EECS 230
ENGINEERING ELECTROMAGNETICS
Leland Pierce

Transmission Lines 6

University of Michigan - Fall 2024

Chapter 2 Overview

What is a transmission line?

Why study transmission lines?

model

diffeq

solve diffeq

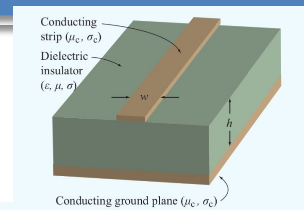
wave propagation

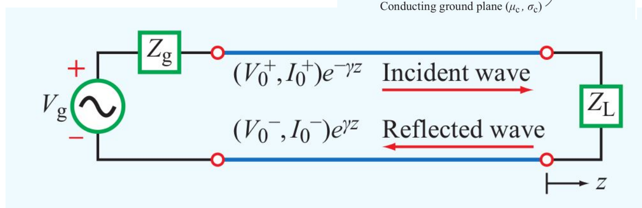
lossless xmission lir

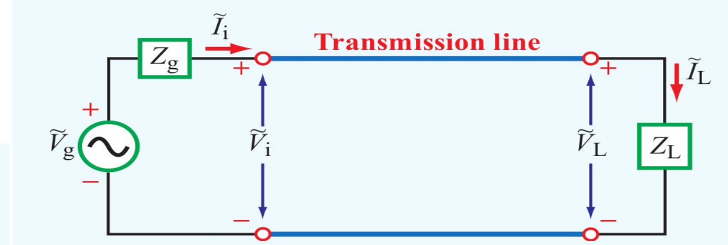
- microstrip lines
- reflections
- standing waves
- impedance
- short, open
- matching
- power flow

smith chart

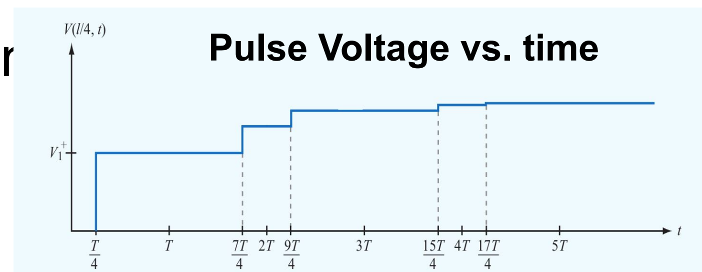
transients

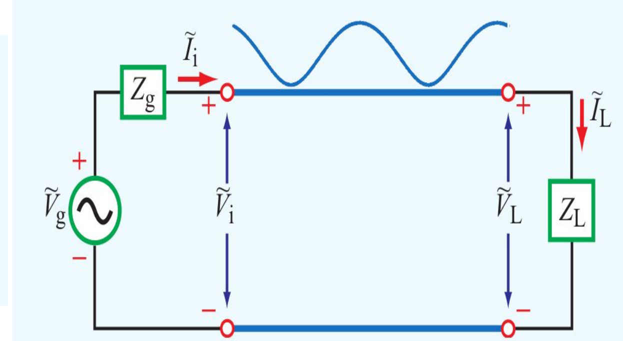




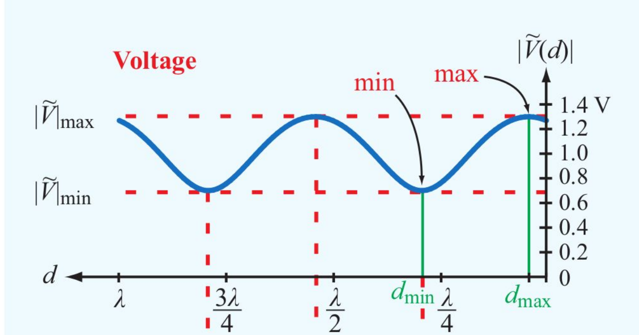


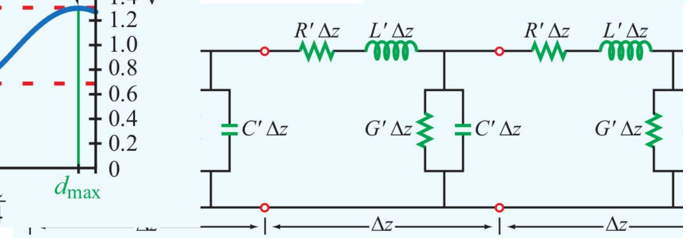
Typical High-Frequency Circuit

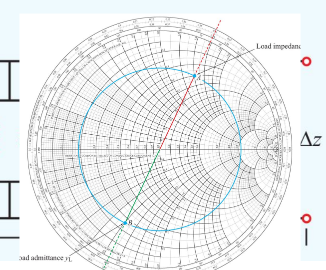




Waves on line: old methods don't work







Today's Lecture Coverage

Review Sections 2-1 through 2-5 of the book:

2-1: What is a transmission line? Why study transmission lines?
2-2: Lumped-Element Model
2-3: Governing Differential Eqns
2-4: Solve the Differential Equations
Properties of the solution: wave propagation
2-5: Lossless Microstrip Line
2-6: Lossless Transmission Lines
2-7: Lossless Transmission Lines: Wave Impedance
2-8: Lossless Transmission Lines: Special Cases
2-9: Lossless Transmission Lines: Power Flow

Section 2-10 of the book:

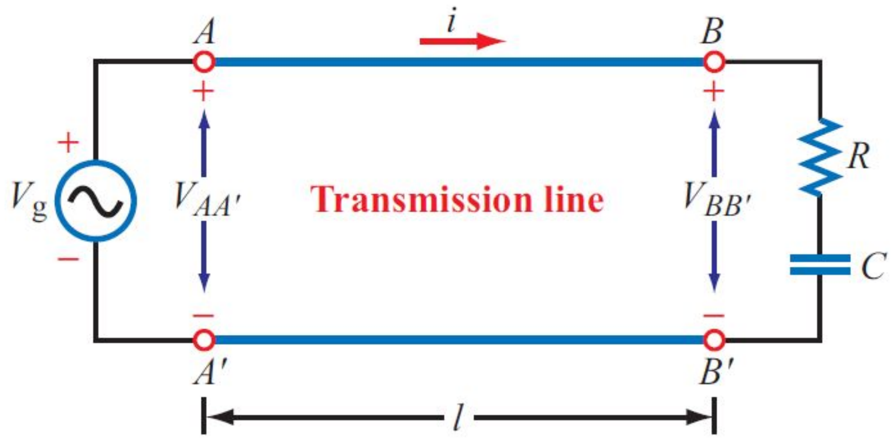
2-10: The Smith Chart

Chapter 2 Review

- A transmission line connects a generator to a load.

Chapter 2 Review

Phase Delay due to length of transmission line:



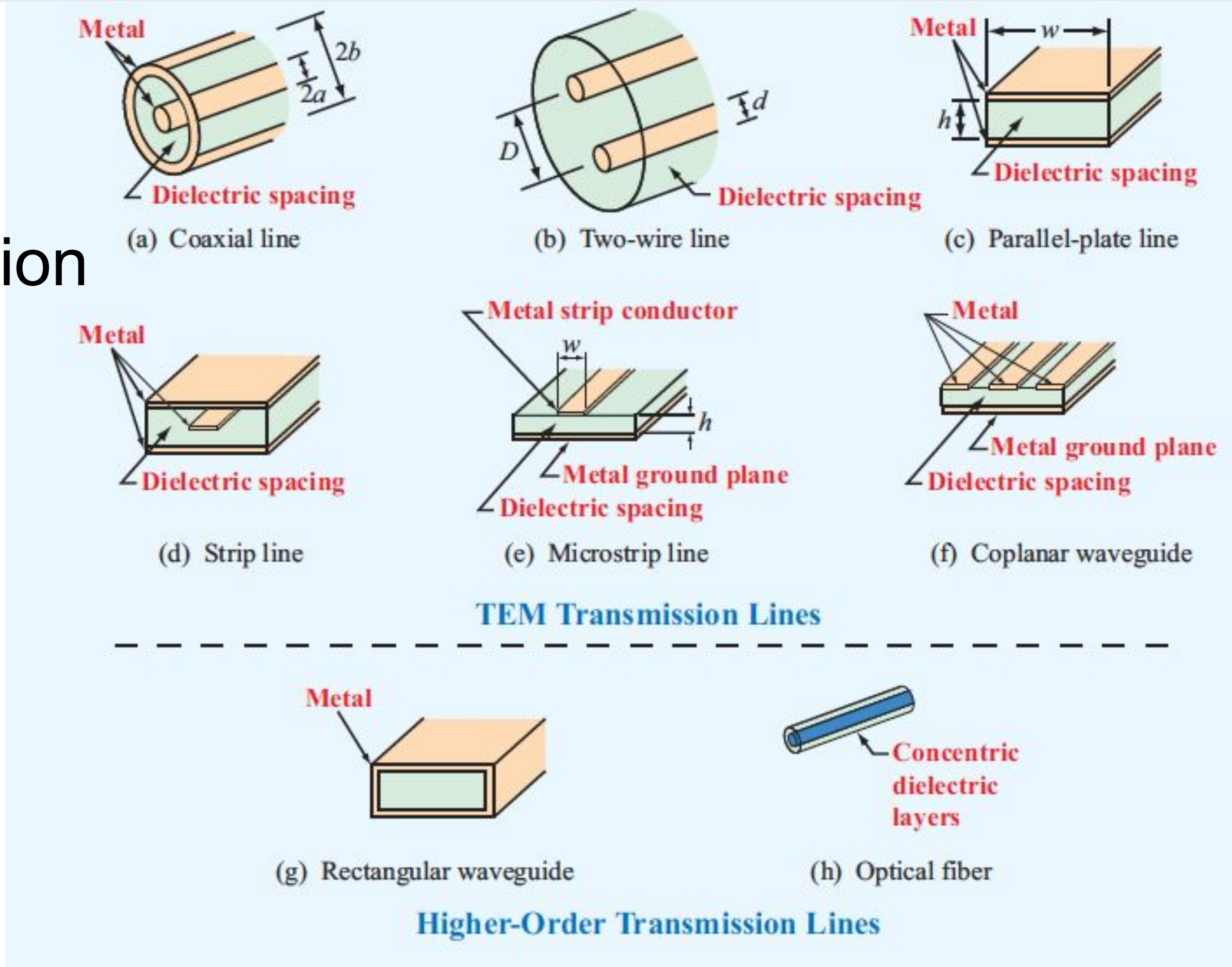
$$\phi_0 = \frac{\omega l}{c} = \frac{2\pi f l}{c} = 2\pi \frac{l}{\lambda} \quad \text{radians.}$$

$l/\lambda \lesssim 0.01$: Can ignore transmission-line effects

$l/\lambda \gtrsim 0.01$: Must deal with phase shift, and other effects...

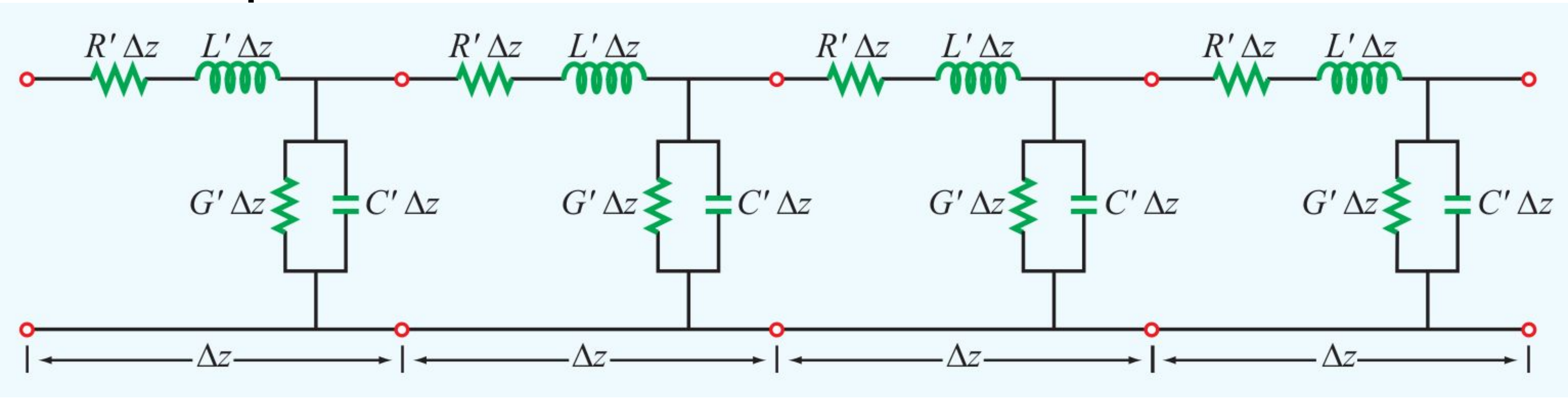
Chapter 2 Review

Different geometries for transmission lines



Chapter 2 Review

Lumped-Element Model:



All parameters are "per unit length":

R': Combined Resistance of BOTH conductors: □/m

L': Combined Inductance of BOTH conductors, H/m

G': Conductance of insulation between inner and outer conductor, S/m

C': Capacitance between inner and outer conductors, F/m

Chapter 2 Review

Lumped-Element Values: geometry/materials/freq

Table 2-1 **Transmission-line parameters R', L', G', and C' for three types of lines.**

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\varepsilon w}{h}$	F/m

$$R_s = \sqrt{\pi f \mu_c / \sigma_c}$$

Chapter 2 Review

Transmission-line governing Differential Equations

$$-\frac{d\widetilde{V}(z)}{dz} = (R' + j\omega L')\,\widetilde{I}(z),$$

$$-\frac{d\widetilde{I}(z)}{dz} = (G' + j\omega C')\,\widetilde{V}(z).$$

(telegrapher's equations in phasor form)

Chapter 2 Review

Transmission-line governing Differential Equation for *V*:

$$\frac{d^2\widetilde{V}(z)}{dz^2} - \gamma^2\,\widetilde{V}(z) = 0,$$

(wave equation for $\widetilde{V}(z)$)

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}\,.$$

(propagation constant)

Chapter 2 Review

Complex propagation constant:

$$\gamma = \alpha + j\beta$$

Where:

γ: Units of 1/m

α: Attenuation constant, units of Np/m (>0 in this class)

β: Phase constant, units of rad/m

Np and radians are both "dimensionless", and are the conventional "units" for these parameters.

Chapter 2 Review

Form of the solution: traveling waves, going in both directions:

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad \text{(V)},$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \qquad \text{(A)}.$$

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \qquad (\Omega),$$

Chapter 2 Review

Solution in time-domain

$$\upsilon(z,t) = |V_0^+|e^{-\alpha z}\cos(\omega t - \beta z + \phi^+) + |V_0^-|e^{+\alpha z}\cos(\omega t + \beta z + \phi^-)$$

Remaining unknowns are determined via specification of source and load.

Chapter 2 Review

- The wave equation for a general Transmission Line.

$$\frac{d^2\widetilde{V}(z)}{dz^2} - \gamma^2\,\widetilde{V}(z) = 0, \qquad (2.21)$$

- General solution of the wave equation
  - *It involves both incident and reflected waves*

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad (\text{V}),$$

$$\widetilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \qquad (\text{A}).$$

Chapter 2 Review

- Useful Relations for lossless Transmission Lines:

$$\alpha = 0 \qquad \text{(lossless line)},$$

$$\beta = \omega\sqrt{L'C'} \qquad \text{(lossless line)}. \qquad (2.45)$$

$$\beta = \omega\sqrt{\mu\varepsilon} \qquad \text{(rad/m)}, \qquad (2.49)$$

$$u_p = \frac{1}{\sqrt{\mu_0\epsilon_0}}\frac{1}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}}$$

$$\lambda = \frac{u_p}{f} = \frac{c}{f}\frac{1}{\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

$$Z_0 = \sqrt{\frac{L'}{C'}} \qquad \text{(lossless line)}, \qquad (2.46)$$

(REAL)

Chapter 2 Review

- Voltage reflection coefficient due to load:

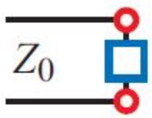
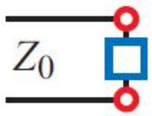
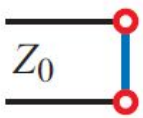
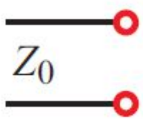
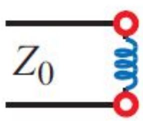
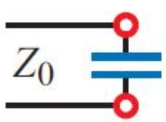
$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{z_L - 1}{z_L + 1}$$

- Load impedance in terms of Γ:

$$Z_L = \frac{1+\Gamma}{1-\Gamma} Z_0$$

Chapter 2 Review

Reflection Coefficient $\Gamma = |\Gamma| e^{j\theta_r}$

Load	$\lvert\Gamma\rvert$	θ_r
 $Z_L = (r + jx)Z_0$	$\left[\dfrac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$	$\tan^{-1}\left(\dfrac{x}{r-1}\right) - \tan^{-1}\left(\dfrac{x}{r+1}\right)$
 Z_0	0 (no reflection)	irrelevant
 (short)	1	$\pm 180°$ (phase opposition)
 (open)	1	0 (in-phase)
 $jX = j\omega L$	1	$\pm 180° - 2\tan^{-1} x$
 $jX = \dfrac{-j}{\omega C}$	1	$\pm 180° + 2\tan^{-1} x$

$$z_L = Z_L / Z_0 = (R + jX)/Z_0 = r + jx$$

Chapter 2 Review

- Concept of standing wave

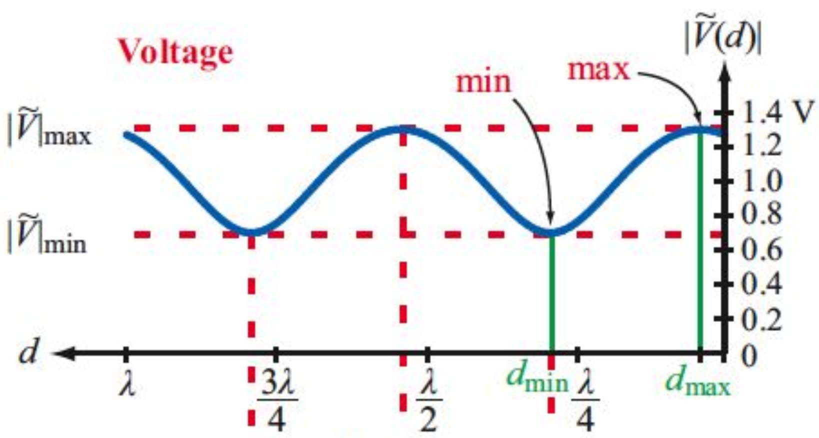
$$\widetilde{V}(z) = V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z}),$$

$$\widetilde{I}(z) = \frac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}).$$

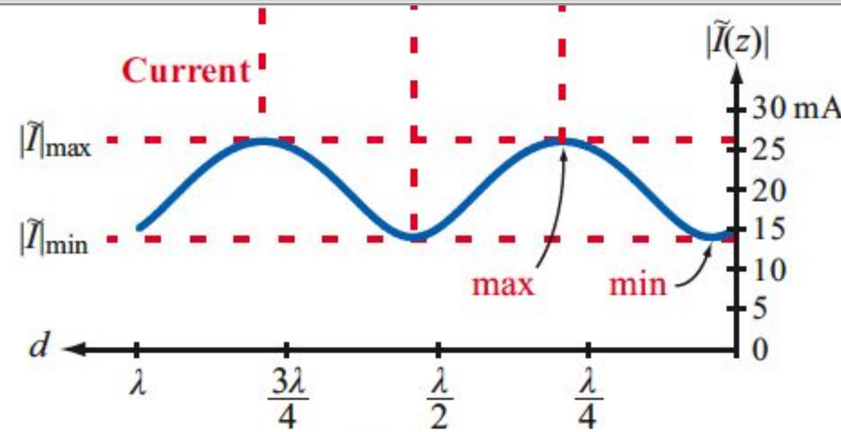
  - Voltage (current) magnitudes at any point on line:

$$|\widetilde{V}(d)| = |V_0^+|\left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)\right]^{1/2}. \quad (2.66)$$

Chapter 2 Review



(a) $|\widetilde{V}(d)|$ versus d



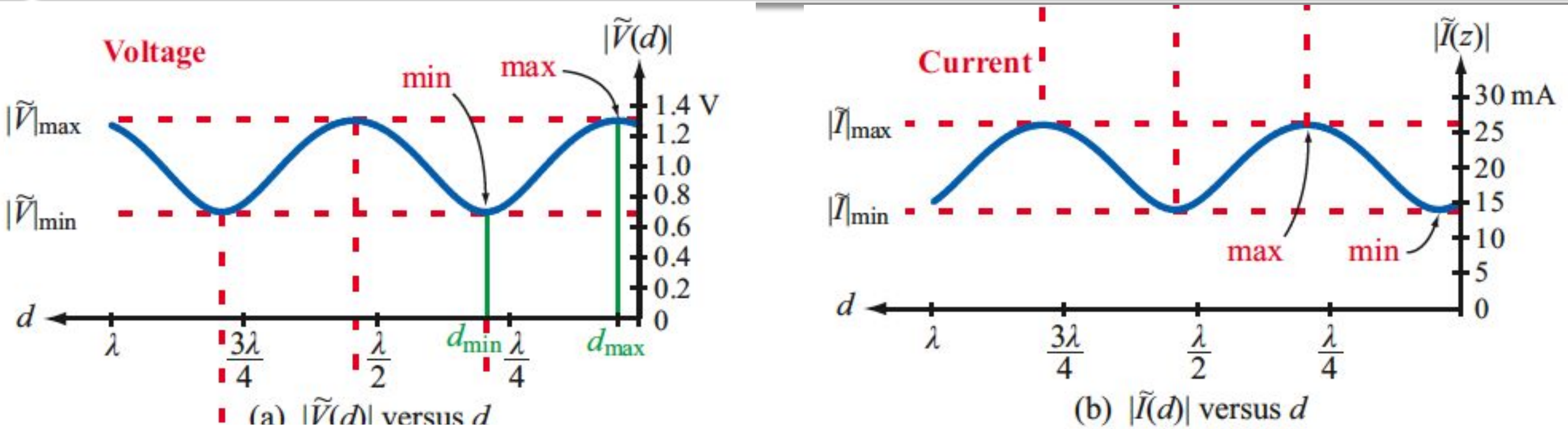
(b) $|\widetilde{I}(d)|$ versus d

- Location of minima / maxima

$$d_{max} = \frac{\theta_r + 2n\pi}{2\beta} = \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2}, \qquad \begin{cases} n = 1, 2, \ldots & \text{if } \theta_r < 0, \\ n = 0, 1, 2, \ldots & \text{if } \theta_r \geq 0, \end{cases} \qquad (2.70)$$

Value of V_{max}: $|\widetilde{V}(d)| = |\widetilde{V}|_{max} = |V_0^+|[1 + |\Gamma|],$

Chapter 2 Review



- Spatial period of standing wave: $\frac{\lambda}{2}$

- Standing wave ratio S:

$$S = \frac{|\widetilde{V}|_{max}}{|\widetilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

Chapter 2 Review

Wave Impedance:

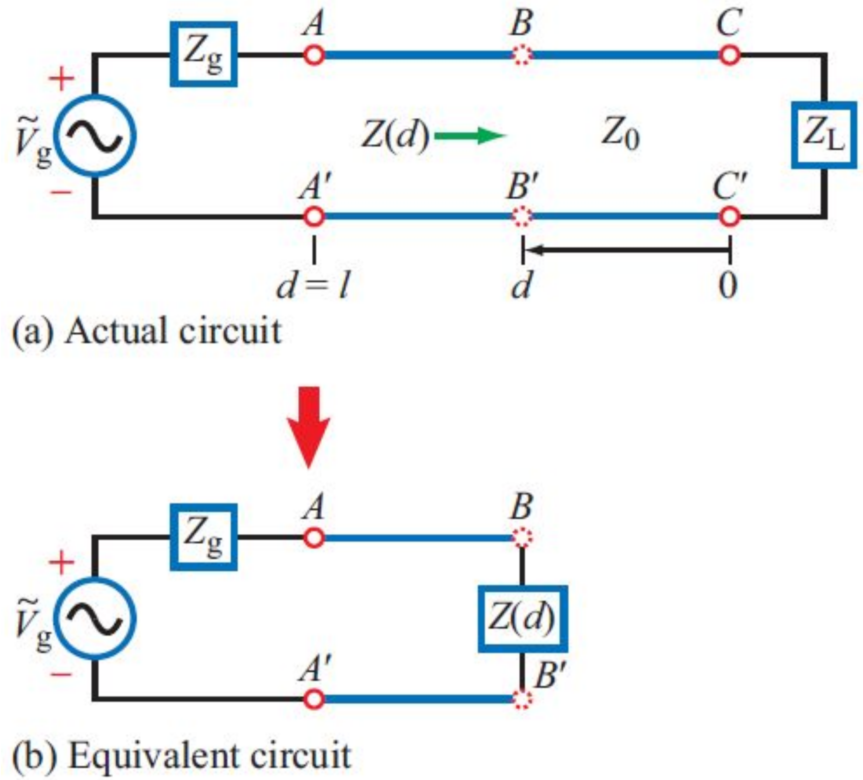
At a distance *d* from the load:

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)}$$

$$= \frac{V_0^+[e^{j\beta d} + \Gamma e^{-j\beta d}]}{V_0^+[e^{j\beta d} - \Gamma e^{-j\beta d}]} Z_0$$

$$= Z_0 \left[\frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}\right]$$

$$= Z_0 \left[\frac{1 + \Gamma_d}{1 - \Gamma_d}\right] \qquad (\Omega),$$

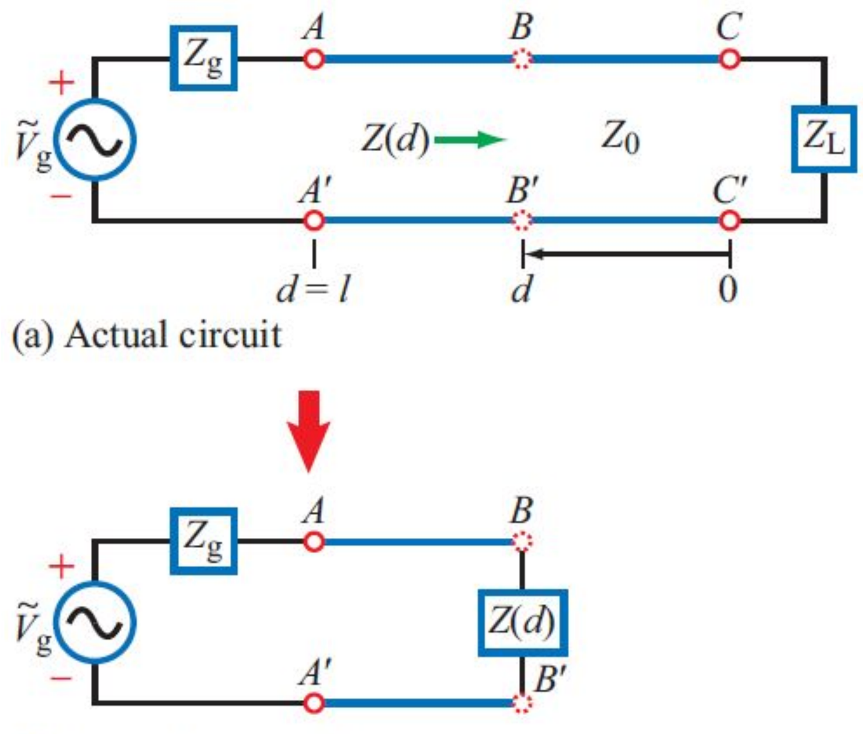


Chapter 2 Review

Define the phase-shifted voltage reflection coefficient:

$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_r} e^{-j2\beta d}$$

$$= |\Gamma| e^{j(\theta_r - 2\beta d)}$$



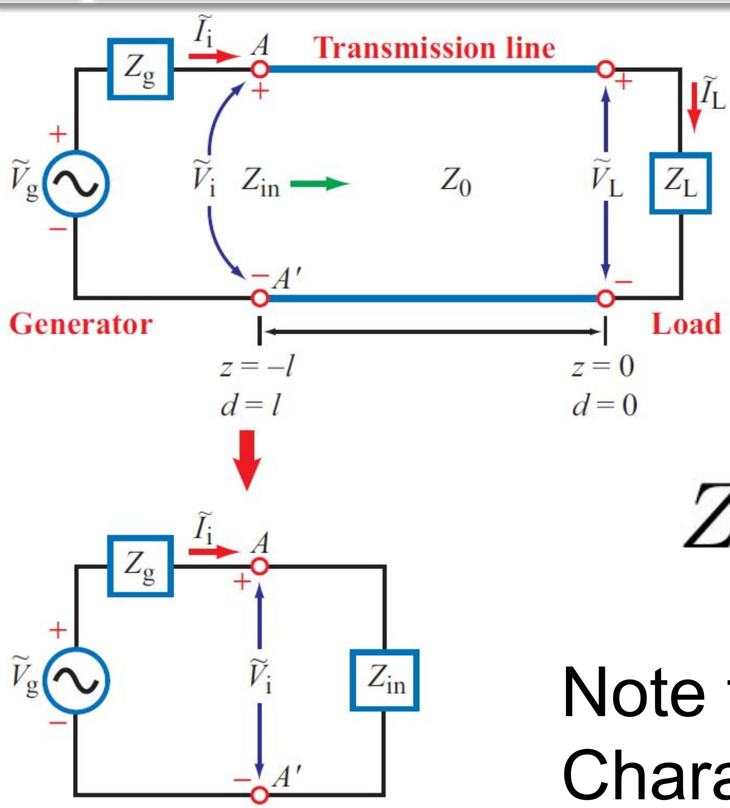
(a) Actual circuit

(b) Equivalent circuit

Z(d) is different than Z_0:
Ratio of Total Voltage and Current

Recall: $Z_0 = \dfrac{V_0^+}{I_0^+}$

Chapter 2 Review



Input impedance:

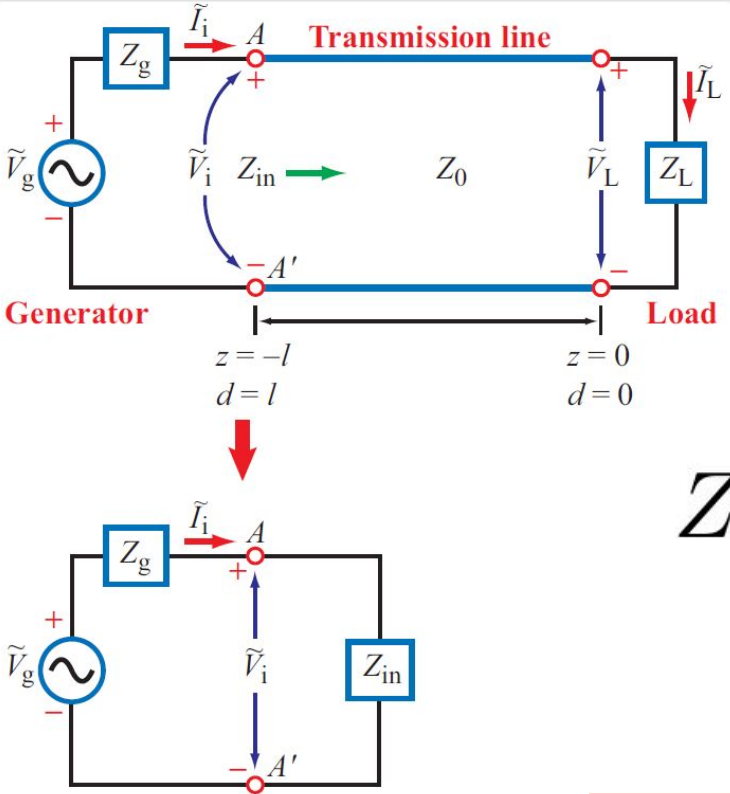
impedance of the transmission line at *d=l:*

$$Z_{\text{in}} = Z(d = l) = \frac{\widetilde{V}(d = l)}{\widetilde{I}(d = l)}$$

Note that Z_{in} is different from the Characteristic Impedance, and is different from the Load Impedance:

$$Z_{\text{in}} \neq Z_0 \qquad Z_{\text{in}} \neq Z_L$$

Chapter 2 Review

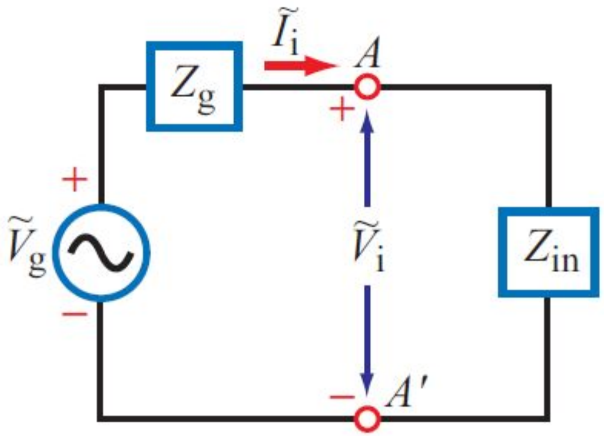


Input impedance:
impedance of the transmission line at *d=l:*

$$Z_{\text{in}} = Z(d=l) = \frac{\widetilde{V}(d=l)}{\widetilde{I}(d=l)}$$

$$Z_{\text{in}} = Z_0 \left[\frac{z_L + j\tan(\beta l)}{1 + jz_L \tan(\beta l)} \right]$$

Chapter 2 Review



Voltage Amplitude:

$$V_0^+ = \left(\frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}} \right) \left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}} \right).$$

We used the boundary conditions (Z_L, Z_g , V_g, l) to solve for the 2 unknowns.

