\,AB\sin\theta_{A}$$



$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \qquad \text{(Cartesian).}$$

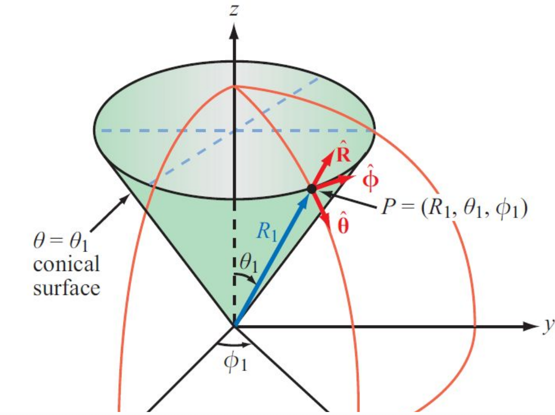


$$\nabla\cdot\mathbf{E} = \text{div}\,\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\int_{\mathcal{V}} \nabla\cdot\mathbf{E}\,d\mathcal{V} = \oint_S \mathbf{E}\cdot d\mathbf{s}.$$

(divergence theorem)

$$\nabla\times\mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$



$$\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s} = \oint_C \mathbf{B}\cdot d\mathbf{l}.$$

$$\nabla^2 V = \nabla\cdot(\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

Lecture Coverage

Today's lecture:

Sections 3-1 thru 3-7 of the book:

3-1: Vector Algebra
3-2: Coordinate Systems
3-3: Coordinate System Transformations
3-4: Gradient
3-5: Divergence
3-6: Curl
3-7: Laplacian

Vector Multiplication: Scalar Product or "Dot Product"

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}$$

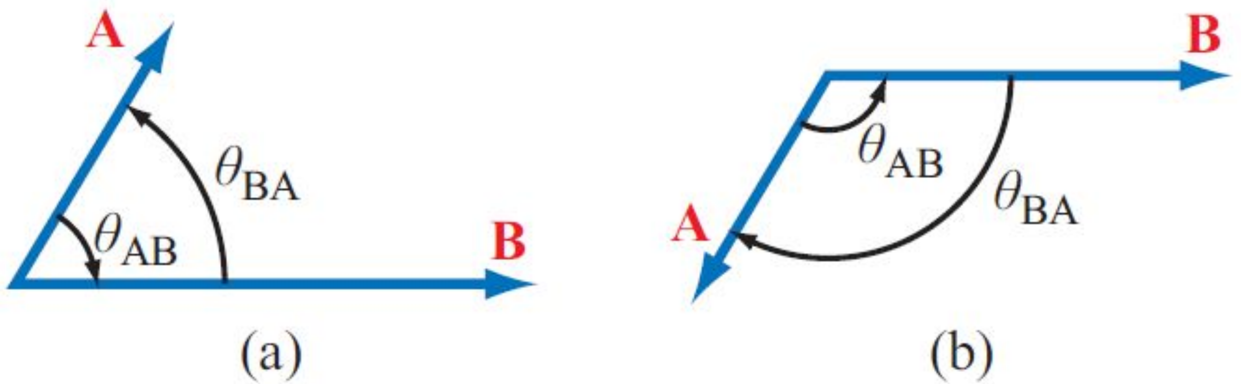


Figure 3-5: The angle θ_{AB} is the angle between **A** and **B**, measured from **A** to **B** between vector tails. The dot product is positive if $0 \le \theta_{AB} < 90°$, as in (a), and it is negative if $90° < \theta_{AB} \le 180°$, as in (b).

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative property)},$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive property)}$$

Vector Multiplication: Scalar Product or "Dot Product"

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta_{AB}$$

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1,$$
$$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0.$$

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A}\cdot\mathbf{A}}$$

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A}\cdot\mathbf{B}}{\sqrt[+]{\mathbf{A}\cdot\mathbf{A}}\ \sqrt[+]{\mathbf{B}\cdot\mathbf{B}}}\right]$$

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

Vector Multiplication: Vector Product or "Cross Product"

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}$$

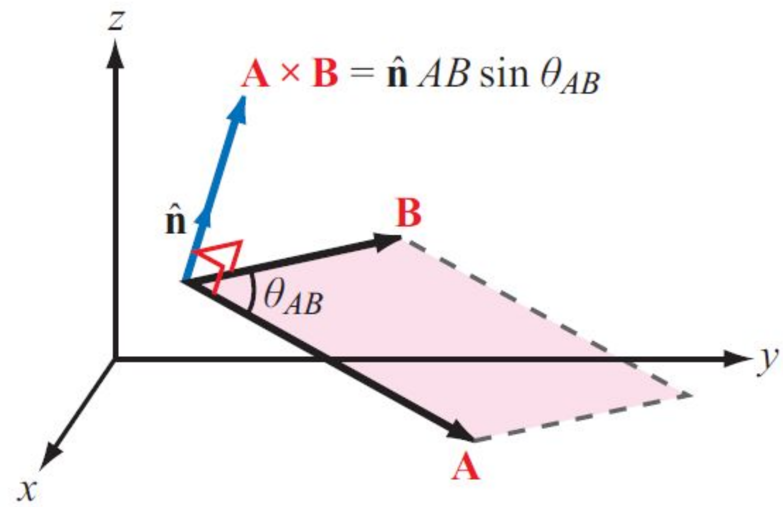
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

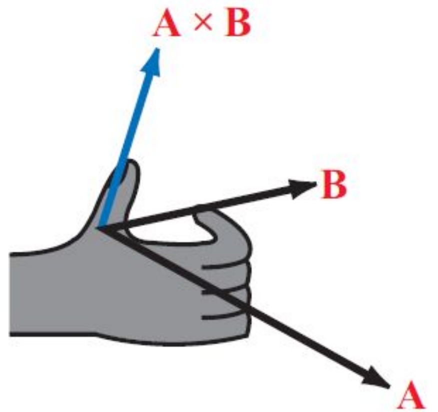
$$\mathbf{A} \times \mathbf{A} = 0$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}. \qquad (3.25)$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$



(a) Cross product



(b) Right-hand rule

Triple Products

Scalar Triple Product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}),$$

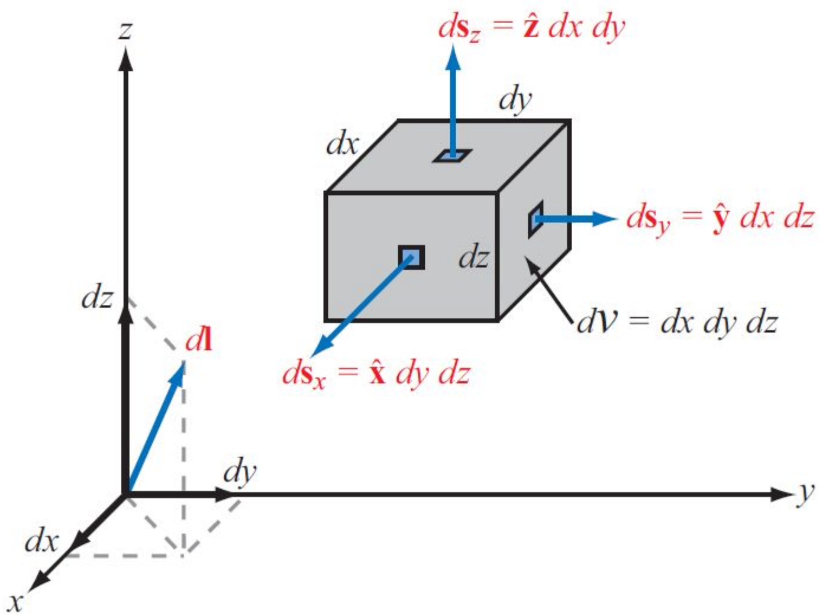
which is known as the "bac-cab" rule.

Cartesian Coordinate System

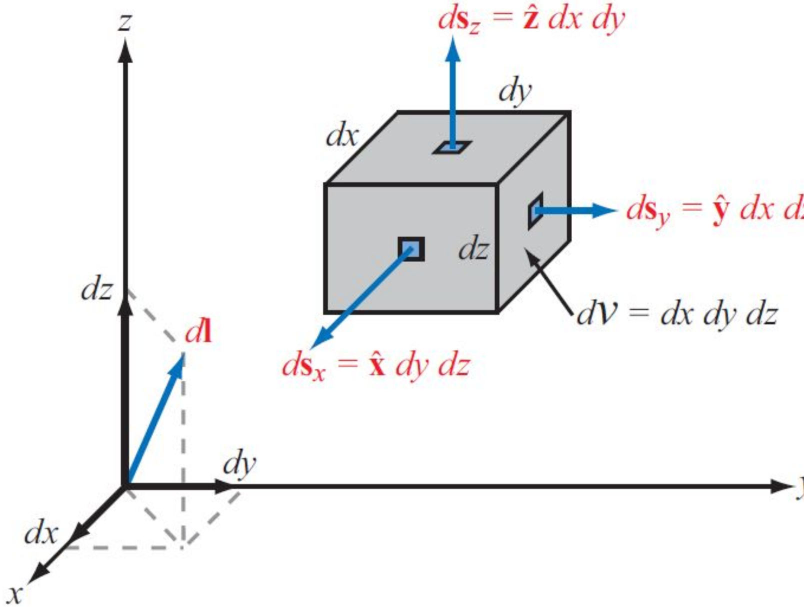
Differential length vector

$$d\mathbf{l} = \hat{\mathbf{x}}\, dl_x + \hat{\mathbf{y}}\, dl_y + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz, \quad (3.34)$$

where $dl_x = dx$ is a differential length along $\hat{\mathbf{x}}$, and similar interpretations apply to $dl_y = dy$ and $dl_z = dz$.



Cartesian Coordinate System



Differential area vectors

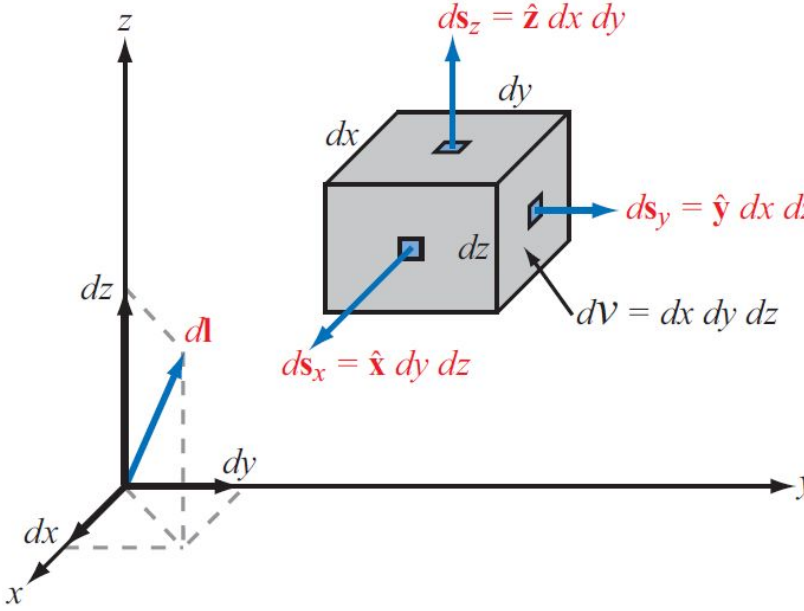
$$d\mathbf{s}_x = \hat{\mathbf{x}}\, dl_y\, dl_z = \hat{\mathbf{x}}\, dy\, dz \qquad (y\text{–}z \text{ plane}), \qquad (3.35a)$$

with the subscript on $d\mathbf{s}$ denoting its direction. Similarly,

$$d\mathbf{s}_y = \hat{\mathbf{y}}\, dx\, dz \qquad (x\text{–}z \text{ plane}), \qquad (3.35b)$$

$$d\mathbf{s}_z = \hat{\mathbf{z}}\, dx\, dy \qquad (x\text{–}y \text{ plane}). \qquad (3.35c)$$

Cartesian Coordinate System

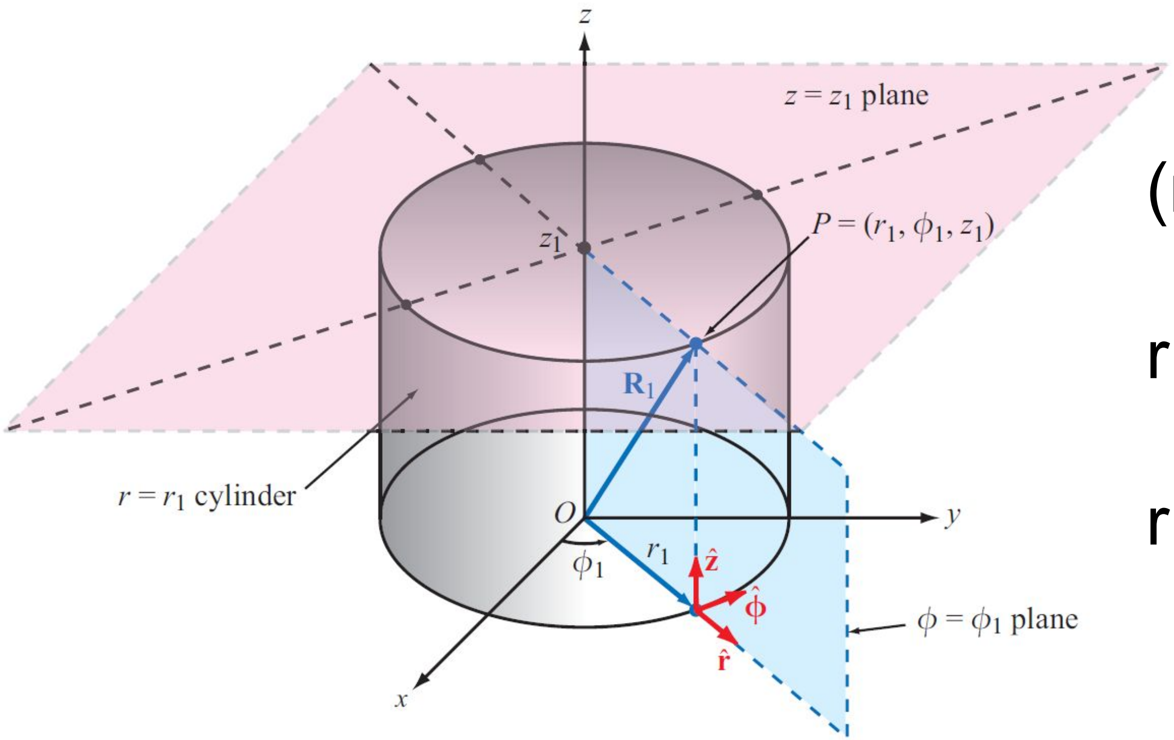


Differential Volume

A ***differential volume*** equals the product of all three differential lengths:

$$dV = dx\; dy\; dz. \qquad (3.36)$$

Cylindrical Coordinate System



(r, ϕ, z)

r is sometimes called ρ

r and ϕ are in x-y plane

Cylindrical Coordinate System

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z,$$

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \qquad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}},$$

$$d\mathbf{l} = \hat{\mathbf{r}}\, dr + \hat{\boldsymbol{\phi}} r\, d\phi + \hat{\mathbf{z}}\, dz$$

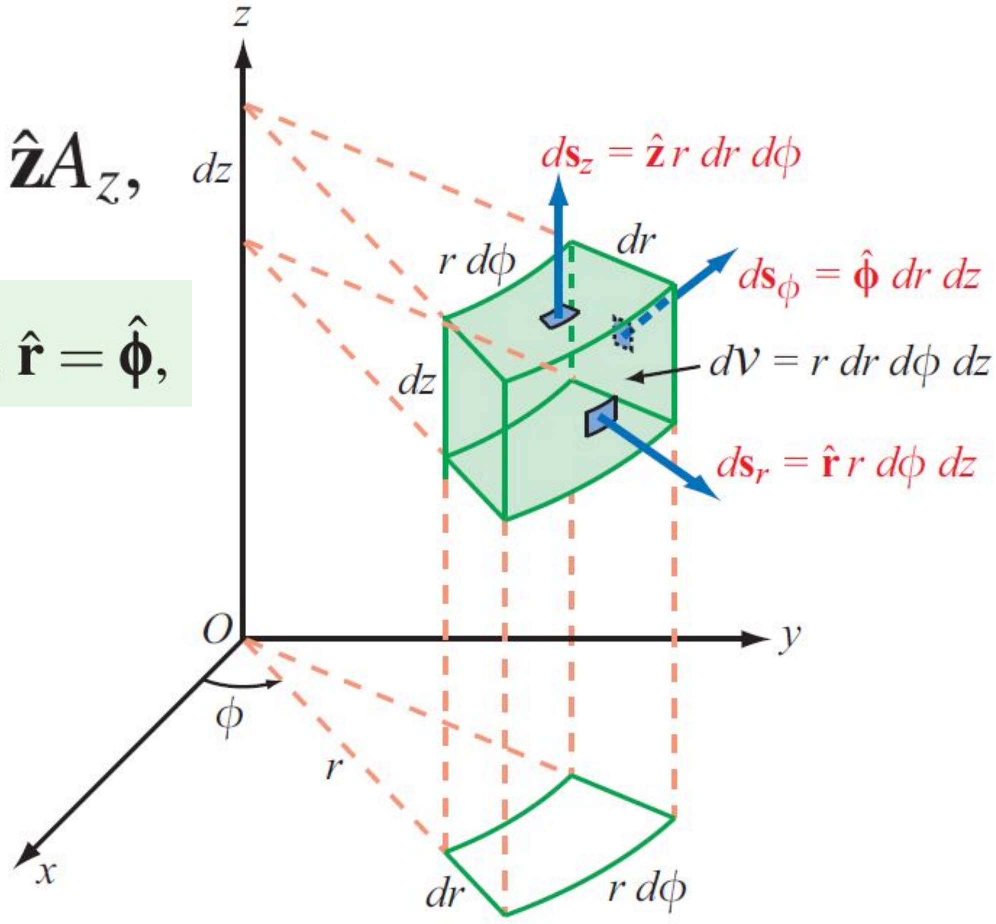
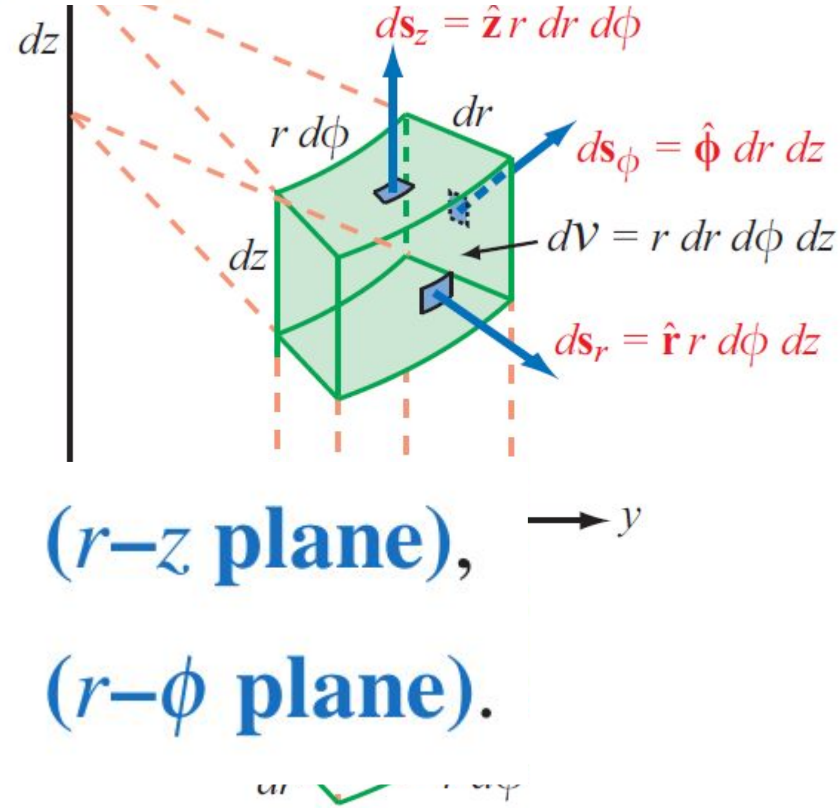


Figure 3-10: Differential areas and volume in cylindrical coordinates.

Cylindrical Coordinate System

$$d\mathbf{s}_r = \hat{\mathbf{r}}\, dl_\phi\, dl_z = \hat{\mathbf{r}} r\, d\phi\, dz \quad (\phi\text{–}z \text{ cylindrical surface}),$$



$$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, dl_r\, dl_z = \hat{\boldsymbol{\phi}}\, dr\, dz \quad (r\text{–}z \text{ plane}),$$

$$d\mathbf{s}_z = \hat{\mathbf{z}}\, dl_r\, dl_\phi = \hat{\mathbf{z}} r\, dr\, d\phi \quad (r\text{–}\phi \text{ plane}).$$

$$d\mathcal{V} = r\, dr\, d\phi\, dz$$

Figure 3-10: Differential areas and volume in cylindrical coordinates.