This completes the solution of the transmission line differential equation.

Chapter 2 Review

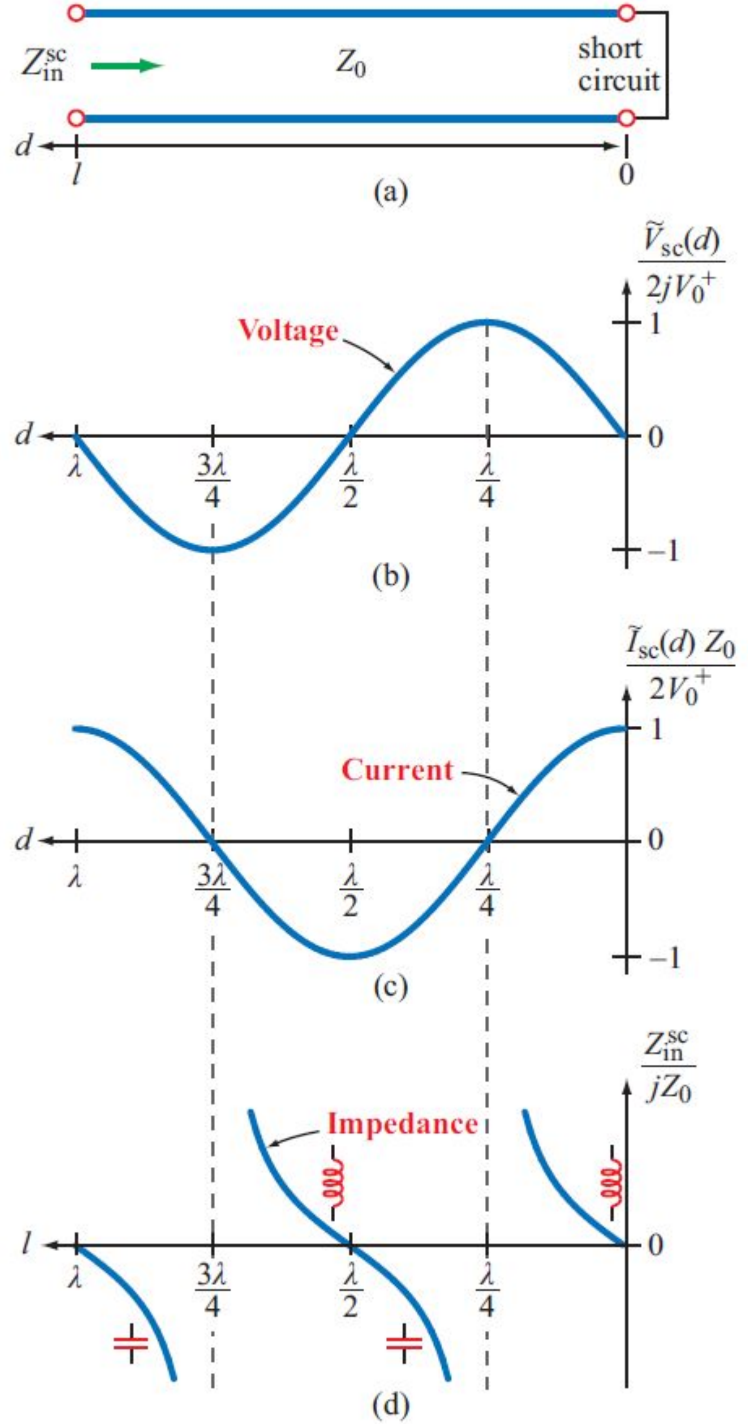
For the short-circuited line:

$$\Gamma = -1$$

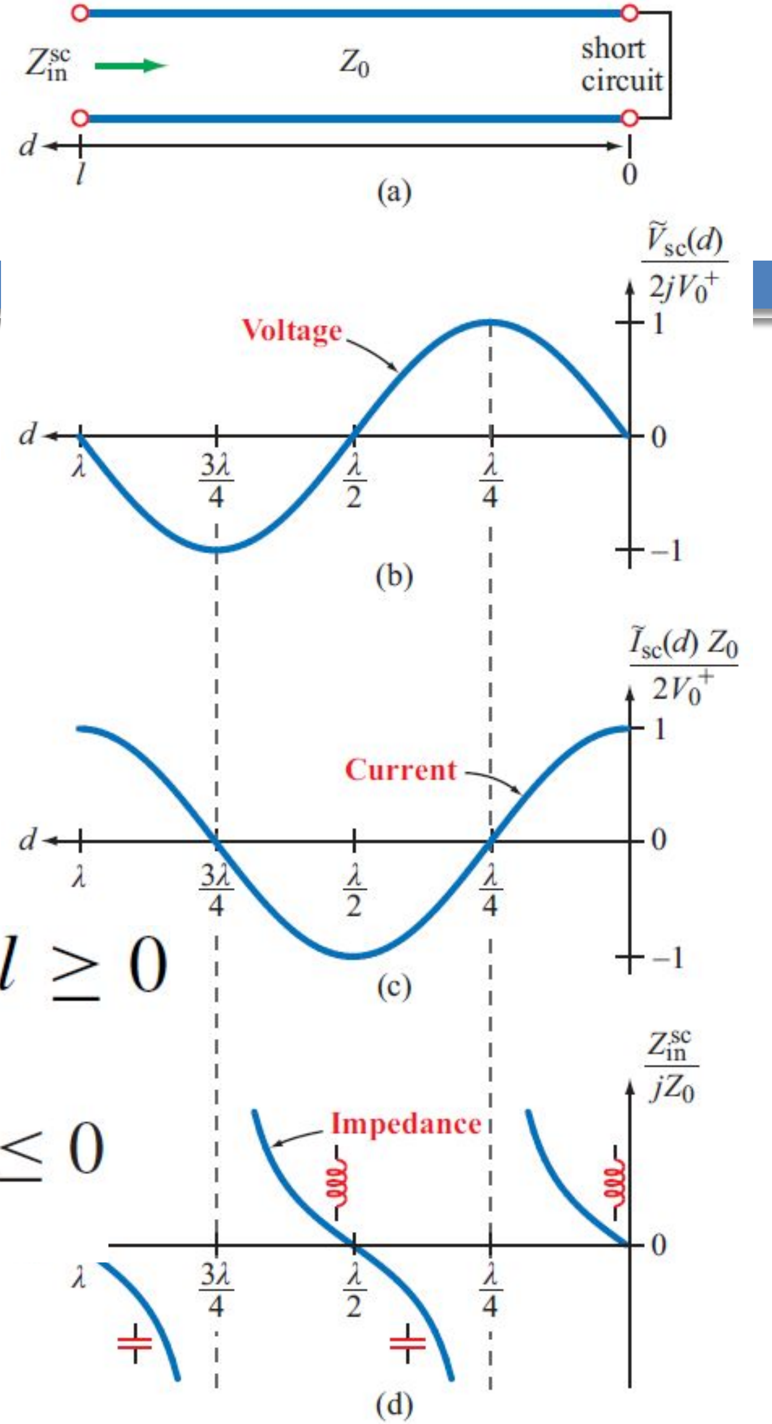
$$\widetilde{V}_{sc}(d) = 2jV_0^+ \sin\beta d,$$

$$\widetilde{I}_{sc}(d) = \frac{2V_0^+}{Z_0}\cos\beta d,$$

$$Z_{sc}(d) = \frac{\widetilde{V}_{sc}(d)}{\widetilde{I}_{sc}(d)} = jZ_0 \tan\beta d.$$



Chapter 2 Review



At its input, the short-circuited line **appears** like an inductor or a capacitor depending on the sign of

$$\tan \beta d$$

$$j\omega L_{eq} = jZ_0 \tan \beta l, \qquad \text{if } \tan \beta l \geq 0$$

$$\frac{1}{j\omega C_{eq}} = jZ_0 \tan \beta l, \qquad \text{if } \tan \beta l \leq 0$$

Chapter 2 Review

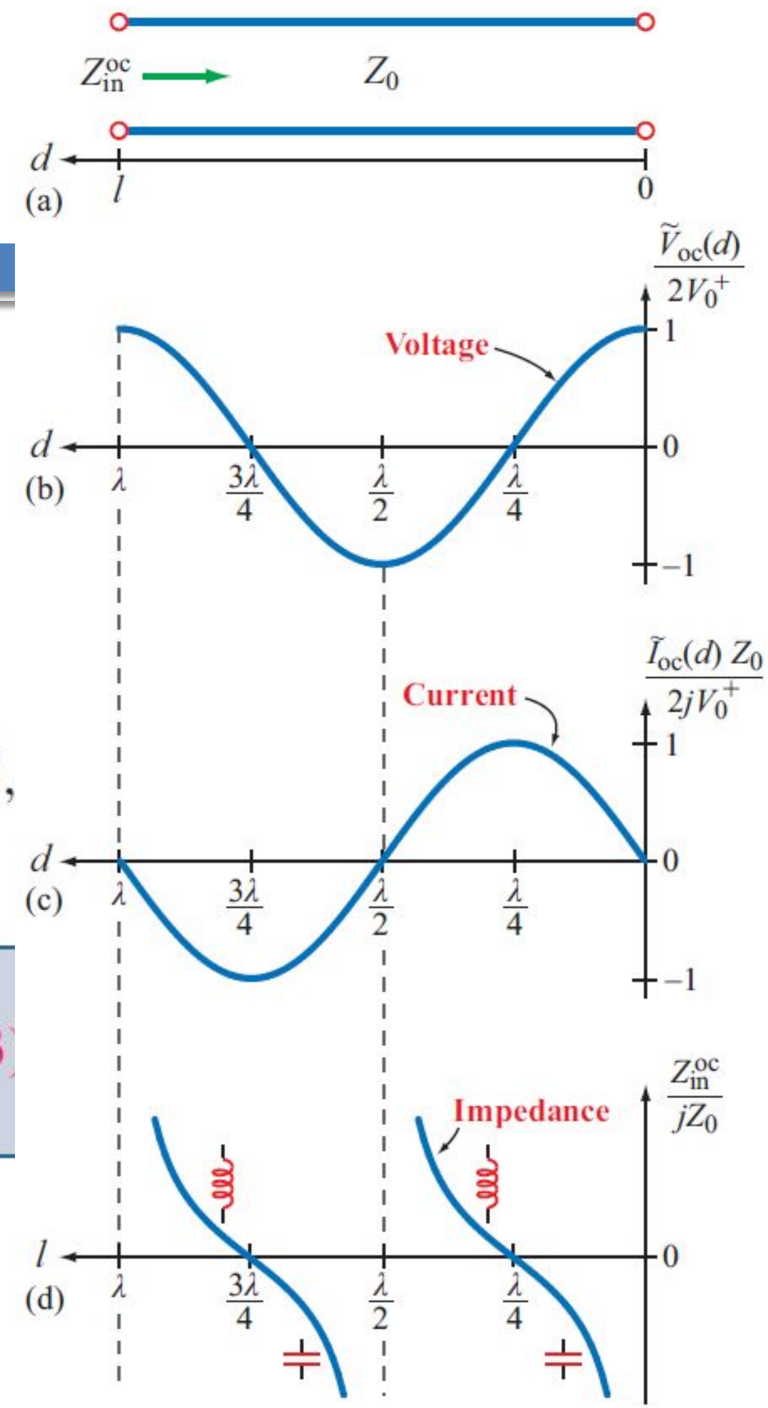
Open-circuited Line:

$$\Gamma = 1$$

$$\widetilde{V}_{oc}(d) = V_0^+[e^{j\beta d} + e^{-j\beta d}] = 2V_0^+ \cos \beta d,$$

$$\widetilde{I}_{oc}(d) = \frac{V_0^+}{Z_0}[e^{j\beta d} - e^{-j\beta d}] = \frac{2jV_0^+}{Z_0} \sin \beta d,$$

$$Z_{in}^{oc} = \frac{\widetilde{V}_{oc}(l)}{\widetilde{I}_{oc}(l)} = -jZ_0 \cot \beta l. \qquad (2.93)$$



Chapter 2 Review

Short-Circuit/Open-Circuit Method:

- Given length *l*
- Measure Z_{in} twice:
  - when terminated in a short
  - when terminated in an open

Use both: get Z_0, β:

$$Z_{in}^{sc} = \frac{\tilde{V}_{sc}(l)}{\tilde{I}_{sc}(l)} = jZ_0 \tan \beta l.$$

$$Z_{in}^{oc} = \frac{\tilde{V}_{oc}(l)}{\tilde{I}_{oc}(l)} = -jZ_0 \cot \beta l.$$

$$Z_0 = +\sqrt{Z_{in}^{sc} \, Z_{in}^{oc}} \, ,$$

$$\tan \beta l = \sqrt{\frac{-Z_{in}^{sc}}{Z_{in}^{oc}}} \, .$$

Chapter 2 Review

Half-Wavelength Line:

$$Z_{\text{in}} = Z_0 \left(\frac{z_L + j\tan(\beta l)}{1 + j z_L \tan(\beta l)} \right)$$

If $l = n\lambda/2$, where n is an integer,

$$\tan \beta l = \tan\left[(2\pi/\lambda)\,(n\lambda/2)\right] = \tan n\pi = 0.$$

Consequently, Eq. (2.79) reduces to

$$Z_{\text{in}} = Z_{\text{L}}, \qquad \text{for } l = n\lambda/2, \qquad (2.96)$$

which means that a half-wavelength line (or any integer multiple of $\lambda/2$) does not modify the load impedance.

Chapter 2 Review

Quarter-Wavelength Line:

$$Z_{\text{in}} = Z_0 \left(\frac{z_L + j\tan(\beta l)}{1 + j z_L \tan(\beta l)} \right)$$

For $l = \lambda/4$, $\beta l = (2\pi/\lambda)(\lambda/4) = \pi/2$

So, as $\beta l \to \pi/2$, $\tan(\beta l) \to \infty$

$$\lim_{\beta l \to \pi/2} Z_{\text{in}} = Z_0 \left(\frac{j\tan(\beta l)}{j z_L \tan(\beta l)} \right) = \frac{Z_0^2}{Z_L}$$

$$Z_{\text{in}} = \frac{Z_0^2}{Z_L}, \qquad \text{for } l = \lambda/4 + n\lambda/2.$$

Chapter 2 Review

Instantaneous Power Flow:

$$P(d,t) = P^{\mathrm{i}}(d,t) + P^{\mathrm{r}}(d,t)$$

The 2 terms are the Incident and Reflected power:

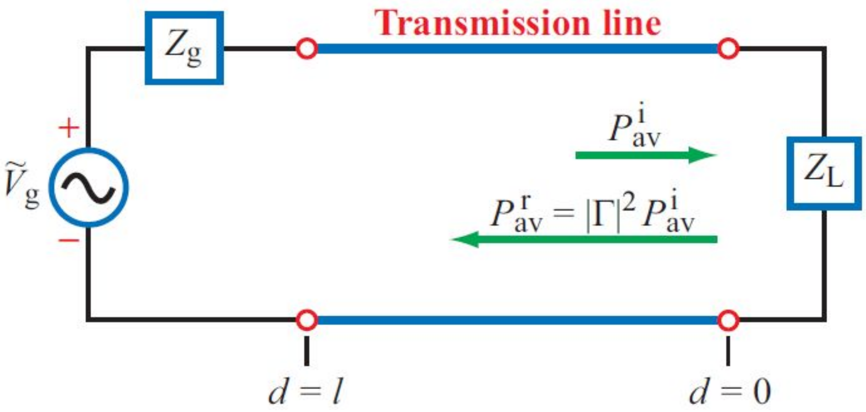
$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t + 2\beta d + 2\phi^+)],$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_{\mathrm{r}})].$$

oscillating at TWICE the frequency of *V* or *I*

Chapter 2 Review

Average Power Flow:



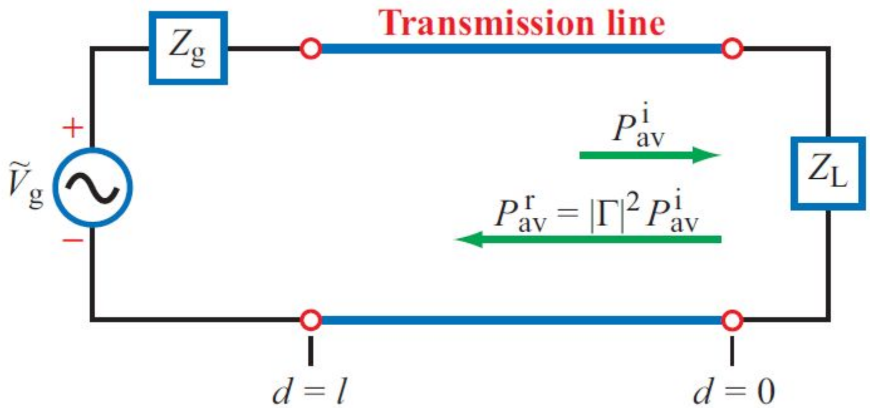
For the incident power, average over one period:

$$P^{i}_{avg} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \frac{|V_0^+|^2}{2Z_0} \left[1 + \cos(2\omega t + 2\beta d + 2\phi^+)\right] dt$$

$$\omega = \frac{2\pi}{T}, \qquad hence \quad T = \frac{2\pi}{\omega}$$

Chapter 2 Review

Average Power Flow:

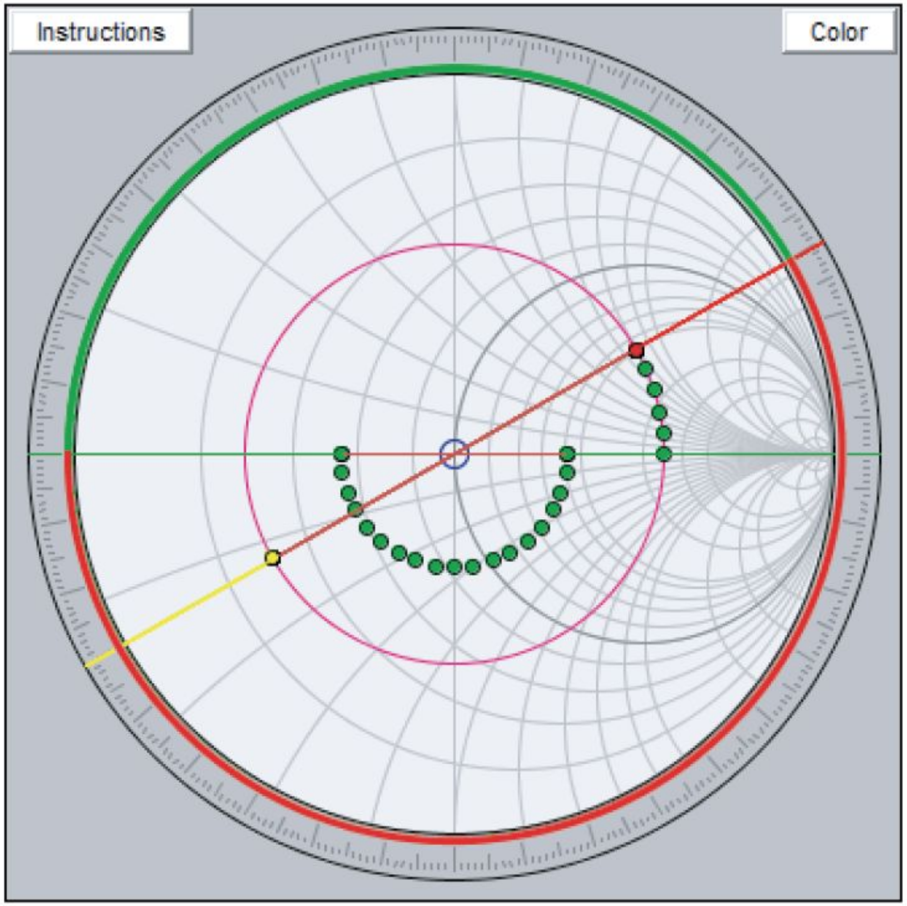


$$P^i_{avg} = \frac{|V_0^+|^2}{2Z_0}$$

$$P^r_{avg} = -|\Gamma|^2 P^i_{avg}$$

2-10 The Smith Chart

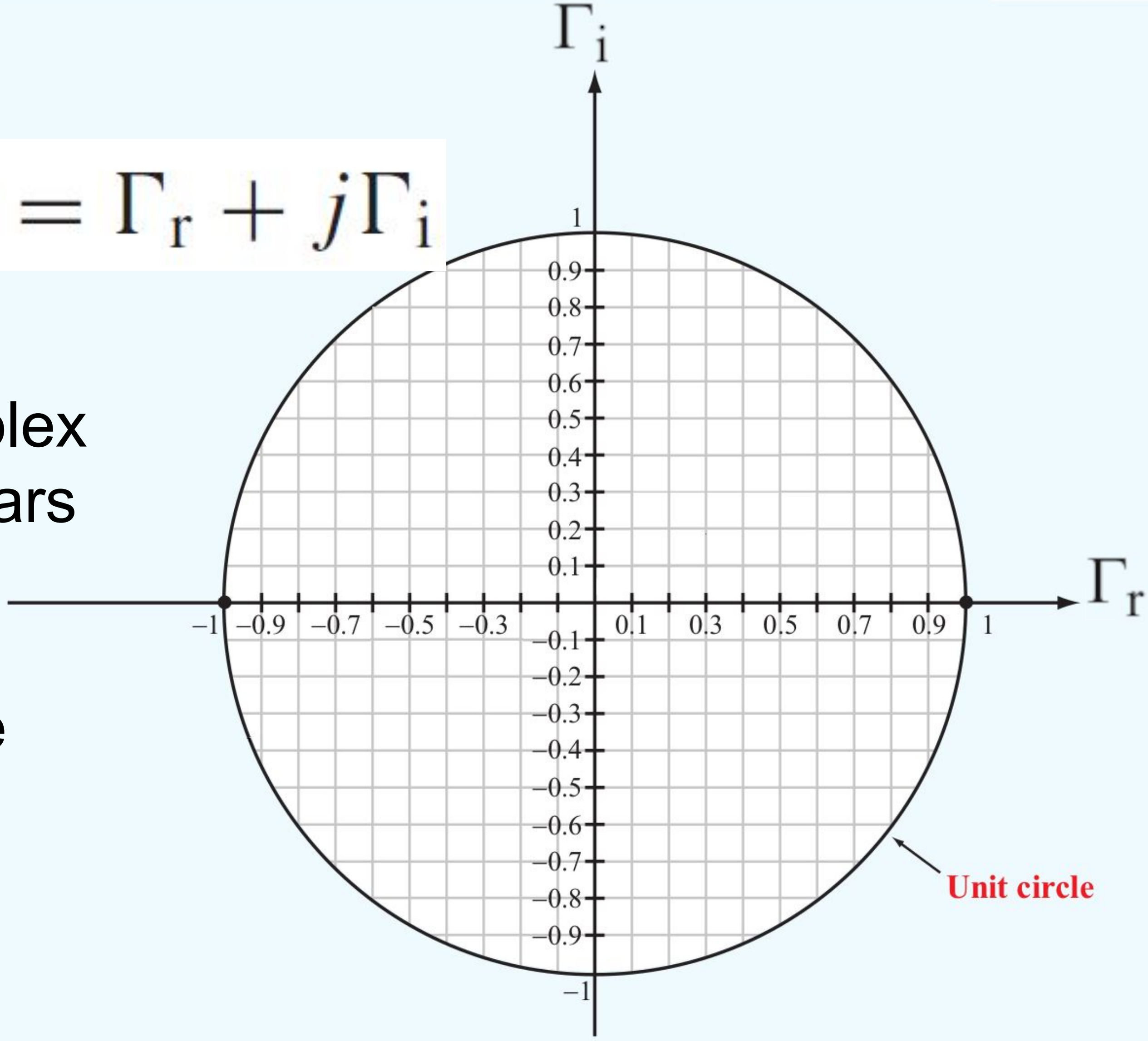
- Developed in 1939 by P. W. Smith as a graphical tool to analyze and design transmission-line circuits
- Today, it is used to characterize the performance of microwave circuits, etc.



2-10 Complex Plane

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

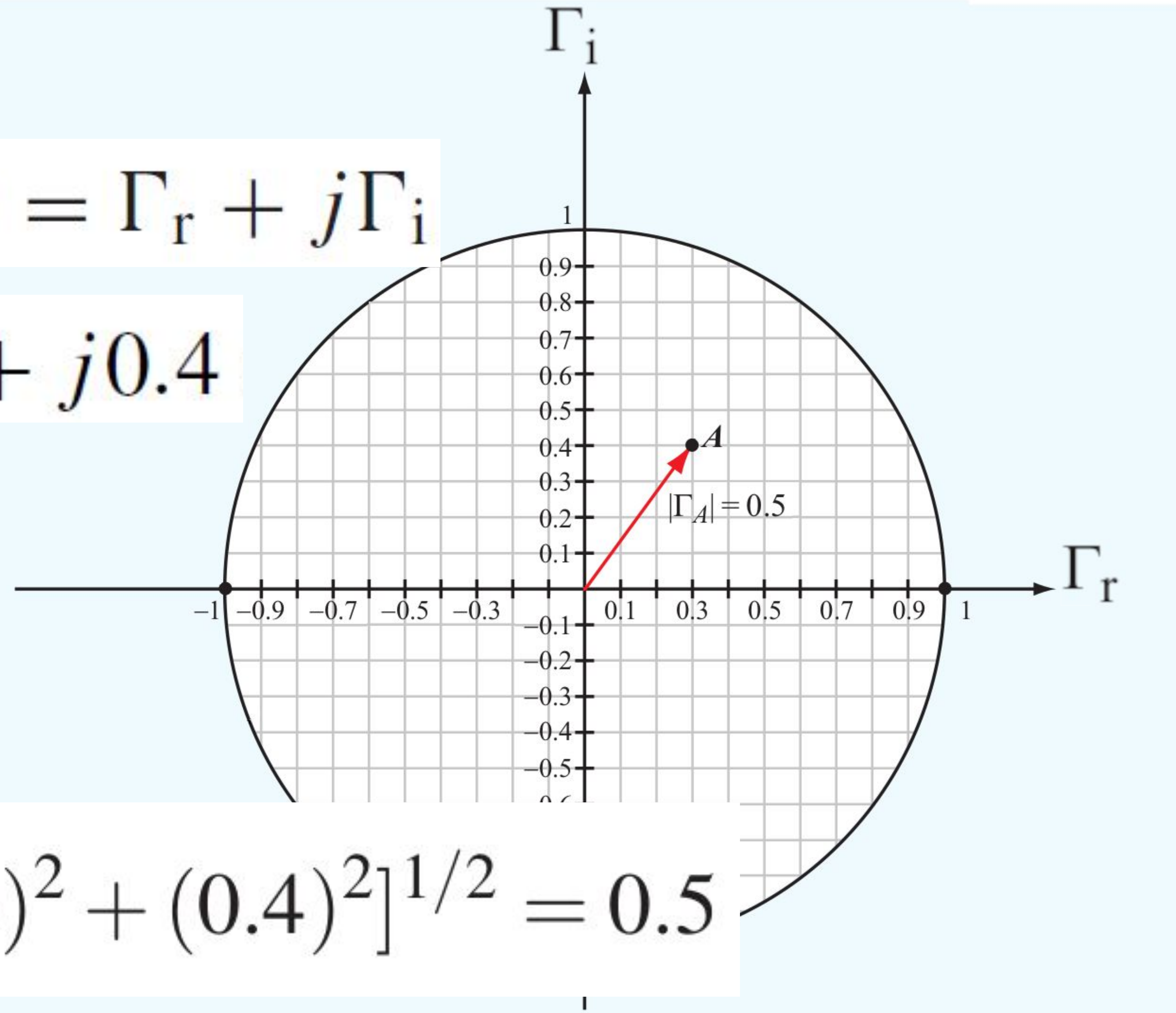
This is a complex number: appears within the unit circle on the complex plane



2-10 Complex Plane

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_A = 0.3 + j0.4$$

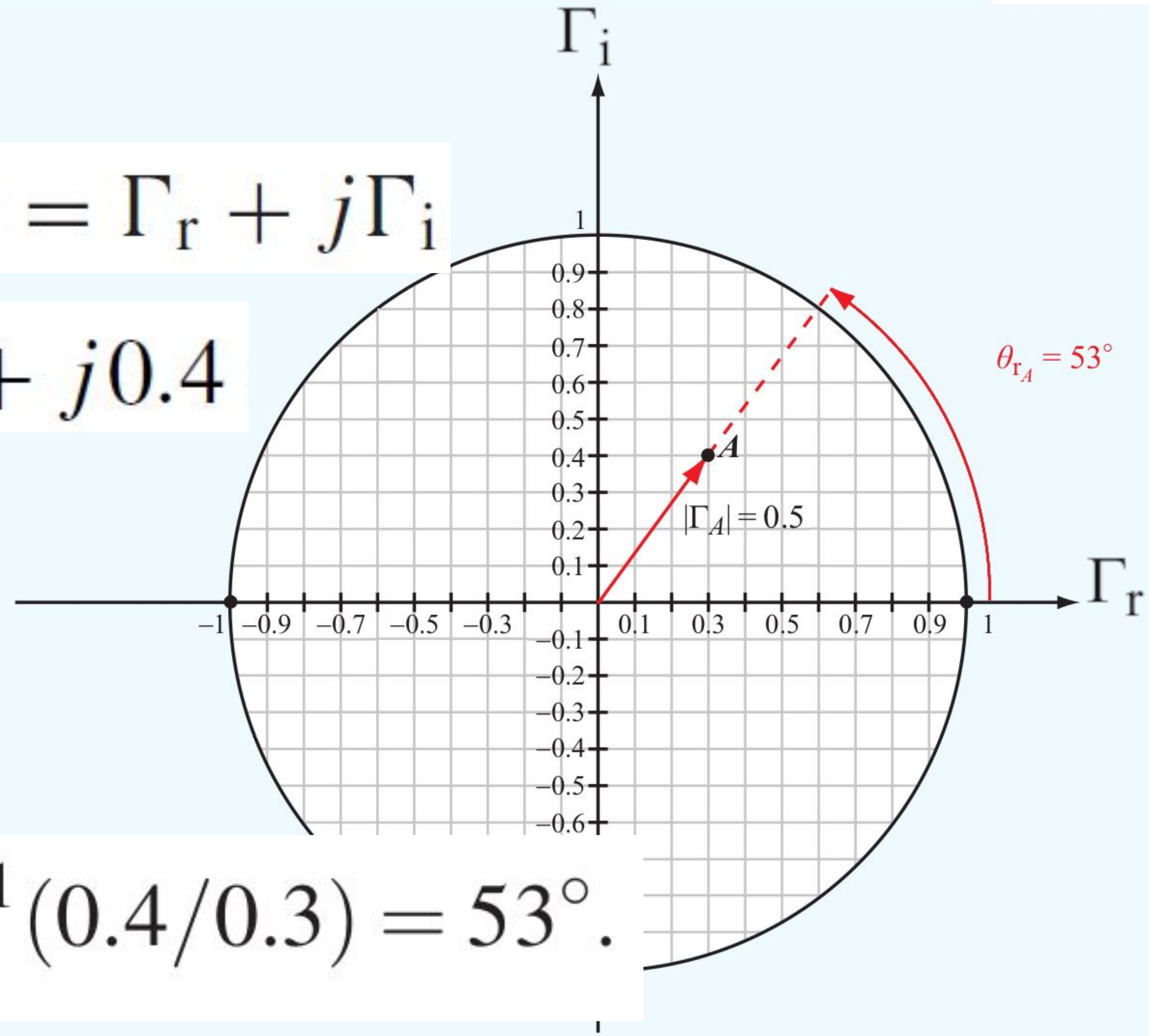


$$|\Gamma_A| = [(0.3)^2 + (0.4)^2]^{1/2} = 0.5$$

2-10 Complex Plane

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_A = 0.3 + j0.4$$



$$\theta_{r_A} = \tan^{-1}(0.4/0.3) = 53°.$$

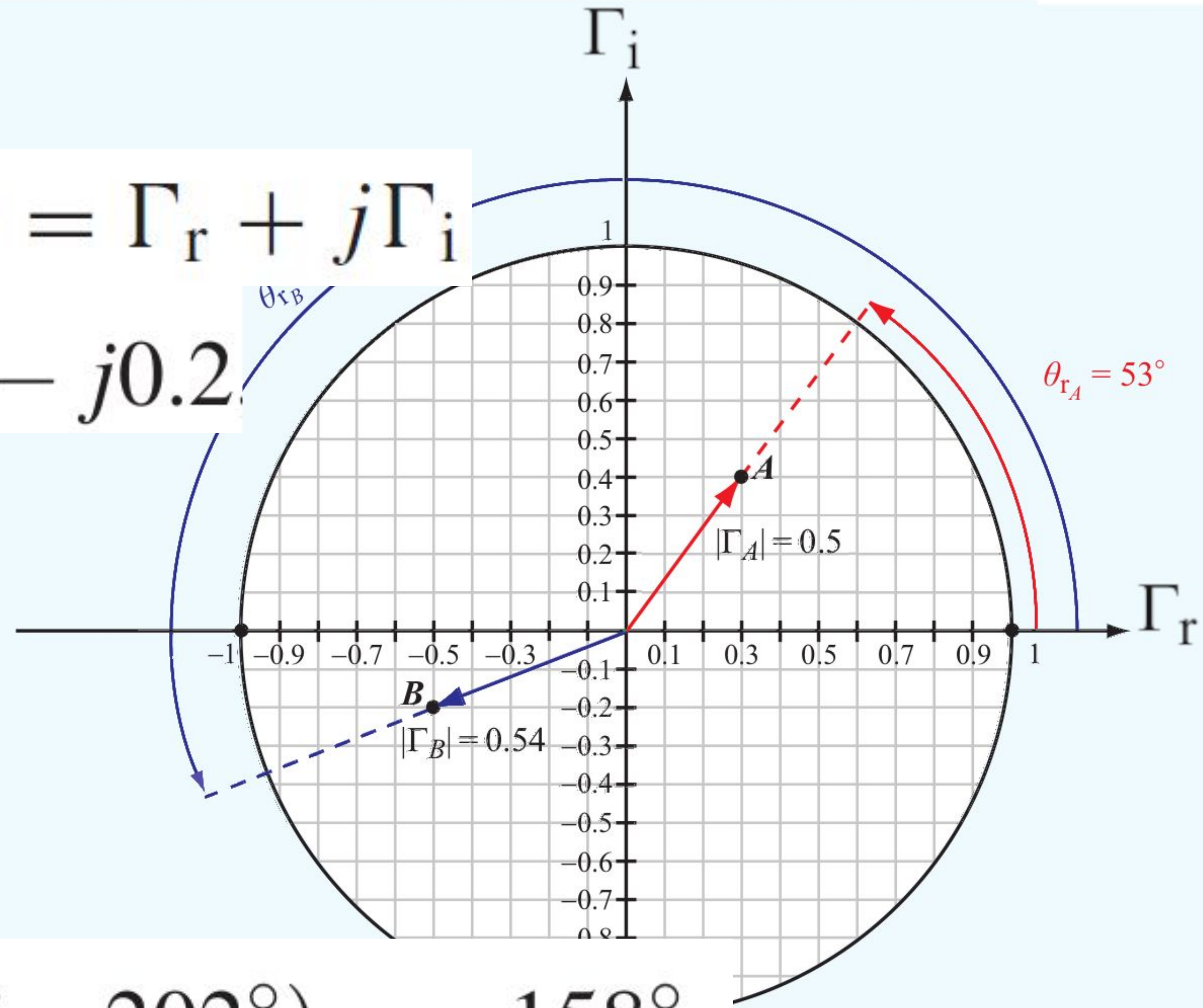
2-10 Complex Plane

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_B = -0.5 - j0.2$$

$$|\Gamma_B| = 0.54$$

$$\theta_{r_B} = 202^\circ$$



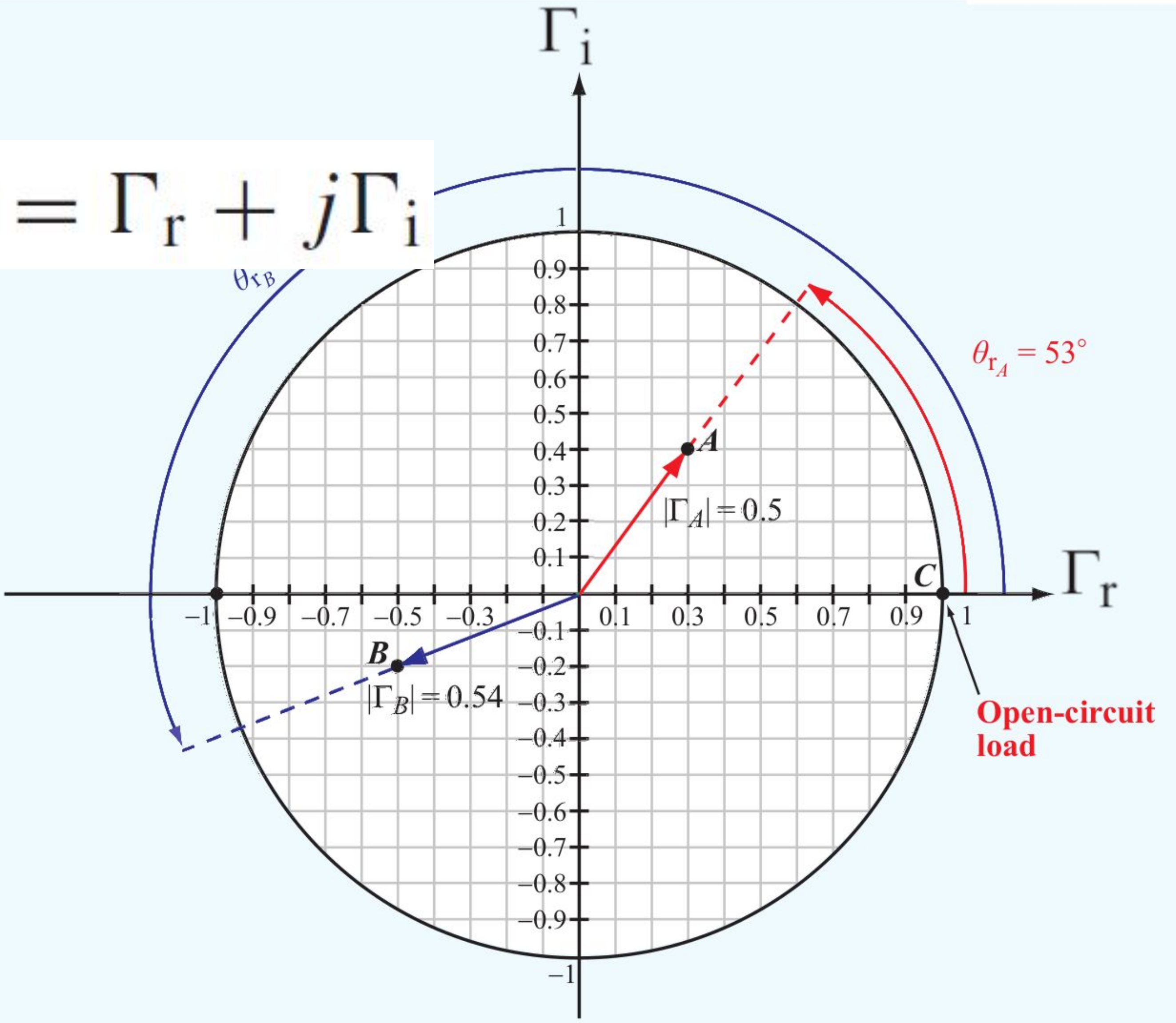
$$\theta_{r_B} = (360^\circ - 202^\circ) = -158^\circ.$$

2-10 Complex Plane

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_C = 1$$

(Open-circuit)

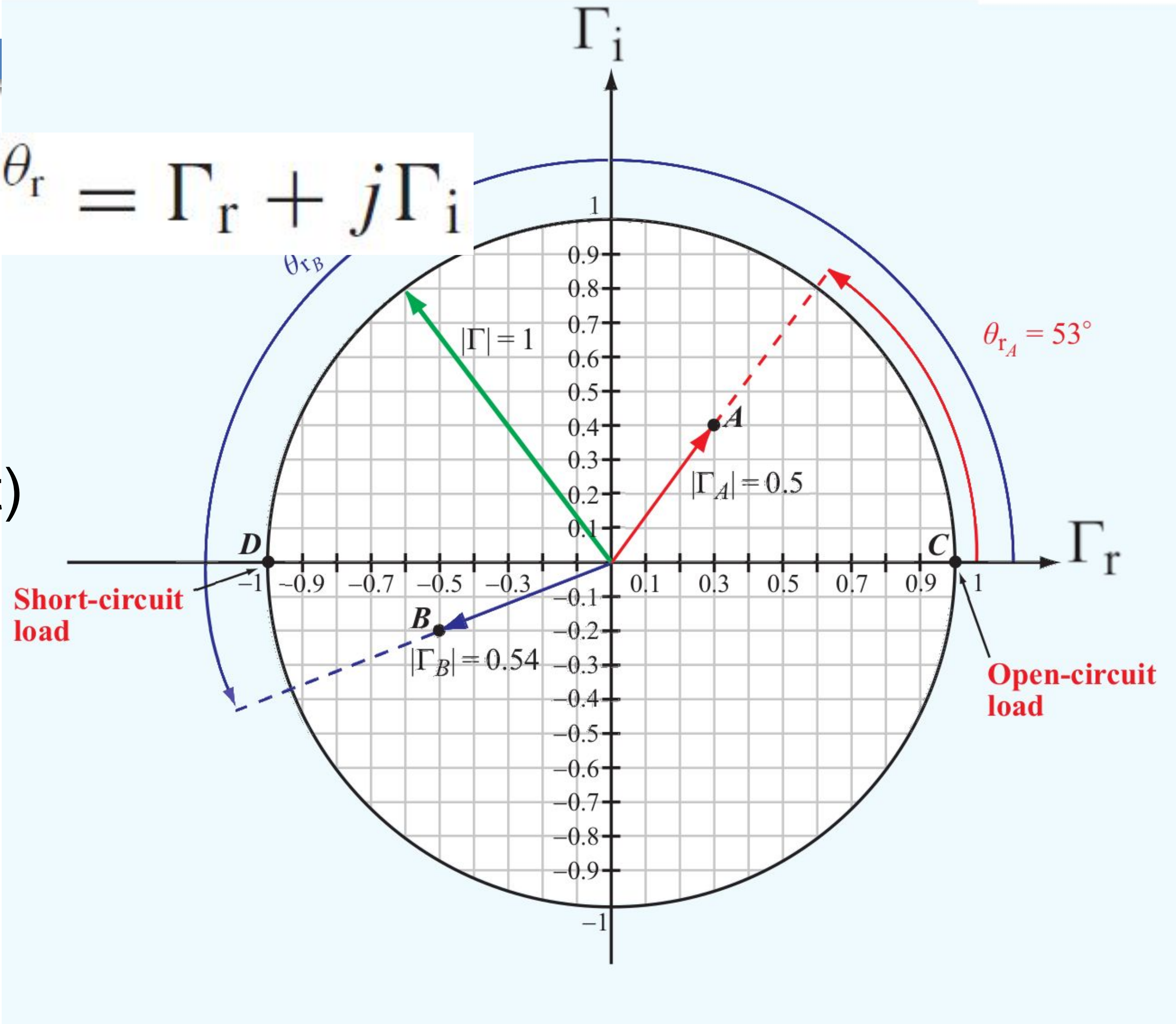


2-10 Complex Plane

$$\Gamma = |\Gamma|e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_D = -1$$

(Short-circuit)

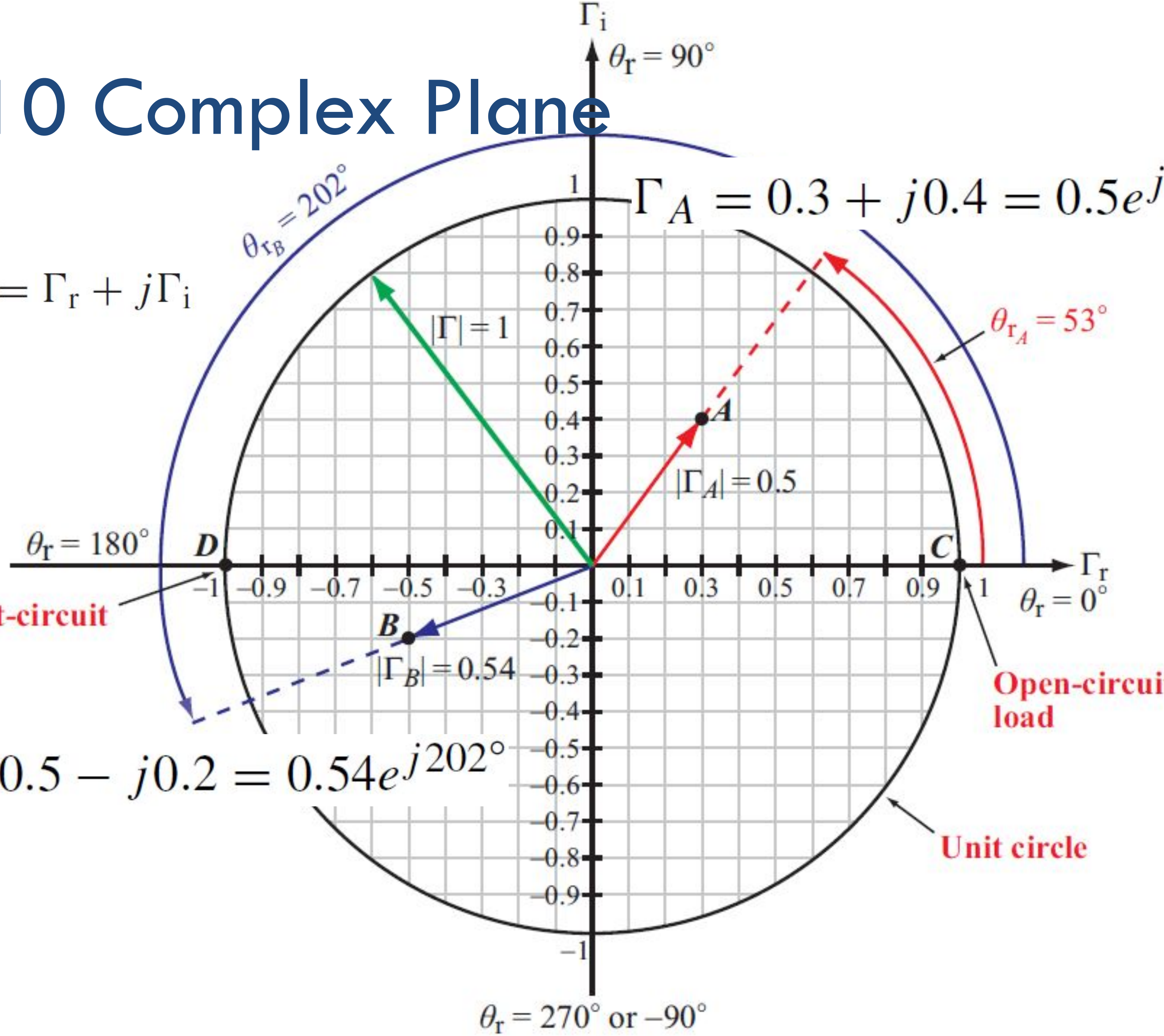


2-10 Complex Plane

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_A = 0.3 + j0.4 = 0.5e^{j53}$$

$$\Gamma_B = -0.5 - j0.2 = 0.54e^{j202^\circ}$$



2-10 Smith Chart Parametric Equations

What other complex parameters can we display this way?

Normalized Load Impedance:

$$z_L = \frac{Z_L}{Z_0}$$

$$z_L = \frac{1+\Gamma}{1-\Gamma}$$

$$z_L = r_L + jx_L$$

$$r_L + jx_L = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i}$$

$$r_L = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2}$$

$$x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}$$

2-10 Smith Chart Parametric Equations

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \qquad x_L = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

Algebraic manipulations can put these into the form of equations for circles:

$$(x - x_0)^2 + (y - y_0)^2 = a^2$$

$$\left(\Gamma_r - \frac{r_L}{1 + r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1 + r_L}\right)^2 \qquad (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$

2-10 Smith Chart Parametric Equations

r_L Circles:

$$\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2$$

centered at $\Gamma_r = r_L/(1+r_L)$ and $\Gamma_i = 0$

radius is $1/(1+r_L)$

2-10 Smith Chart Parametric Equations

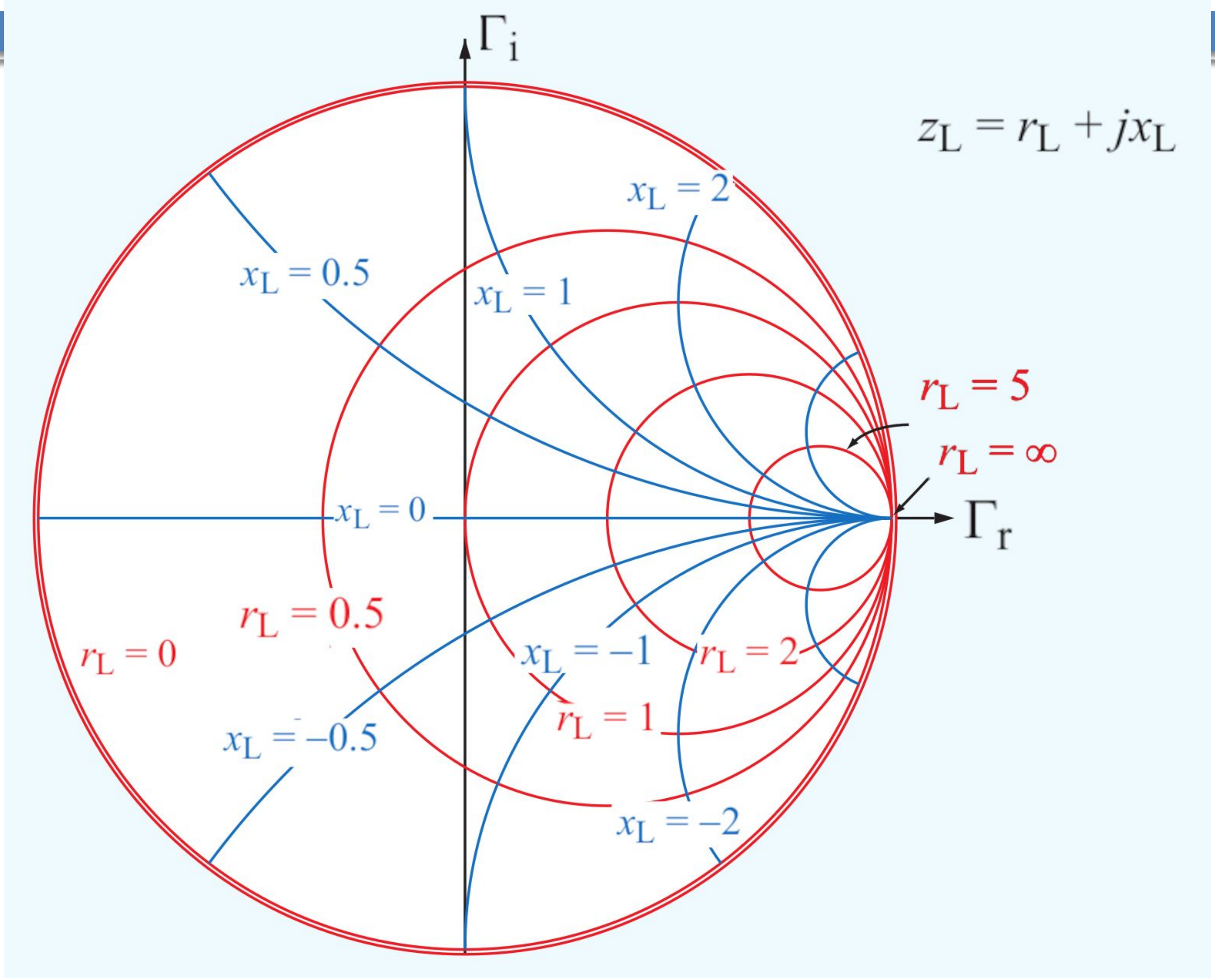
x_L Circles:

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$

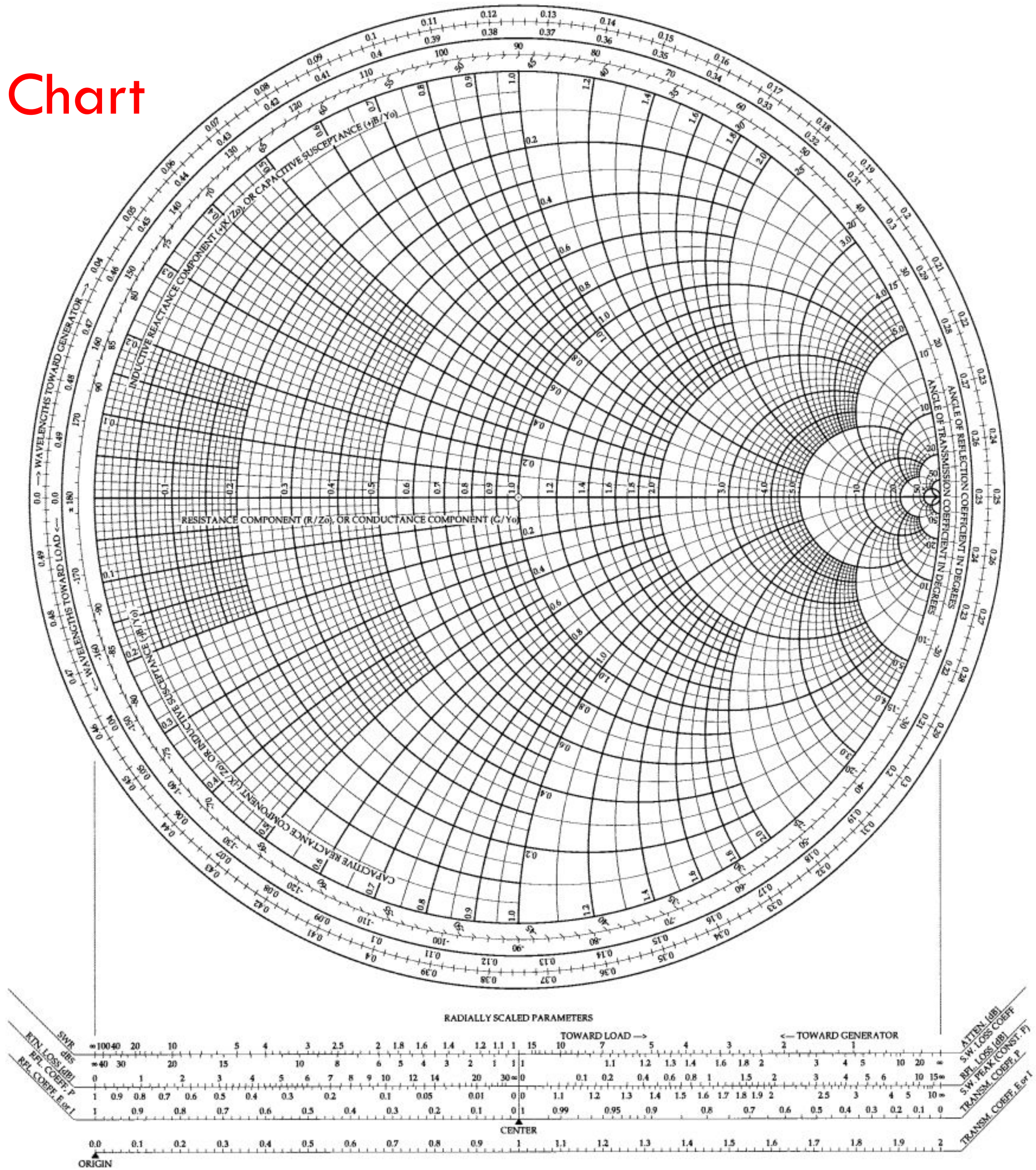
centered at $\Gamma_r = 1$ and $\Gamma_i = 1/x_L$