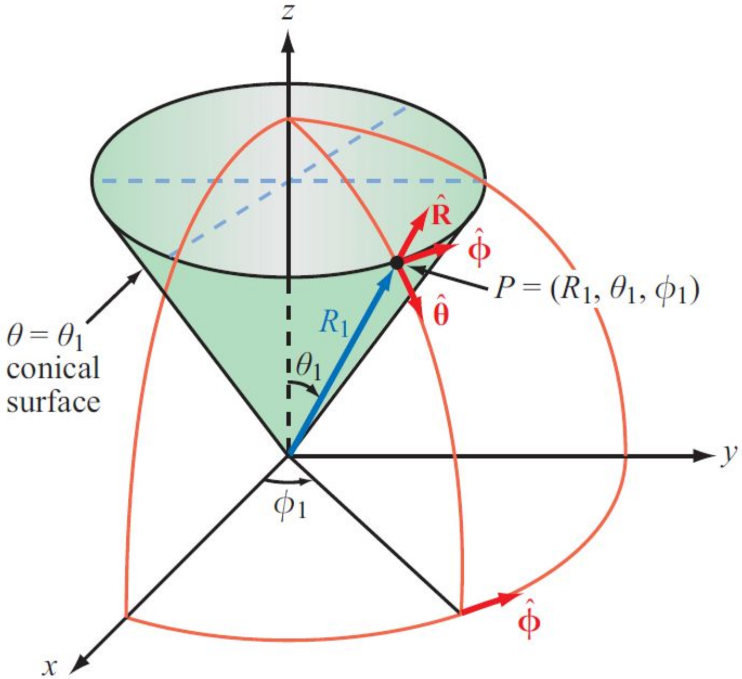
Spherical Coordinate System

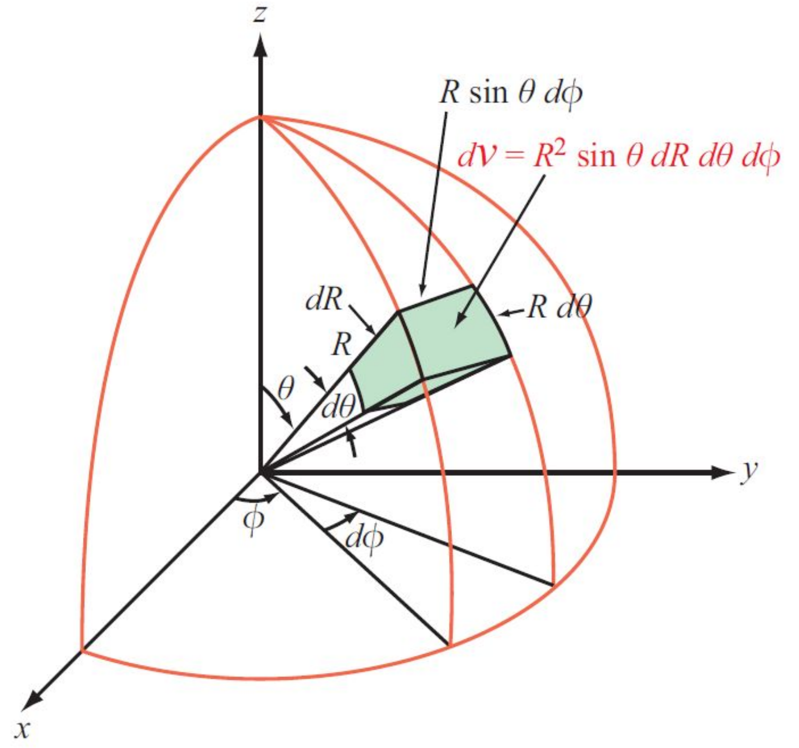
(R, θ, ϕ)

R (to distinguish from r)
radial in all directions

θ angle from "up" (+z-axis)

ϕ same as in cylindrical



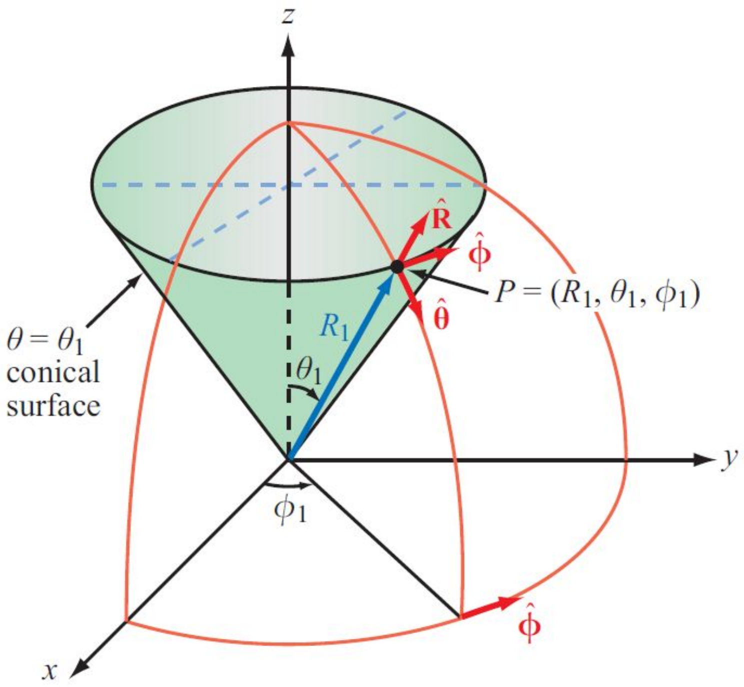


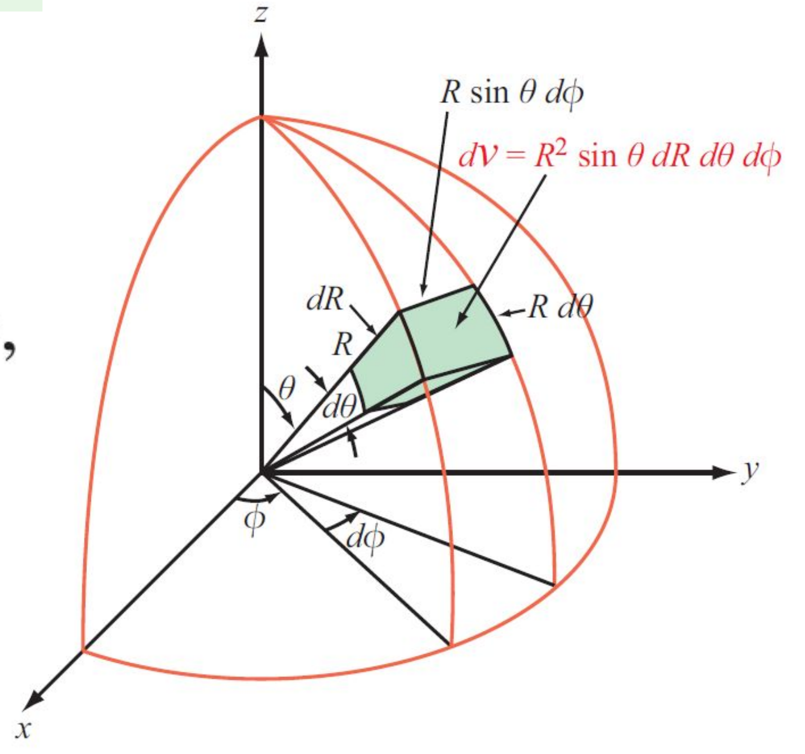
Spherical Coordinate System

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi,$$

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}.$$

$$d\mathbf{l} = \hat{\mathbf{R}}\,dl_R + \hat{\boldsymbol{\theta}}\,dl_\theta + \hat{\boldsymbol{\phi}}\,dl_\phi$$
$$= \hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi,$$





Spherical Coordinate System

$$d\mathbf{s}_R = \hat{\mathbf{R}}\, dl_\theta\, dl_\phi = \hat{\mathbf{R}} R^2 \sin\theta\, d\theta\, d\phi$$

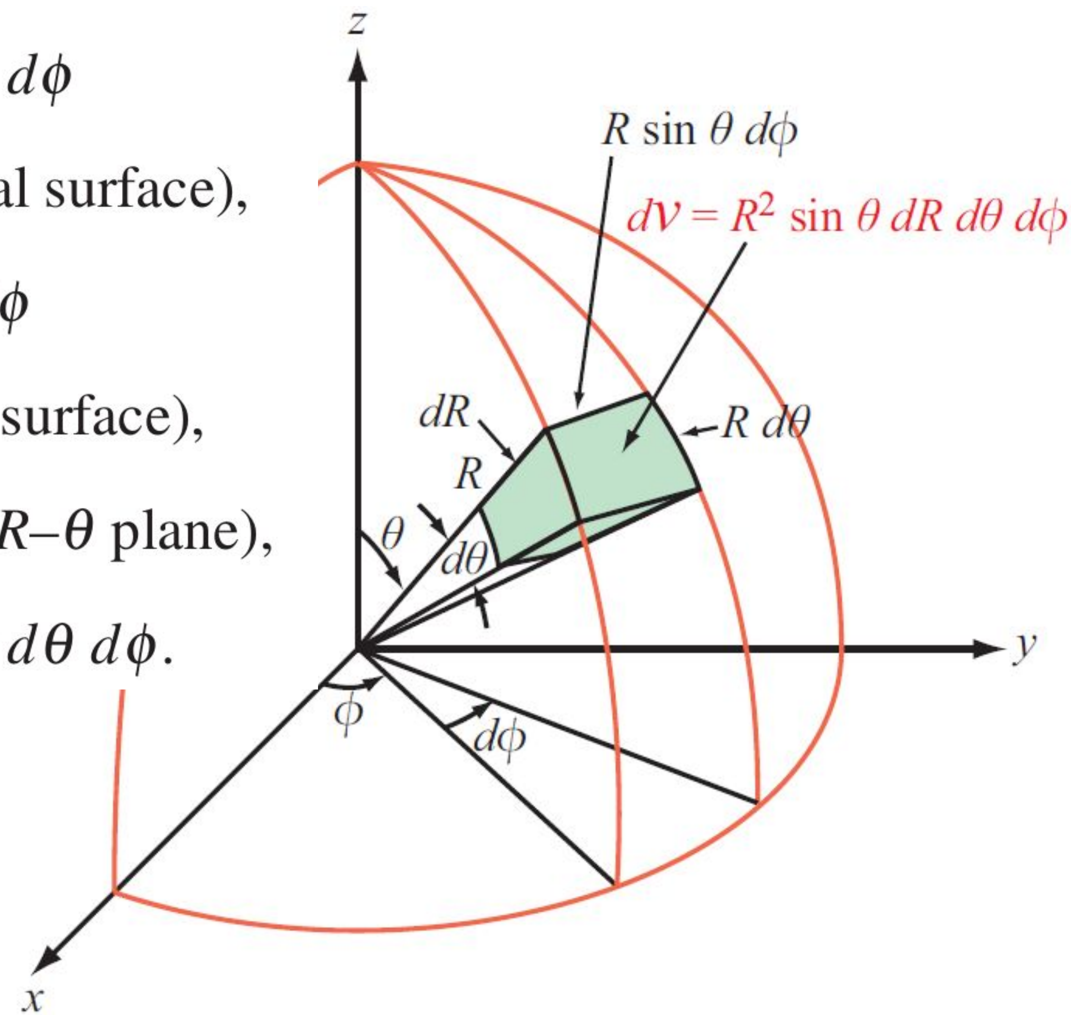
(θ–ϕ spherical surface),

$$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\, dl_R\, dl_\phi = \hat{\boldsymbol{\theta}} R \sin\theta\, dR\, d\phi$$

(R–ϕ conical surface),

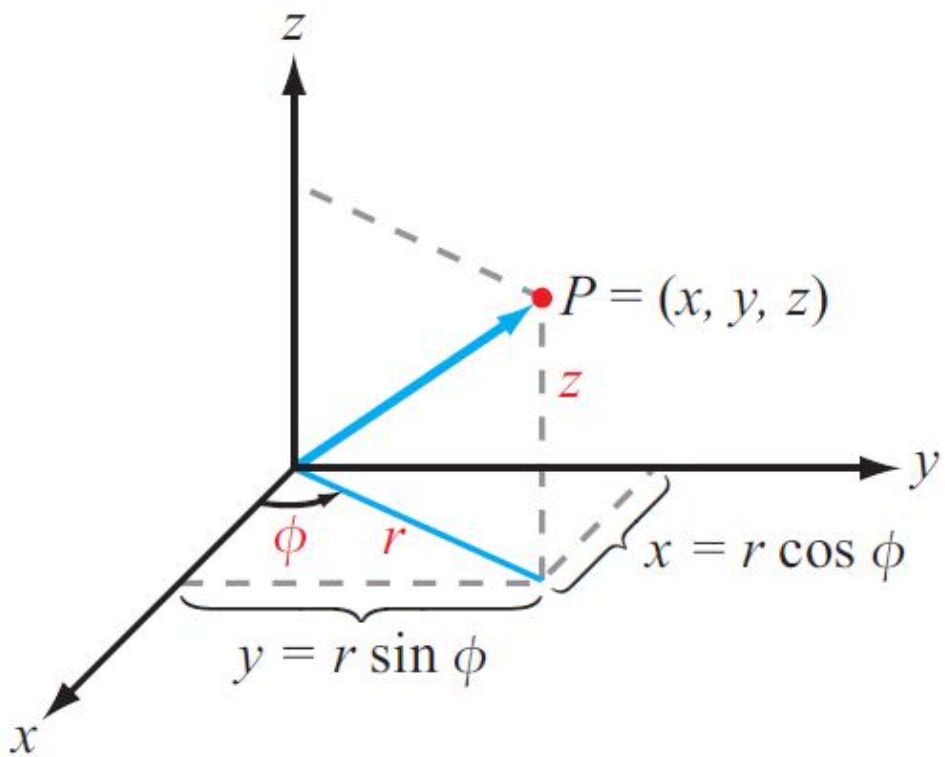
$$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, dl_R\, dl_\theta = \hat{\boldsymbol{\phi}} R\, dR\, d\theta \quad (R\text{–}\theta \text{ plane}),$$

$$d\upsilon = dl_R\, dl_\theta\, dl_\phi = R^2 \sin\theta\, dR\, d\theta\, d\phi.$$



Coordinate Transformations: Coordinates

Cylindrical



$$r = \sqrt[+]{x^2 + y^2},$$

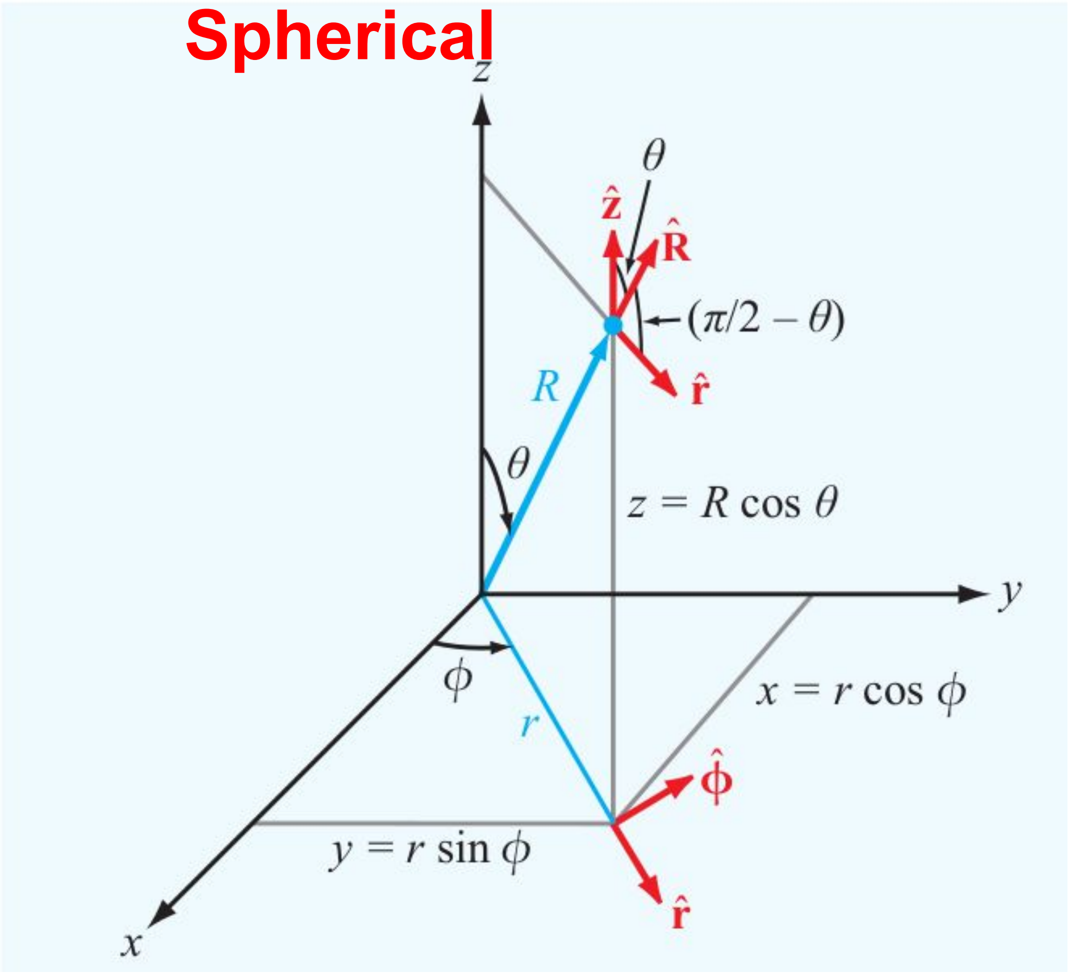
$$\phi = \tan^{-1}\left(\frac{y}{x}\right),$$

$$x = r\cos\phi,$$

$$y = r\sin\phi.$$

$$z = z$$

Coordinate Transformations: Coordinates



$$R = \sqrt[+]{x^2 + y^2 + z^2},$$

$$\theta = \tan^{-1}\left[\frac{\sqrt[+]{x^2 + y^2}}{z}\right],$$

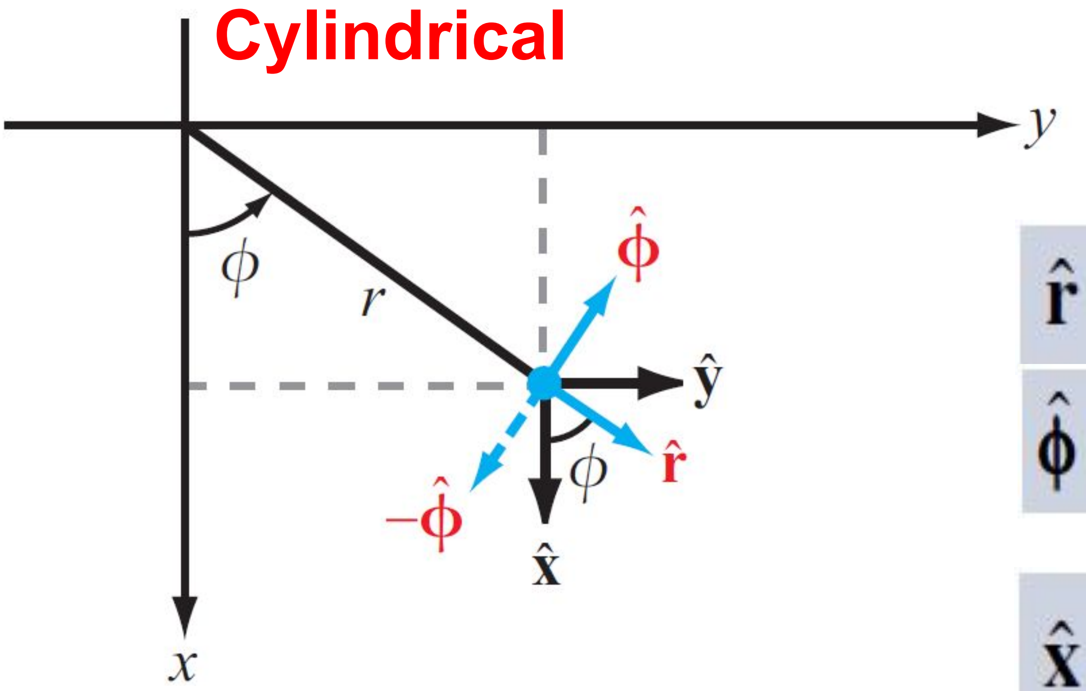
$$\phi = \tan^{-1}\left(\frac{y}{x}\right).$$

$$x = R\sin\theta\cos\phi,$$

$$y = R\sin\theta\sin\phi,$$

$$z = R\cos\theta.$$

Coordinate Transformations: Unit Vectors



$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi.$$

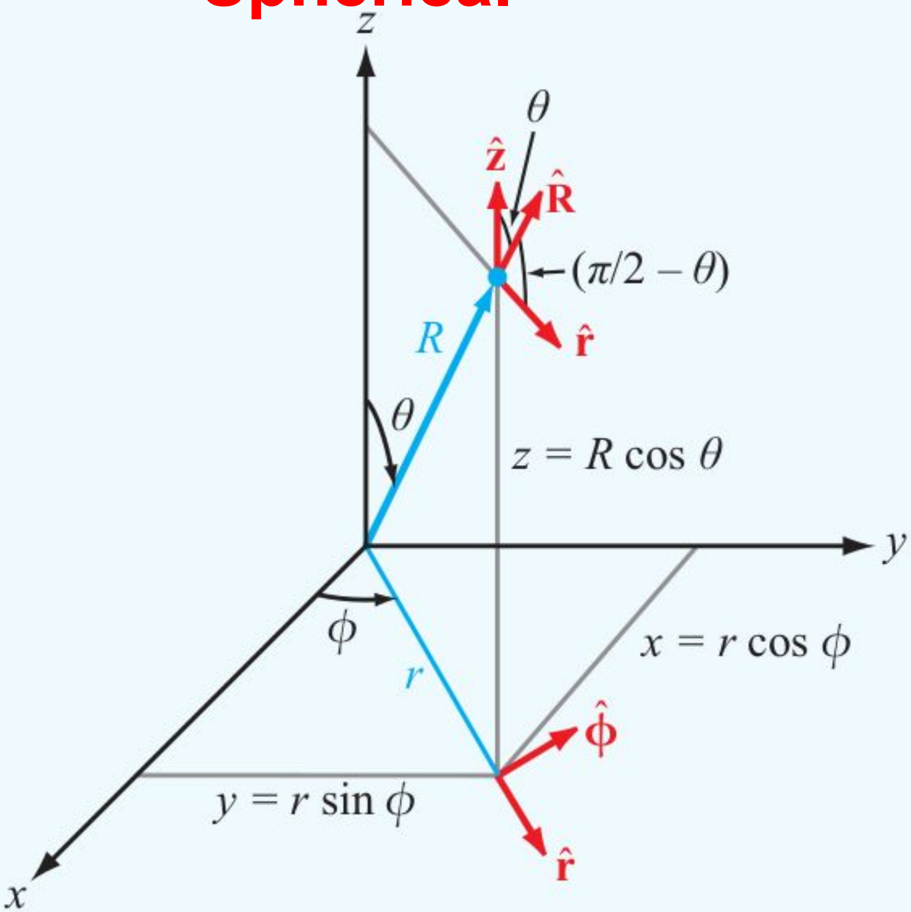
$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi.$$

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi,$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi.$$

Coordinate Transformations: Unit Vectors

Spherical



$$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta \cos\phi + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta.$$

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta \cos\phi + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta.$$

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi.$$

$$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta \cos\phi + \hat{\boldsymbol{\theta}} \cos\theta \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi,$$

$$\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta \sin\phi + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi,$$

$$\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta.$$

Distance Between 2 Points

$$d = |\mathbf{R}_{12}|$$
$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \quad (3.66)$$

$$d = \left[(r_2 \cos\phi_2 - r_1 \cos\phi_1)^2 + (r_2 \sin\phi_2 - r_1 \sin\phi_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$
$$= \left[r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2}$$
$$\text{(cylindrical).} \quad (3.67)$$

$$d = \left\{R_2^2 + R_1^2 - 2R_1 R_2[\cos\theta_2 \cos\theta_1 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\right\}^{1/2}$$
$$\text{(spherical).} \quad (3.68)$$

Vectors in 3 Coordinate Systems

Table 3-1 **Summary of vector relations.**

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vector properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $d\upsilon =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Transformations in 3 Coordinate Systems

Table 3-2 Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

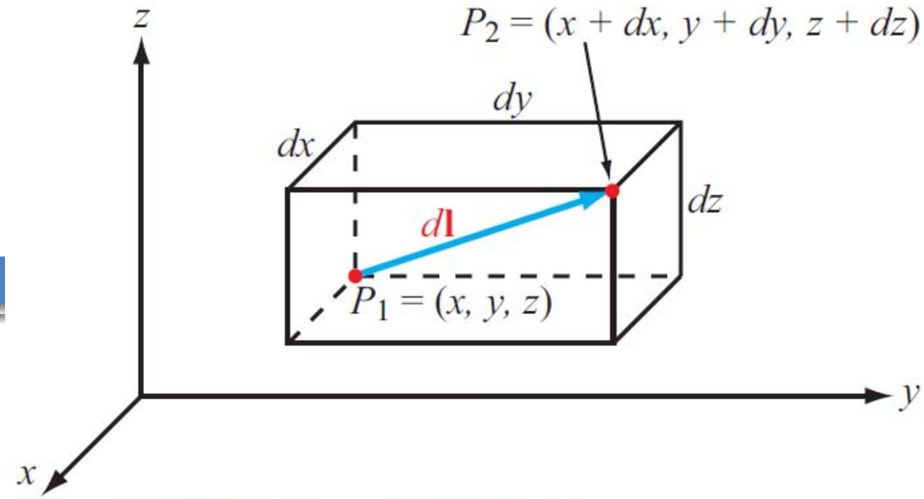
Coordinate Systems: Notation

There are different ways of naming the coordinate directions, and naming of unit vectors.