radius is $1/x_L$

2-10 Smith Chart Parametric Equations

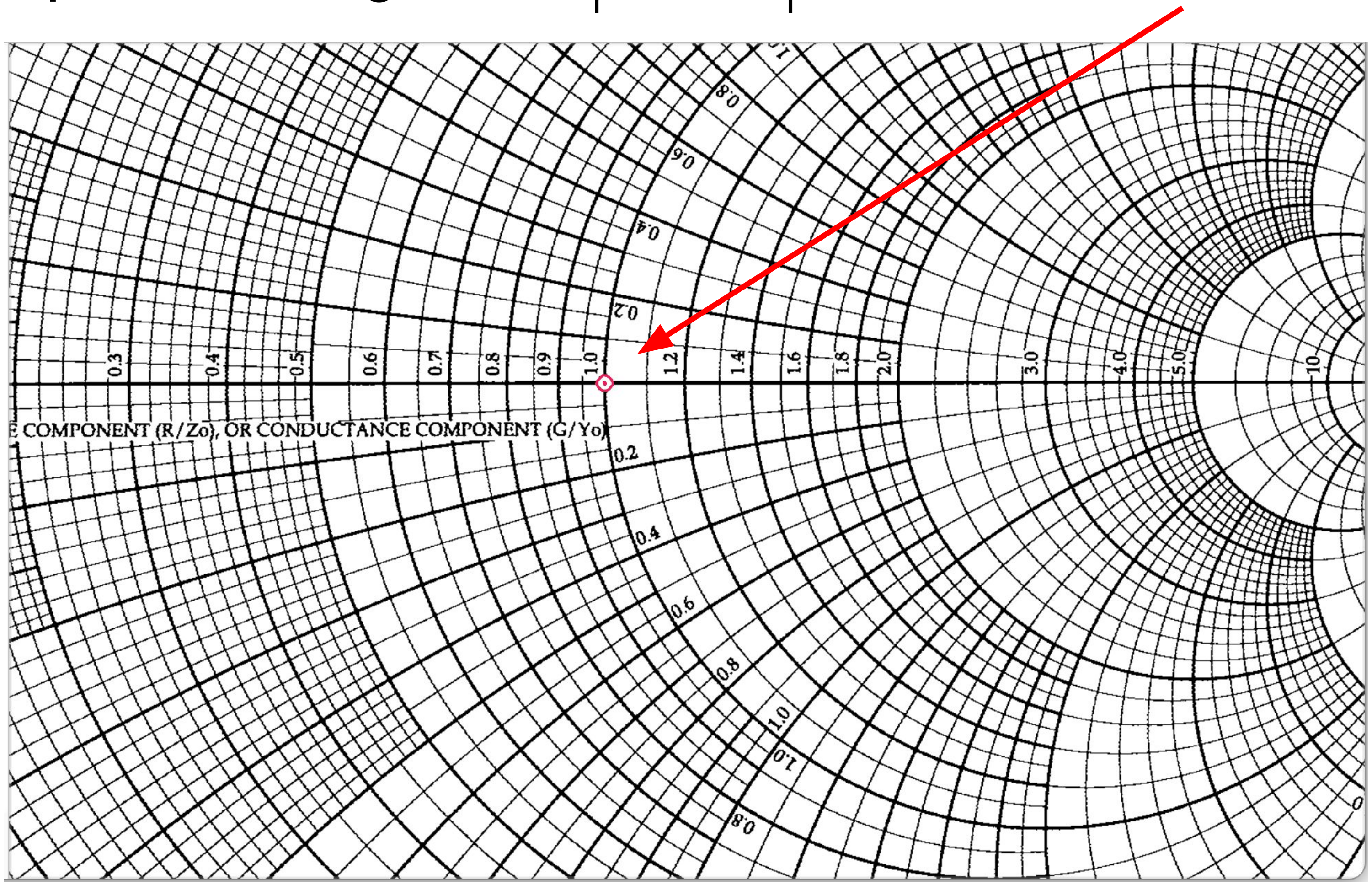


2-10 Complete Smith Chart



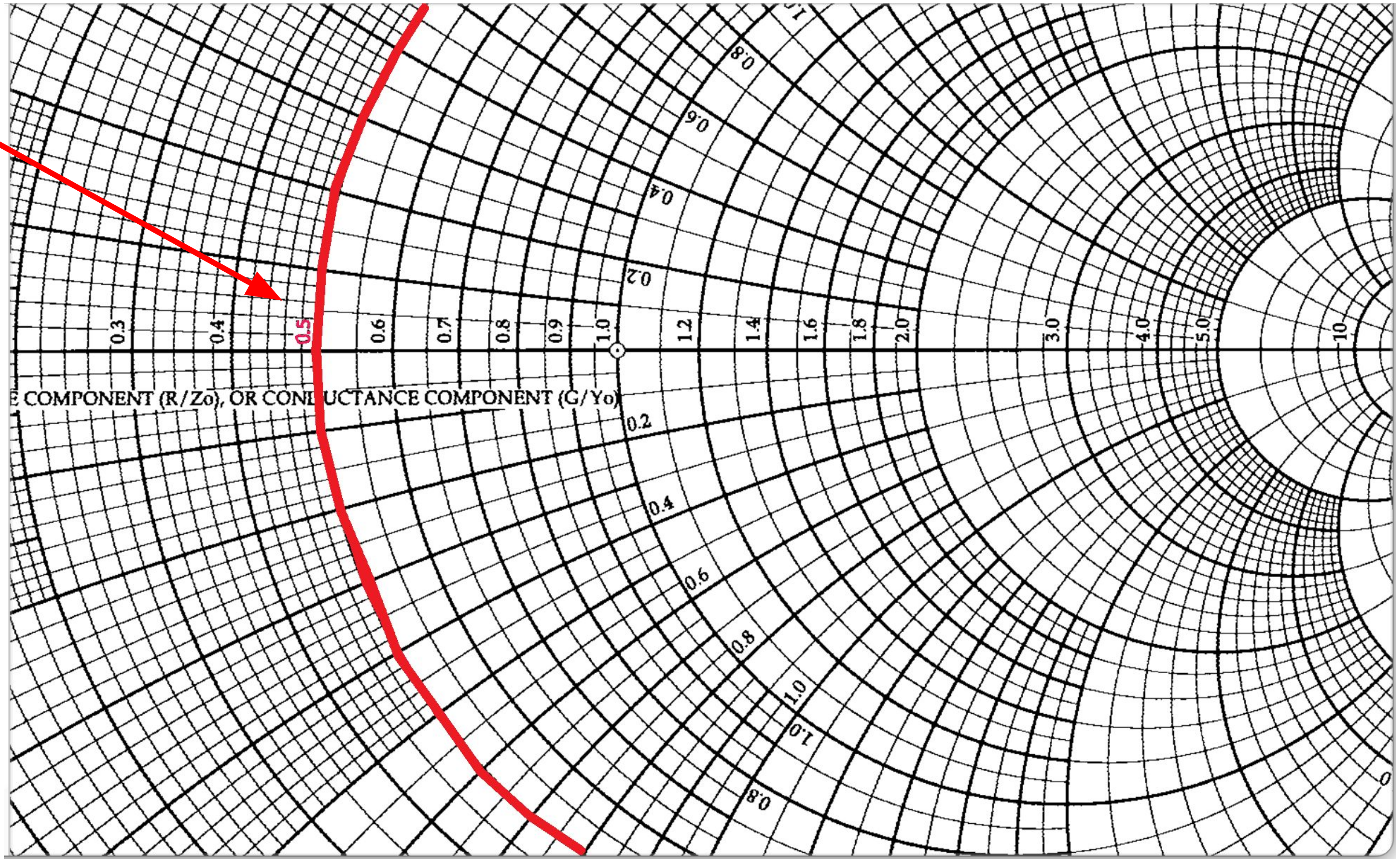
2-10 Smith Chart Details

Central part has origin of Γ_r and Γ_i axes



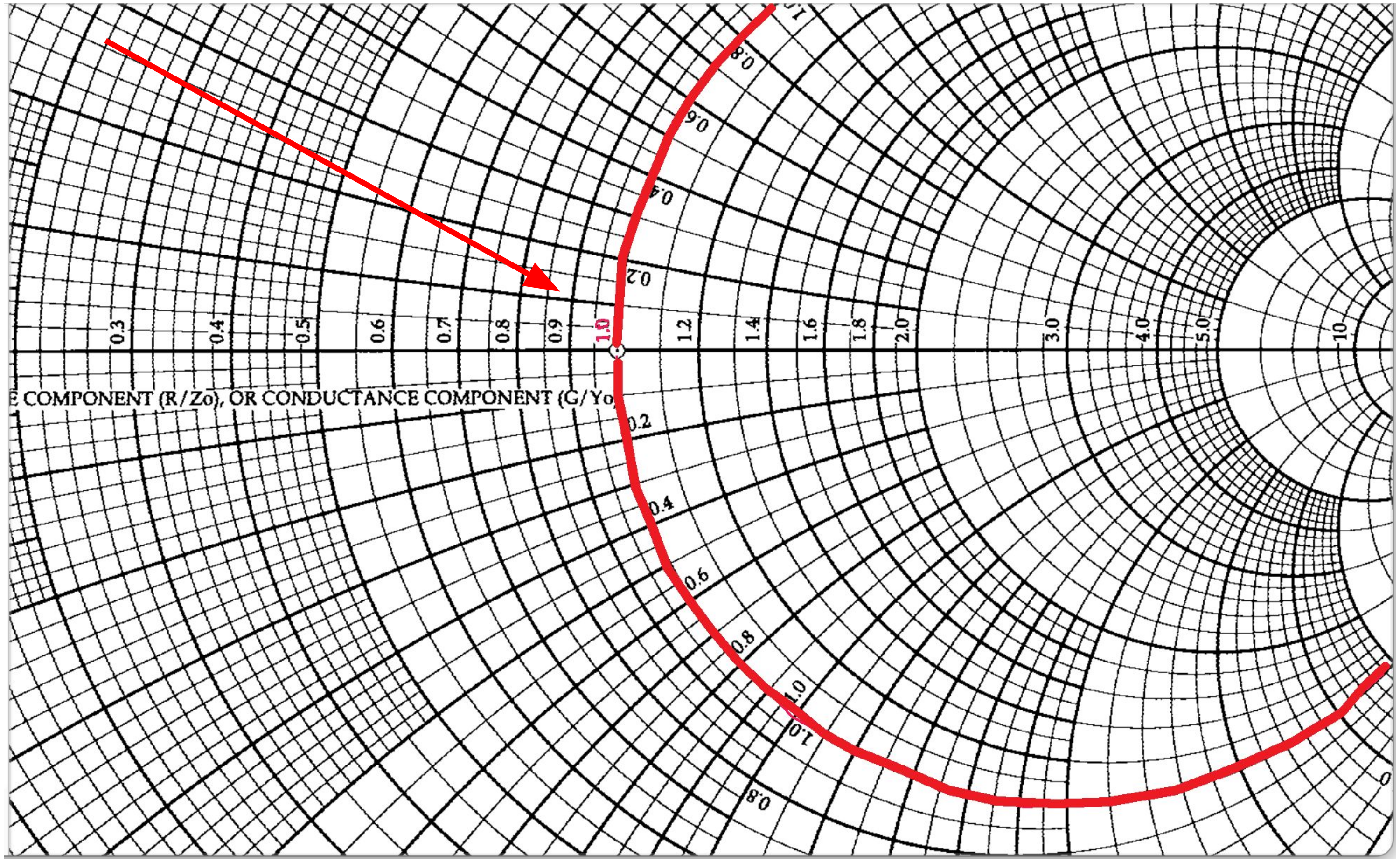
2-10 Smith Chart Details

r_L = 0.5 circle



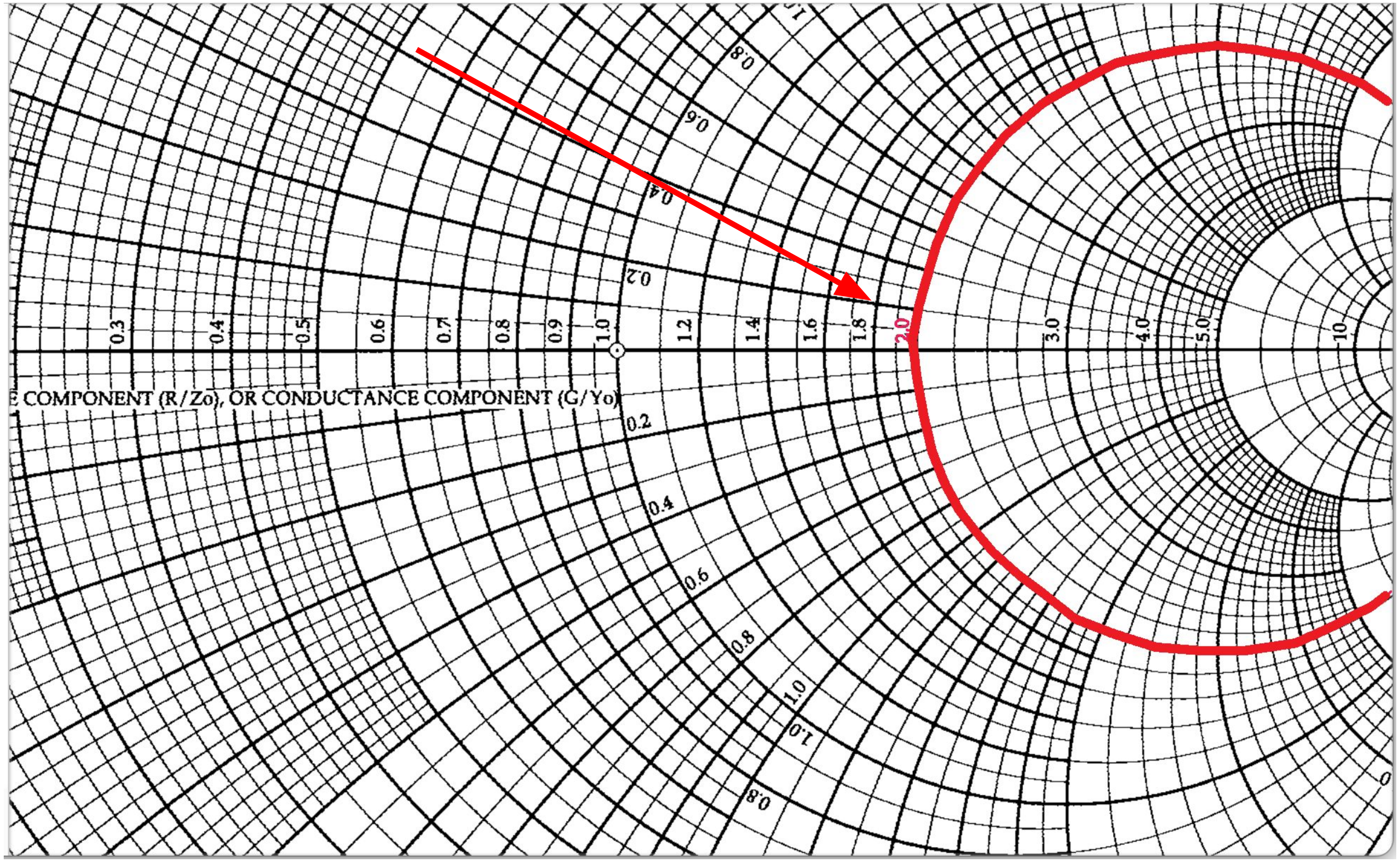
2-10 Smith Chart Details

r_L = 1.0 circle



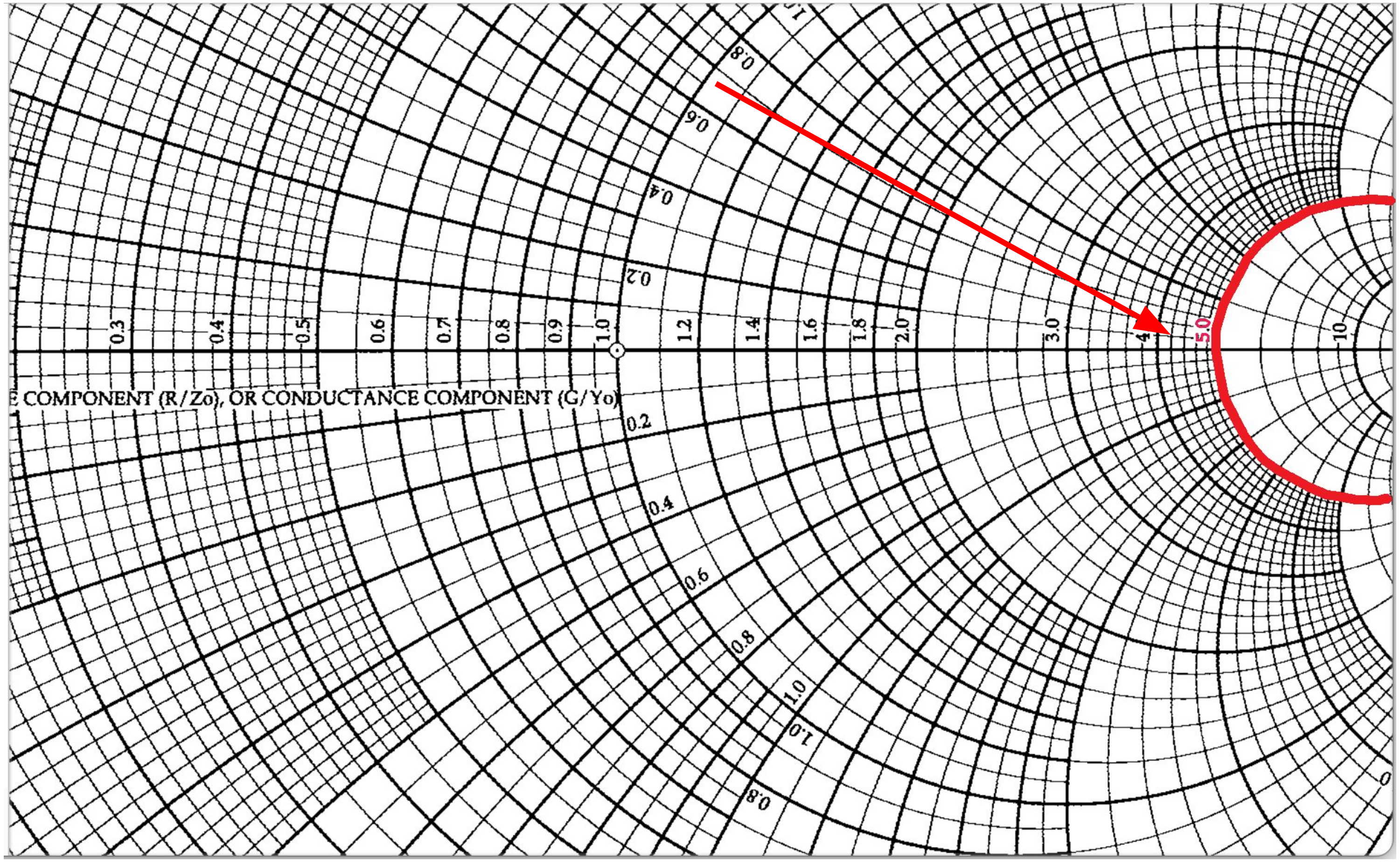
2-10 Smith Chart Details

r_L = 2.0 circle



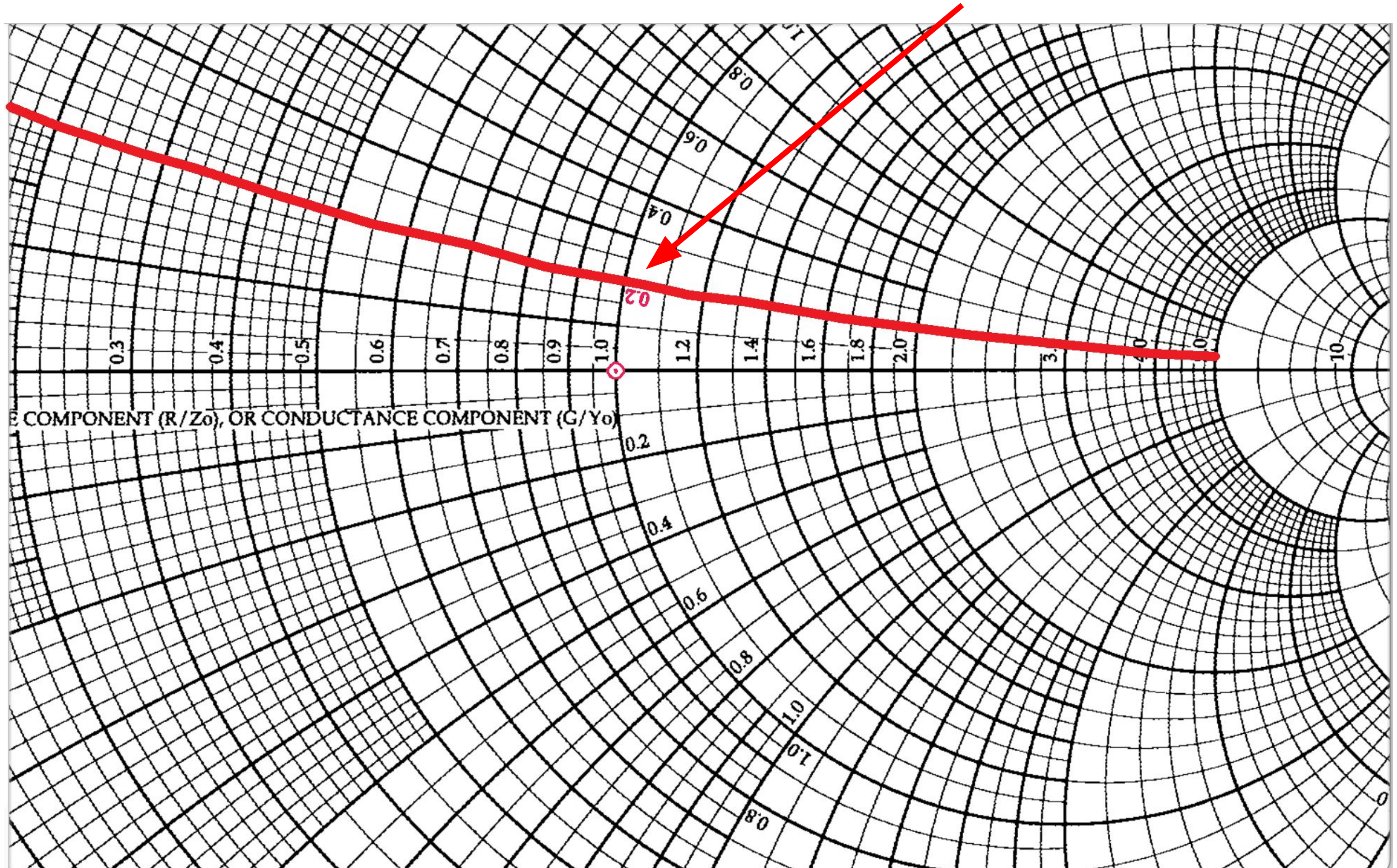
2-10 Smith Chart Details

r_L = 5.0 circle



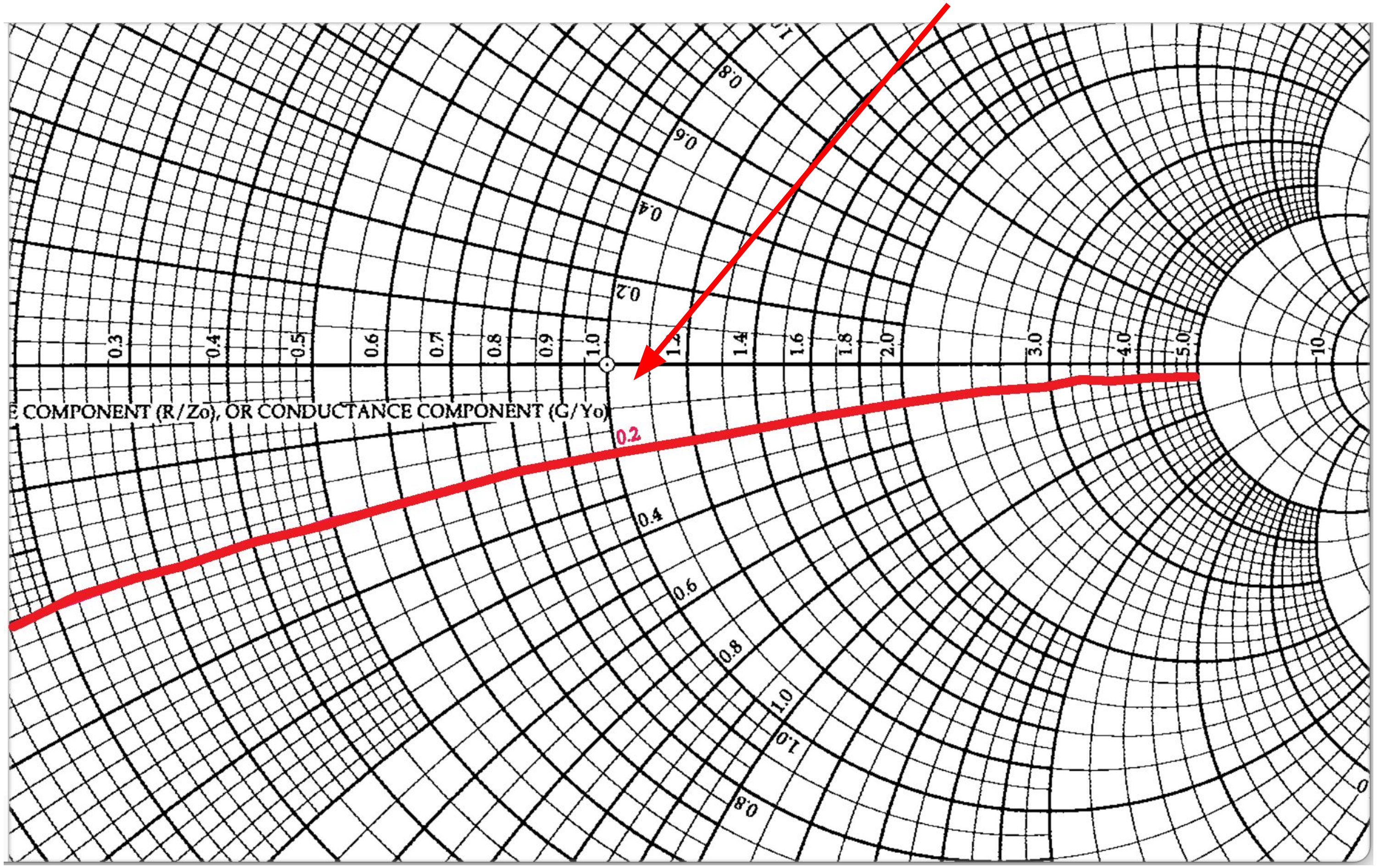
2-10 Smith Chart Details

x_L = +0.2 circle



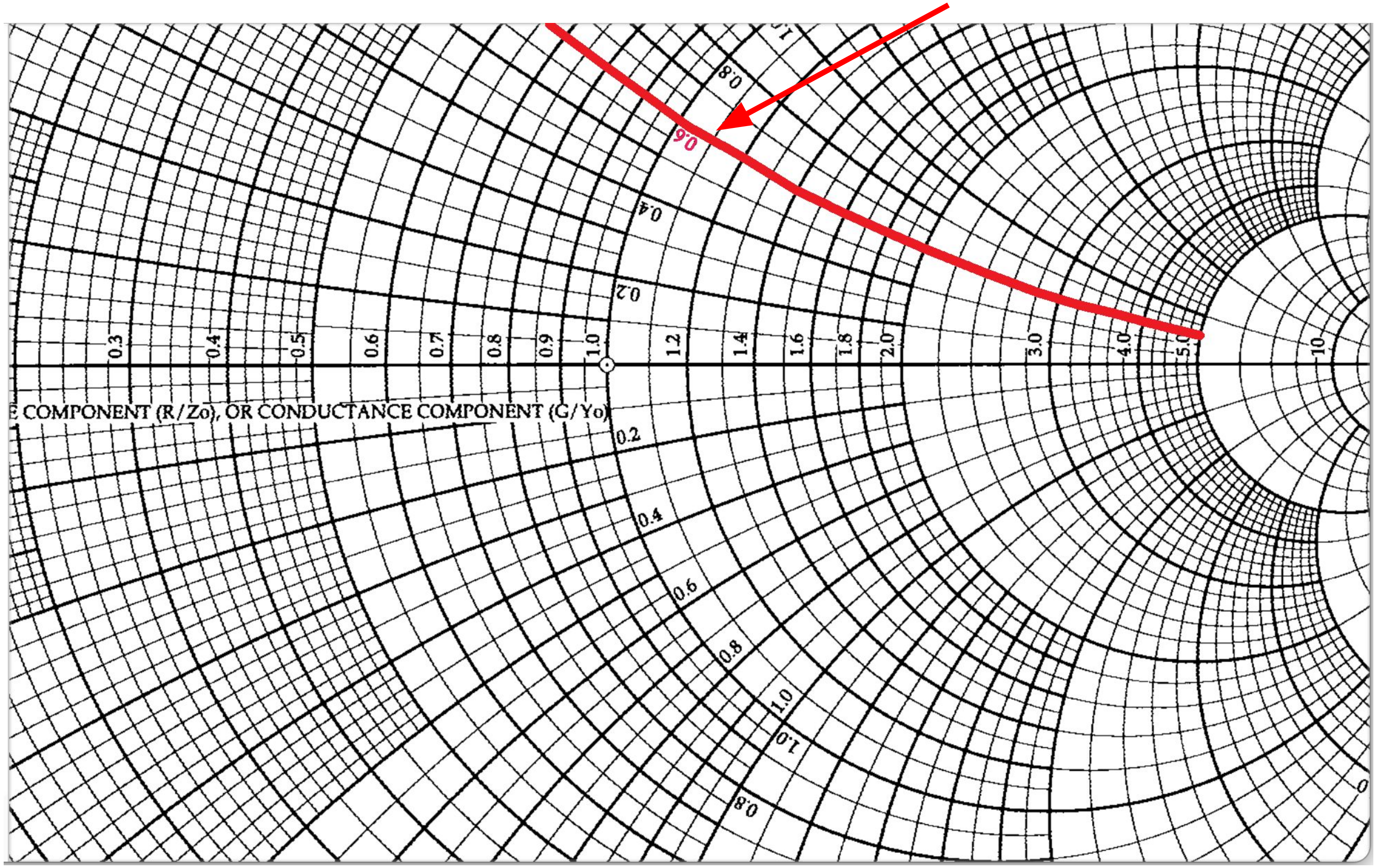
2-10 Smith Chart Details

$x_L = -0.2$ circle



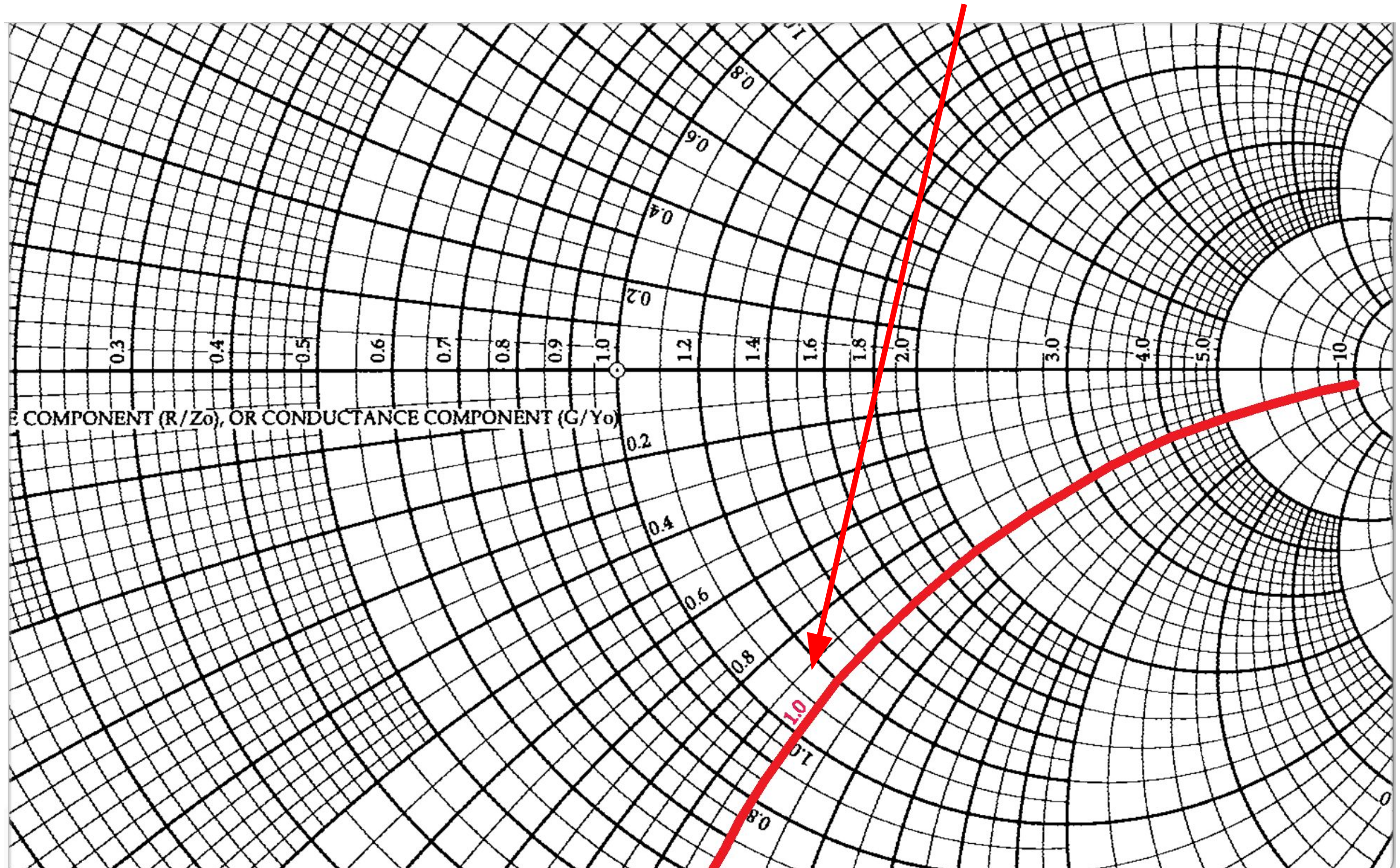
2-10 Smith Chart Details

x_L = +0.6 circle



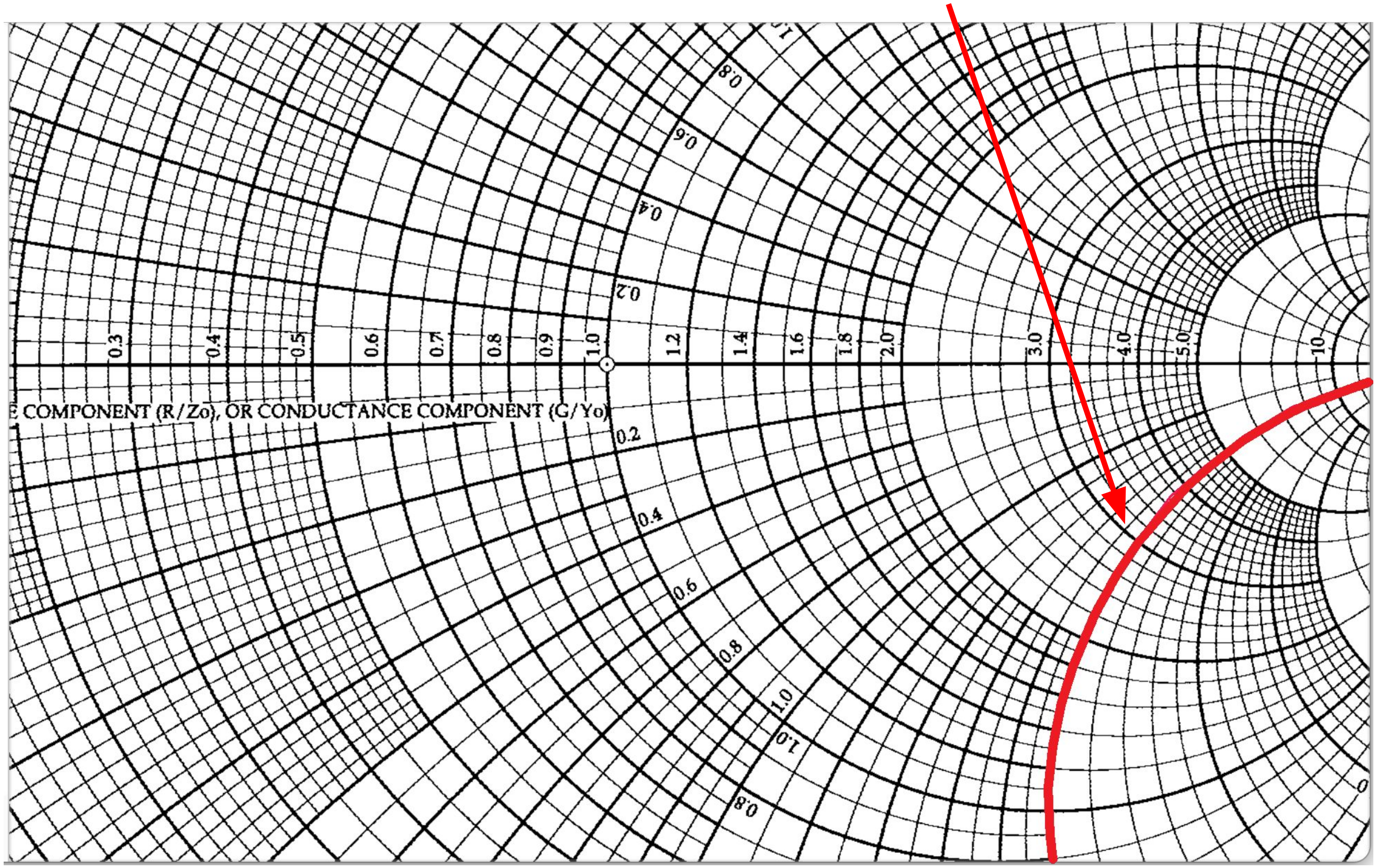
2-10 Smith Chart Details

$x_L = -1.0$ circle



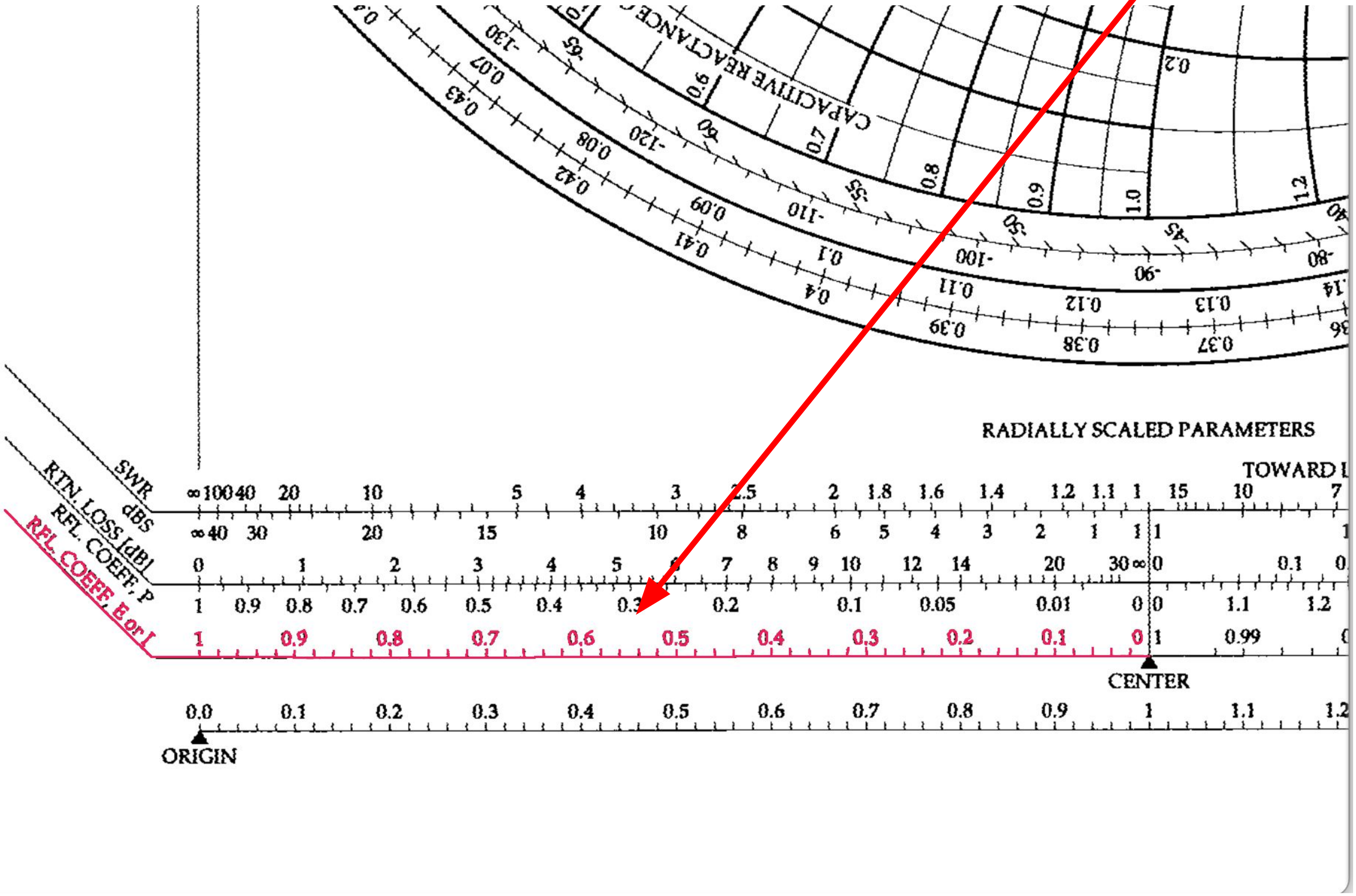
2-10 Smith Chart Details

x_L = -2.0 circle



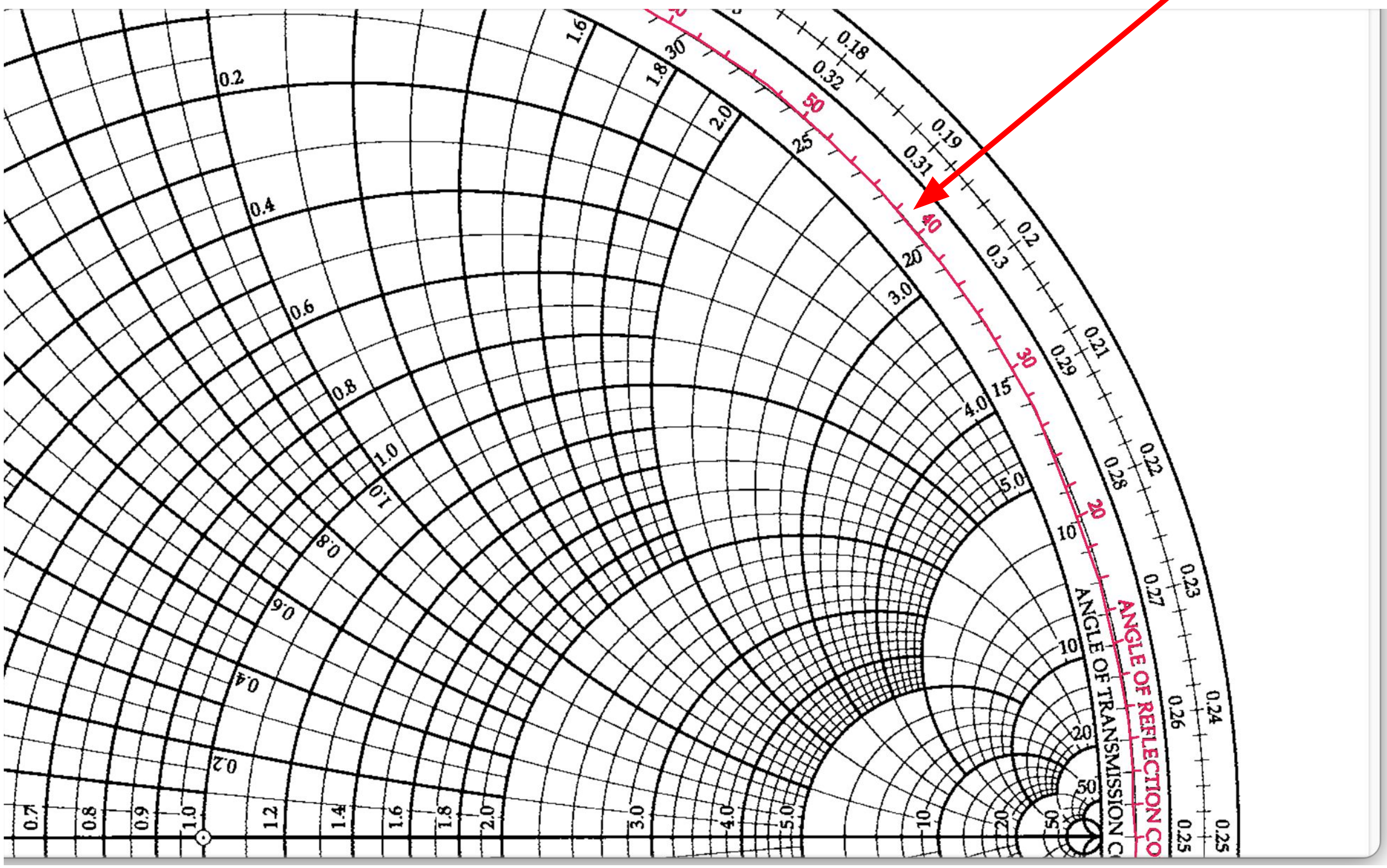
2-10 Smith Chart Details

Γ_L magnitude: radial scale near the bottom



2-10 Smith Chart Details

Γ_L phase in degrees: scale around the edge

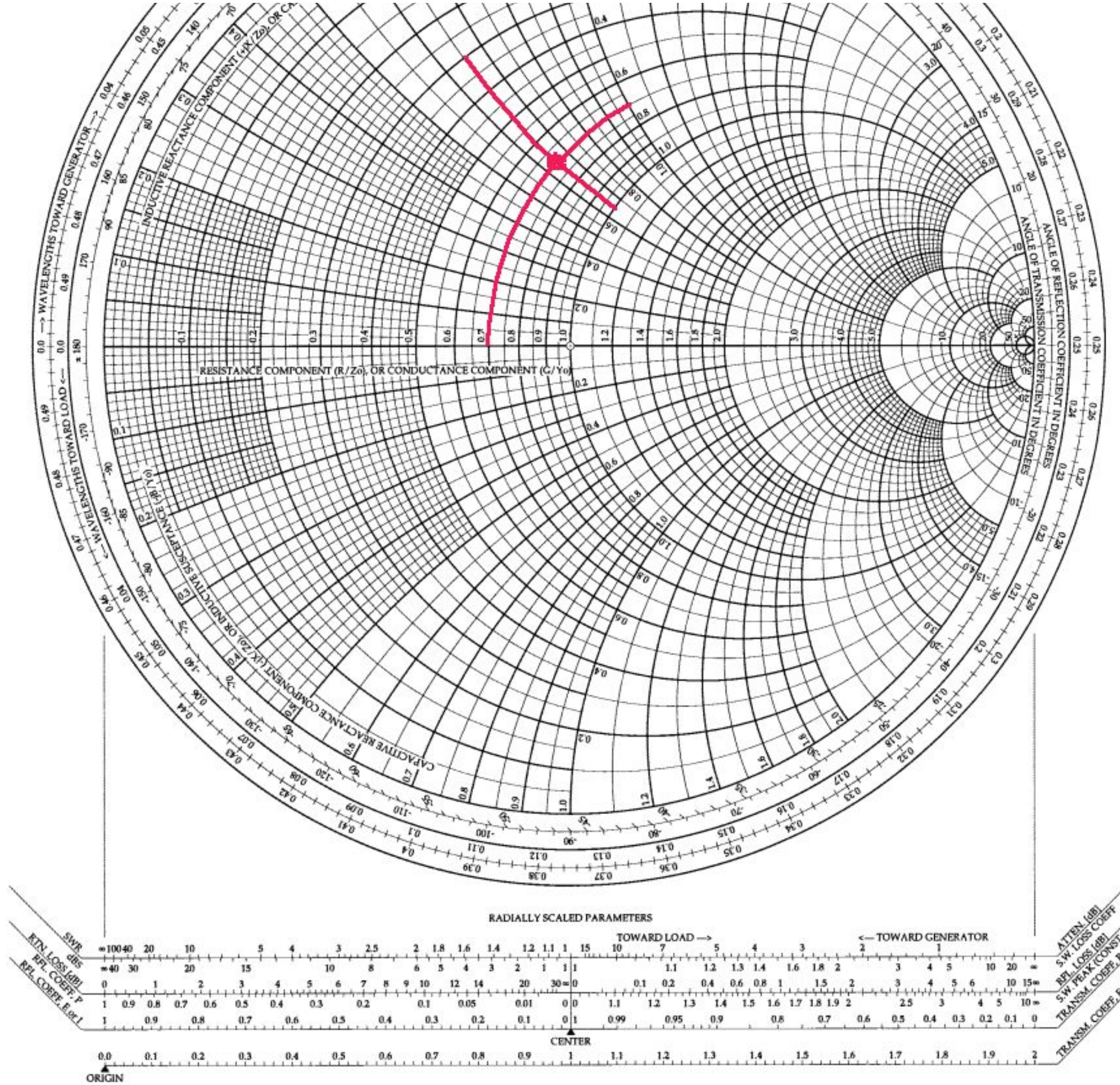


2-10 Smith Chart Details

Obtain values for r_L, x_L from the nearest circles:

$z_L = (0.7, 0.65)$

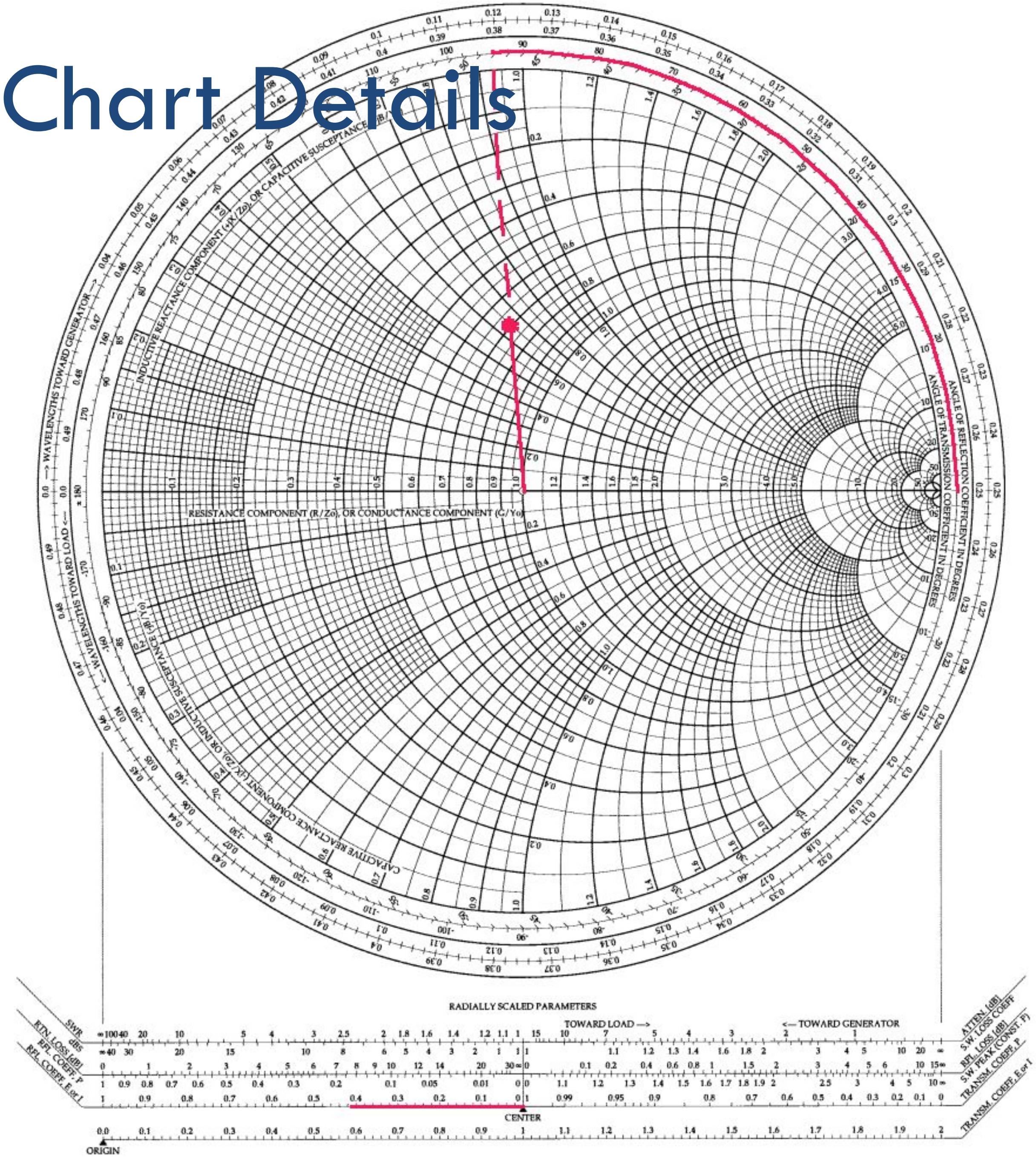
To get Z_L:
Multiply z_L by Z_0



2-10 Smith Chart Details

Obtain values for Γ_L from the angle scale and magnitude scale:

0.41, 94°

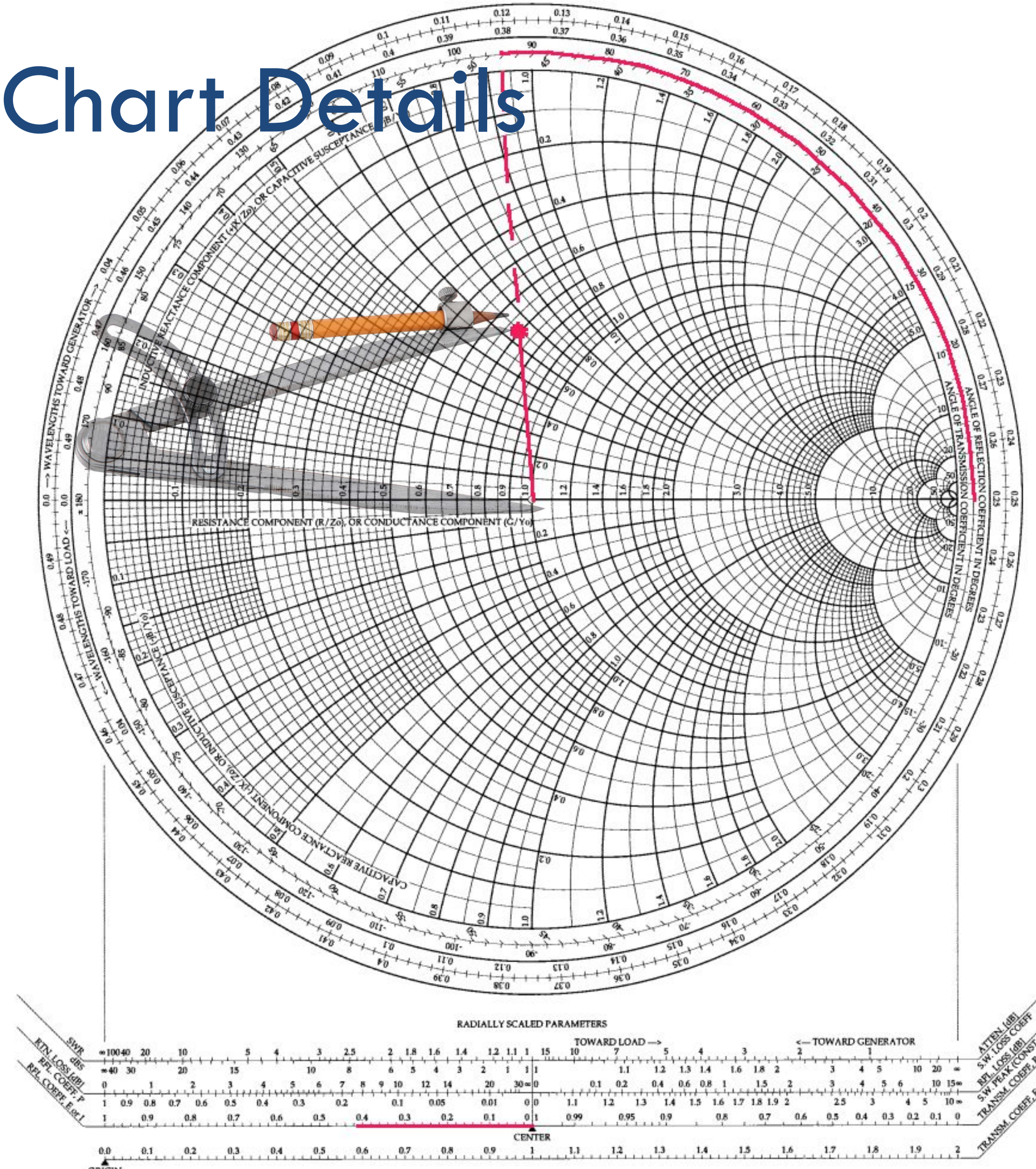


2-10 Smith Chart Details

Obtain values for Γ_L from the angle scale and magnitude scale:

0.41, 94°

Measure dist from origin using compass



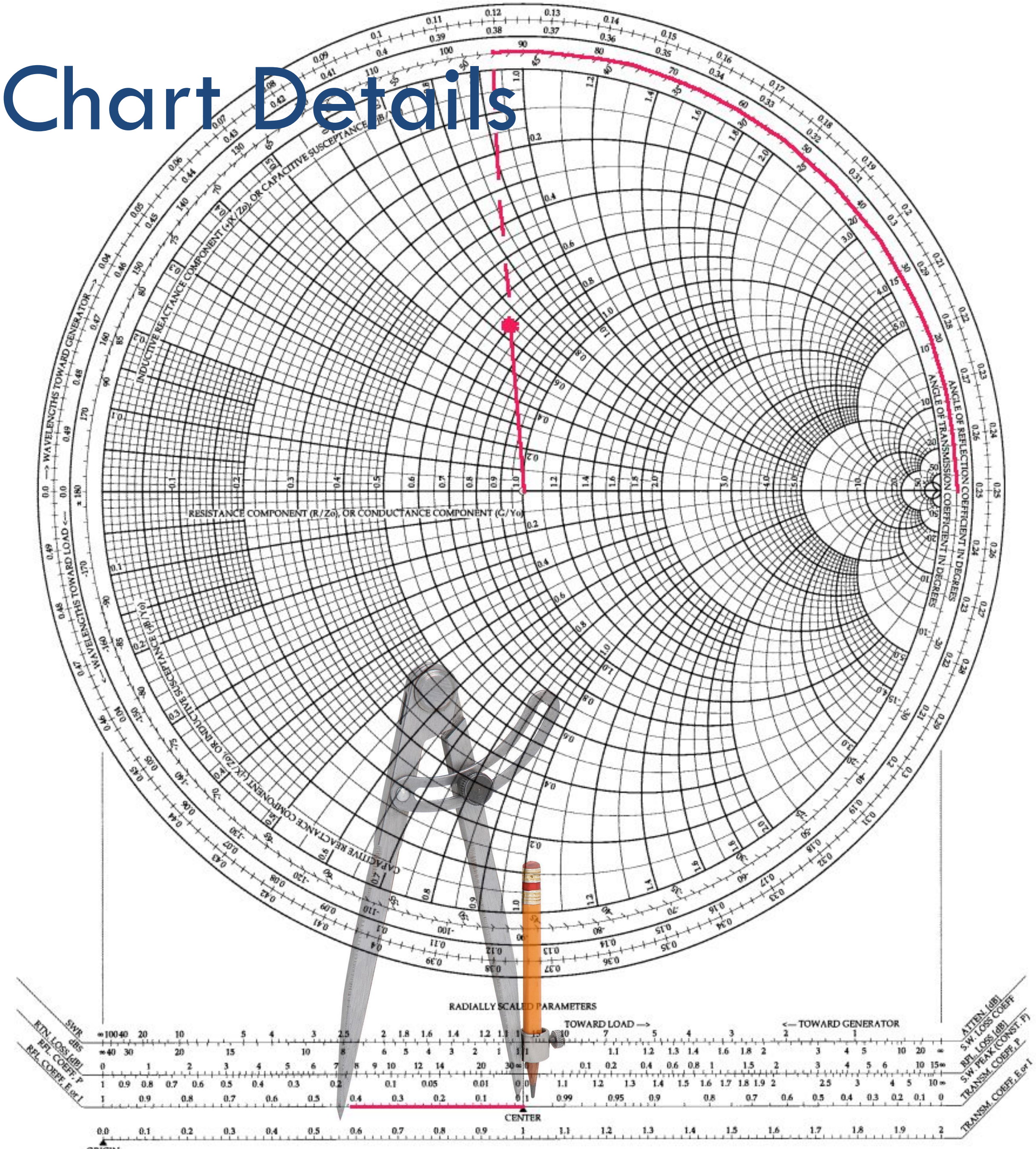
(compass pic: harborfreight.com)

2-10 Smith Chart Details

Obtain values for Γ_L from the angle scale and magnitude scale:

0.41, 94°

Measure dist from origin using compass, then lay that along scale near bottom to get magnitude



2-10 Smith Chart Example 1

Given: z_L=2-j1

Find: the voltage reflection coefficient Γ_r?

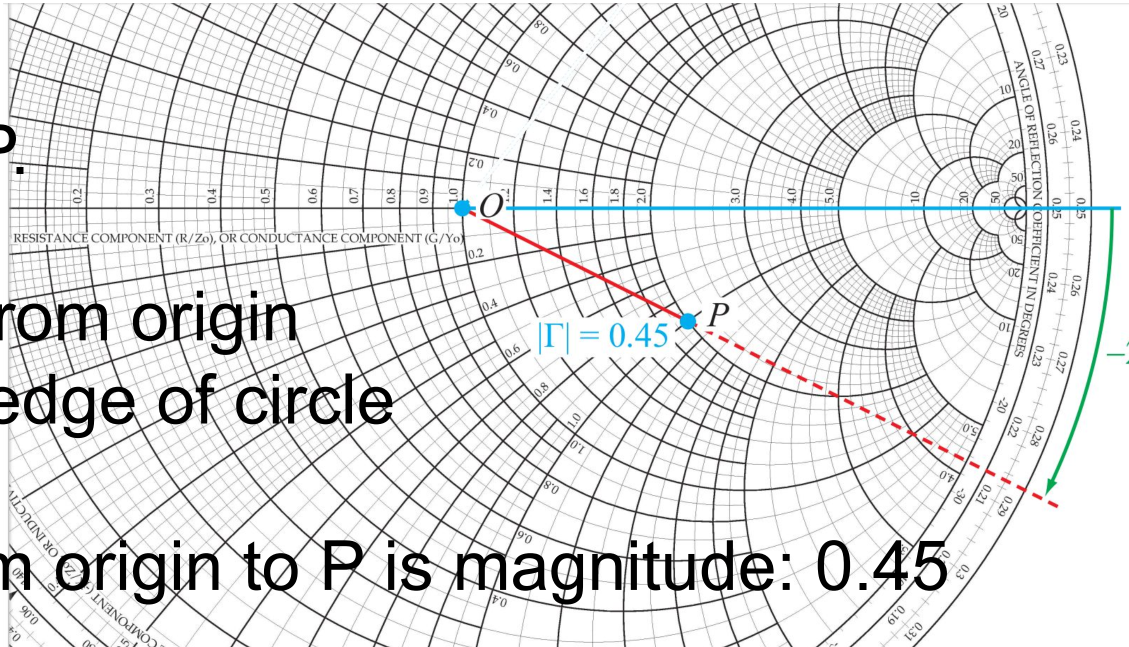
Solution:

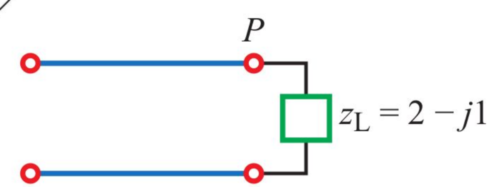
1. Plot point P.