In this class you must use the notation from the book, as presented in this lecture.

It is wrong if you don't.

Gradient of A Scalar Field



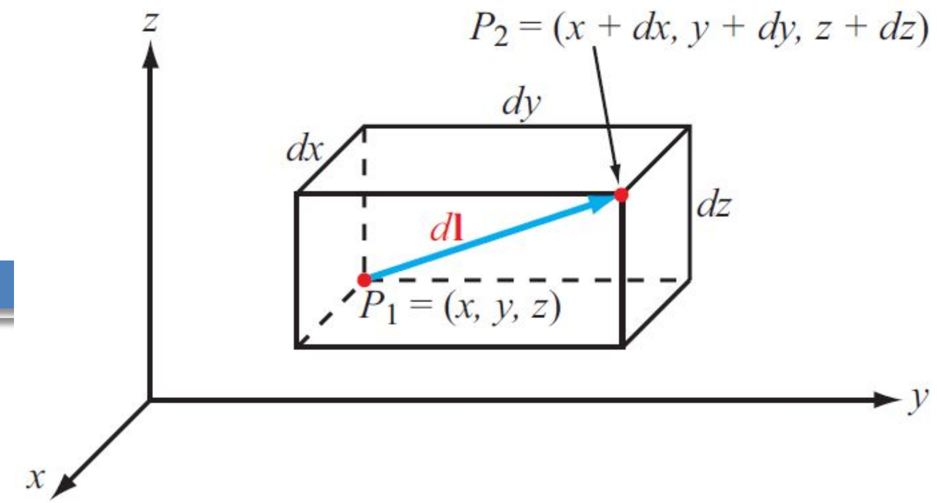
From differential calculus, the temperature difference between points P_1 and P_2, $dT = T_2 - T_1$, is

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz. \qquad (3.70)$$

Because $dx = \hat{\mathbf{x}} \cdot d\mathbf{l}$, $dy = \hat{\mathbf{y}} \cdot d\mathbf{l}$, and $dz = \hat{\mathbf{z}} \cdot d\mathbf{l}$, Eq. (3.70) can be rewritten as

$$\begin{aligned} dT &= \hat{\mathbf{x}}\frac{\partial T}{\partial x} \cdot d\mathbf{l} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} \cdot d\mathbf{l} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \cdot d\mathbf{l} \\ &= \left[\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right] \cdot d\mathbf{l}. \qquad (3.71) \end{aligned}$$

Gradient of A Scalar Field



$$\nabla T = \text{grad } T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} . \qquad (3.72)$$

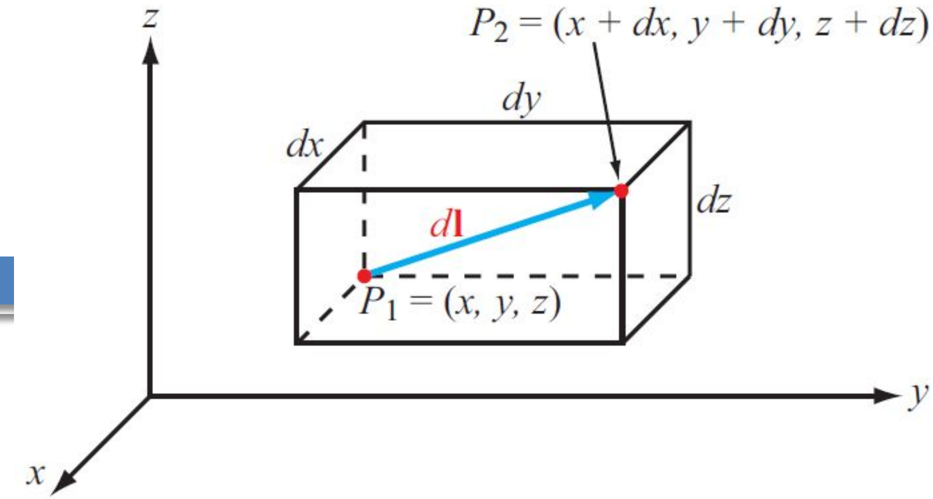
Equation (3.71) can then be expressed as

$$dT = \nabla T \cdot d\mathbf{l}. \qquad (3.73)$$

The symbol ∇ is called the ***del*** or ***gradient operator*** and is defined as

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \qquad \text{(Cartesian).} \qquad (3.74)$$

Gradient of A Scalar Field



$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \qquad \text{(Cartesian).} \qquad (3.74)$$

- The gradient operator itself has no physical meaning
- It attains a physical meaning once it operates on a scalar quantity.

- The result of the operation is a vector with **magnitude** equal to the maximum rate of change of the physical quantity per unit distance and **pointing** in the direction of maximum increase.

Gradient: Directional Derivative

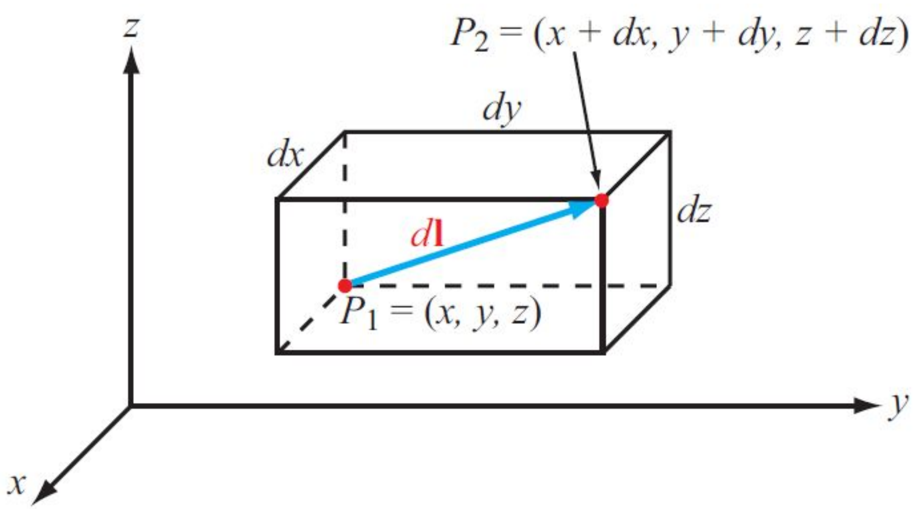
Definition of directional derivative:

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l. \qquad (3.75)$$

where:

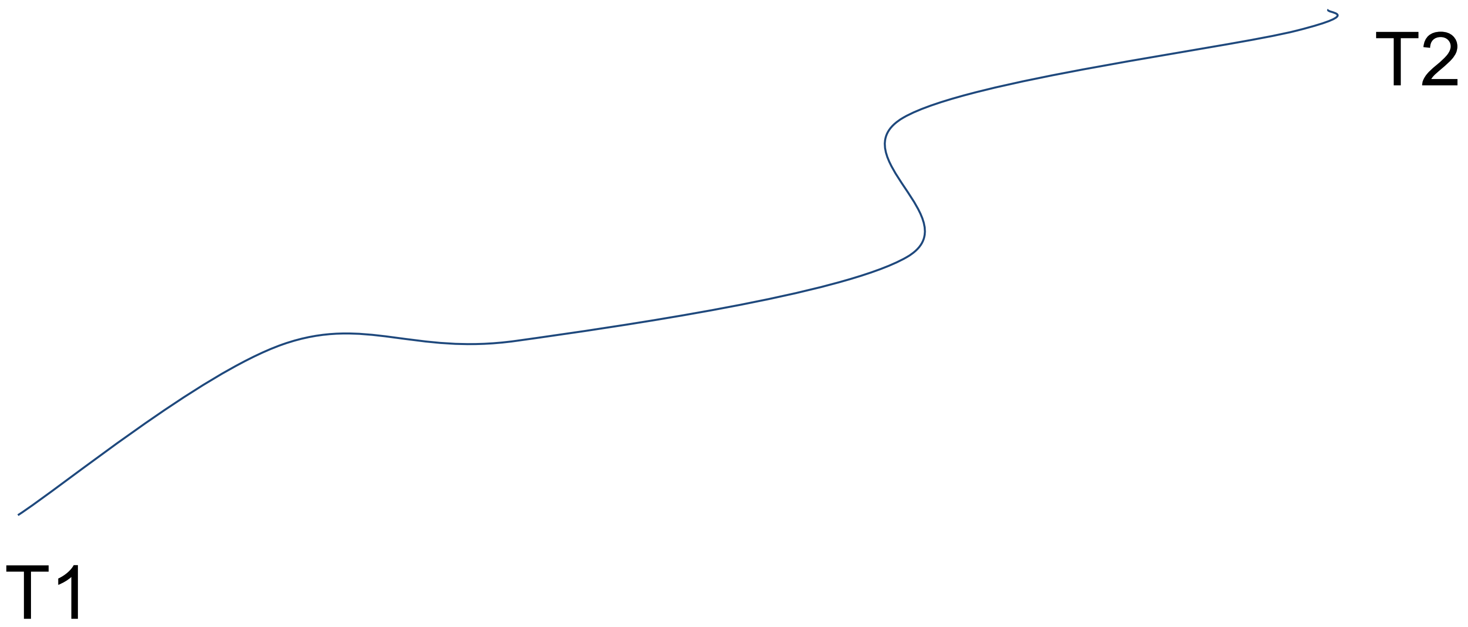
$$d\mathbf{l} = \hat{\mathbf{a}}_l \, dl,$$

is the direction along which we want the derivative:



Gradient: Directional Derivative

Use the directional derivative to find the difference in values between 2 ends of a path:



For some fields, T, the path matters

Gradient: Directional Derivative

Since we know: $dT = \nabla T \cdot d\mathbf{l}.$

integrating both sides:

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\mathbf{l}.$$

Example: Directional Derivative

Given: $T = x^2 + y^2 z$

Find: directional deriv along ***l*** = $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$, evaluate it at the point (1,-1,2)

Solution:

Step1: find the gradient of *T*:

$$\nabla T = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) (x^2 + y^2 z)$$

$$= \hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2.$$

Example: Directional Derivative

Solution:

Step2: find the unit vector of l: $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$

$$\hat{\mathbf{a}}_l = \frac{\mathbf{l}}{|\mathbf{l}|} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}$$

Example: Directional Derivative

Solution:

Step3: find the directional derivative:

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l = (\hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2) \cdot \left(\frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}} \right)$$

$$\boxed{\frac{dT}{dl} = \frac{4x + 6yz - 2y^2}{\sqrt{17}}}$$

Example: Directional Derivative

Solution:

Step4: evaluate the directional derivative:

$$\frac{dT}{dl} = \frac{4x + 6yz - 2y^2}{\sqrt{17}}$$

At (1, −1, 2),

$$\left.\frac{dT}{dl}\right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}} .$$

$$= \boxed{-2.43}$$

Divergence of a Vector Field

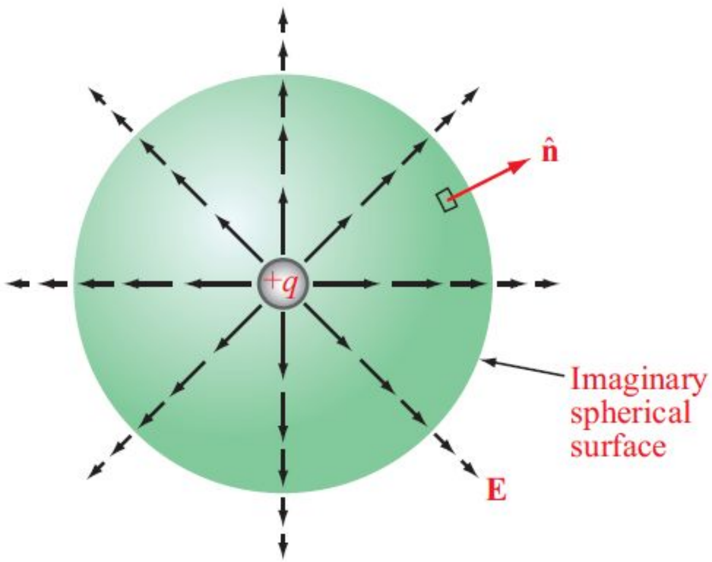


Figure 3-20: Flux lines of the electric field **E** due to a positive charge q.

Total flux crossing a closed surface, *S*, such as a sphere:

$$\text{Total flux} = \oint_S \mathbf{E} \cdot d\mathbf{s}.$$

where **E** is a vector field.

$d\mathbf{s} = \hat{\mathbf{n}}\, ds$

$\hat{\mathbf{n}}$ = outward normal

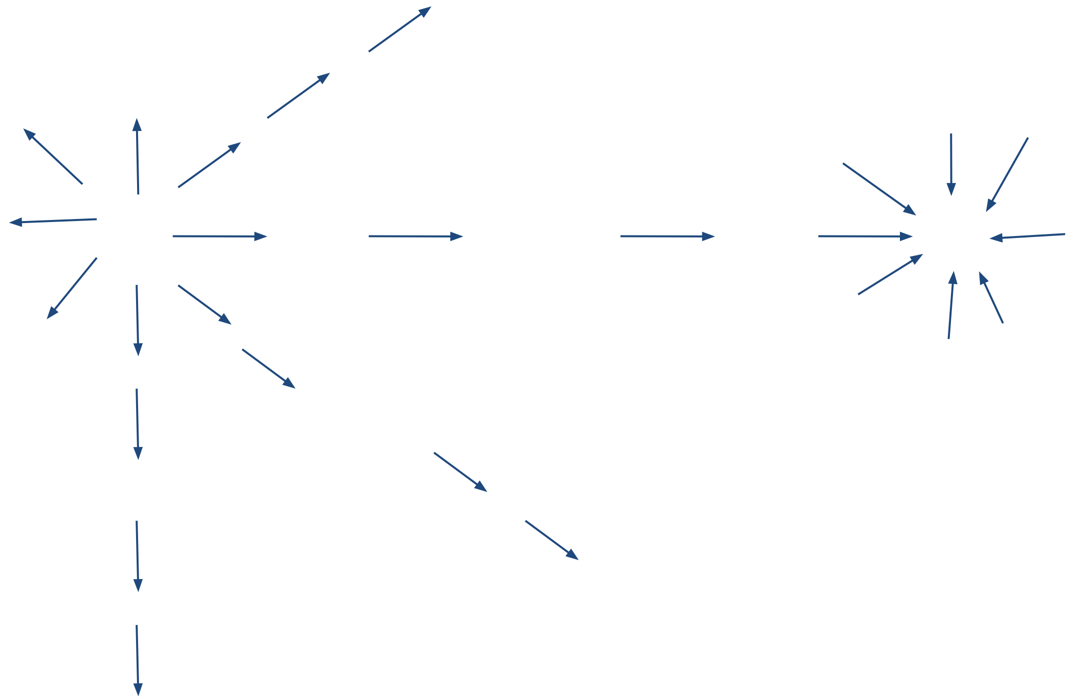
ds = differential surf area

Divergence of a Vector Field

Example in 2D:

Underground pipe with water flowing out of it.

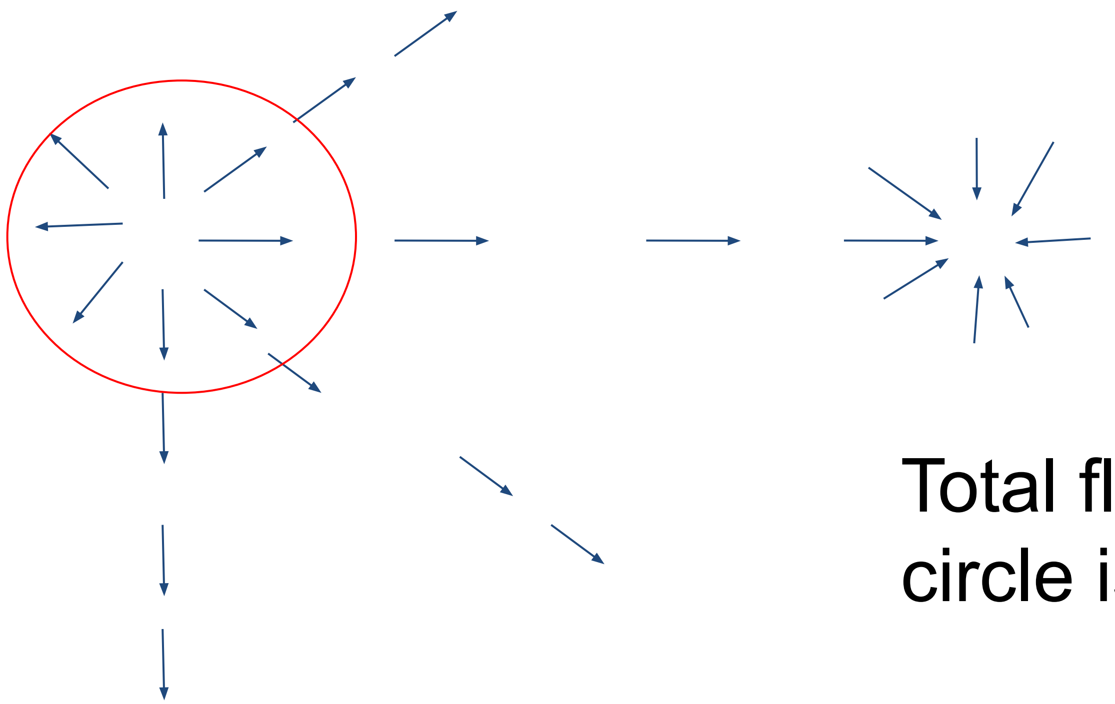
Drain nearby. lines of flow:



Divergence of a Vector Field

Example in 2D:

Underground pipe with water flowing out of it. Drain nearby. lines of flow:



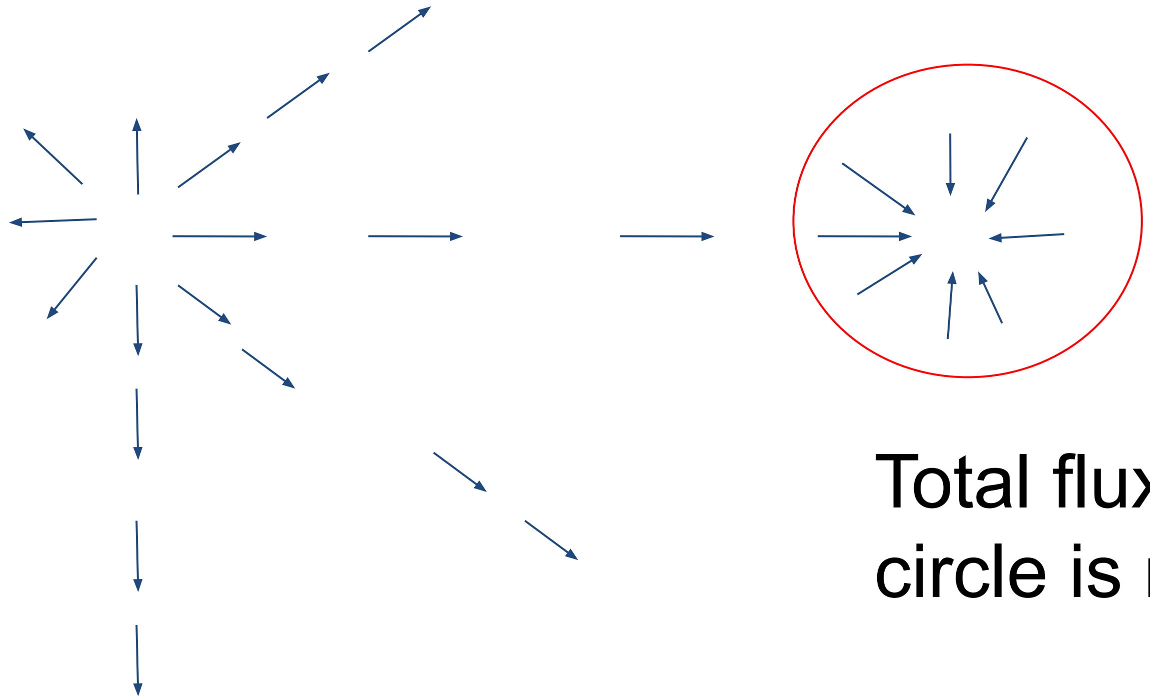
Total flux around red circle is positive

Divergence of a Vector Field

Example in 2D:

Underground pipe with water flowing out of it.

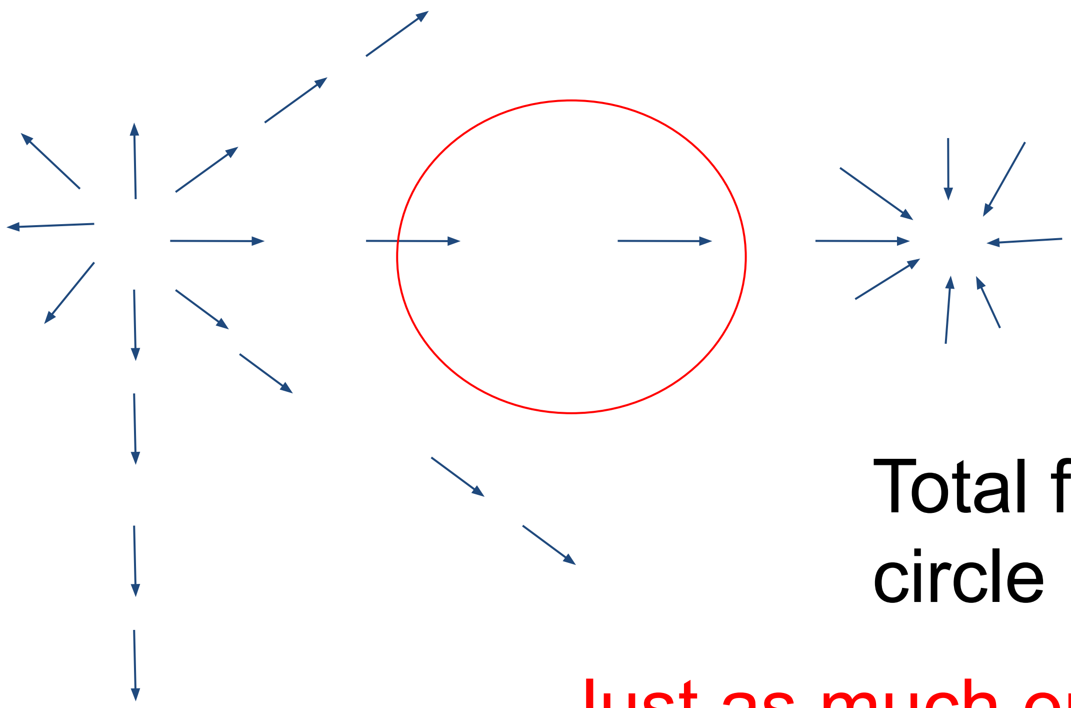
Drain nearby. lines of flow:



Total flux around red circle is negative

Divergence of a Vector Field

Example in 2D:
Underground pipe with water flowing out of it.
Drain nearby. lines of flow:



Total flux around red circle is zero

Just as much entering as leaving

Divergence of a Vector Field

Intuition:
Total flux is positive for a source
Total flux is negative for a sink
Total flux is zero otherwise.