2. Draw line from origin through P to edge of circle

3. Length from origin to P is magnitude: 0.45

4. Angle from $+\Gamma_r$ axis is phase: -26.6°





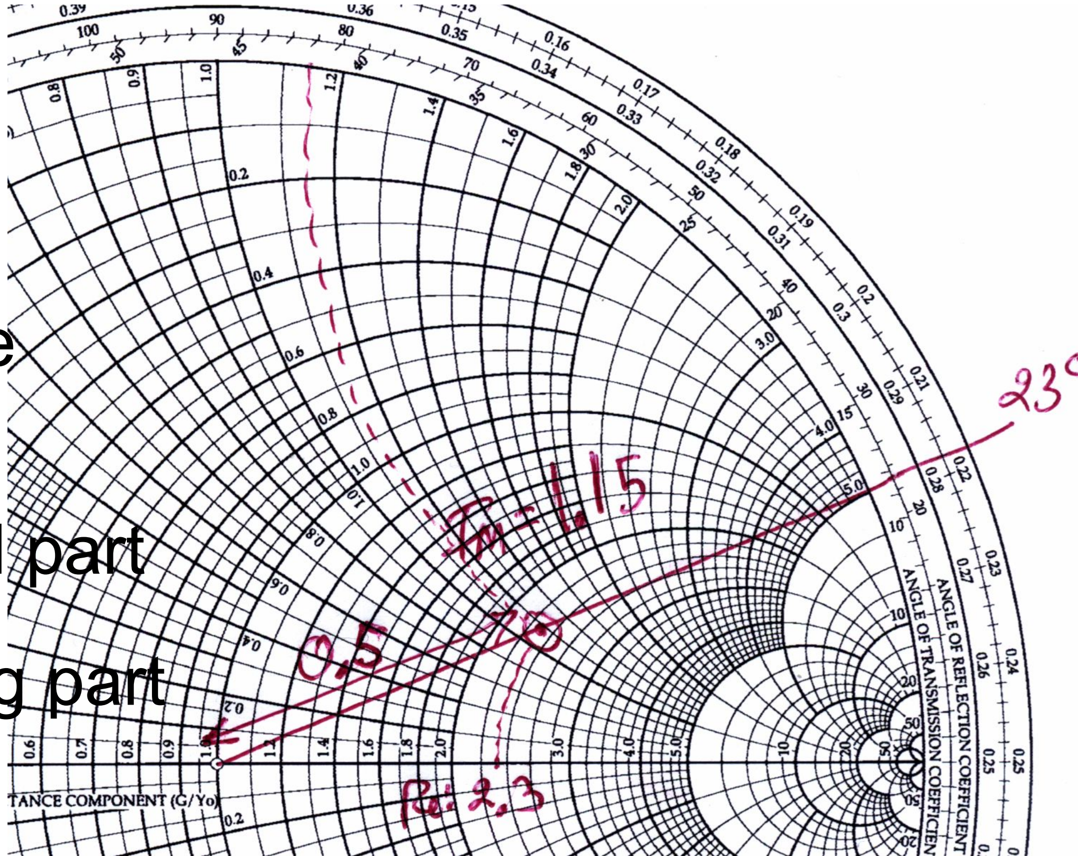
2-10 Smith Chart Example 2

Given: the voltage reflection coefficient $\Gamma_r = 0.5e^{j23deg}$

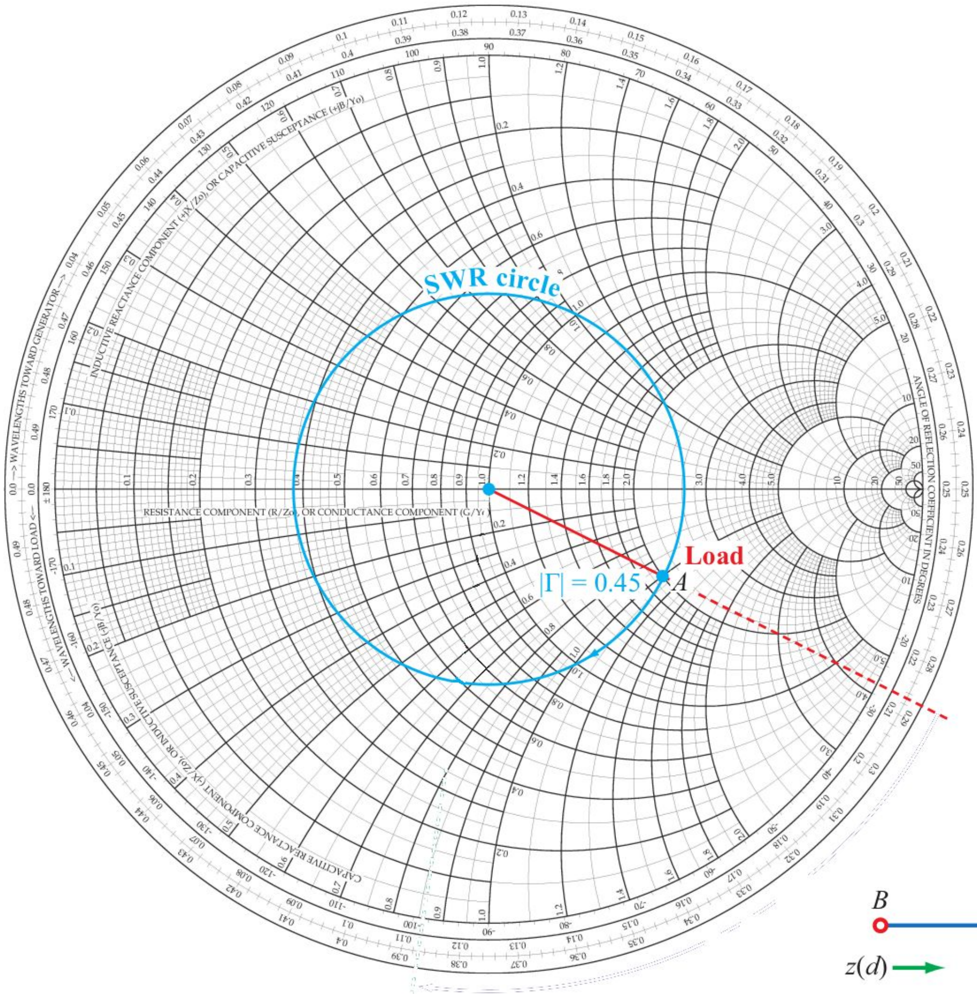
Find: normalized load impedance: z_L

Solution:

1. Draw radial line from origin to angle=23°
2. draw point on that line dist of 0.5 from origin
3. nearest XR line = real part
4. nearest XL line = imag part
5. $z_L = 2.3 + j1.15$



2.10 Constant-$|\Gamma|$ Circle

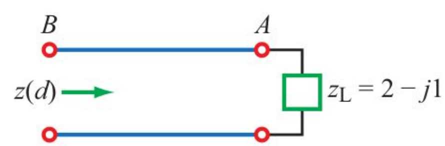


All points on a circle centered at the origin have the same $|\Gamma|$.

Since

$$S = \frac{1+|\Gamma|}{1-|\Gamma|}$$

this is a also a circle of constant S (VSWR).



2.10 Input Impedance

Since the normalized wave impedance is:

$$z(d) = \frac{Z(d)}{Z_0} = \frac{1+\Gamma_d}{1-\Gamma_d};$$

where:

$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$$

Can plot $\boldsymbol{\Gamma}_L$, then read off z_L
Change the phase of $\boldsymbol{\Gamma}$, then read off z_d while staying on the constant $|\boldsymbol{\Gamma}|$ circle.

How much to rotate around the circle, and in what direction, given a certain value for d?

2.10 Input Impedance

Knowing the value of d, need to rotate by: $-2\beta d$ (radians)

Since negative: rotate CLOCKWISE

Since an entire rotation is 2π, solve for *d*:

$$2\beta d = 2\,\frac{2\pi}{\lambda}\,d = 2\pi$$

So, one rotation around the Smith Chart corresponds to:

$$d = \lambda/2$$

2.10 Input Impedance

So, one rotation around the Smith Chart corresponds to:

$$d = \lambda/2$$

The scale around the perimeter of the Smith Chart:

Labeled "Wavelengths Toward Generator"

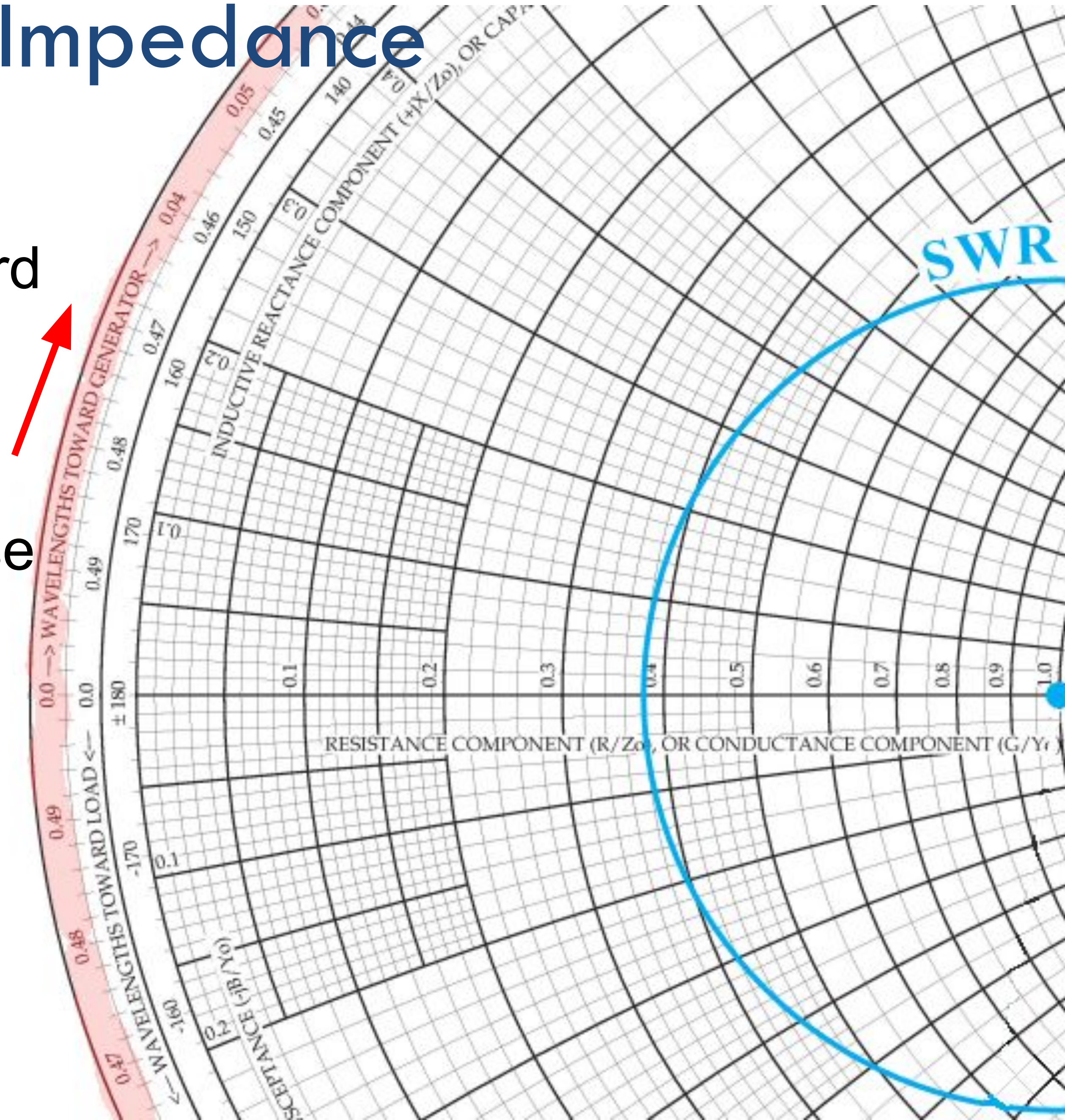
has been calibrated with values as a fraction of λ,

from 0 to $\lambda/2$

2.10 Input Impedance

Wavelengths Toward Generator

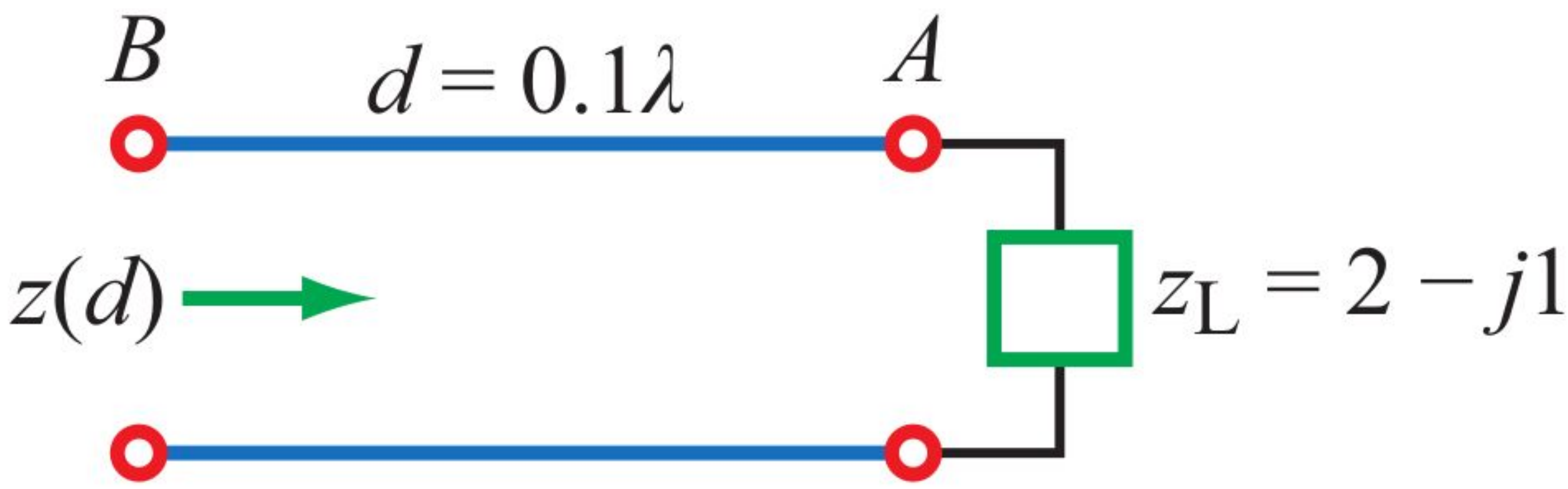
from 0 to $\lambda/2$
increasing clockwise



2.10 Input Impedance: Example

Given: Z_0=50Ω
Z_L=(100-j50)Ω

Find: Z(d) at d=0.1λ

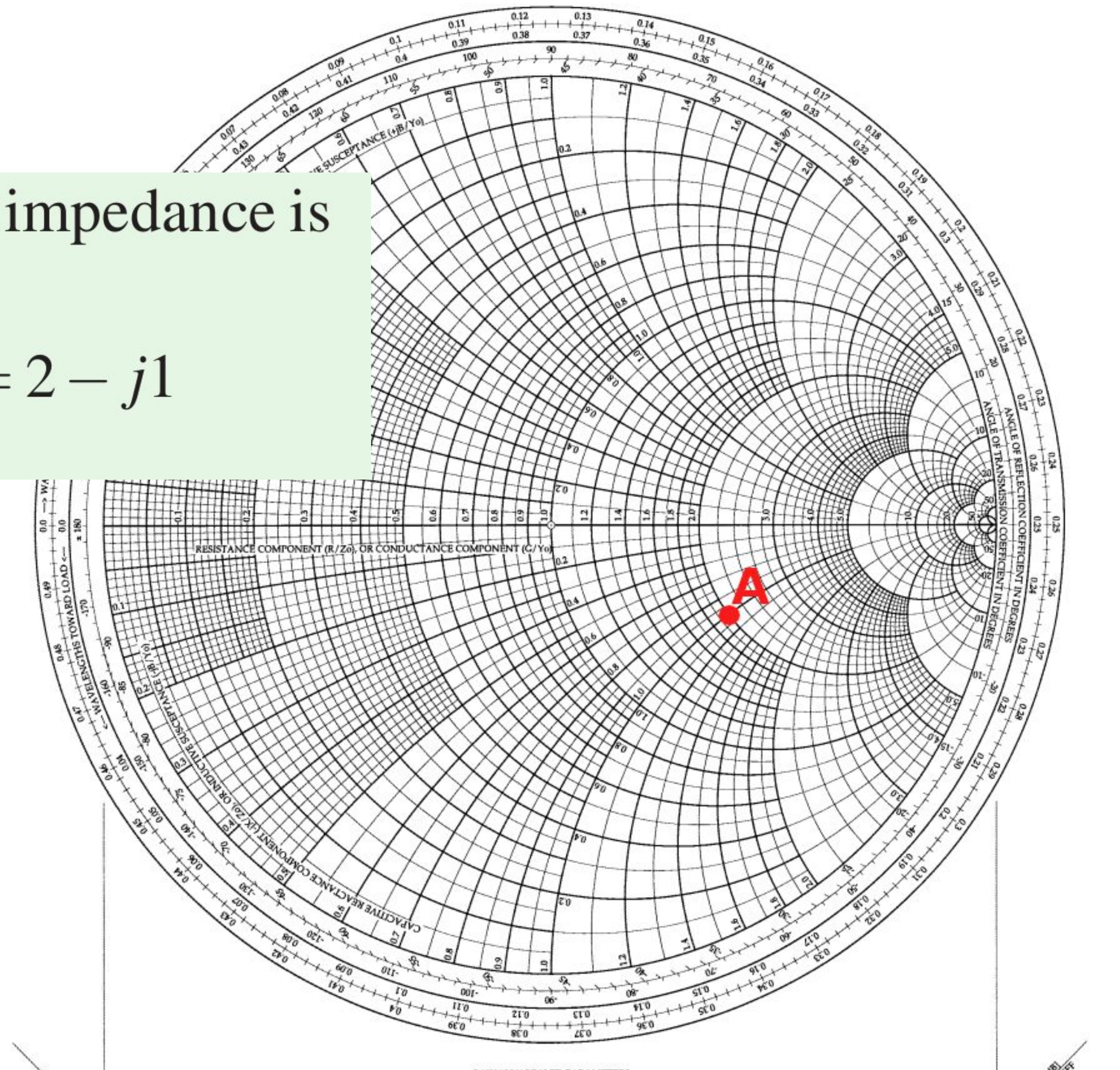


2.10 Input Impedance: Example

Solution:

1. The normalized load impedance is

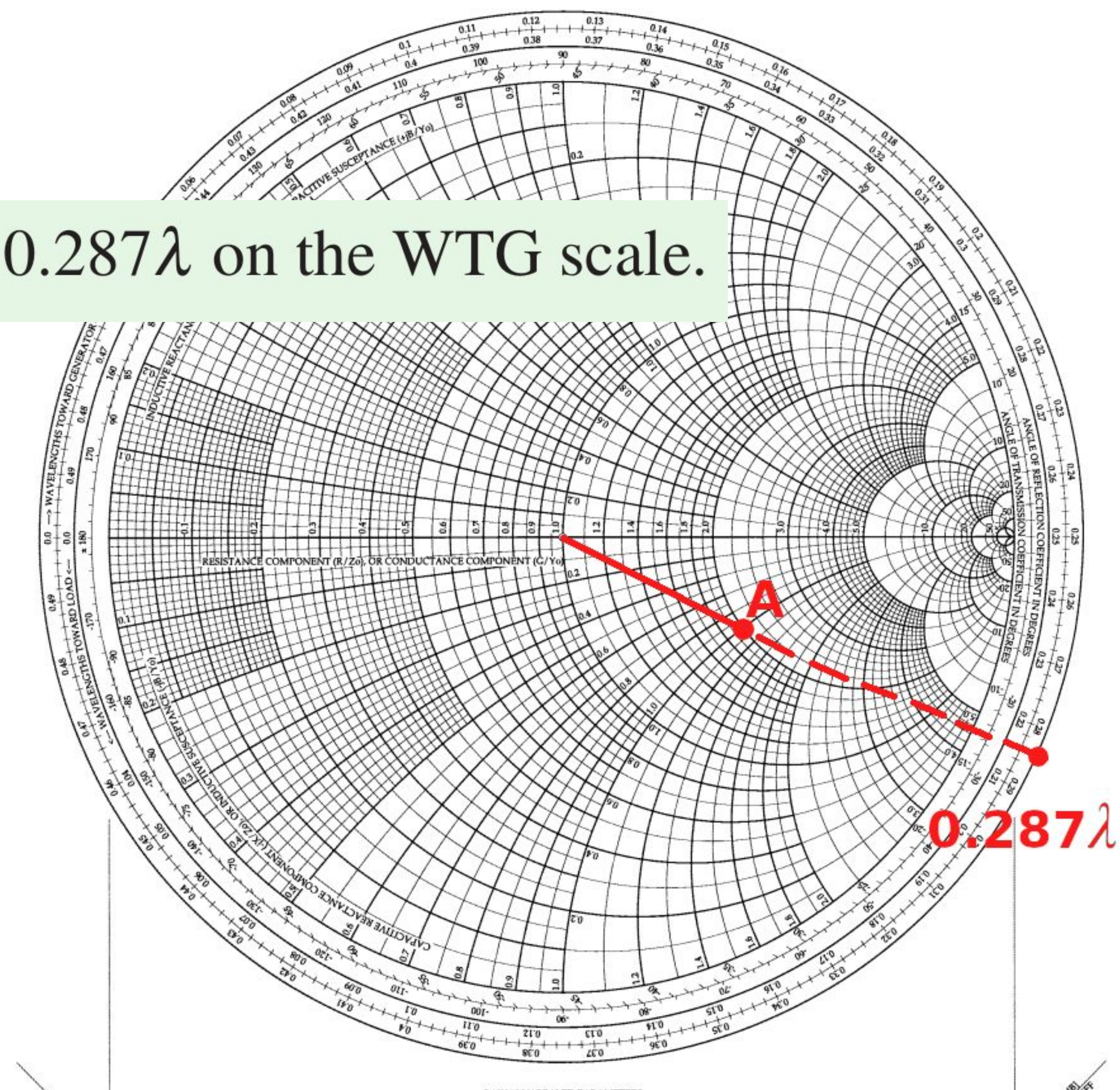
$$z_L = \frac{Z_L}{Z_0} = \frac{100 - j50}{50} = 2 - j1$$



2.10 Input Impedance: Example

Solution:

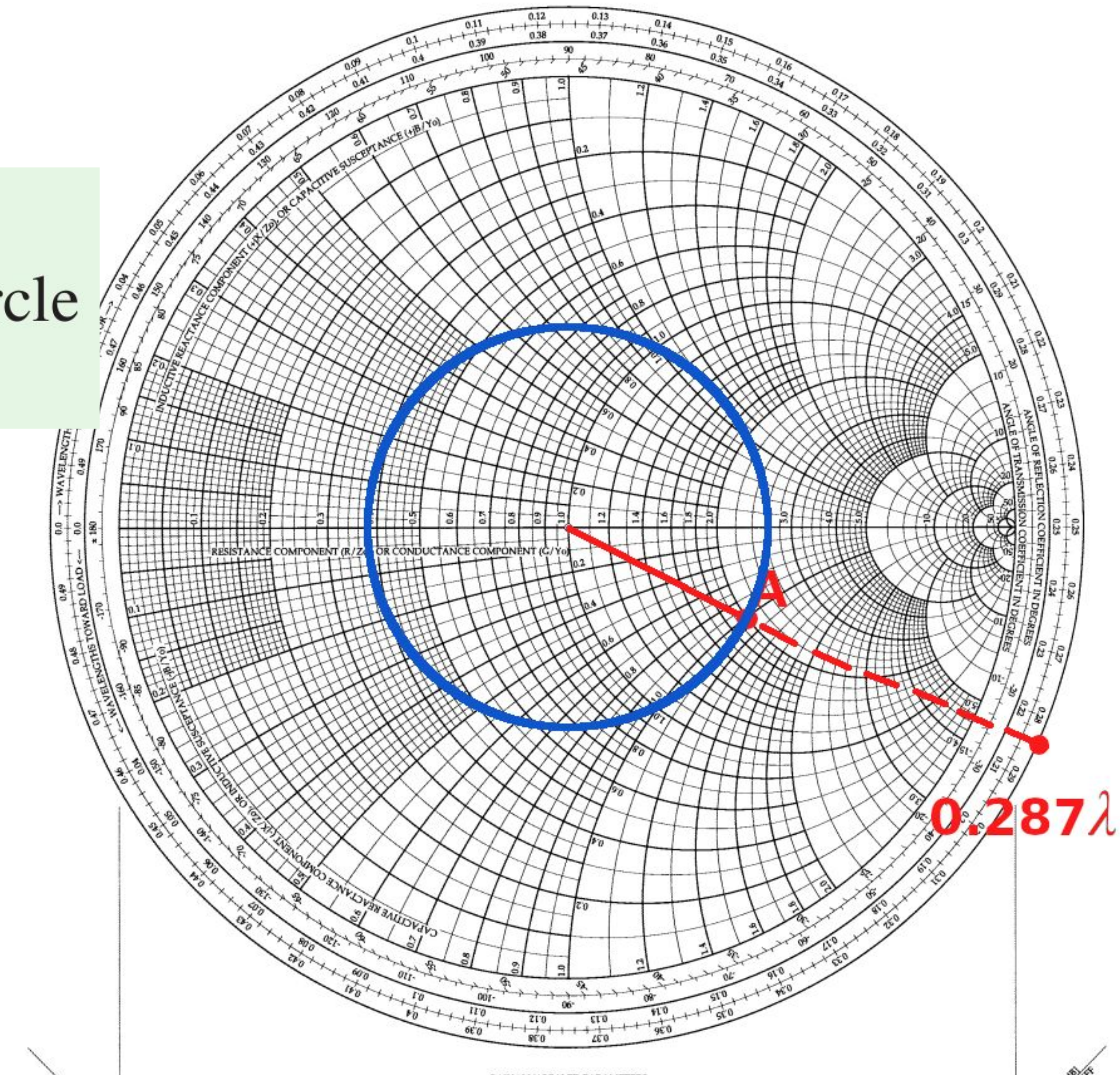
2. Point *A* is located at 0.287λ on the WTG scale.



2.10 Input Impedance: Example

Solution:

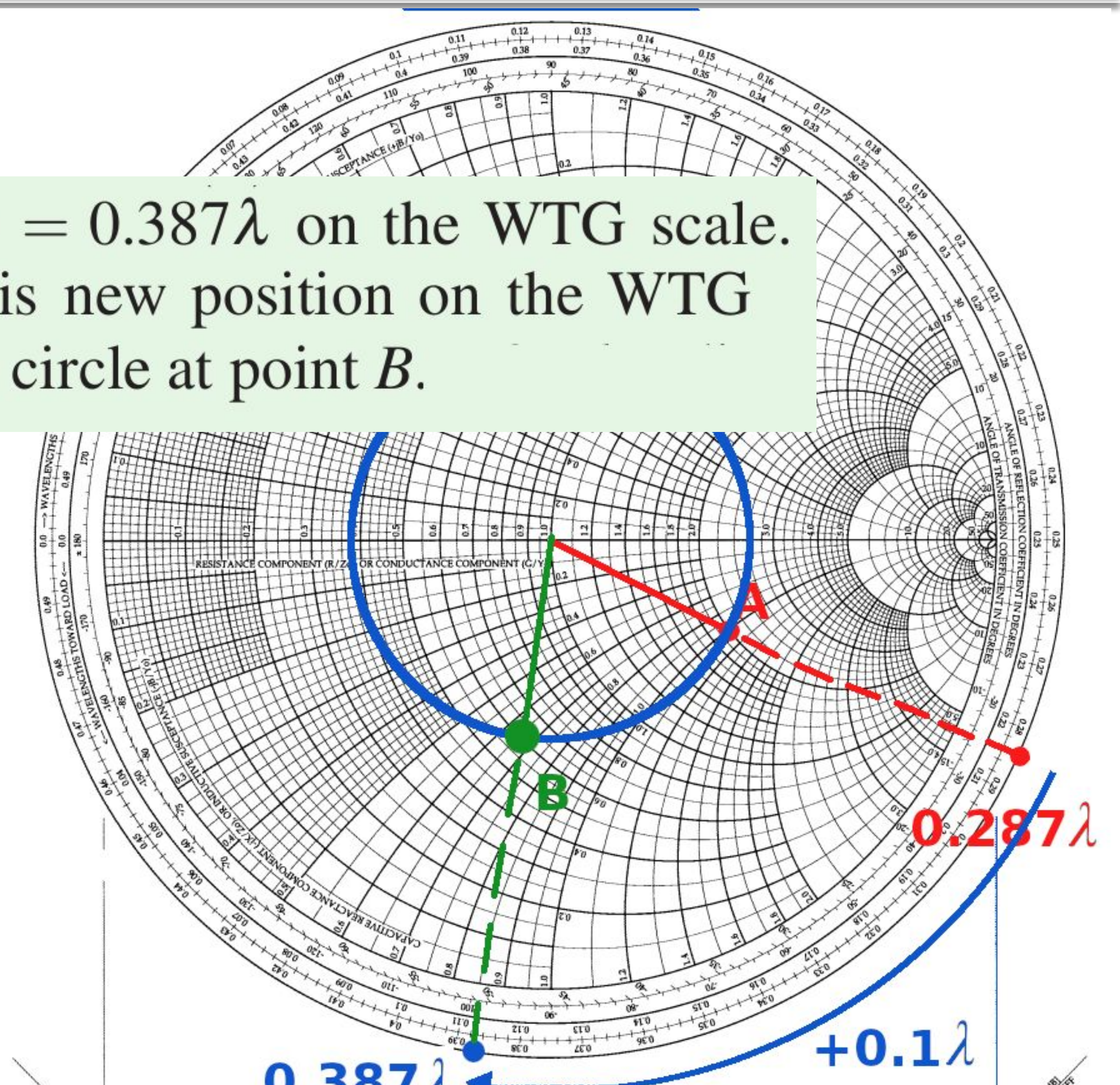
3. Using a compass, construct the SWR circle through *A*.



2.10 Input Impedance: Example

Solution:

4. Locate $0.287\lambda + 0.1\lambda = 0.387\lambda$ on the WTG scale. A radial line through this new position on the WTG scale intersects the SWR circle at point B.