Relation to Electromagnetics:
Source of **E** field is +charges
So: positive total flux: enclosed +charges
negative total flux: enclosed -charges
zero total flux: no enclosed charges

Divergence of a Vector Field

Divergence is defined as the Total Flux per volume:

$$\operatorname{div} \mathbf{E} \triangleq \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta V},$$

which can be expressed as:

$$\nabla \cdot \mathbf{E} = \operatorname{div} \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Divergence Theorem

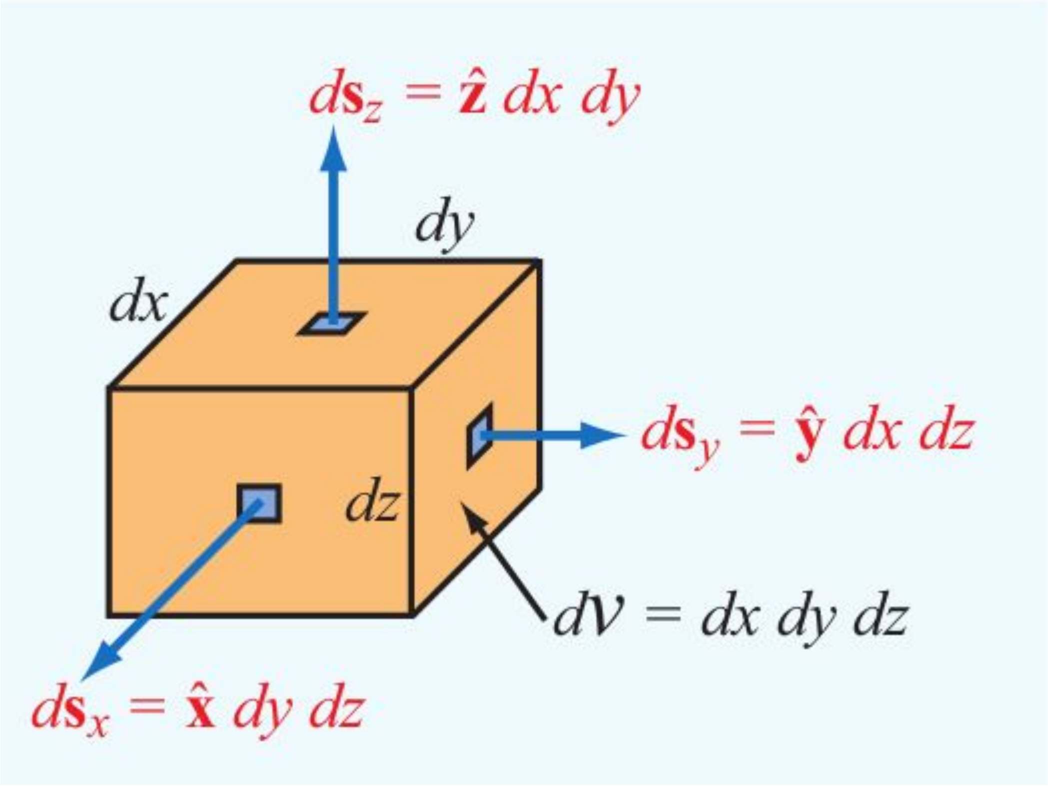
From previous equation we can say:

$$\int_{v} \nabla \cdot \mathbf{E} \, dv = \oint_{S} \mathbf{E} \cdot d\mathbf{s}.$$

We will use this equation in order to solve problems in the following chapters.

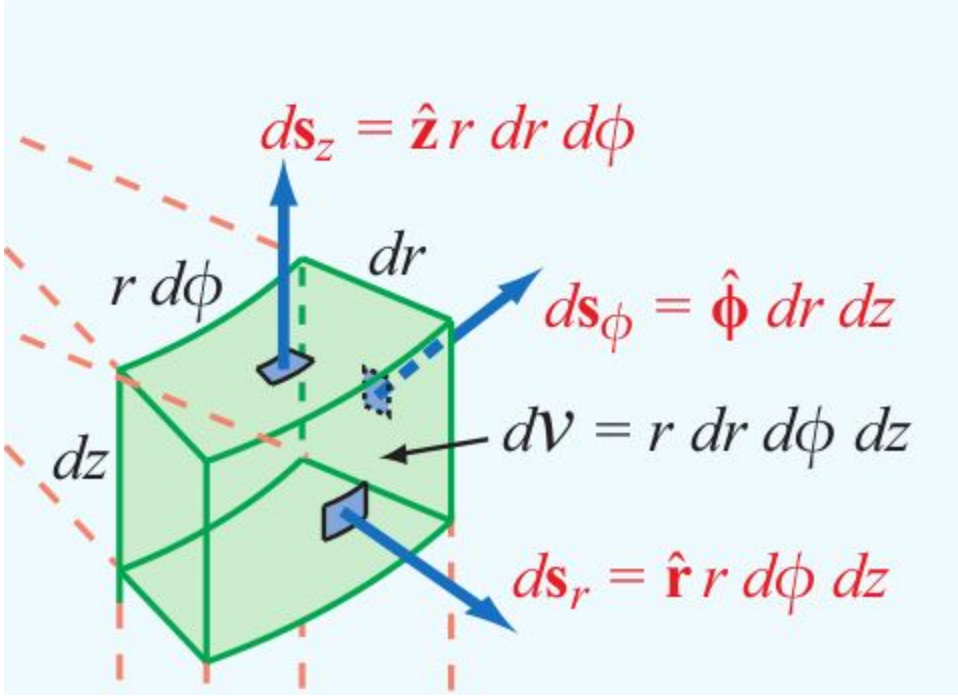
Divergence Theorem

Examples of $d\upsilon$ and $d\mathbf{s}$:



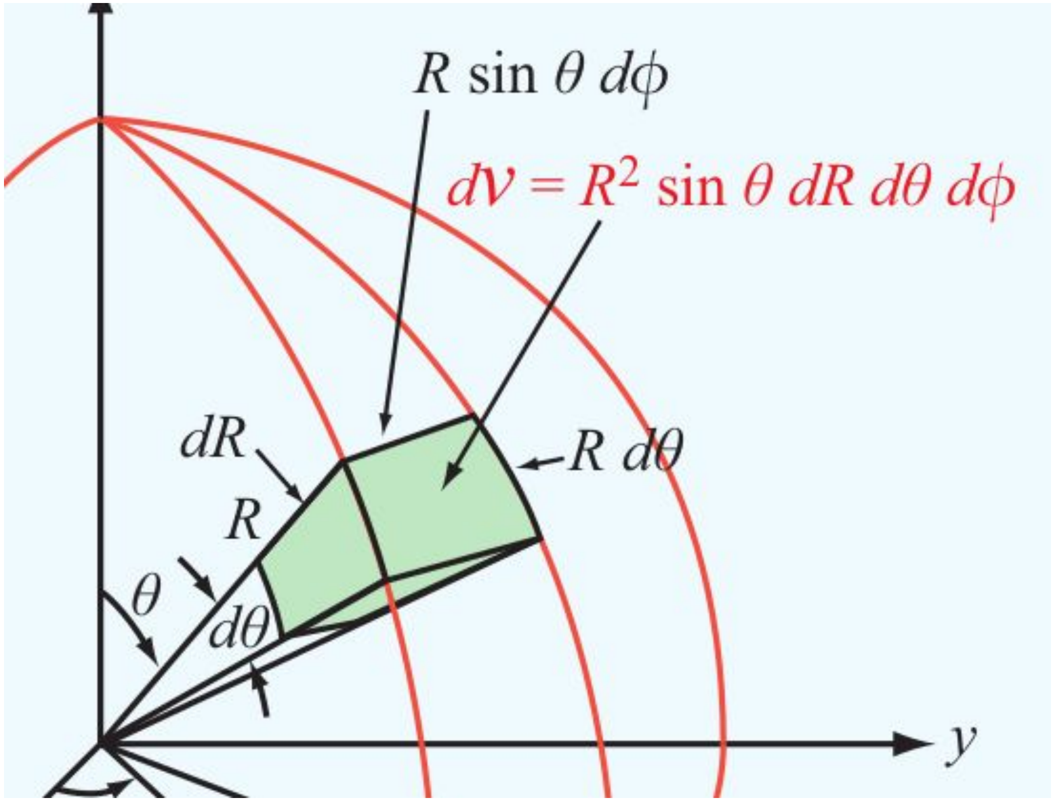
Divergence Theorem

Examples of $d\upsilon$ and $d\mathbf{s}$:



Divergence Theorem

Examples of $d\upsilon$ and $d\mathbf{s}$:



Example 3-13: Divergence

Given: $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2z$

Find: divergence of **E**, evaluate at (2,-2,0)

Solution:

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2z)$$

$$= 6x + 0 + x^2$$

$$\boxed{\nabla \cdot \mathbf{E} = x^2 + 6x.}$$

Example 3-13: Divergence

Solution:

Evaluate: at (2,-2,0):

$$\nabla \cdot \mathbf{E} = 2^2 + 6 \cdot 2 = \boxed{16}$$

Example 3-13: Divergence

Given: $\mathbf{E} = \hat{\mathbf{R}}(a^3 \cos\theta / R^2) - \hat{\boldsymbol{\theta}}(a^3 \sin\theta / R^2)$

Find: divergence of **E**, evaluate at (a/2,0,π)

Solution: from appendix C:

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(a^3\cos\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{a^3\sin^2\theta}{R^2}\right)$$

$$= 0 - \frac{2a^3\cos\theta}{R^3}$$

$$\nabla \cdot \mathbf{E} = -\frac{2a^3\cos\theta}{R^3}.$$

Example 3-13: Divergence

Solution:

Evaluate: at (a/2,0,π):

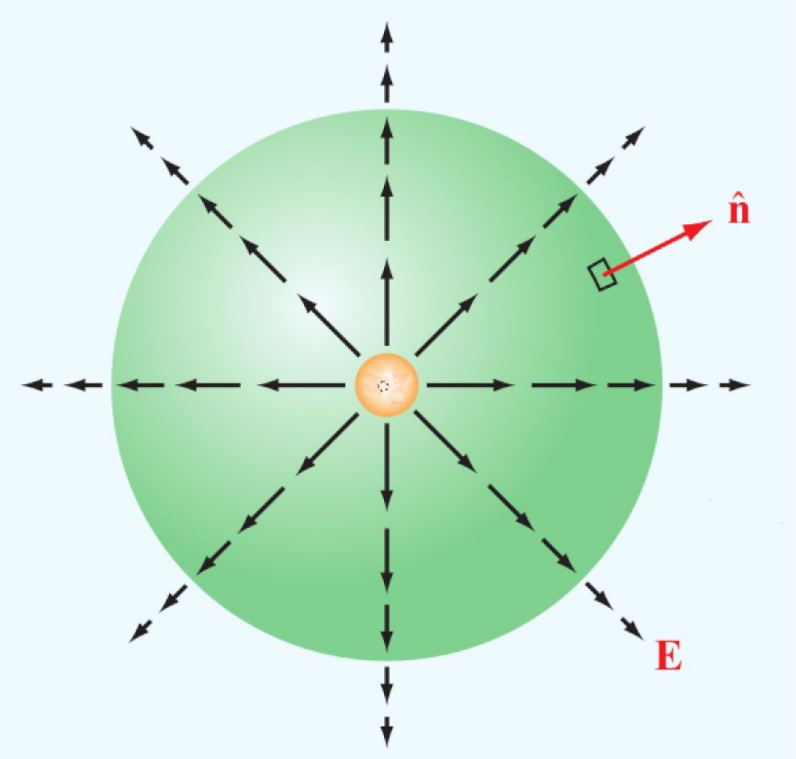
$$\nabla \cdot \mathbf{E} = -\frac{2a^3 \cos\theta}{R^3}$$

$$\nabla \cdot \mathbf{E} = -\frac{2a^3 \cos(0)}{(a/2)^3}$$

$$\nabla \cdot \mathbf{E} = -\frac{2a^3}{a^3/8}$$

$$\nabla \cdot \mathbf{E} = -16$$

Exercise 3-15: Total Flux



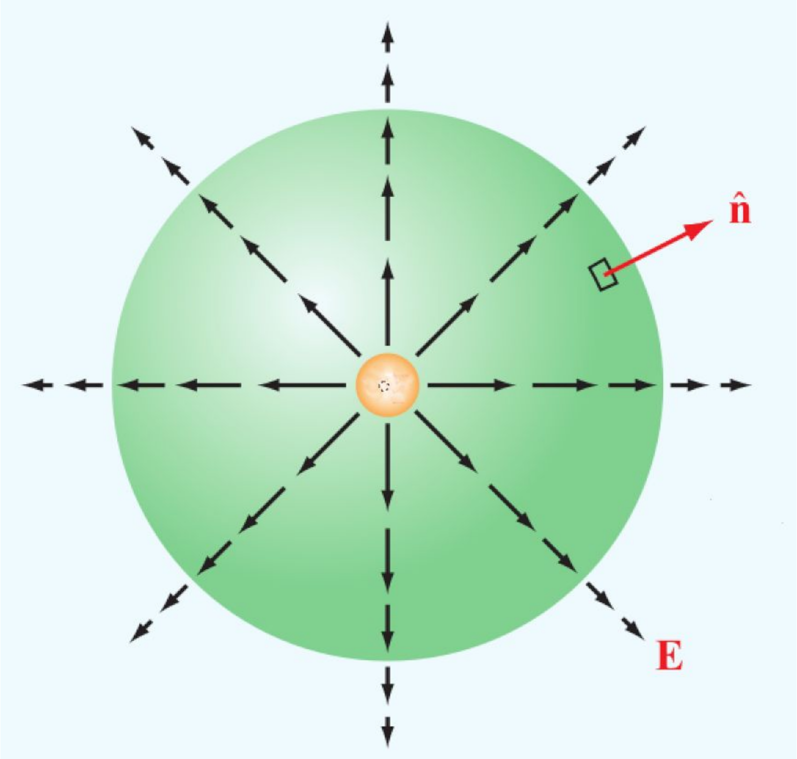
Given: $\mathbf{E} = \hat{\mathbf{R}}AR$

Find: The total flux of **E** through a spherical surface of radius *a* centered at the origin.

Solution:

$$\text{Total flux} = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

Exercise 3-15: Total Flux



Solution:

$$\text{Total flux} = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

look up in table 3.1:

$$d\mathbf{s} = \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi$$

To integrate over entire sphere:

θ ranges from 0 to π

ϕ ranges from 0 to 2 π

Exercise 3-15: Total Flux

Solution:

$$\text{Total flux} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \mathbf{E}(R=a) \cdot \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi \, \Big|_{R=a}$$

$$\text{Total flux} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}} A a \cdot \hat{\mathbf{R}} a^2 \sin\theta \, d\theta \, d\phi$$

$$\text{Total flux} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} A a a^2 \sin\theta \, d\theta \, d\phi$$

$$\text{Total flux} = \int_{\theta=0}^{\pi} 2\pi A a^3 \sin\theta \, d\theta$$

Exercise 3-15: Total Flux

Solution:

$$\text{Total flux} = \int_{\theta=0}^{\pi} 2\pi A a^3 \sin\theta \, d\theta$$

$$\text{Total flux} = 2\pi A a^3 \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$\text{Total flux} = 2\pi A a^3 \Big[-\cos\theta \Big]_{\theta=0}^{\pi}$$

$$\text{Total flux} = 2\pi A a^3 \left[-\cos\pi - (-\cos 0) \right]$$

$$\text{Total flux} = 2\pi A a^3 \left[-(-1) - (-1) \right]$$

Exercise 3-15: Total Flux

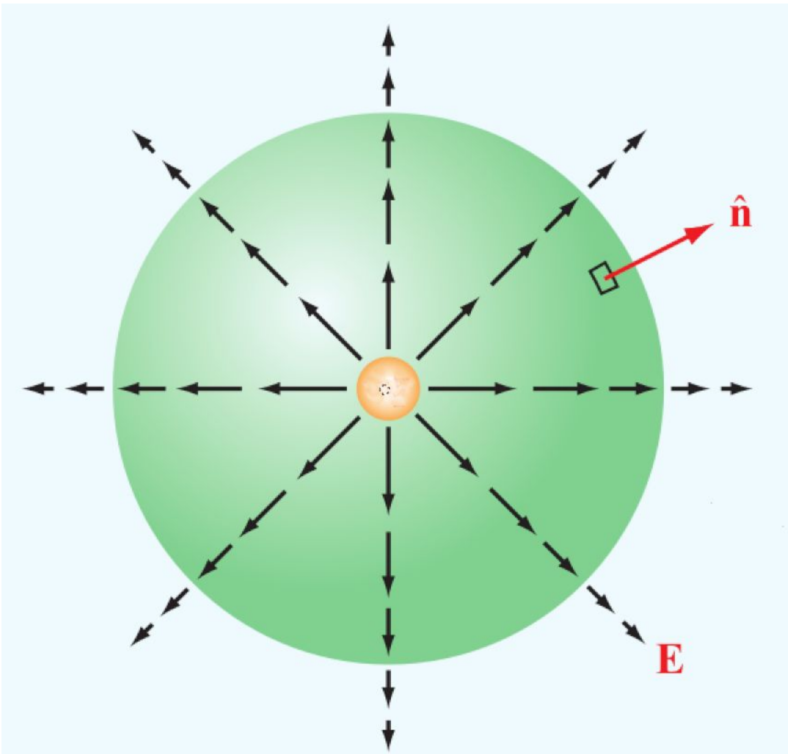
Solution:

$$\text{Total flux} = 2\pi A a^3 \left[-(-1) - (-1)\right]$$

$$\text{Total flux} = 2\pi A a^3 \left[1 + 1\right]$$

$$\boxed{\text{Total flux} = 4\pi A a^3}$$

Exercise 3-16: Divergence Theorem



Given: $\mathbf{E} = \hat{\mathbf{R}}AR$

Find: $\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V}$

where the Volume is of the sphere with radius *a*

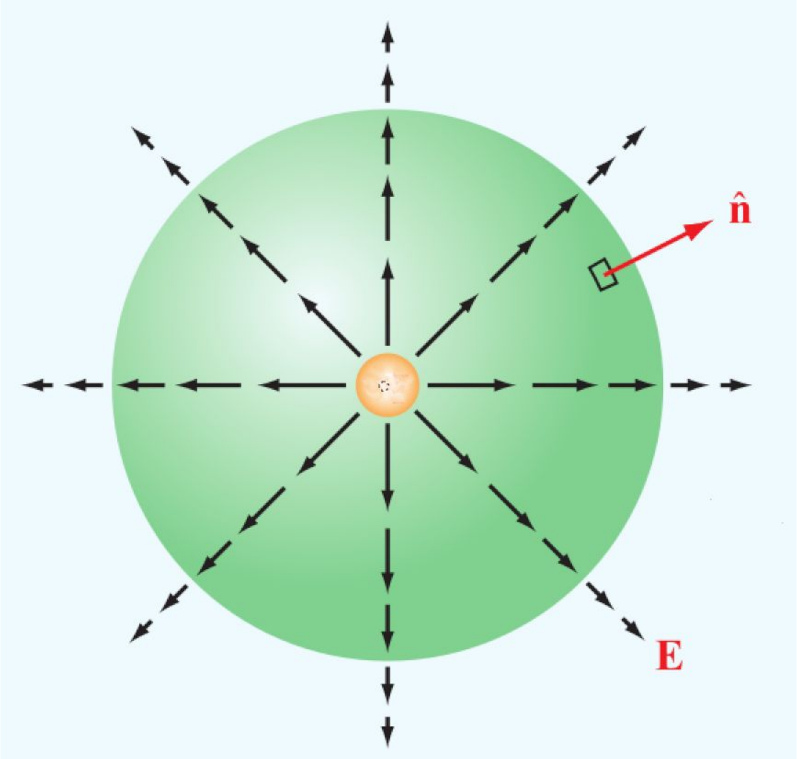
Solution:

step1: calc **∇· E:**

From appendix C:

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

Exercise 3-16: Divergence Theorem



Solution:

step1: calc **∇· E:**

since:

$$E_\theta = 0, \qquad E_\phi = 0$$

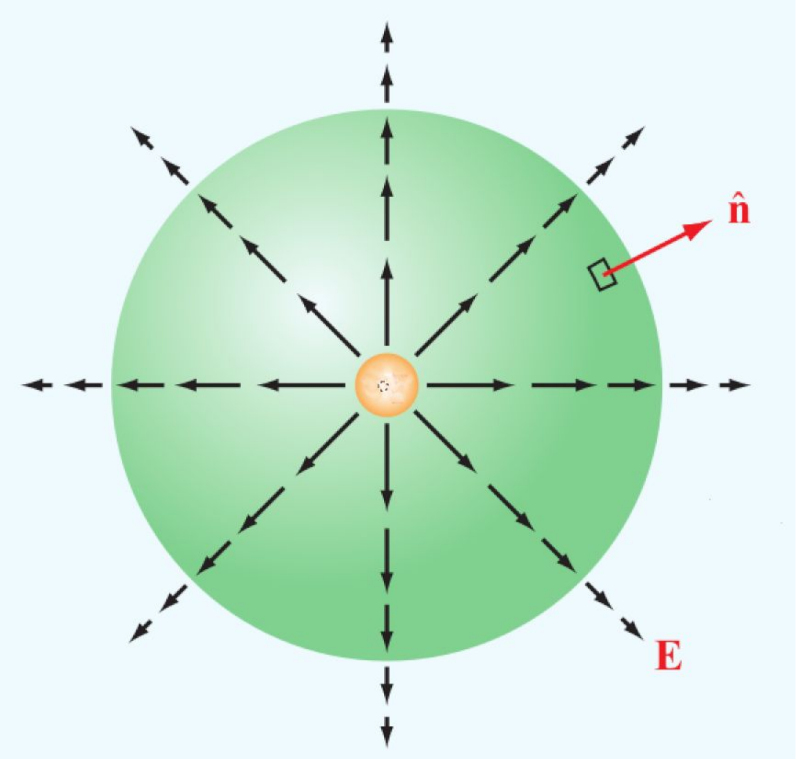
we get:

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R)$$

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 AR)$$

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(AR^3)$$

Exercise 3-16: Divergence Theorem



Solution:

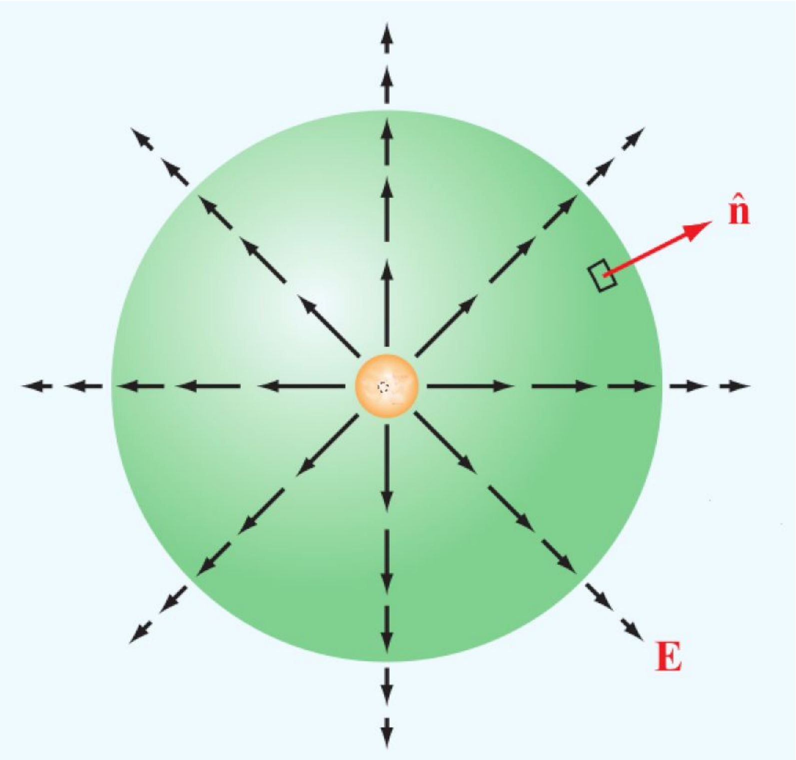
step1: calc **∇· E:**

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2} \frac{\partial}{\partial R}(AR^3)$$

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2}(3AR^2)$$

$$\nabla \cdot \mathbf{E} = 3A$$

Exercise 3-16: Divergence Theorem



Solution:

step2: expression for d$\mathcal{V}$:

$$d\mathcal{V} = R^2 \sin\theta \, dR \, d\theta \, d\phi$$

step3: range of integration:

R: 0 to a

θ: 0 to π

ϕ: 0 to 2π

step4: Plug in:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E} \, d\mathcal{V} = \int_{R=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 3A \, R^2 \sin\theta \, dR \, d\theta \, d\phi$$

Exercise 3-16: Divergence Theorem

Solution:

step5: integrate:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = \int_{R=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 3A\, R^2 \sin\theta\, dR\, d\theta\, d\phi$$

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = \int_{R=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 3A\, R^2 \sin\theta\, dR\, d\theta\, d\phi$$

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = 3A \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{R^3}{3} \Bigg|_{R=0}^{a} \sin\theta\, d\theta\, d\phi$$

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = 3A \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (a^3/3) \sin\theta\, d\theta\, d\phi$$

Exercise 3-16: Divergence Theorem

Solution:

step5: integrate:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = 3A \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (a^3/3) \sin\theta \, d\theta \, d\phi$$

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = Aa^3 \int_{\phi=0}^{2\pi} \Big[-\cos\theta \Big]_{\theta=0}^{\pi} d\phi$$

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = Aa^3 \int_{\phi=0}^{2\pi} [-(-1) - (-1)] \, d\phi$$

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = 2Aa^3 \int_{\phi=0}^{2\pi} d\phi$$

Exercise 3-16: Divergence Theorem

Solution:

step5: integrate:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E} \, d\mathcal{V} = 2Aa^3 \int_{\phi=0}^{2\pi} d\phi$$

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E} \, d\mathcal{V} = 4\pi Aa^3$$

Exercise 3-16: Divergence Theorem

Solution:

This is the same as the solution to exercise 3-15

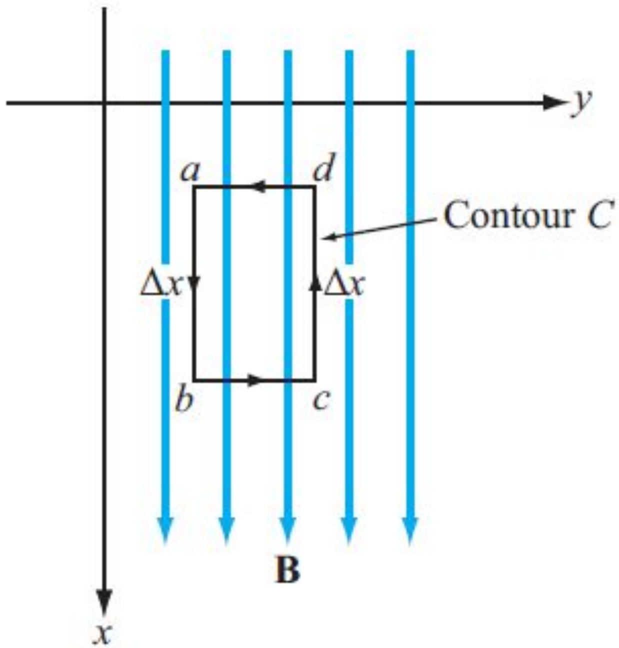
So, for this example:

$$\int_{\upsilon} \nabla \cdot \mathbf{E} \, d\upsilon = \oint_S \mathbf{E} \cdot d\mathbf{s}.$$

Curl of a Vector Field

The curl of a vector **B** describes its rotational property, or **Circulation**:

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$



(a) Uniform field

where :

B is a vector field,

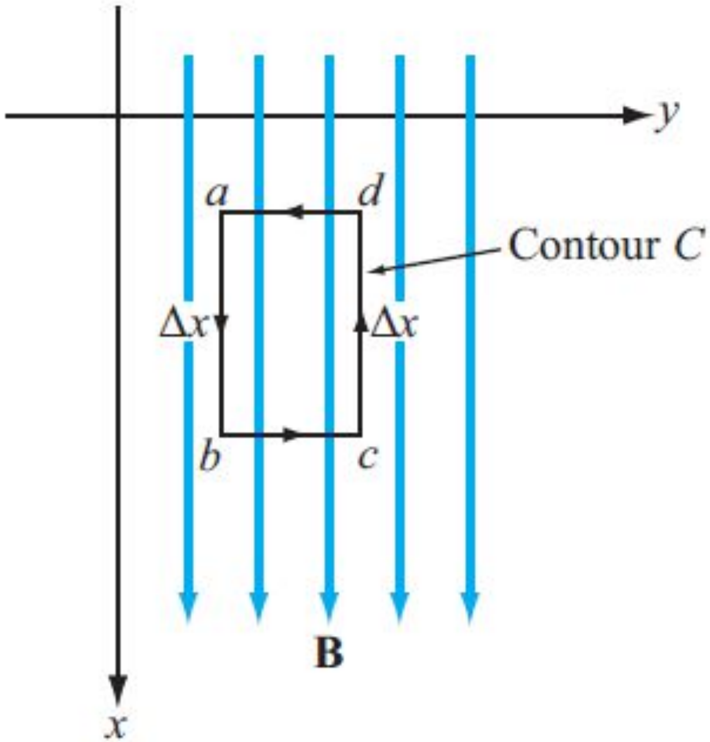
$d\mathbf{l} = \hat{\mathbf{a}}_l \, dl,$

$\hat{\mathbf{a}}_l$ is the unit vector of $d\mathbf{l}$,

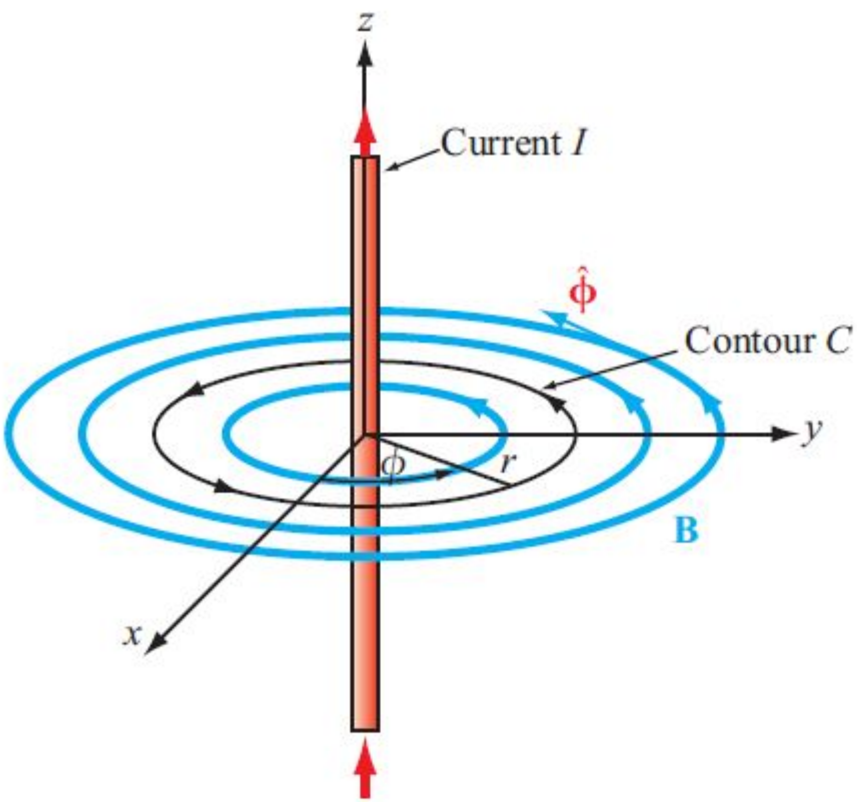
dl is the differential length along the path: *C*

Curl of a Vector Field

Examples:



(a) Uniform field



(b) Azimuthal field

Circulation of uniform field is zero.

Circulation of azimuthal field is non-zero in azimuth plane

Curl of a Vector Field

The curl is defined as the Circulation per unit area:

$$\nabla \times \mathbf{B} = \text{curl}\, \mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max}.$$

with the path, or contour, *C*, oriented so that the circulation is maximum.

Which can be expressed as:

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}} \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right)$$

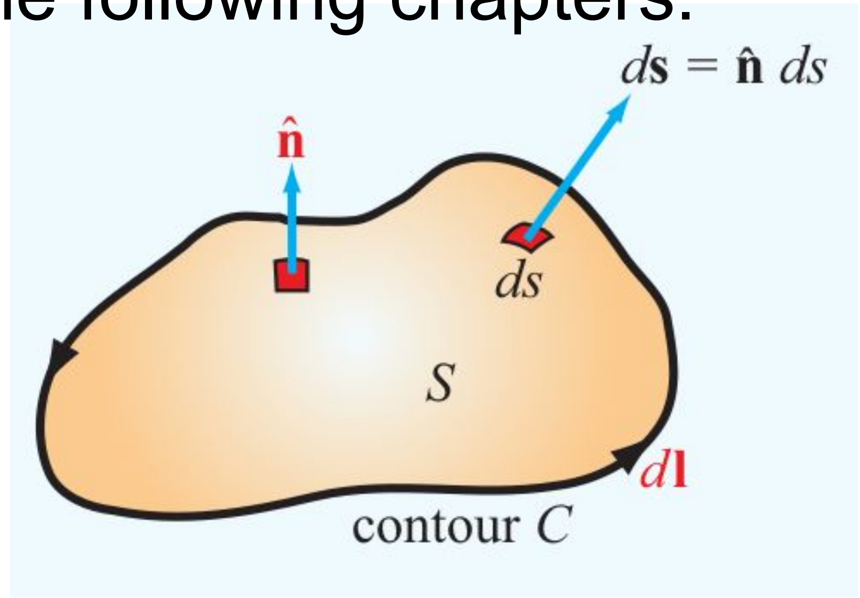
Curl of a Vector Field

From the previous equation we can say:

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$

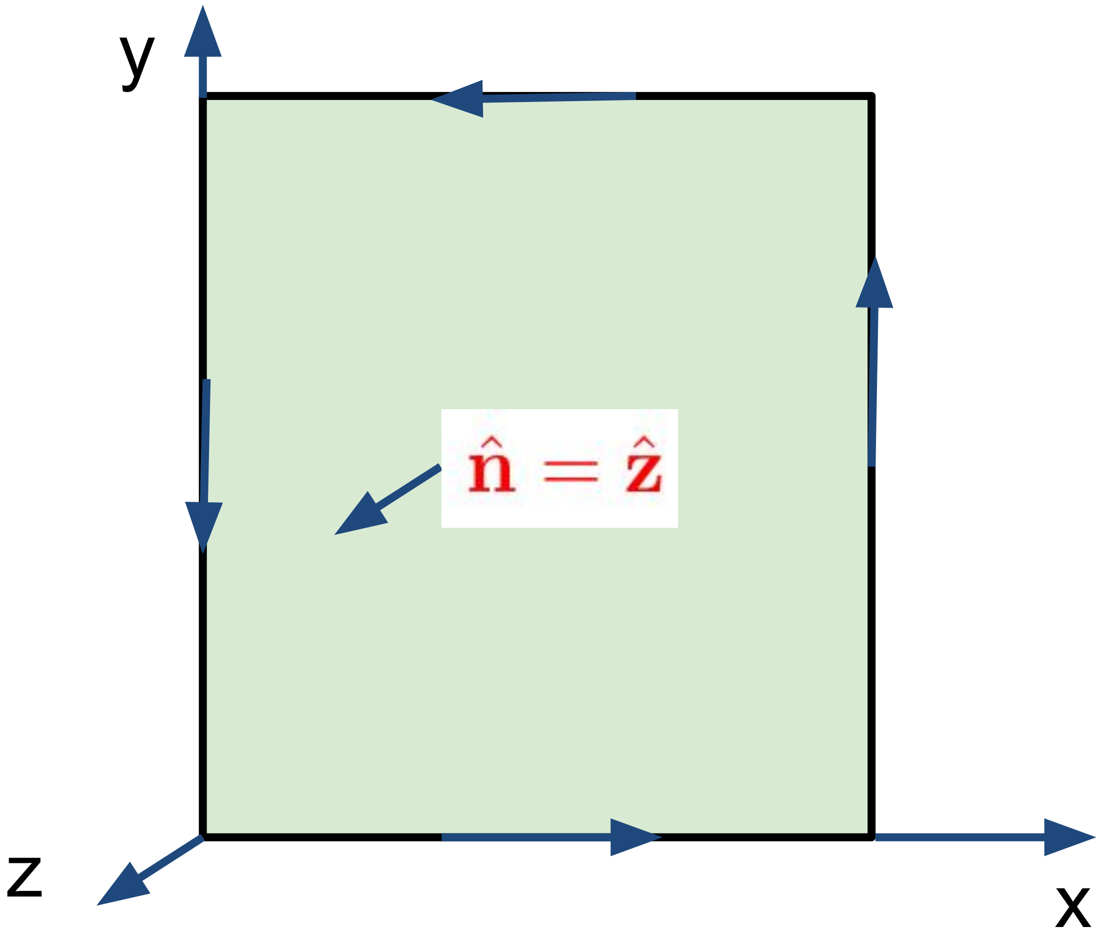
We will use this equation (Stokes' Theorem) in order to solve problems in the following chapters.

This figure shows the geometry relating $d\mathbf{l}$ and $d\mathbf{s}$



Curl of a Vector Field

Examples of *d***s** and *d***l**:

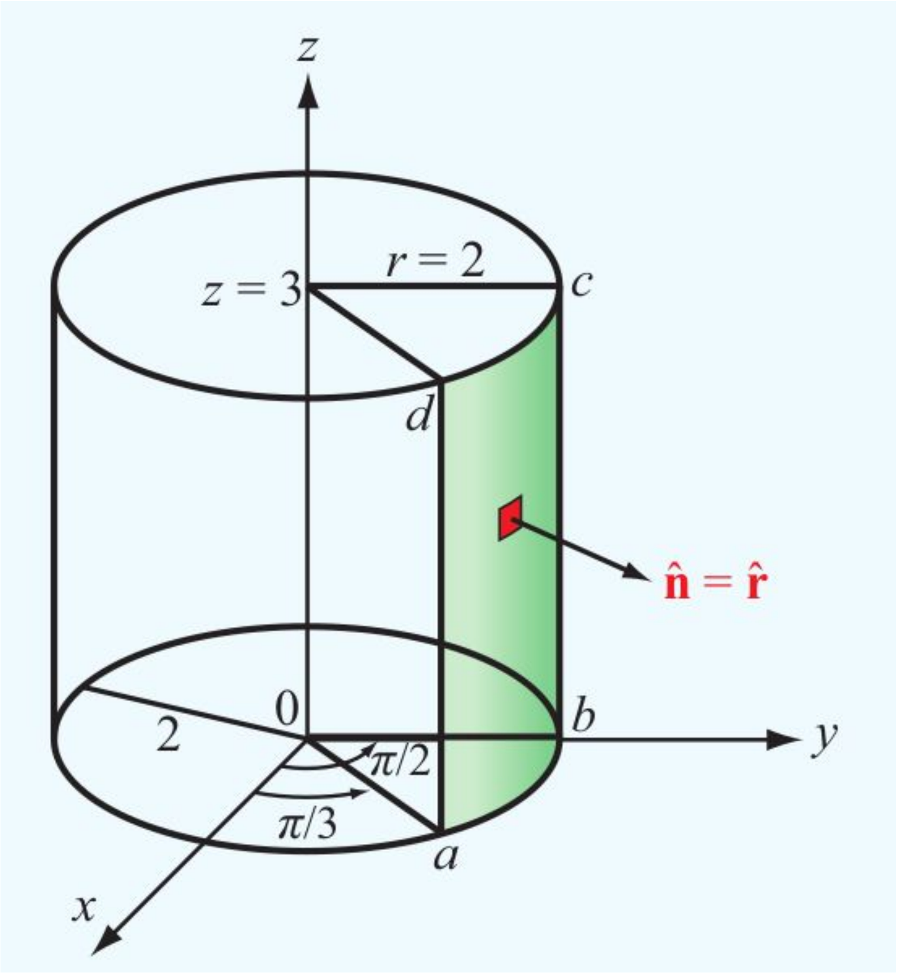


*d***s** is the area, with normal $\hat{\mathbf{z}}$

*d***l** is all 4 line segments that outline the rectangle

Curl of a Vector Field

Examples of $d\mathbf{s}$ and $d\mathbf{l}$:

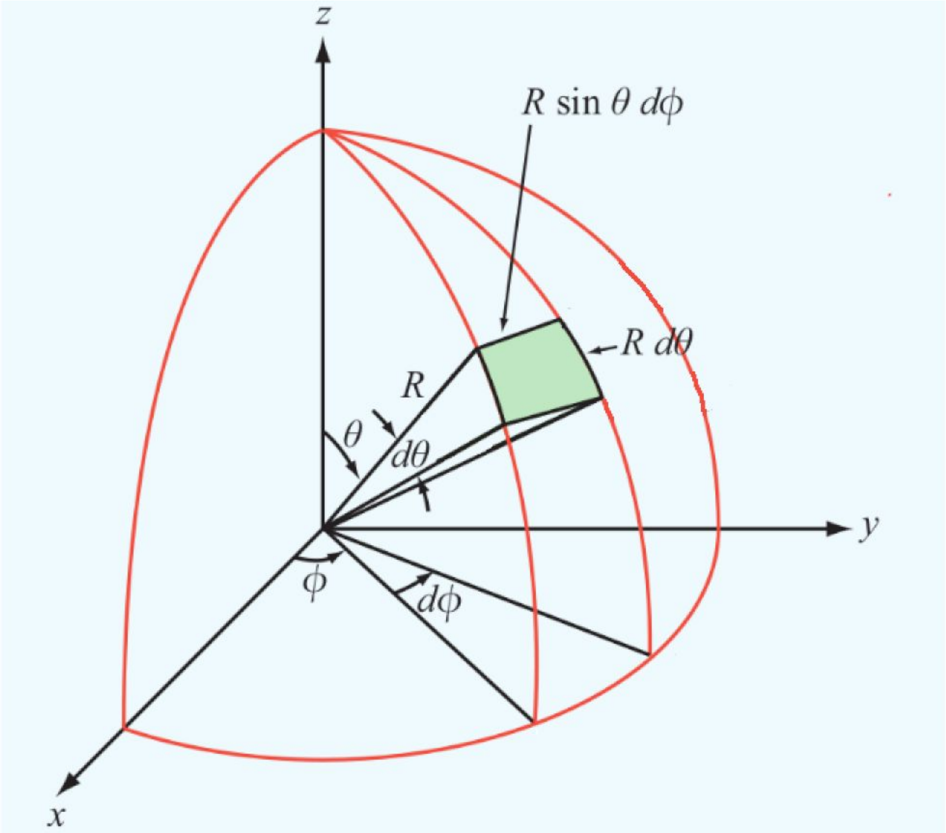


$d\mathbf{s}$ is the curved area, with normal $\hat{\mathbf{r}}$.

$d\mathbf{l}$ is all 4 line segments that outline the curved rectangle

Curl of a Vector Field

Examples of $d\mathbf{s}$ and $d\boldsymbol{l}$:



$d\mathbf{s}$ is the curved area, with normal $\hat{\mathbf{R}}$

$d\boldsymbol{l}$ is all 4 line segments that outline the curved rectangle

Example 3-14 Stokes' Theorem

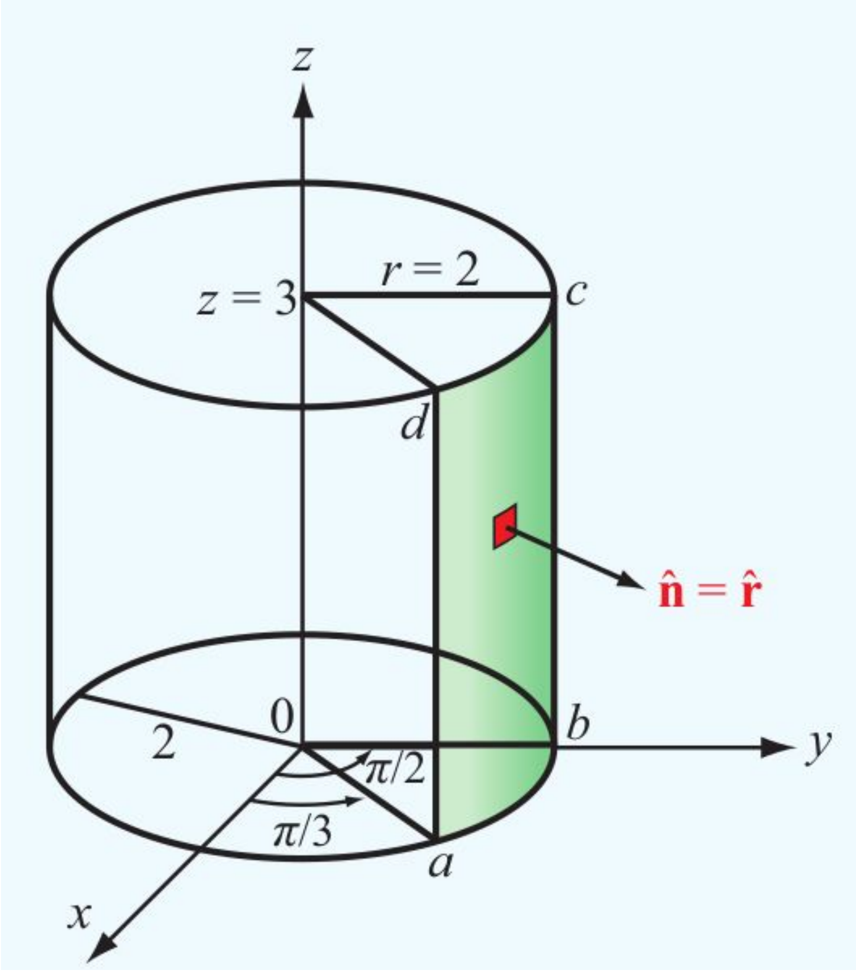
Given: $\mathbf{B} = \hat{\mathbf{z}} \cos\phi / r,$

Find: the Circulation over a segment of a cylindrical surface, defined by:

r = 2

ϕ: π/3 to π/2

z: 0 to 3



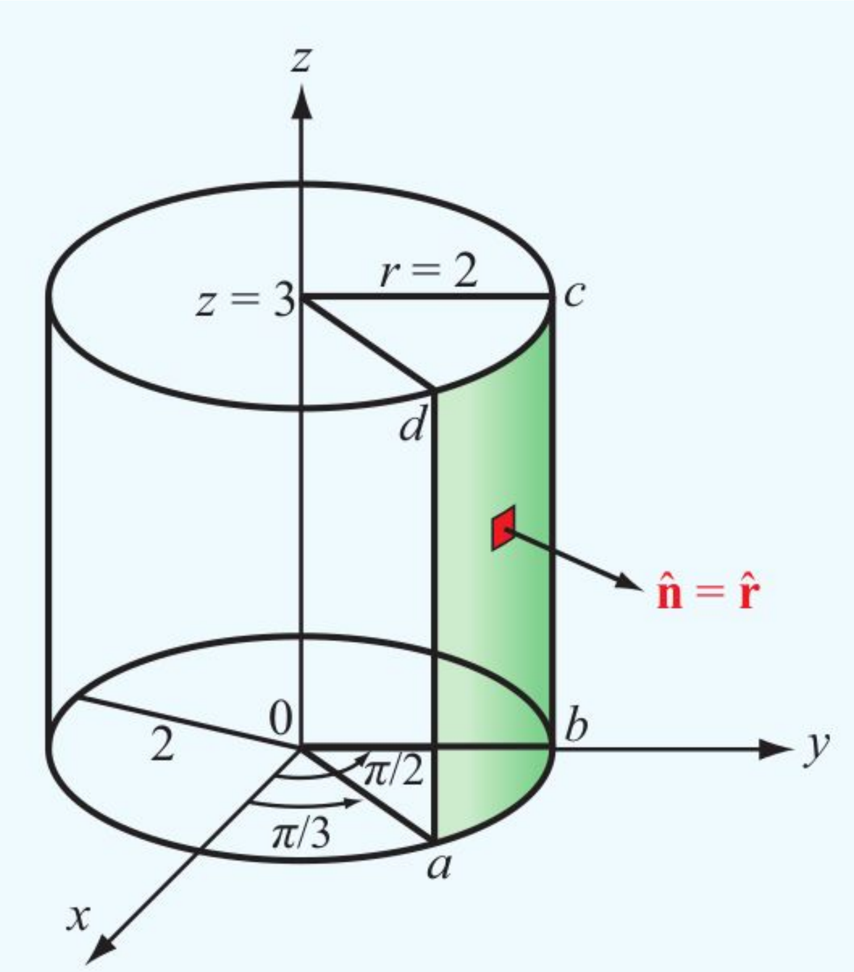
Example 3-14 Stokes' Theorem

Solution:

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

step1: identify *C*:
C is the closed path that is on the edge of the green surface

C has a direction, that goes from start to end continuously.



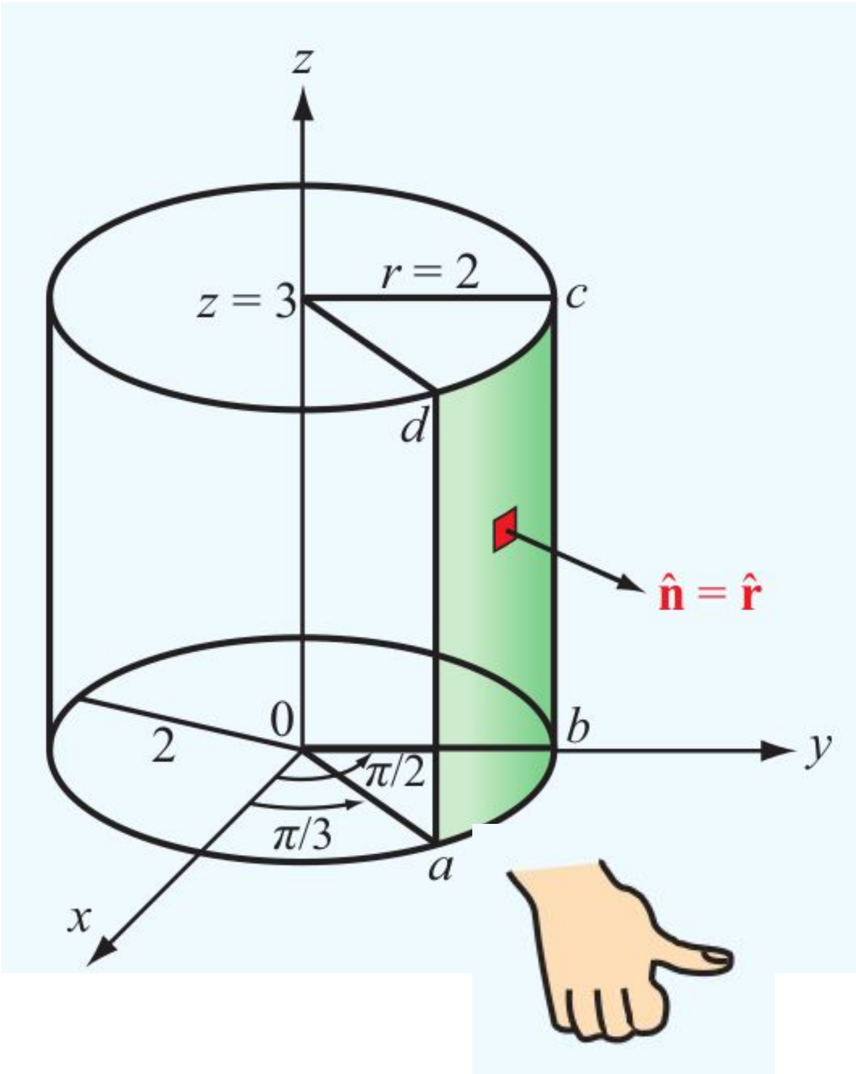
Example 3-14 Stokes' Theorem

Solution:

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

step1: identify *C*:

The direction of C is such that the right hand-rule, with fingers in the direction of C, has the thumb pointing in the direction of the surface normal.



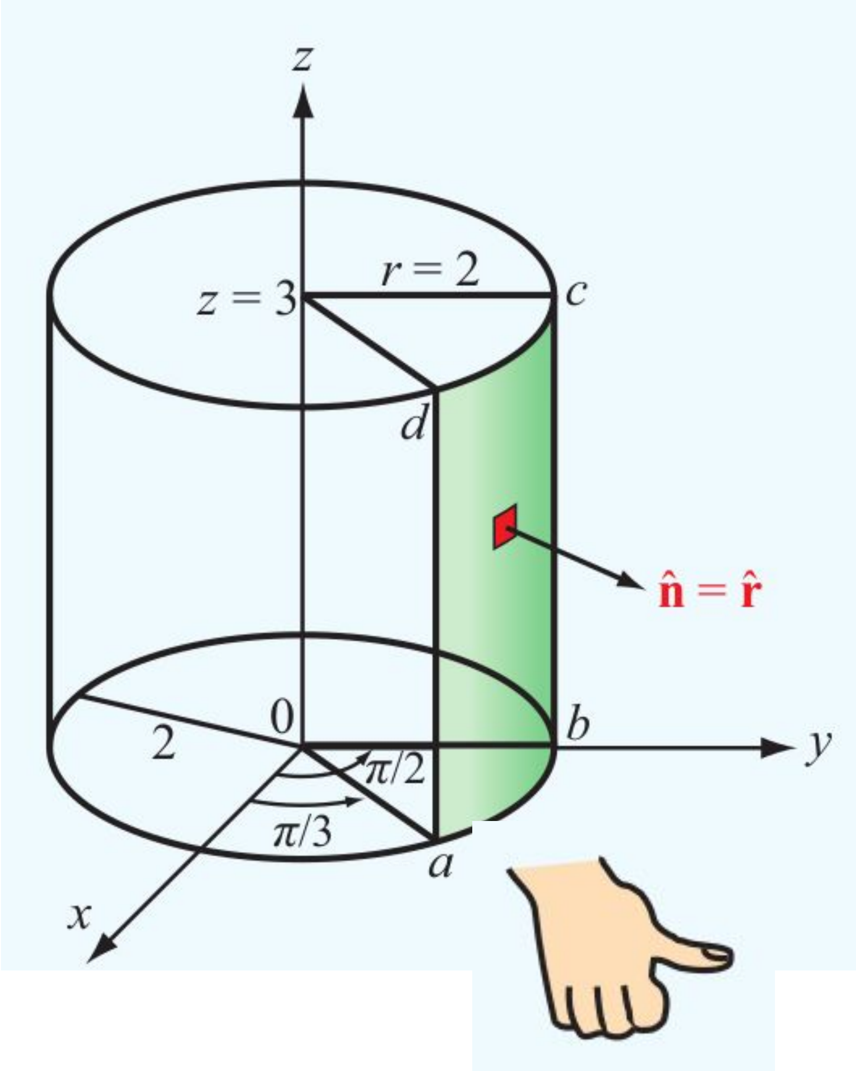
Example 3-14 Stokes' Theorem

Solution:

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

step1: identify C:

Hence, to get the normal to be $\hat{\mathbf{r}}$ we use the closed path: a-b-c-d-a



$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l} + \int_b^c \mathbf{B}_{bc} \cdot d\mathbf{l} + \int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l} + \int_d^a \mathbf{B}_{da} \cdot d\mathbf{l},$$

Example 3-14 Stokes' Theorem

Solution:

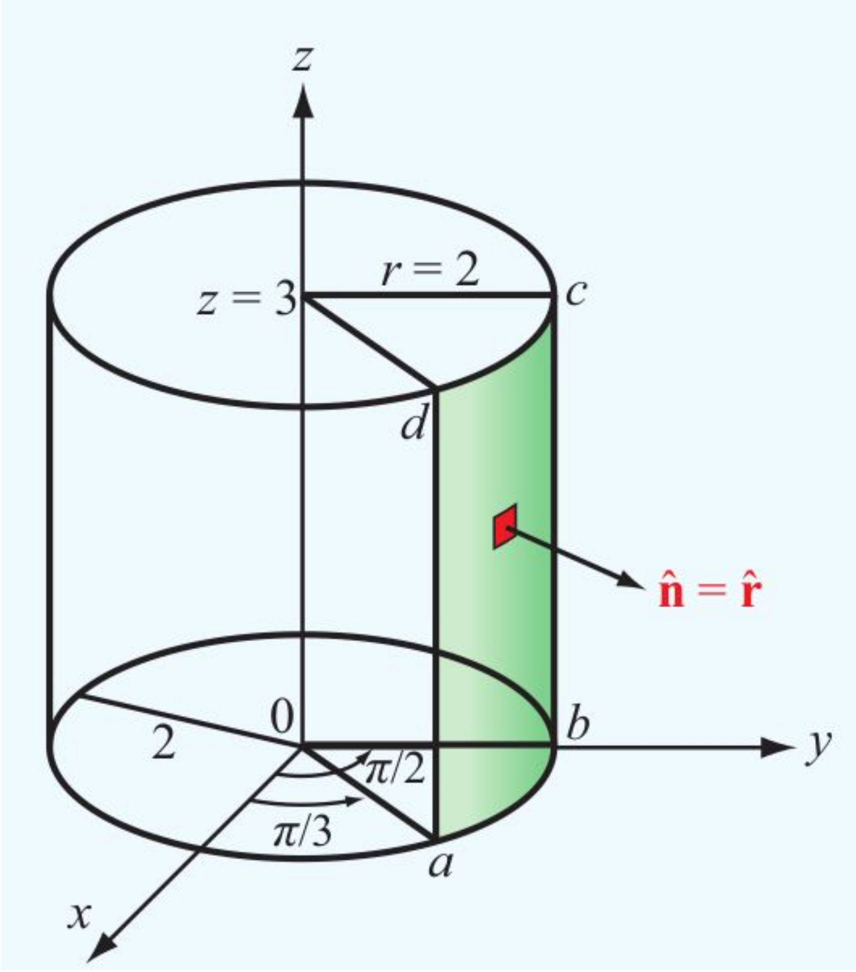
$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

step1: identify *C*:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l} + \int_b^c \mathbf{B}_{bc} \cdot d\mathbf{l} + \int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l} + \int_d^a \mathbf{B}_{da} \cdot d\mathbf{l},$$

where:

$\mathbf{B}_{ab}$ is **B** *along* segment *ab* etc....



Example 3-14 Stokes' Theorem

Solution:

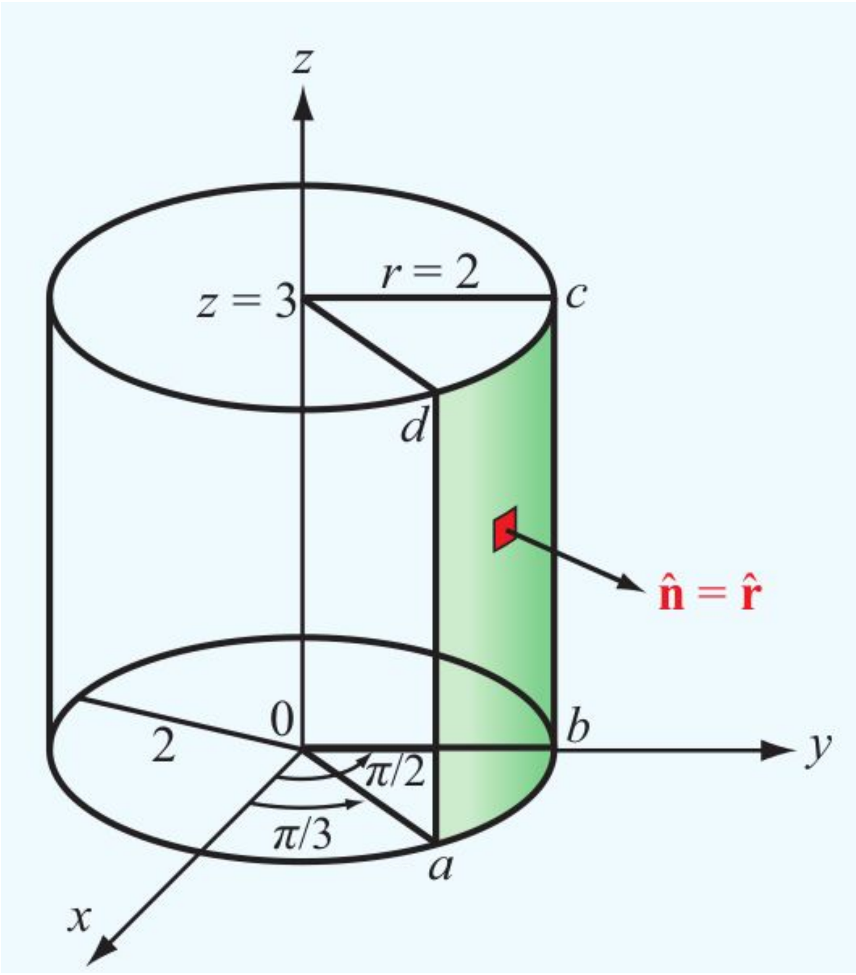
$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

step2: evaluate each integral:

$$\int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l}$$

$\mathbf{B}_{ab}$ along arc from *a* to *b*:
r=2, ϕ varies, *z*=0,   so:

$$\mathbf{B}_{ab} = \hat{\mathbf{z}} \frac{\cos\phi}{r} = \hat{\mathbf{z}} \frac{\cos\phi}{2}$$



Example 3-14 Stokes' Theorem

Solution:
step2: evaluate each integral:

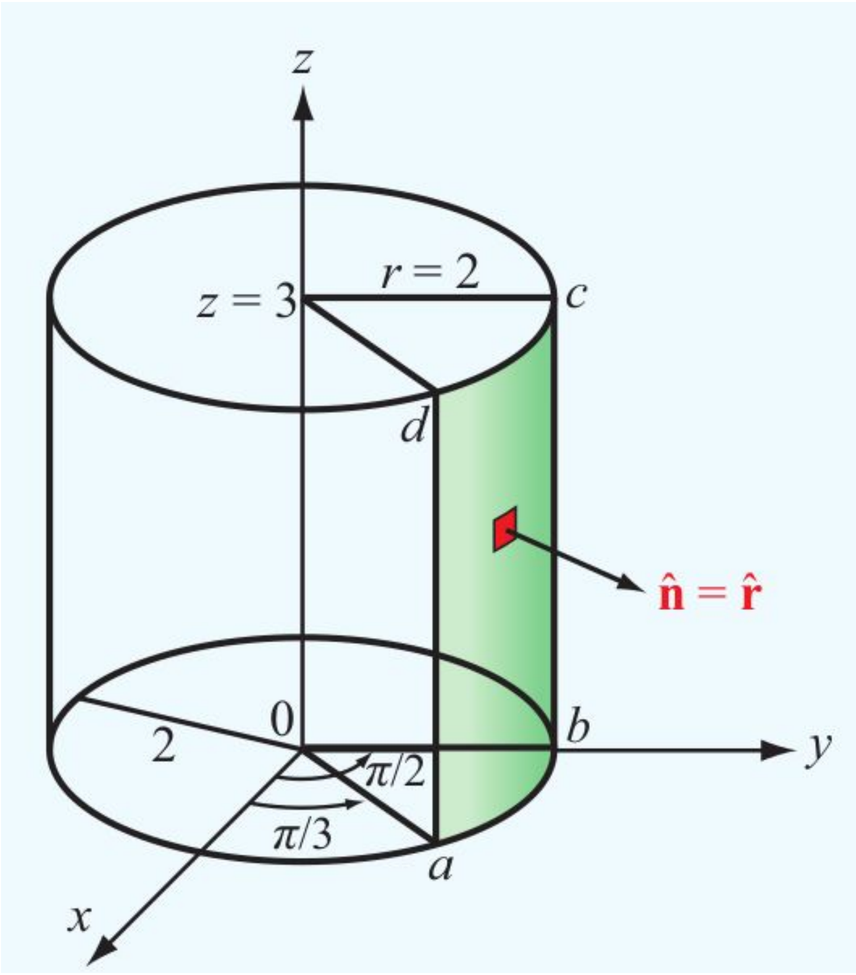
$$\int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l}$$

From table 3-1:

$$d\mathbf{l} = \hat{\mathbf{r}}dr + \hat{\boldsymbol{\phi}}\, r d\phi + \hat{\mathbf{z}}dz$$

since only ϕ is varying:

$$d\mathbf{l} = \hat{\boldsymbol{\phi}}\, r d\phi$$



Example 3-14 Stokes' Theorem

Solution:

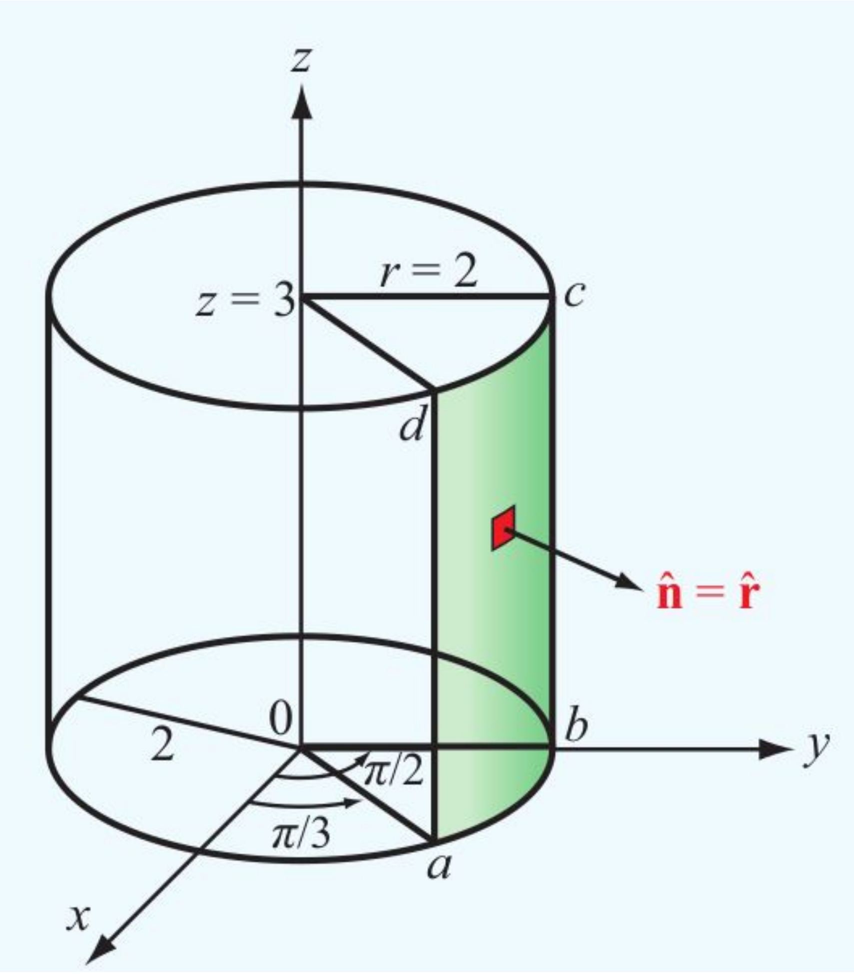
step2: evaluate each integral:

$$\int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l}$$

plug in:

$$\int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l} = \hat{\mathbf{z}}\frac{\cos\phi}{2} \cdot \hat{\boldsymbol{\phi}}\, r\, d\phi$$

$$\int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l} = 0$$



Example 3-14 Stokes' Theorem

Solution:

step2: evaluate each integral:

$$\int_b^c \mathbf{B}_{bc} \cdot d\mathbf{l}$$

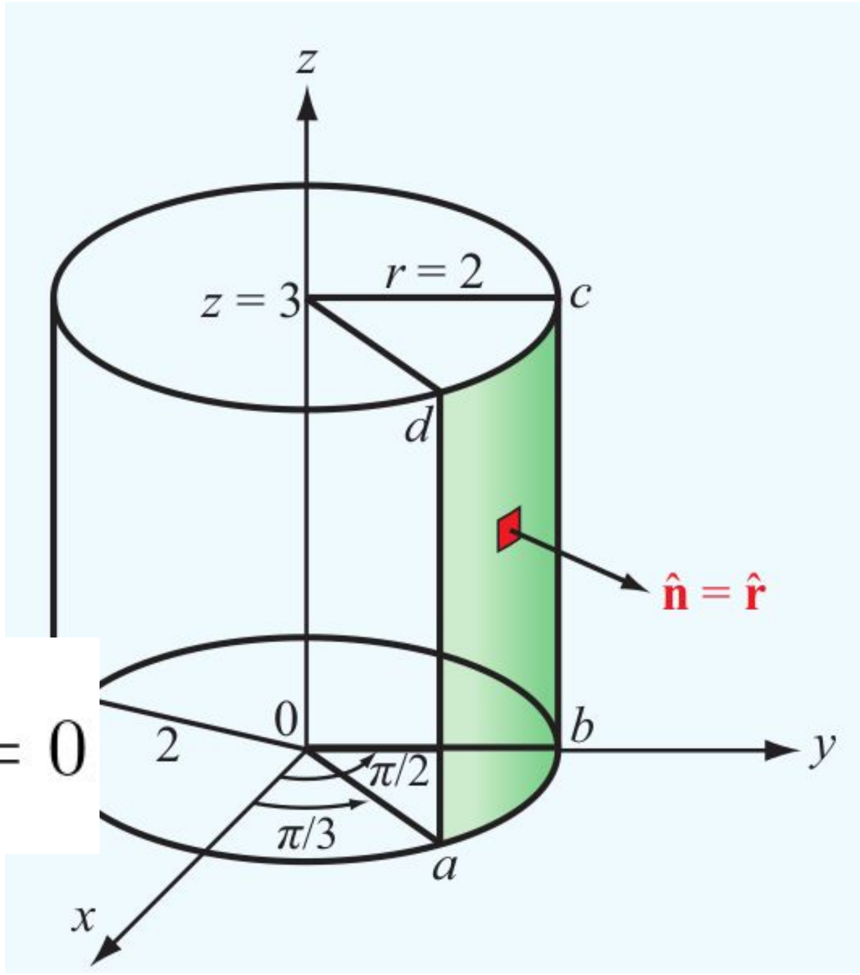
$\mathbf{B}_{bc}$ along line from *b* to *c*:

r=2, ϕ=π/2, z varies, so:

$$\mathbf{B}_{bc} = \hat{\mathbf{z}}\frac{\cos\phi}{r} = \hat{\mathbf{z}}\frac{\cos(\pi/2)}{2} = 0$$

So:

$$\boxed{\int_b^c \mathbf{B}_{bc} \cdot d\mathbf{l} = 0}$$



Example 3-14 Stokes' Theorem

Solution:

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

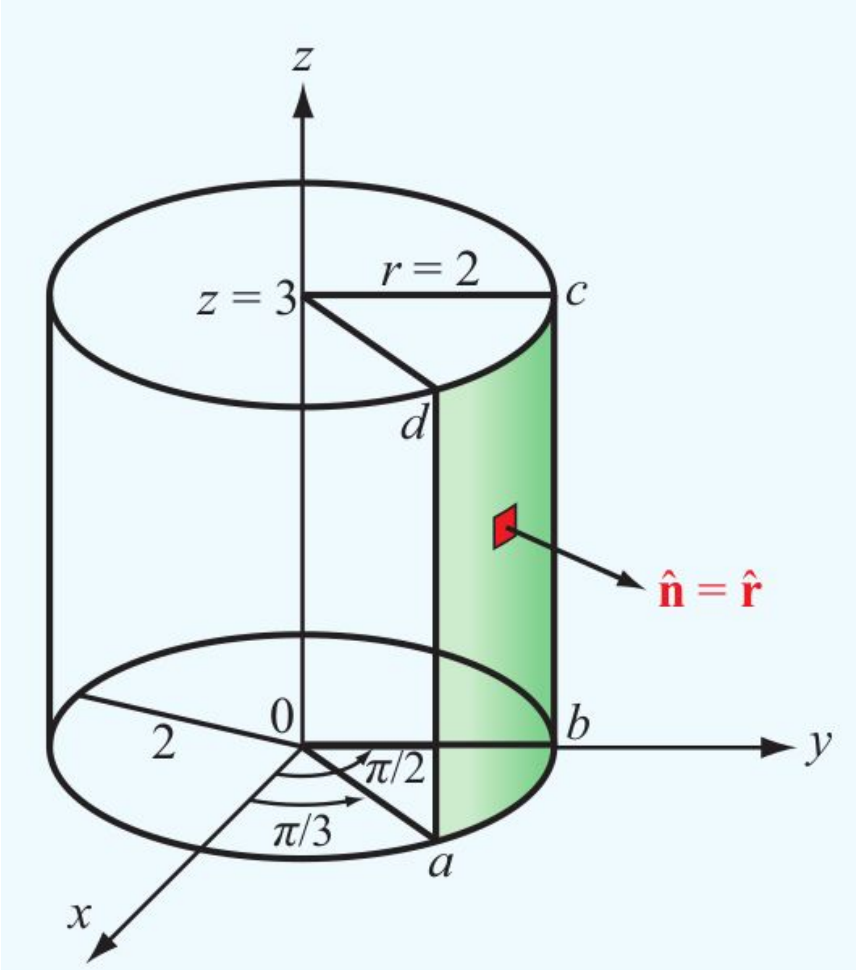
step2: evaluate each integral:

$$\int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l}$$

$\mathbf{B}_{cd}$ along arc from *c* to *d*:

r=2, ϕ varies, z=3, so:

$$\mathbf{B}_{cd} = \hat{\mathbf{z}} \frac{\cos\phi}{r} = \hat{\mathbf{z}} \frac{\cos\phi}{2}$$



Example 3-14 Stokes' Theorem

Solution:

step2: evaluate each integral:

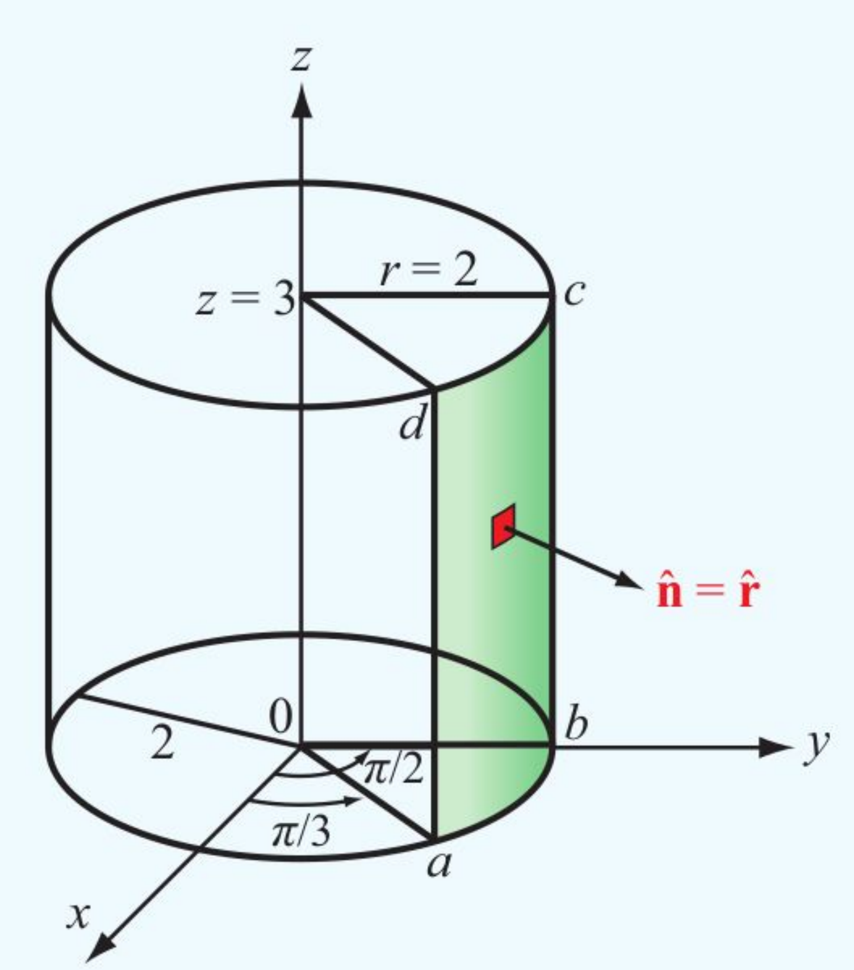
$$\int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l}$$

From table 3-1:

$$d\mathbf{l} = \hat{\mathbf{r}} dr + \hat{\boldsymbol{\phi}} r d\phi + \hat{\mathbf{z}} dz$$

since only ϕ is varying:

$$d\mathbf{l} = \hat{\boldsymbol{\phi}} r d\phi$$



Example 3-14 Stokes' Theorem

Solution:

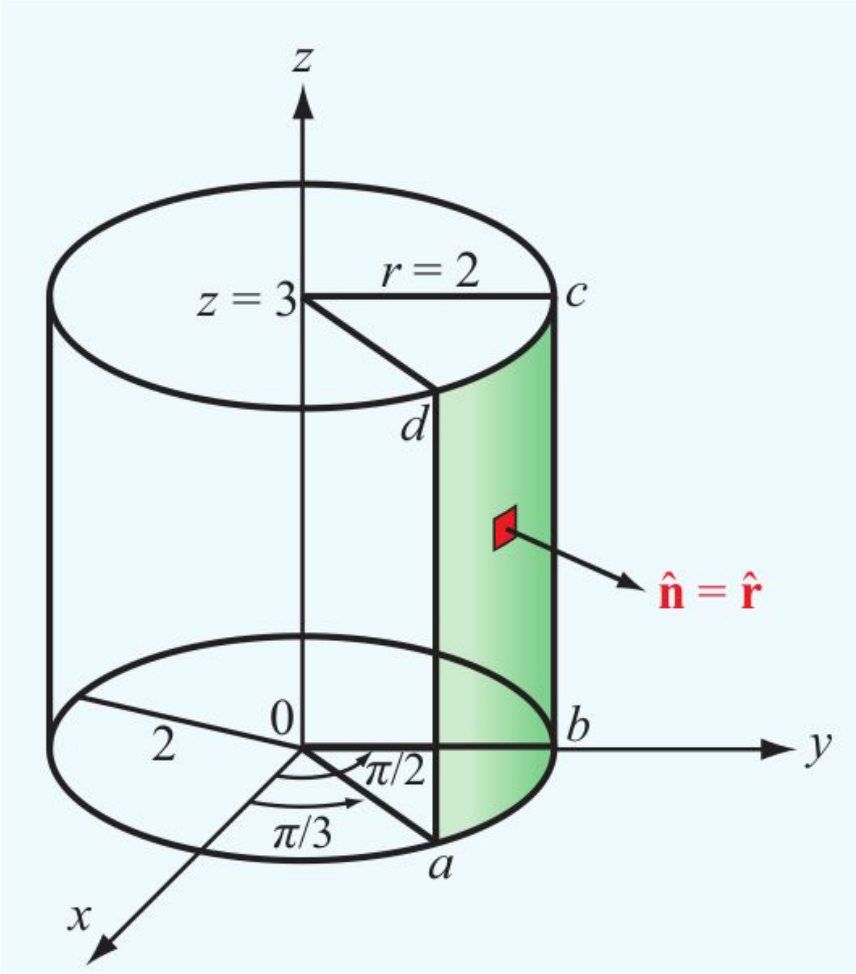
step2: evaluate each integral:

$$\int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l}$$

plug in:

$$\int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l} = \hat{\mathbf{z}} \frac{\cos\phi}{2} \cdot \hat{\boldsymbol{\phi}}\, r\, d\phi$$

$$\int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l} = 0$$



Example 3-14 Stokes' Theorem

Solution:

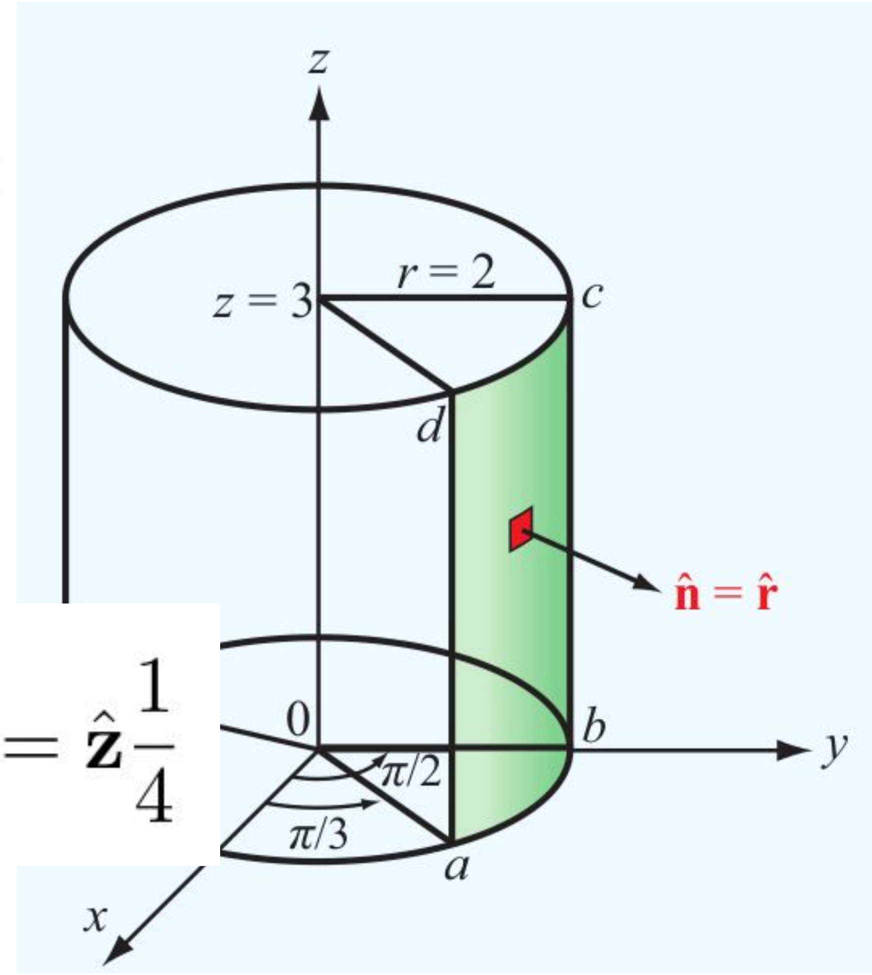
step2: evaluate each integral:

$$\int_d^a \mathbf{B}_{da} \cdot d\mathbf{l}$$

$\mathbf{B}_{da}$ along line from *d* to *a*:

r=2, ϕ=π/3, *z* varies, so:

$$\mathbf{B}_{da} = \hat{\mathbf{z}}\frac{\cos\phi}{r} = \hat{\mathbf{z}}\frac{\cos(\pi/3)}{2} = \hat{\mathbf{z}}\frac{1}{4}$$



Example 3-14 Stokes' Theorem

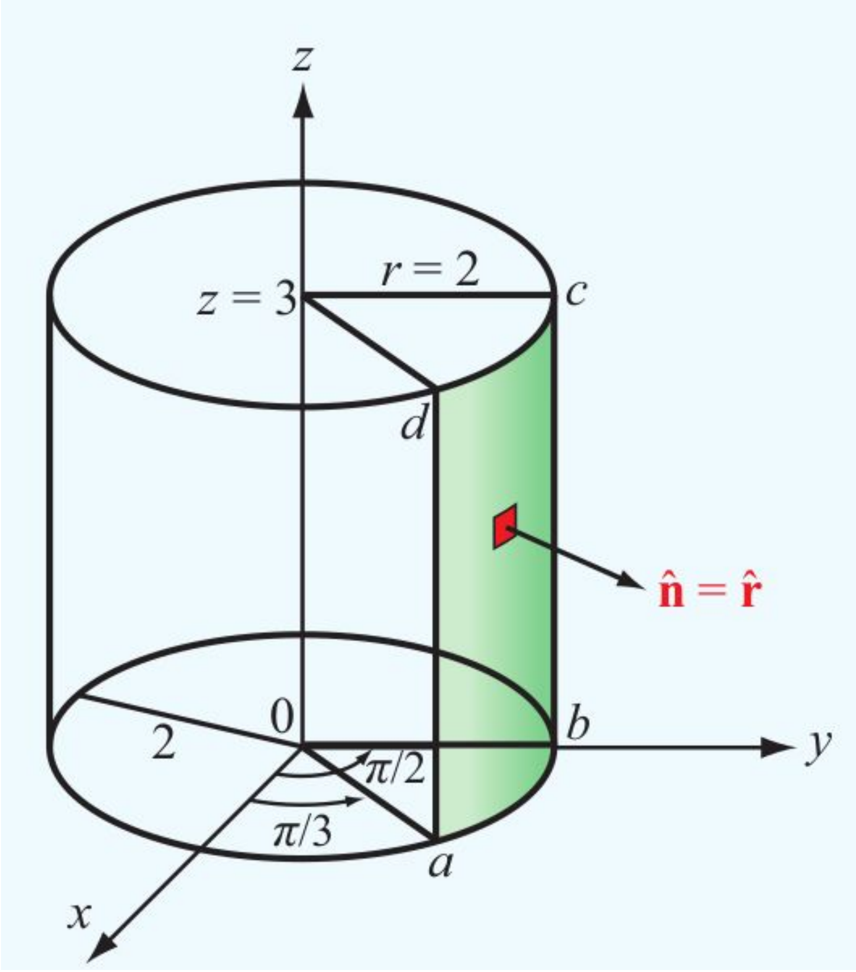
Solution:

step2: evaluate each integral:

$$\int_d^a \mathbf{B}_{da} \cdot d\mathbf{l}$$

$$d\mathbf{l} = \hat{\mathbf{r}}dr + \hat{\boldsymbol{\phi}}\, r\, d\phi + \hat{\mathbf{z}}dz$$

$$d\mathbf{l} = \hat{\mathbf{z}}dz$$



Example 3-14 Stokes' Theorem

Solution:

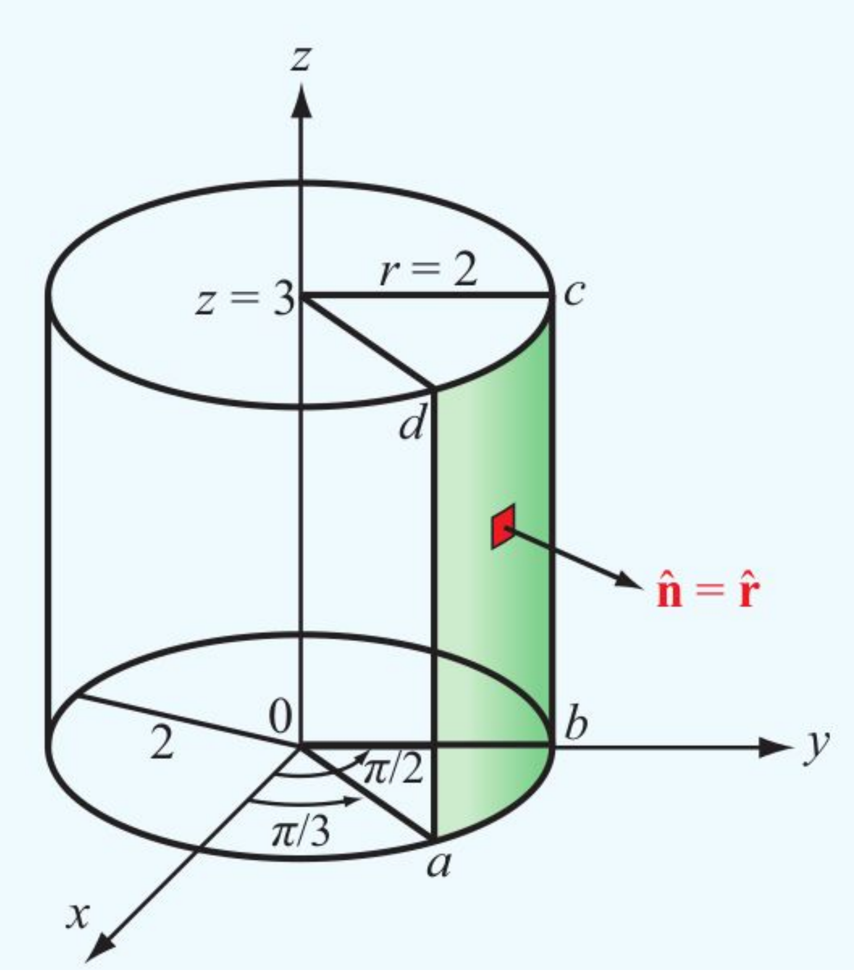
step2: evaluate each integral:

plug in:

$$\int_d^a \mathbf{B}_{da} \cdot d\mathbf{l} = \int_{z=3}^{0} \hat{\mathbf{z}}\frac{1}{4} \cdot \hat{\mathbf{z}}dz$$

$$\int_d^a \mathbf{B}_{da} \cdot d\mathbf{l} = \frac{1}{4}\int_{z=3}^{0} dz$$

$$\int_d^a \mathbf{B}_{da} \cdot d\mathbf{l} = \frac{1}{4}\Big[z\Big]_{z=3}^{0}$$



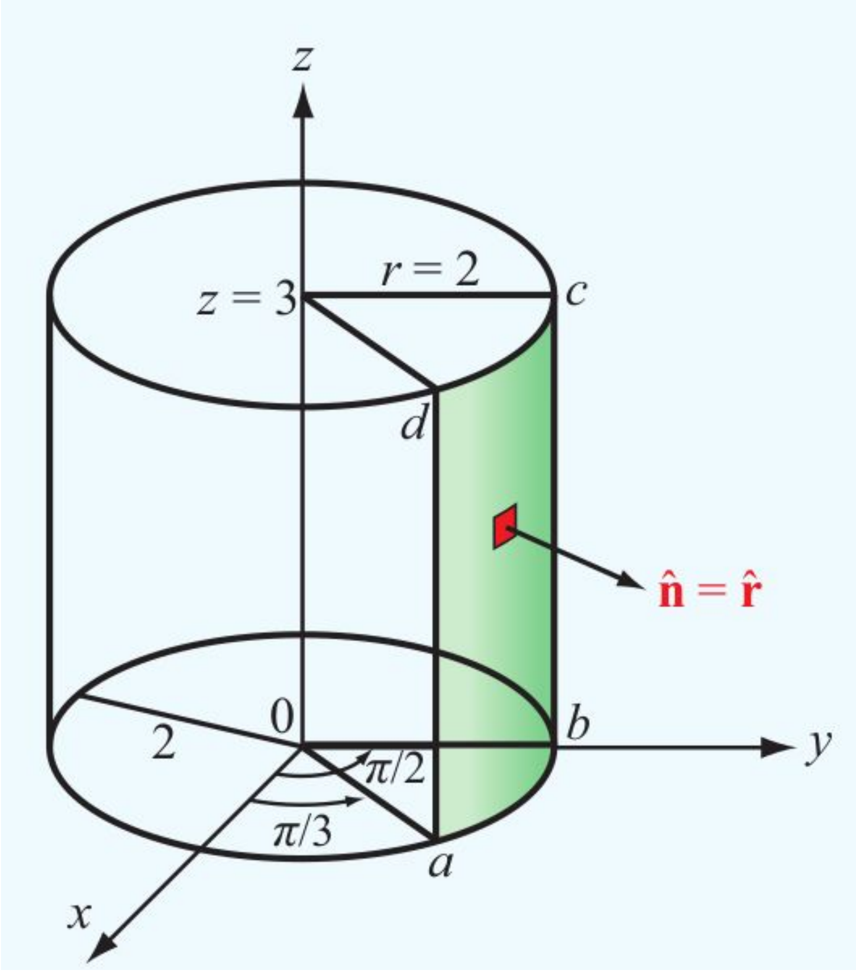
Example 3-14 Stokes' Theorem

Solution:
step2: evaluate each integral:

$$\int_d^a \mathbf{B}_{da} \cdot d\mathbf{l} = \frac{1}{4}\left[z\right]_{z=3}^{0}$$

$$\int_d^a \mathbf{B}_{da} \cdot d\mathbf{l} = \frac{1}{4}\left[0 - 3\right]$$

$$\int_d^a \mathbf{B}_{da} \cdot d\mathbf{l} = -\frac{3}{4}$$



Example 3-14 Stokes' Theorem

Solution:

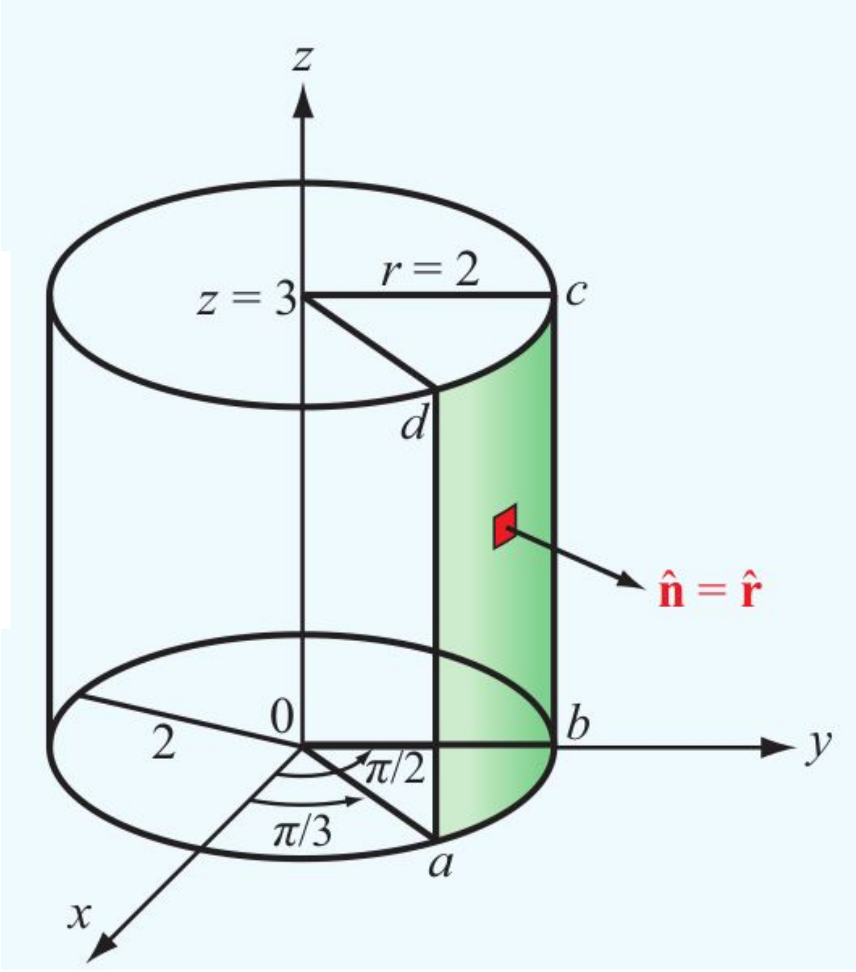
step3: sum integrals:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l} + \int_b^c \mathbf{B}_{bc} \cdot d\mathbf{l} + \int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l} + \int_d^a \mathbf{B}_{da} \cdot d\mathbf{l},$$

So:

Circulation = 0+0+0-3/4

Circulation = -3/4

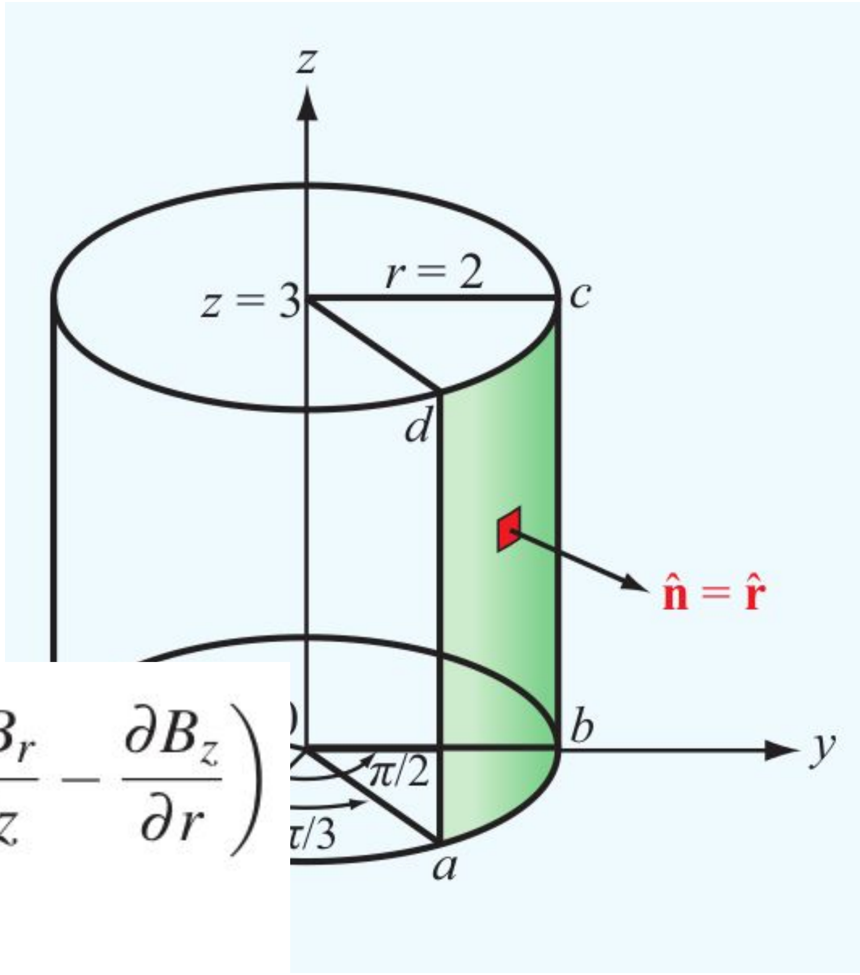


Example 3-14 Stokes' Theorem

Part 2:

Find:

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s}$$



Solution:

step1: find **∇**x**B:**

From appendix C:

$$\nabla \times \mathbf{B} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(rB_\phi) - \frac{\partial B_r}{\partial \phi}\right)$$

Example 3-14 Stokes' Theorem

Solution:

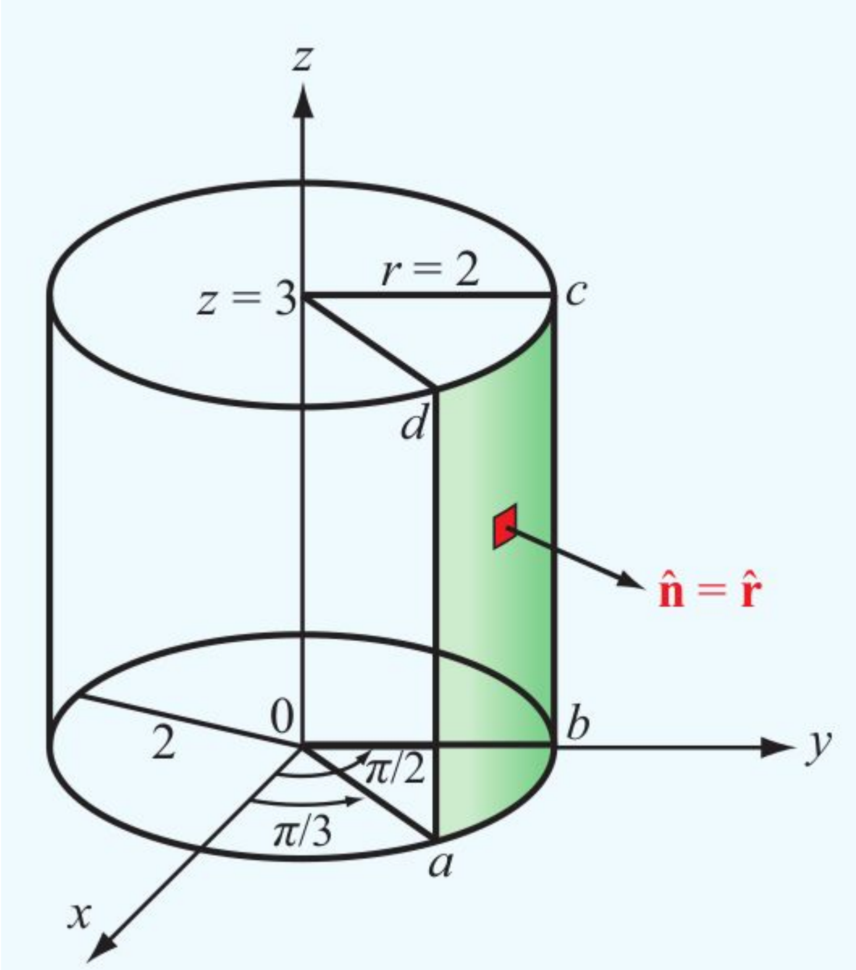
step1: find **∇xB:**

Since only B_z is non-zero:

∇xB=

$$= \hat{\mathbf{r}}\frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r}\right) - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial r}\left(\frac{\cos\phi}{r}\right)$$

∇xB $= -\hat{\mathbf{r}}\frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}.$



Example 3-14 Stokes' Theorem

Solution:

step2: find $d\mathbf{s}$:

From table 3-1:

use expression with $\hat{\mathbf{r}}$:

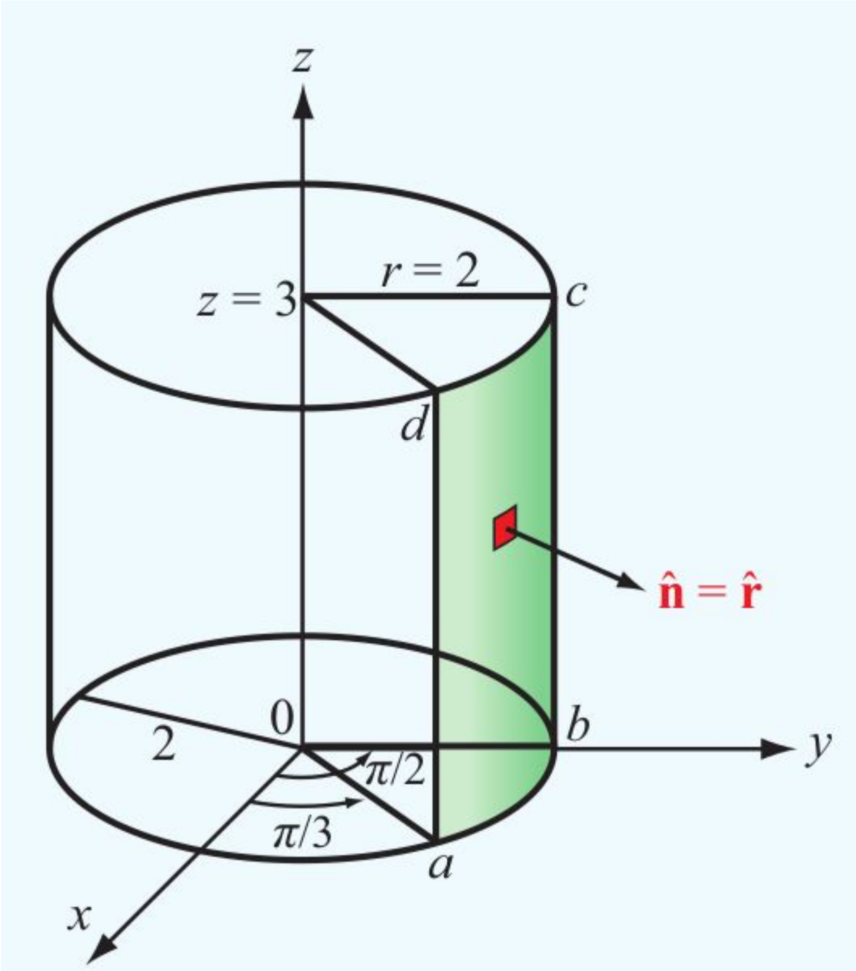
$$d\mathbf{s} = \hat{\mathbf{r}}\, r\, d\phi\, dz$$

step3: range of integration:

$r = 2$

ϕ: $\pi/3$ to $\pi/2$

z: 0 to 3



Example 3-14 Stokes' Theorem

Solution:

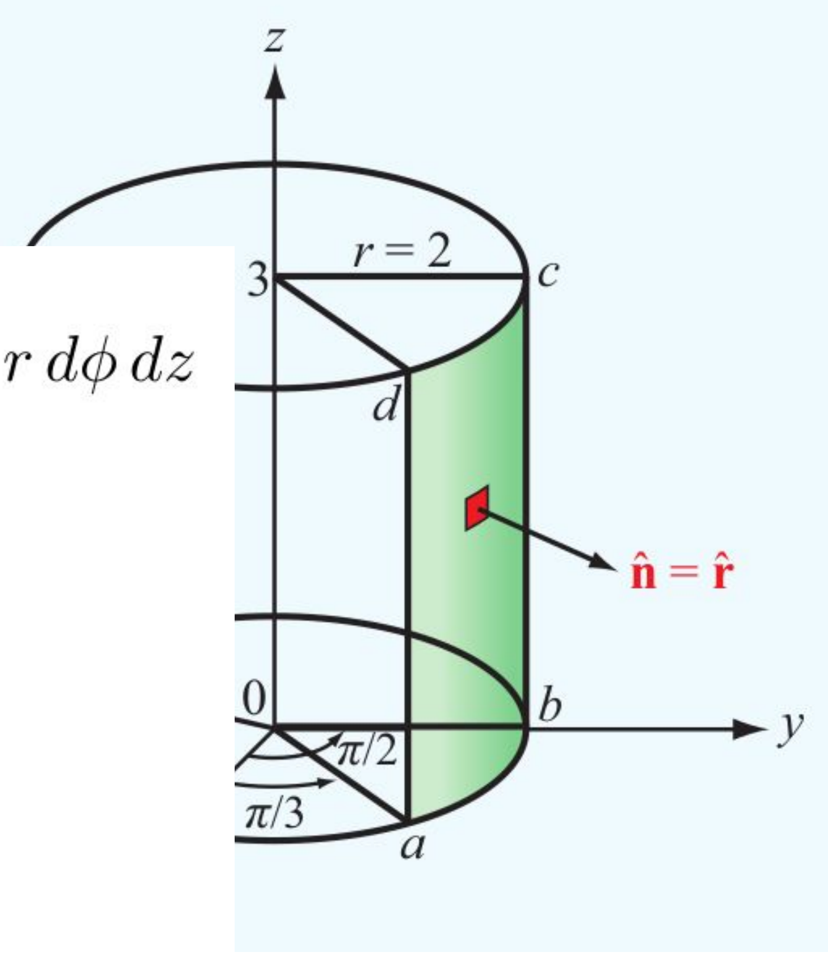
step3: plug in:

$$\int_{z=0}^{3}\int_{\phi=\pi/3}^{\pi/2}\left(-\hat{\mathbf{r}}\frac{\sin\phi}{r^2}+\hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\cdot\hat{\mathbf{r}}\,r\,d\phi\,dz$$

$$\int_{z=0}^{3}\int_{\phi=\pi/3}^{\pi/2}-\frac{\sin\phi}{r^2}\,r\,d\phi\,dz$$

$$3\int_{\phi=\pi/3}^{\pi/2}-\frac{\sin\phi}{r^2}\,r\,d\phi$$

$$\frac{3}{r}\int_{\phi=\pi/3}^{\pi/2}-\sin\phi\,d\phi$$



Example 3-14 Stokes' Theorem

Solution:

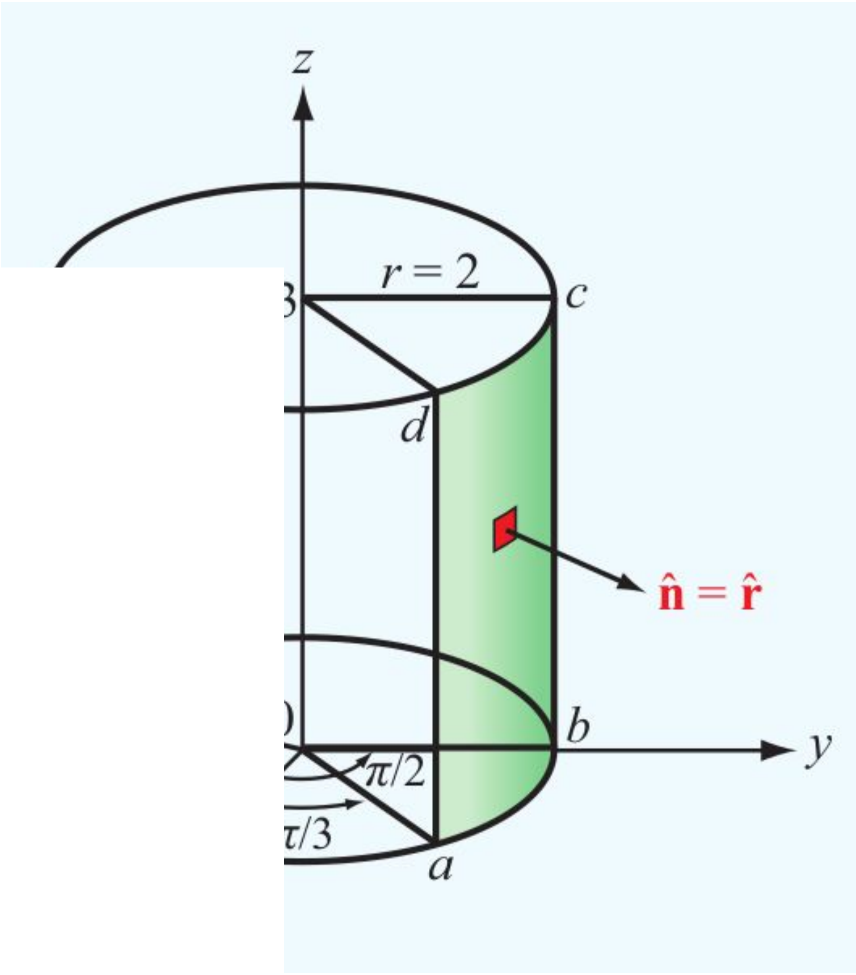
step3: plug in:

$$\frac{3}{r}\int_{\phi=\pi/3}^{\pi/2} -\sin\phi\, d\phi$$

$$\frac{3}{r}\Big[\cos\phi\Big]_{\phi=\pi/3}^{\pi/2}$$

$$\frac{3}{r}\Big[\cos(\pi/2)-\cos(\pi/3)\Big]$$

$$\frac{3}{r}\left[0-\frac{1}{2}\right]$$



Example 3-14 Stokes' Theorem

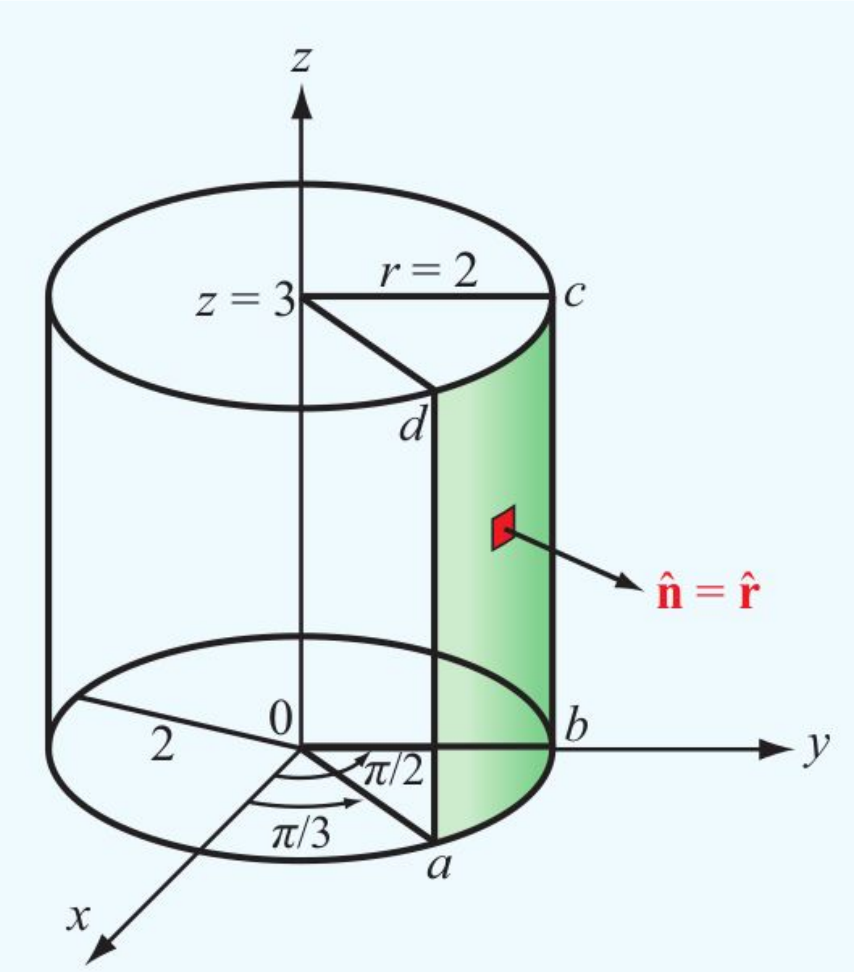
Solution:

step3: since the surface is at r=2:

$$\frac{3}{r}\left[0-\frac{1}{2}\right]$$

$$\frac{3}{2}\left[-\frac{1}{2}\right]$$

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = -\frac{3}{4}$$

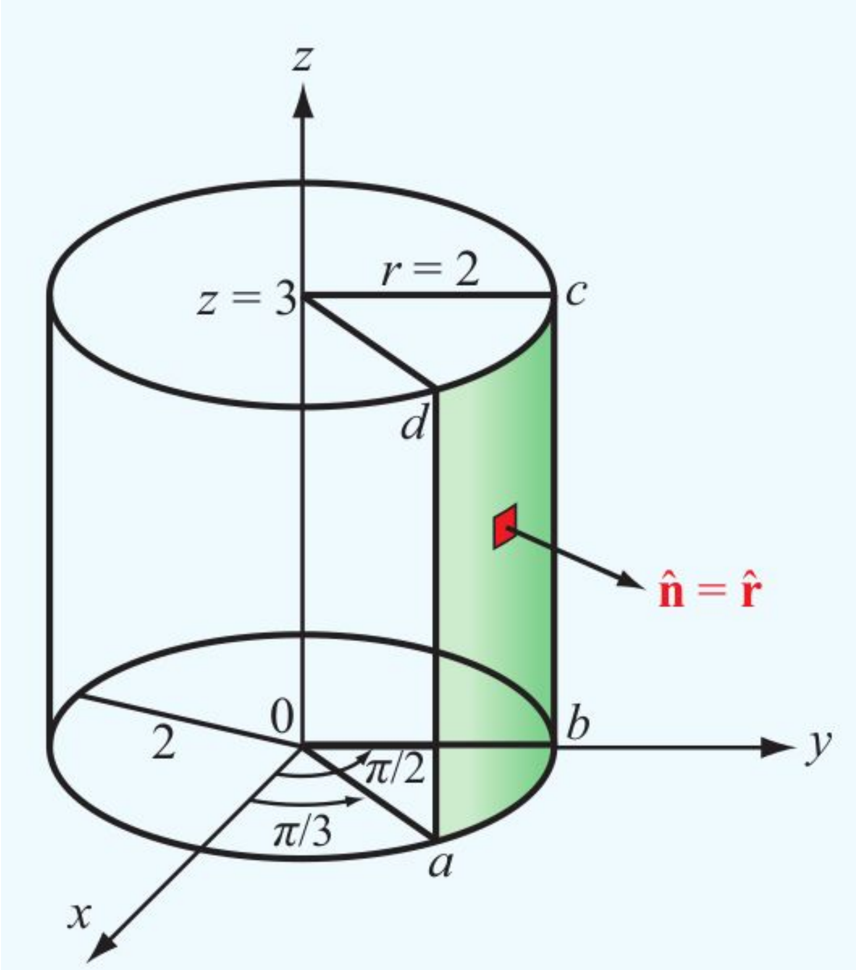


Example 3-14 Stokes' Theorem

Solution:

So, for this example,
Stokes' Theorem holds:

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$



Laplacian Operator

Laplacian of a Scalar Field

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} . \qquad (3.110)$$

Laplacian of a Vector Field

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E}$$

$$= \hat{\mathbf{x}} \nabla^2 E_x + \hat{\mathbf{y}} \nabla^2 E_y + \hat{\mathbf{z}} \nabla^2 E_z$$

We will use this equation in order to solve problems in the following chapters.

Homework

Homework 11 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Sections 4-1 through 4-4:

Maxwell's Equations
Charge and Current Distributions
Coulomb's Law
Gauss' Law