2.10 Input Impedance: Example

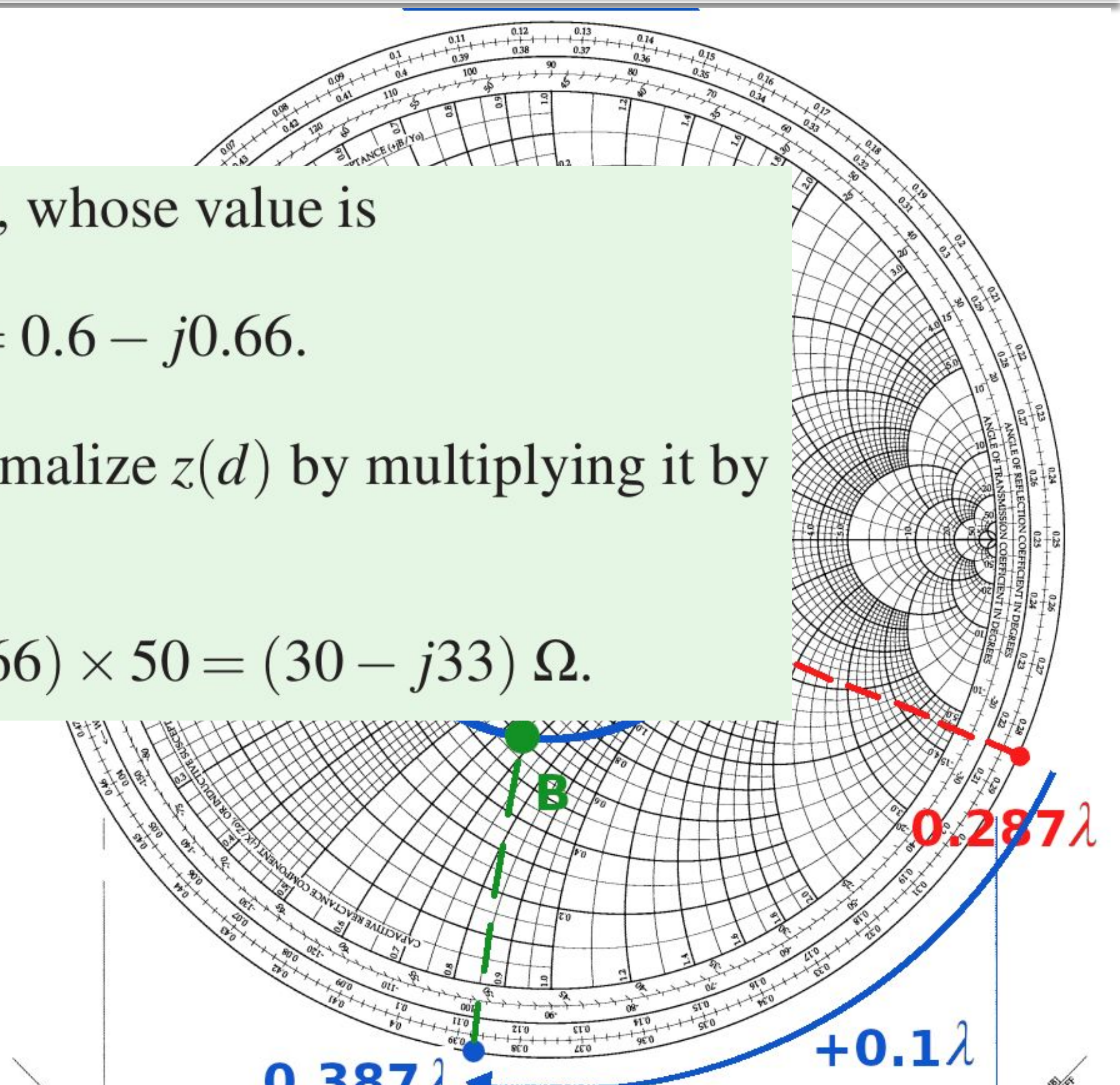
Solution:

5. Point B represents $z(d)$, whose value is

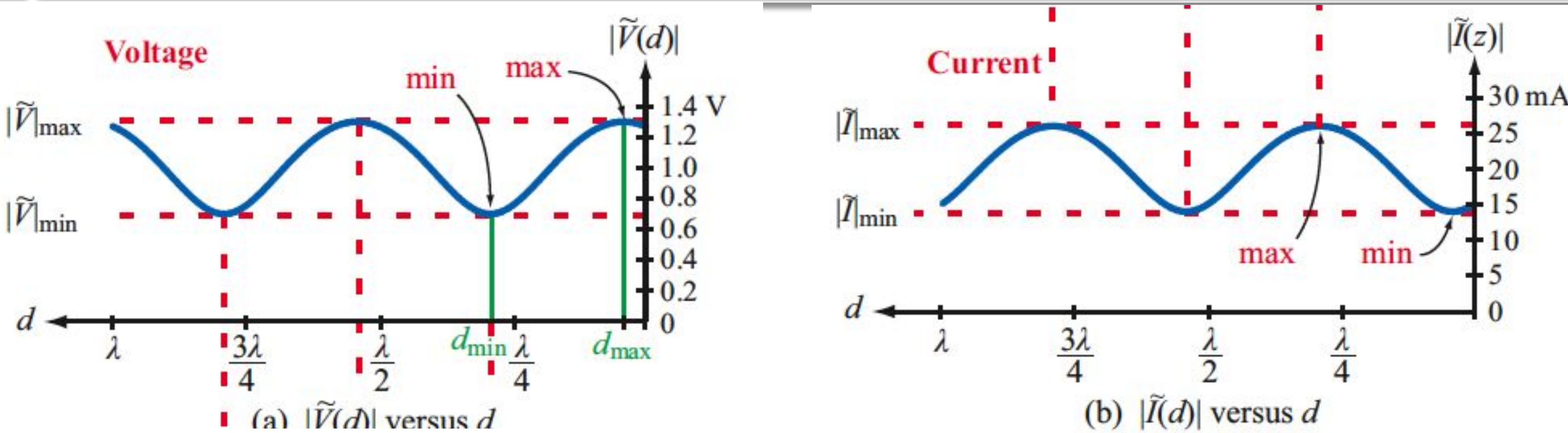
$$z(d) = 0.6 - j0.66.$$

To obtain $Z(d)$, we unnormalize $z(d)$ by multiplying it by $Z_0 = 50\ \Omega$:

$$Z(d) = (0.6 - j0.66) \times 50 = (30 - j33)\ \Omega.$$



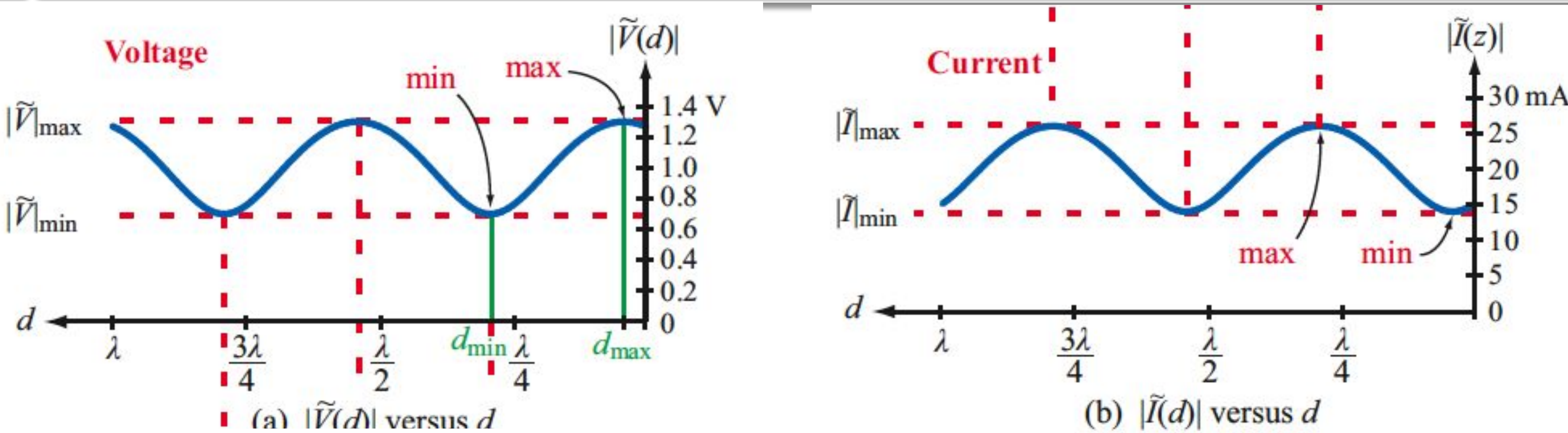
2.10 Voltage Maxima and Minima



- Recall the Voltage on the line is:

$$|\widetilde{V}(d)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)\right]^{1/2}$$

2.10 Voltage Maxima and Minima

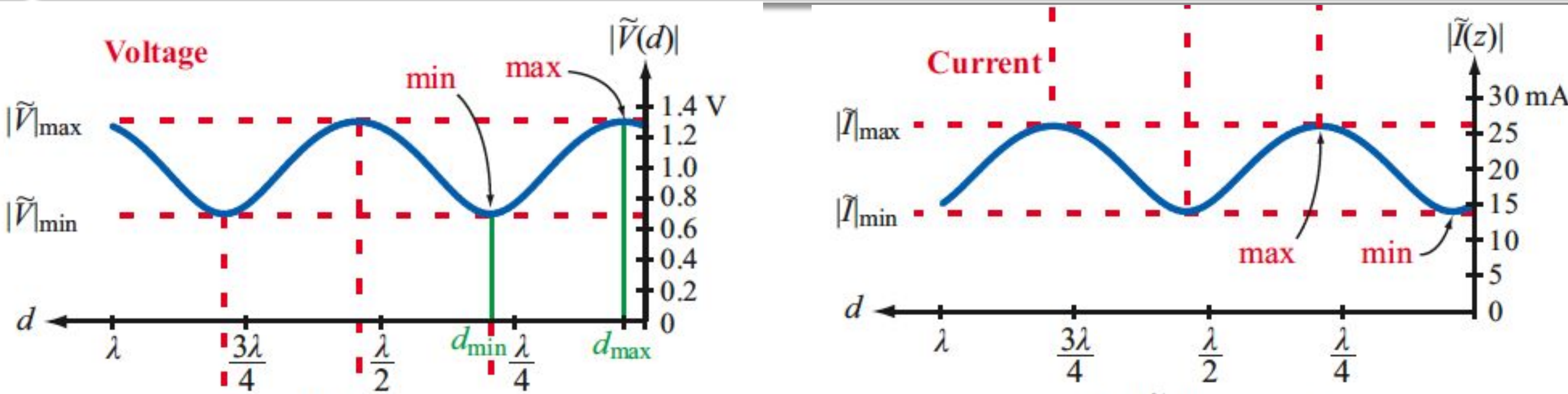


- Location of voltage maxima

$$|\tilde{V}(d)| = |\tilde{V}|_{max} = |V_0^+|[1 + |\Gamma|],$$

$$d_{max} = \frac{\theta_r + 2n\pi}{2\beta} = \frac{\theta_r\lambda}{4\pi} + \frac{n\lambda}{2}, \quad \begin{cases} n = 1, 2, \ldots & \text{if } \theta_r < 0, \\ n = 0, 1, 2, \ldots & \text{if } \theta_r \geq 0, \end{cases} \qquad (2.70)$$

2.10 Voltage Maxima and Minima



(a) $|\widetilde{V}(d)|$ versus d

(b) $|\widetilde{I}(d)|$ versus d

- Location of voltage minima

$$|\widetilde{V}|_{\min} = |V_0^+|[1 - |\Gamma|],$$

$$d_{\min} = \begin{cases} d_{\max} + \lambda/4, & \text{if } d_{\max} < \lambda/4, \\ d_{\max} - \lambda/4, & \text{if } d_{\max} \geq \lambda/4. \end{cases}$$

2.10 Voltage Maxima and Minima

How does this correspond with locations on the Smith Chart?

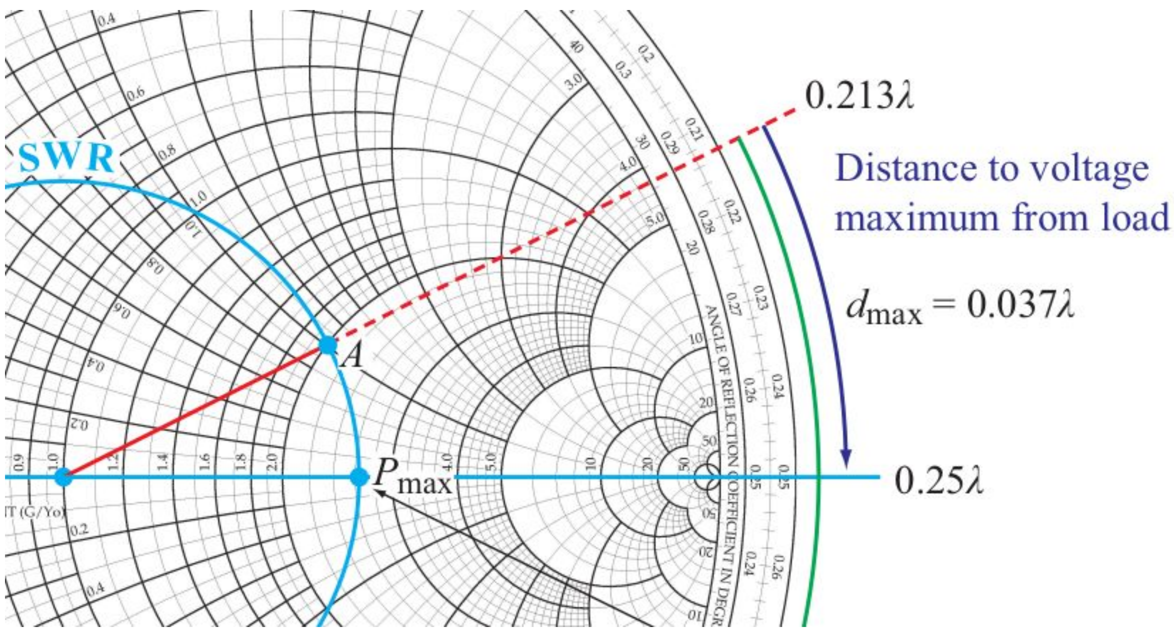
At a Voltage maximum, $2\beta d - \theta_r$ must be $2n\pi$ (so that cos()=1)

At a Voltage minimum, $2\beta d - \theta_r$ must be $(2n+1)\pi$ (so that cos()=-1)

2.10 Voltage Maxima and Minima

If $\theta_r > 0$, then moving clockwise until we get to the $+\Gamma_R$ axis makes this phase value $2n\pi$: d at V_{MAX}

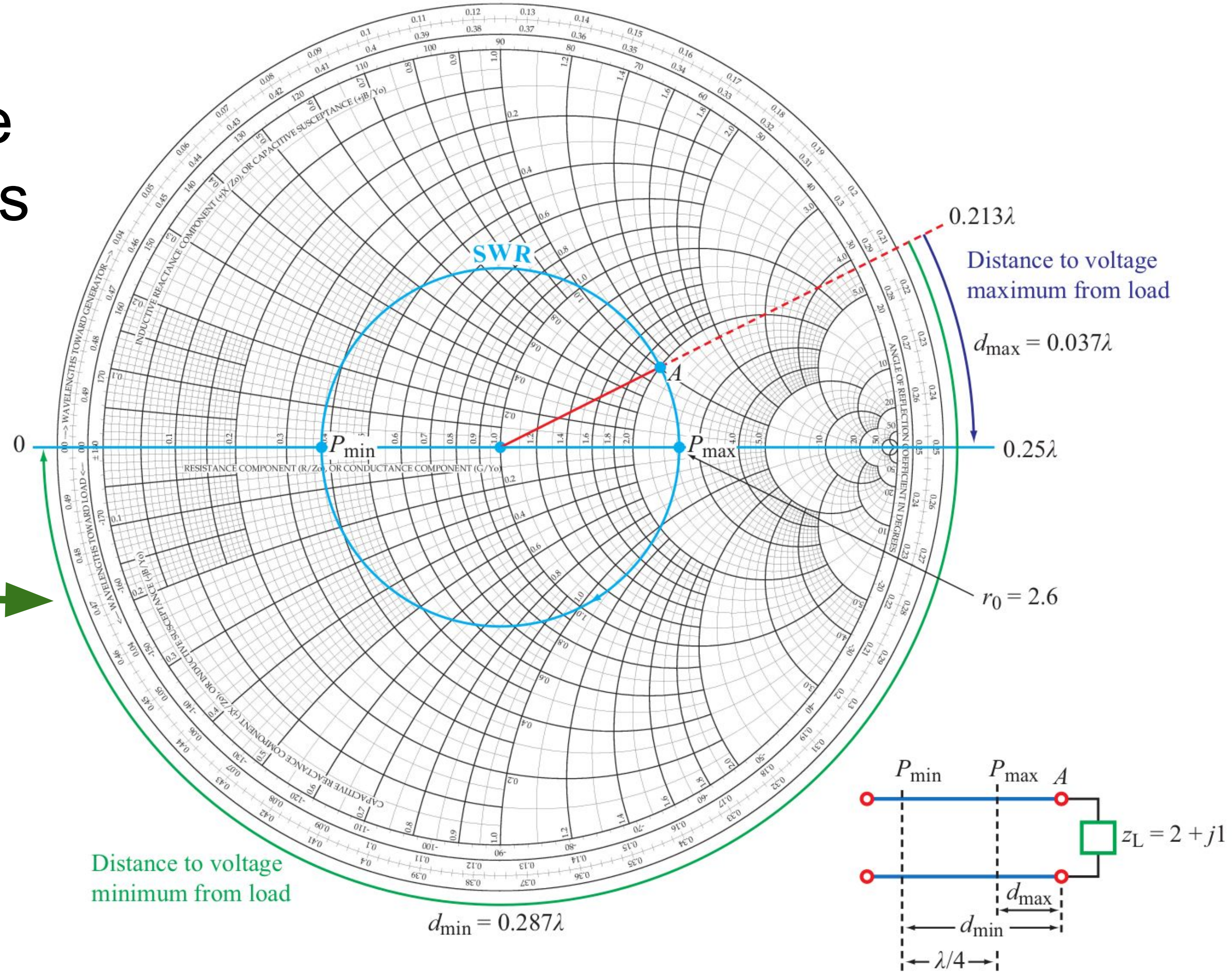
For example: $z_L = 2+j1$



2.10 Voltage Maxima and Minima

Then moving clockwise until we get to the $-\Gamma_R$ axis makes this phase value $(2n+1)\pi$:
d at V_{MIN}

The green arc. →



2.10 Voltage Maxima and Minima

If θ_r <0, then moving clockwise until we get to the $-\Gamma_R$ axis makes this phase value $(2n+1)\pi$: d at V_{MIN},

Then continue moving clockwise until we get to the $+\Gamma_R$ axis makes this phase value $2n\pi$: d at V_{MAX}

2.10 VSWR

Where the VSWR circle intersects the $+\Gamma_R$ axis:
$z(d)$ is real
Γ is real (and positive)

Since the reflection coefficient is defined as:

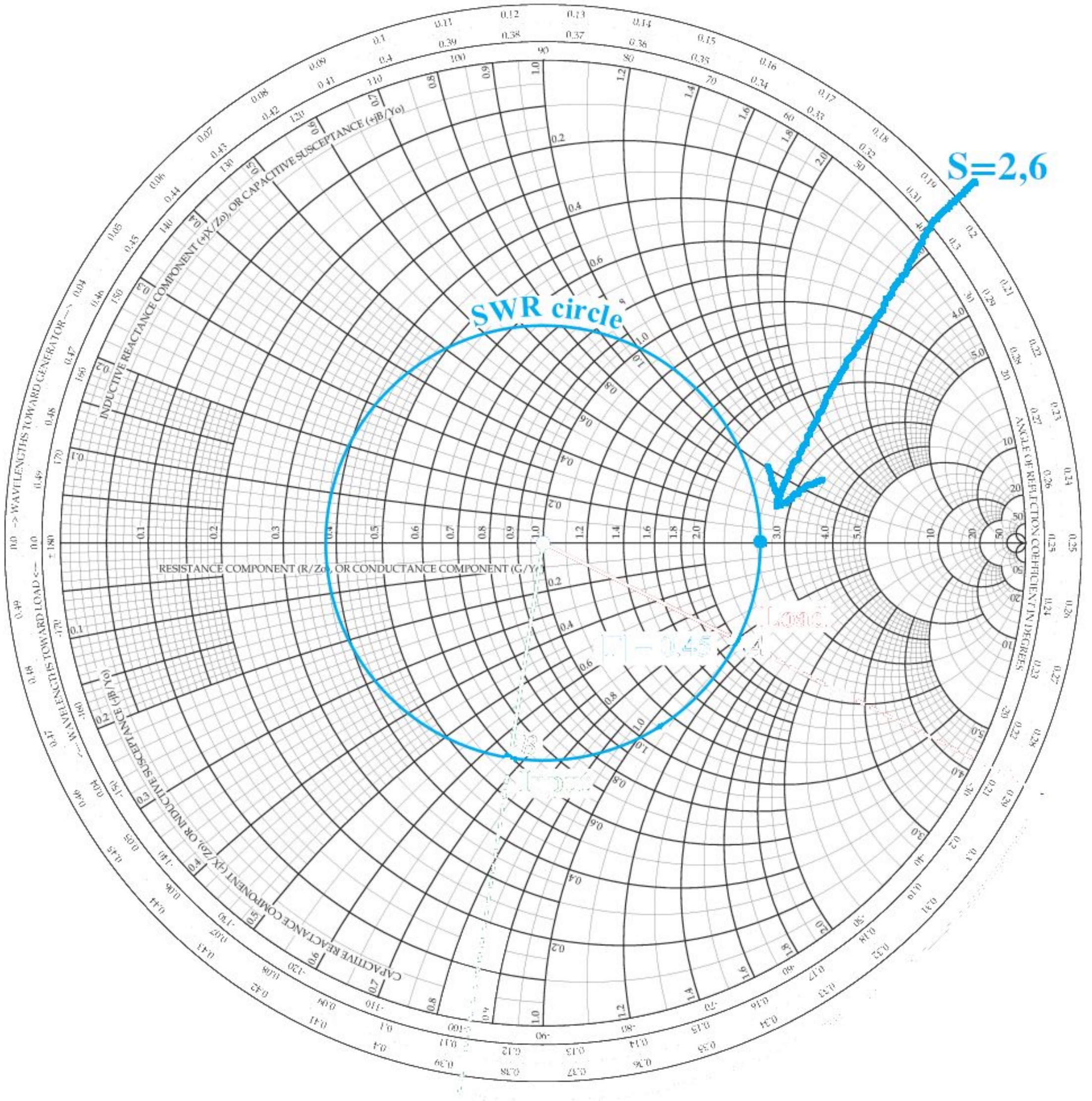
$$\Gamma = \frac{z_L - 1}{z_L + 1}$$

and

$$|\Gamma| = \frac{S - 1}{S + 1}$$

then this real value of $z(d)$ must be equal to S.

2.10 VSWR


S=2,6
SWR circle
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo) OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
<— WAVELENGTHS TOWARD LOAD <—
—> WAVELENGTHS TOWARD GENERATOR —>

2.10 Impedance to Admittance Xform

We know:

$$z_L = \frac{1+\Gamma}{1-\Gamma} \qquad y_L = \frac{1}{z_L} = \frac{1-\Gamma}{1+\Gamma}$$

2.10 Impedance to Admittance Xform

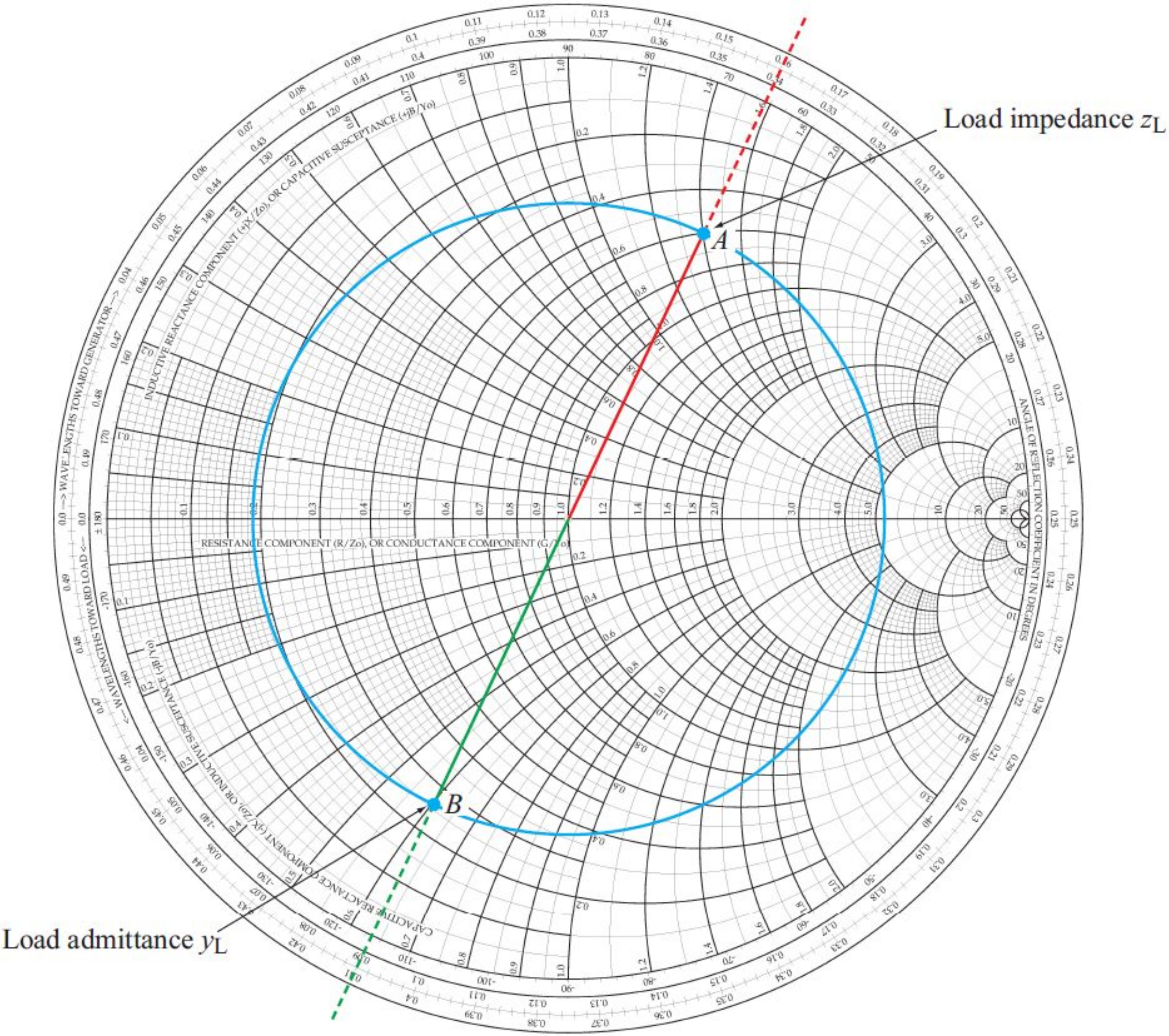
$$z(d) = \frac{1+\Gamma_d}{1-\Gamma_d} = \frac{1+\Gamma e^{-j2\beta d}}{1-\Gamma e^{-j2\beta d}}$$

$$\text{If } -j2\beta d = -j\pi, e^{-j\pi} = -1, \text{ then:}$$

$$z(d) = \frac{1-\Gamma_d}{1+\Gamma_d} = y(d)$$

this happens at $d=\lambda/4$

So, on the Smith Chart, just move a quarter-wavelength toward the generator:



Load impedance z_L
Load admittance y_L
A
B

Example 2.11

Given: Z_0=50Ω, lossless, length=3.3λ, Z_L= (25 + j50) Ω

Find: Using the Smith Chart:

a) voltage reflection coeff: Γ

b) voltage standing wave ratio: S

c) dist of voltage min/max from load

d) input impedance: Z_{in}

e) input admittance: Y_{in}

Example 2.11

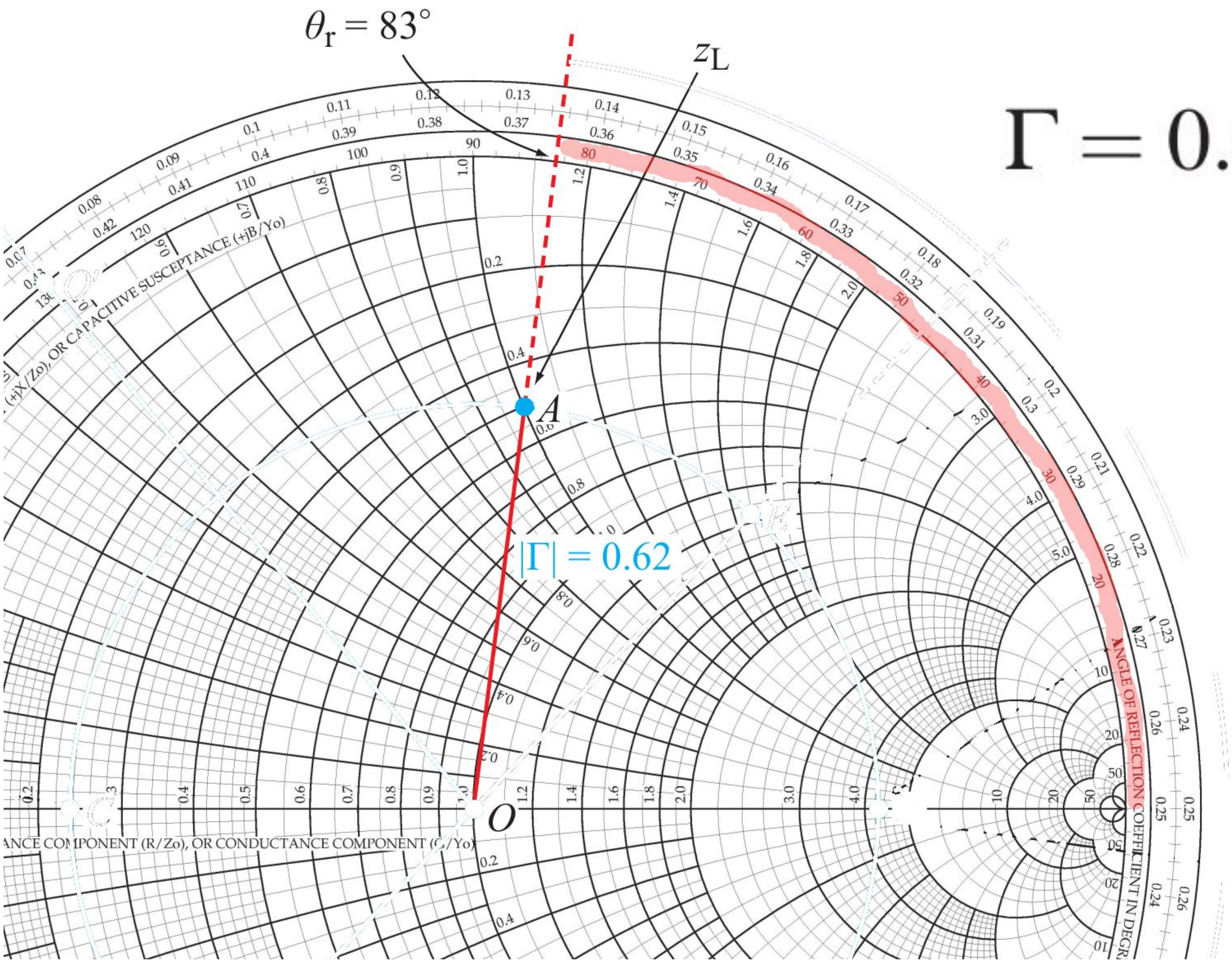
Solution:

a) voltage reflection coeff: Γ

plot the normalized load impedance:

$$z_L = \frac{Z_L}{Z_0} = \frac{25 + j50}{50} = 0.5 + j1,$$

Example 2.11



$$\Gamma = 0.62\underline{/83°}.$$

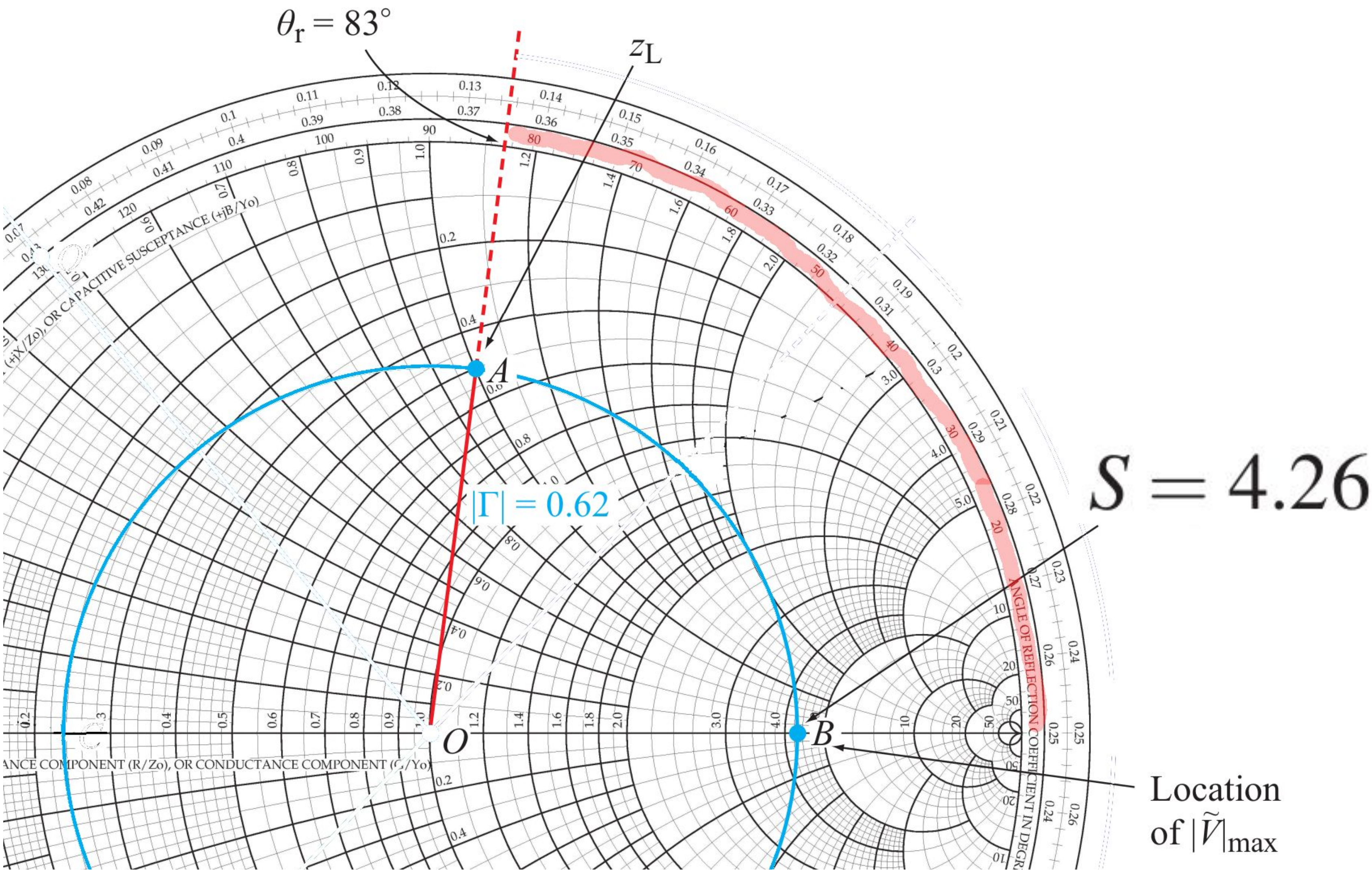
Example 2.11

Solution:

b) voltage standing wave ratio: S

Draw circle centered at origin, going through z_L

Example 2.11



Example 2.11

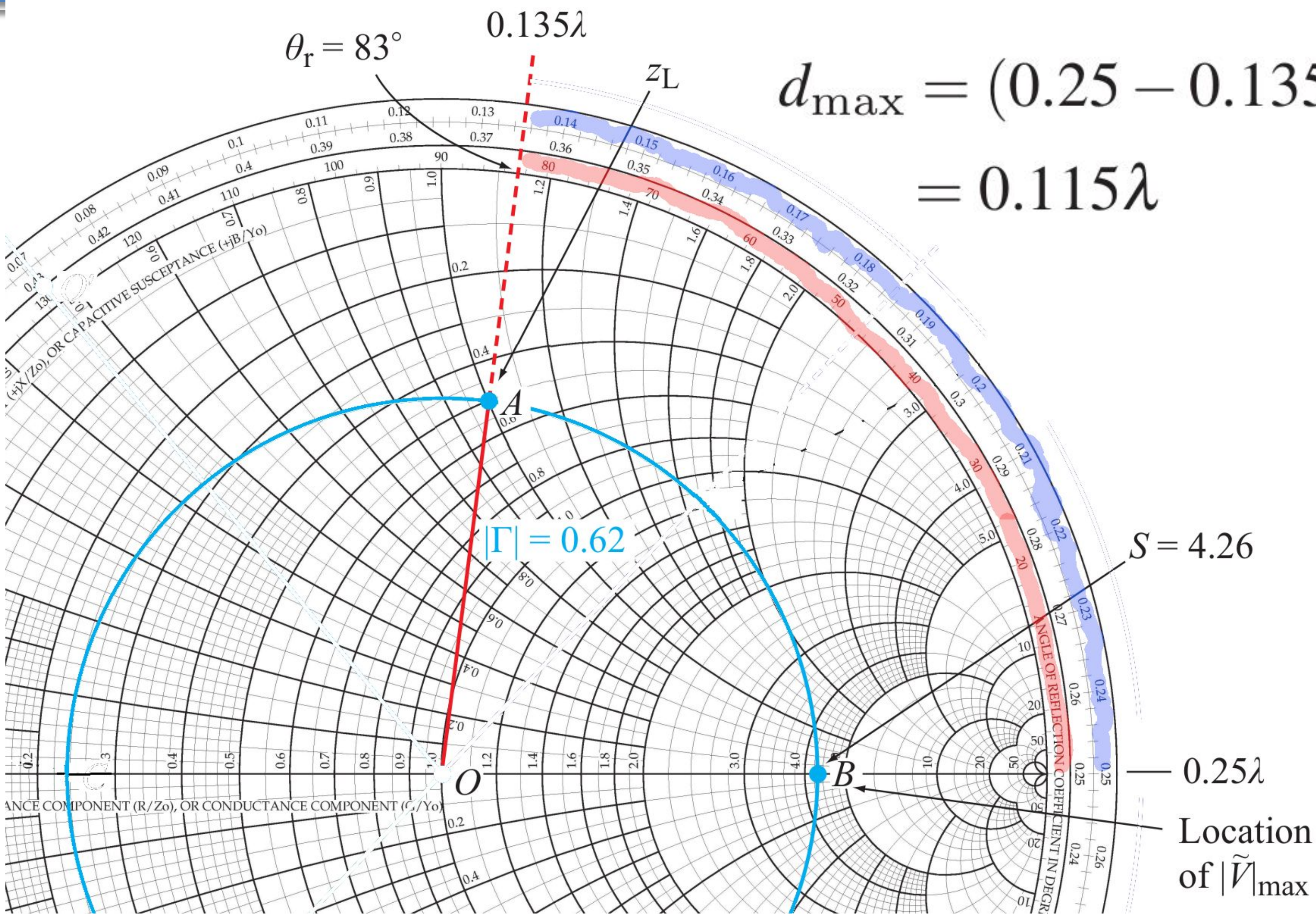
Solution:

c) dist of voltage min/max from load

Distance of Voltage Maximum:

Take the difference in the WTG scale reading between the load and the $+\Gamma_R$ axis.

Example 2.11



$$d_{max} = (0.25 - 0.135)\lambda$$
$$= 0.115\lambda$$

Example 2.11

Solution:

c) dist of voltage min/max from load

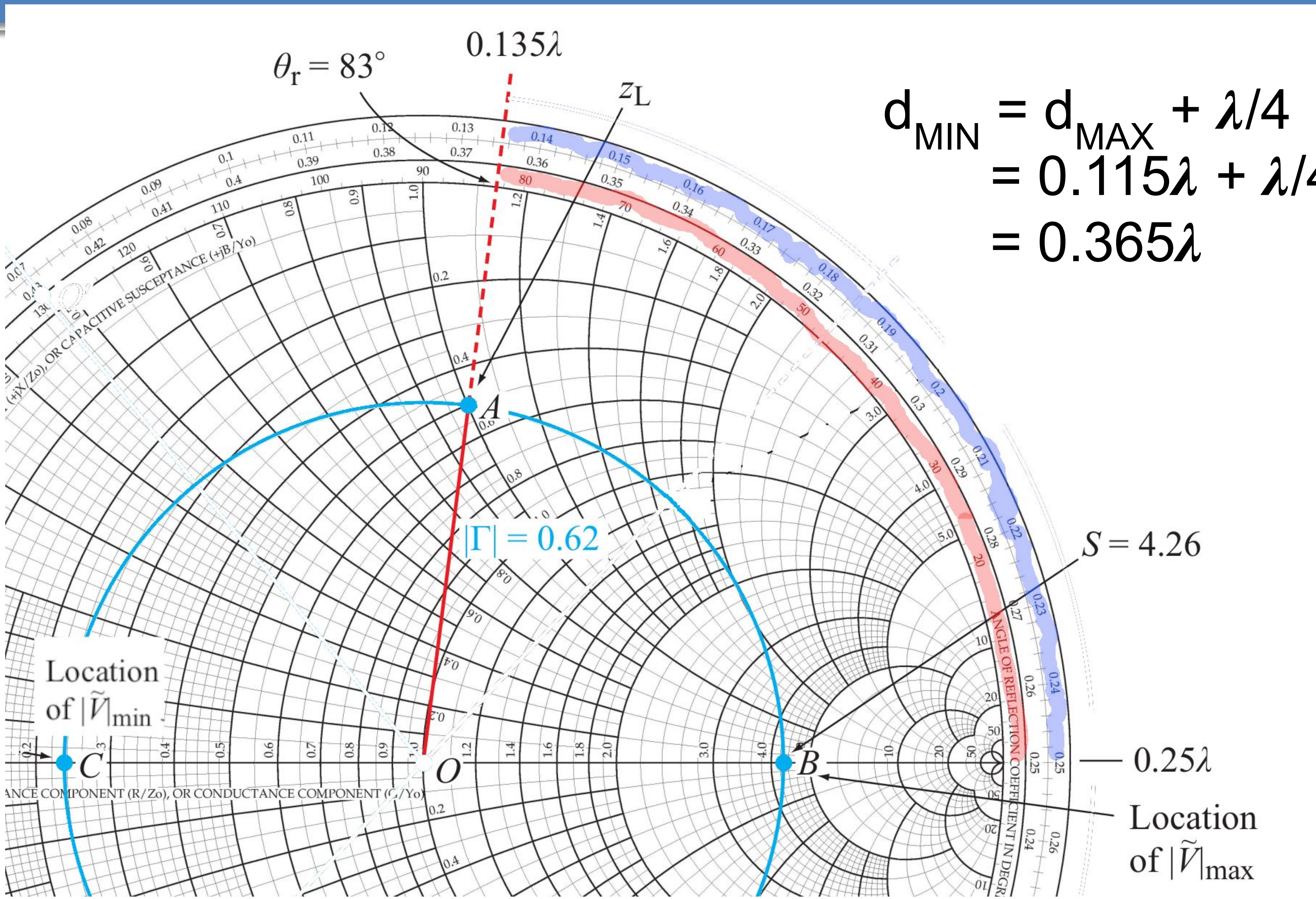
Distance of Voltage Minimum:

Add $\lambda/4$ to value for d_{MAX}:

$d_{MIN} = d_{MAX} + \lambda/4$

$d_{MIN} = 0.115\lambda + \lambda/4 = 0.365\lambda$

Example 2.11



$$d_{MIN} = d_{MAX} + \lambda/4$$
$$= 0.115\lambda + \lambda/4$$
$$= 0.365\lambda$$

Example 2.11

Solution:

d) input impedance: Z_{in}

Line length is 3.3λ

So, need to move 3.3λ toward the generator

Use the WTG scale.

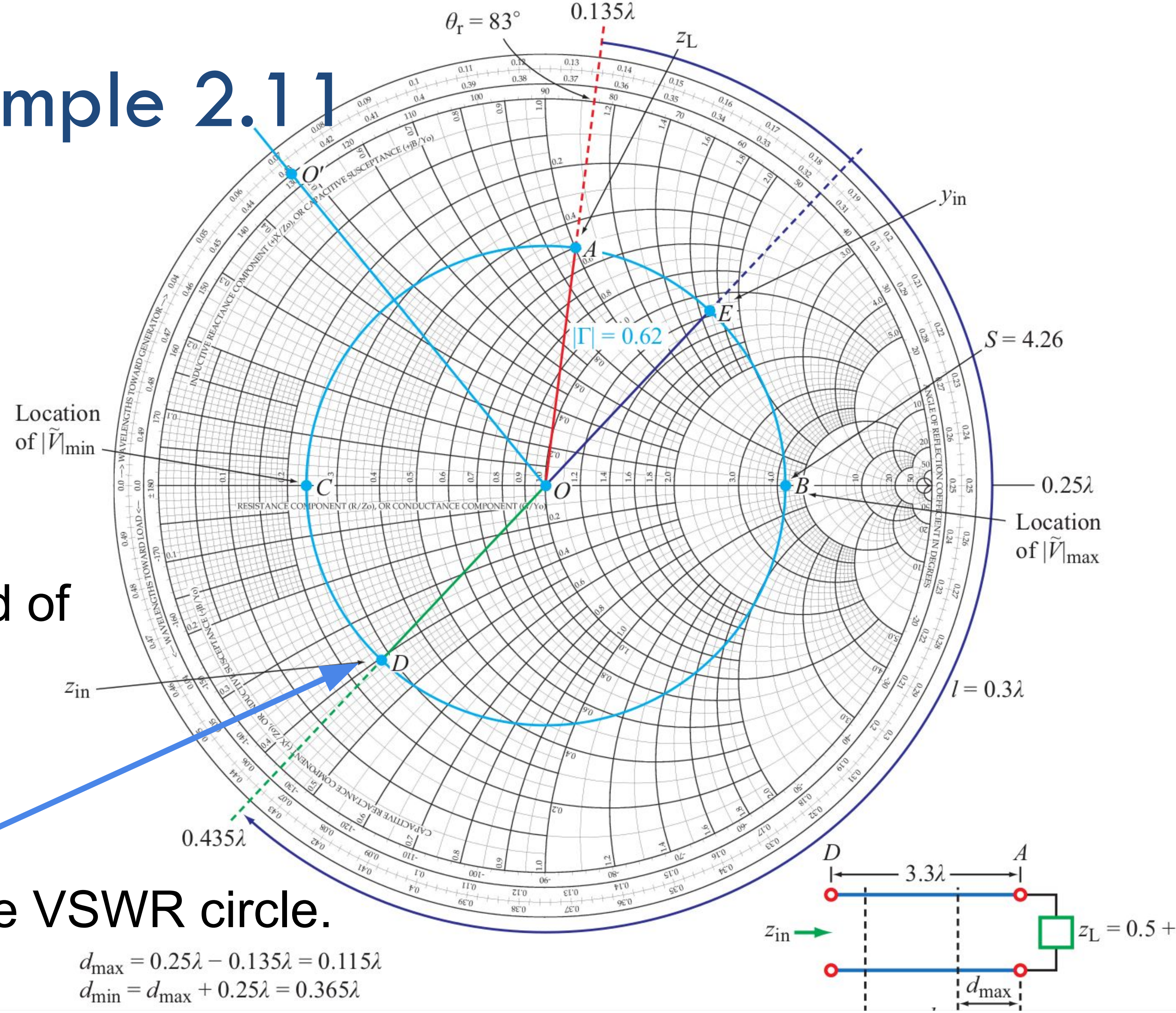
Notice that one complete revolution is 0.5λ, so subtract multiples of 0.5λ from 3.3λ until result is less than 0.5λ:

Need to move 0.3λ toward the generator.

Example 2.11

Load at: 0.135λ

add 0.3λ to get to input end of the line: 0.435λ



z_{IN} on the VSWR circle.

$d_{max} = 0.25\lambda - 0.135\lambda = 0.115\lambda$
$d_{min} = d_{max} + 0.25\lambda = 0.365\lambda$

Example 2.11

Solution:

d) input impedance: Z_{in}

From the Smith Chart:

$$z_{\text{in}} = 0.28 - j0.40,$$

So:

$$Z_{\text{in}} = z_{\text{in}} Z_0 = (0.28 - j0.40)50 = (14 - j20)\ \Omega.$$

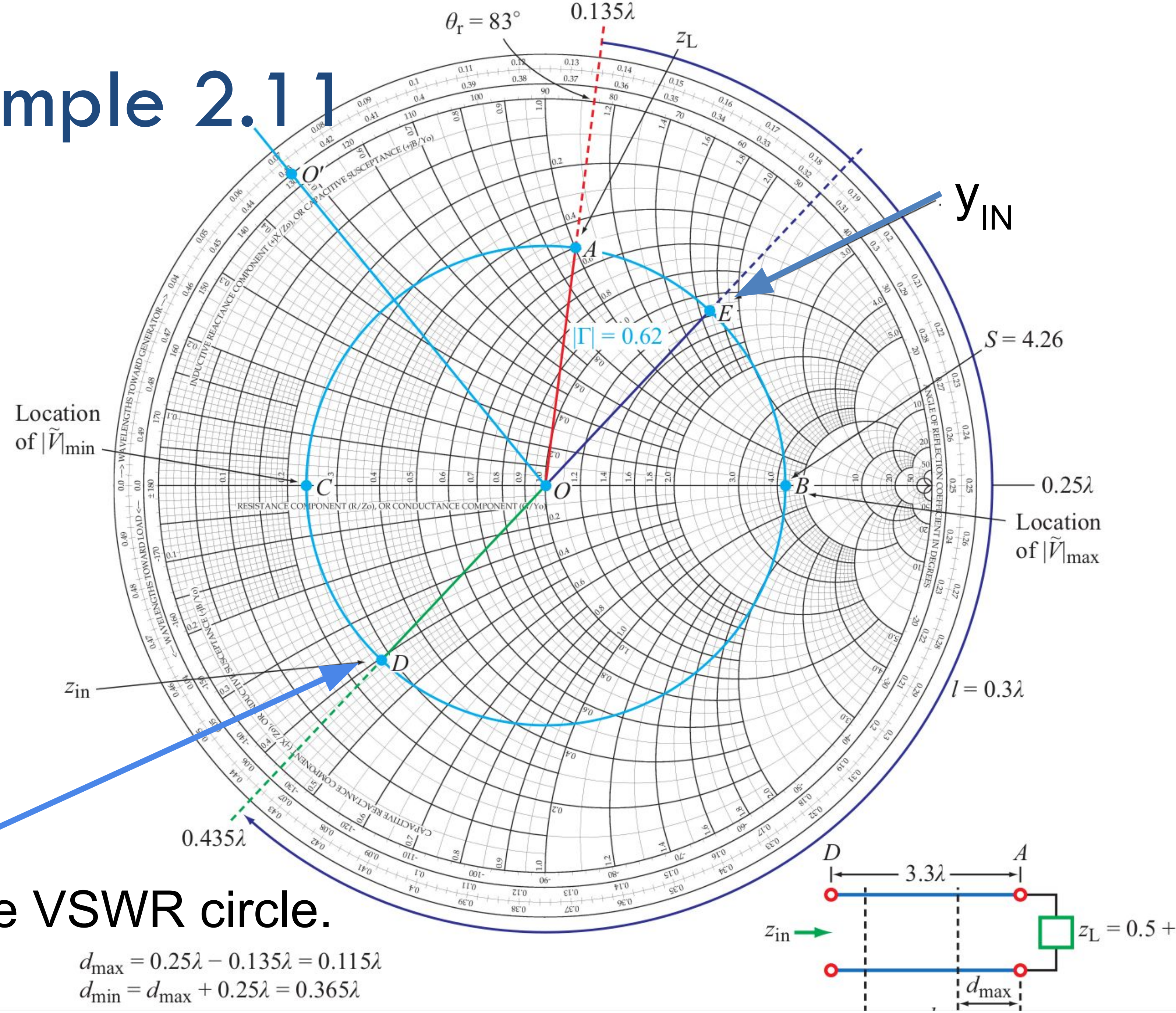
Example 2.11

Solution:

e) input admittance: Y_{in}

Move $\lambda/4$ around the chart to get $1/z_{IN}$

Example 2.11



z_{IN} on the VSWR circle.

Example 2.11

Solution:

e) input admittance: Y_{in}

From the Smith Chart:

$$y_{in} = 1.15 + j1.7,$$

So:

$$Y_{in} = y_{in} Y_0 = \frac{y_{in}}{Z_0} = \frac{1.15 + j1.7}{50} = (0.023 + j0.034)\ \text{S}.$$

Example 2.12

Given: Z_0=50Ω, lossless, S =3,
d_{MIN}=5cm, Δd=20cm
where Δd=dist. betw. successive V_{max} positions.

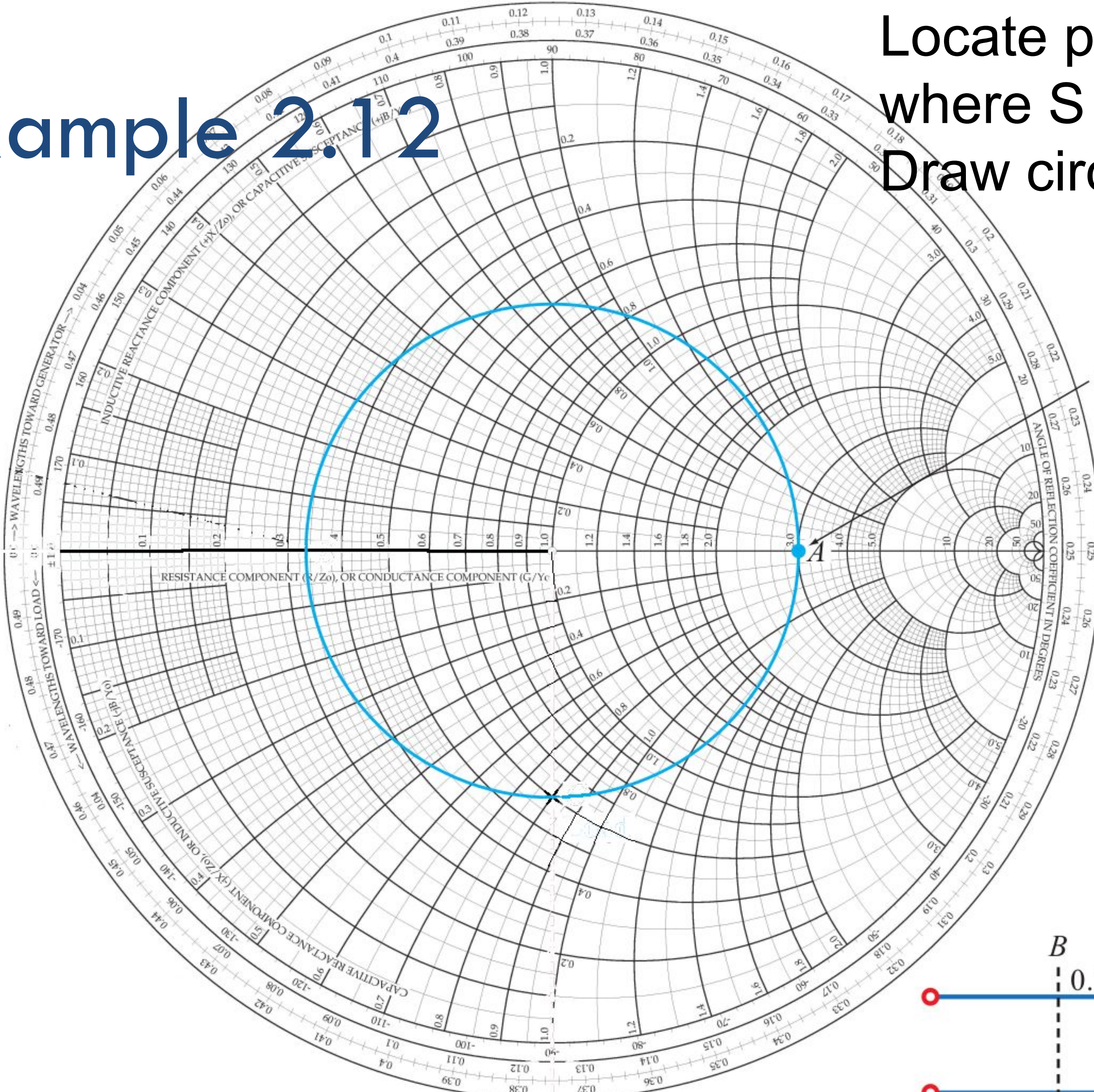
Find: Using the Smith Chart: Z_L

Solution:
Since $\Delta d = \lambda/2$, λ=40cm
and so:

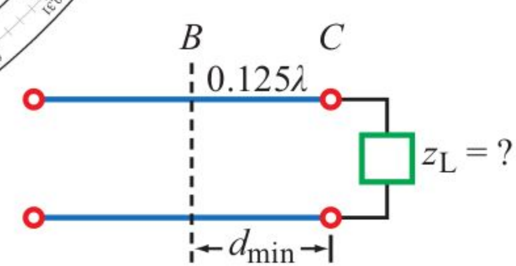
$$d_{MIN} = 5\text{cm} / (40\text{cm}/\lambda) = 0.125\lambda$$

Example 2.12

Locate point A, where S is defined. Draw circle.



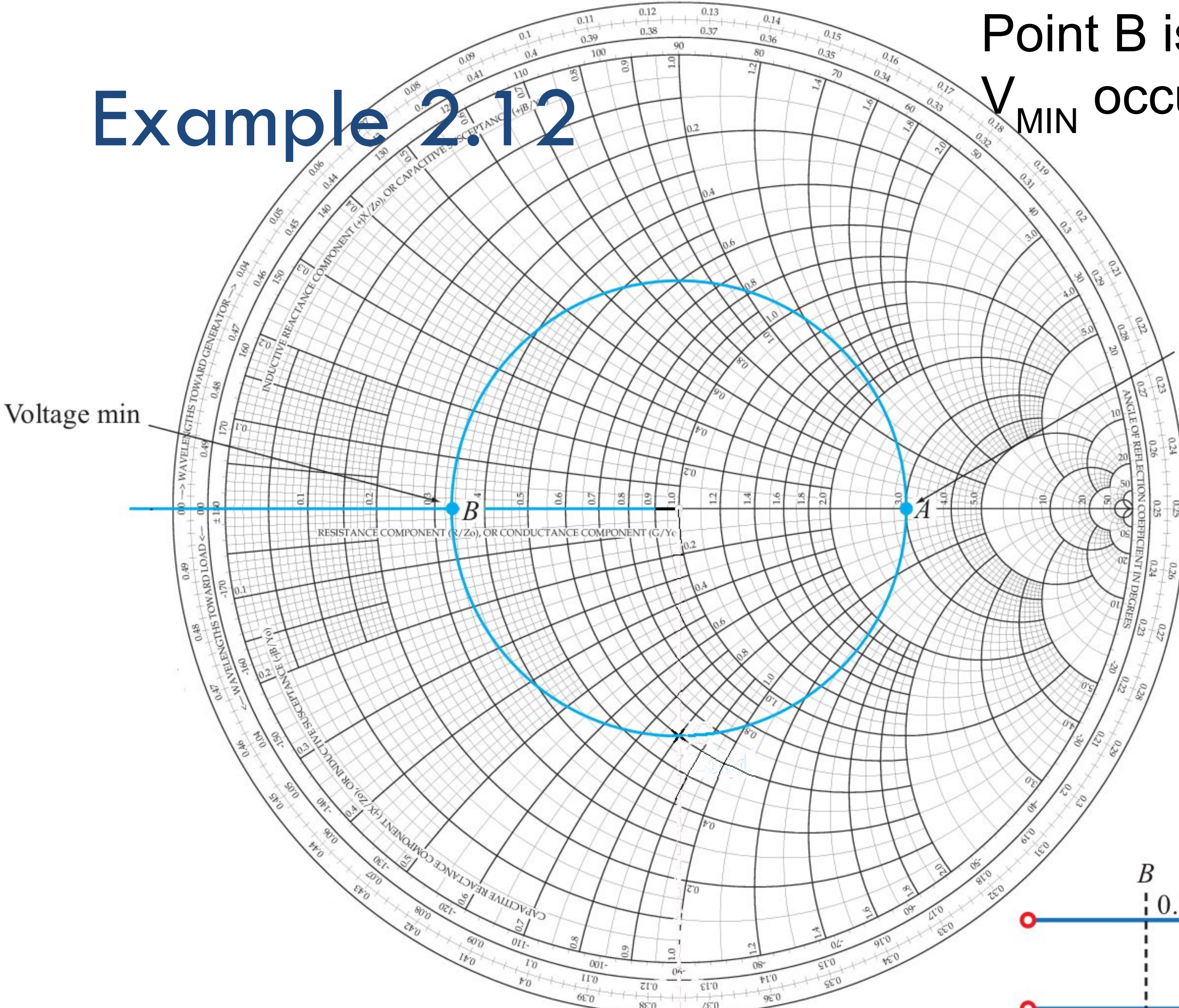
S=3.0

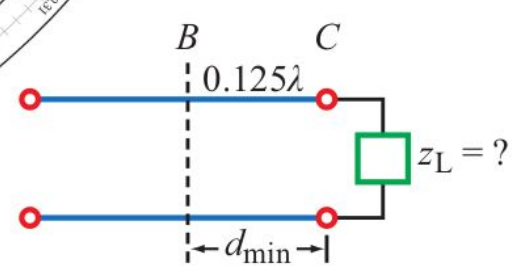


Example 2.12

Point B is where V_{MIN} occurs.

S=3.0

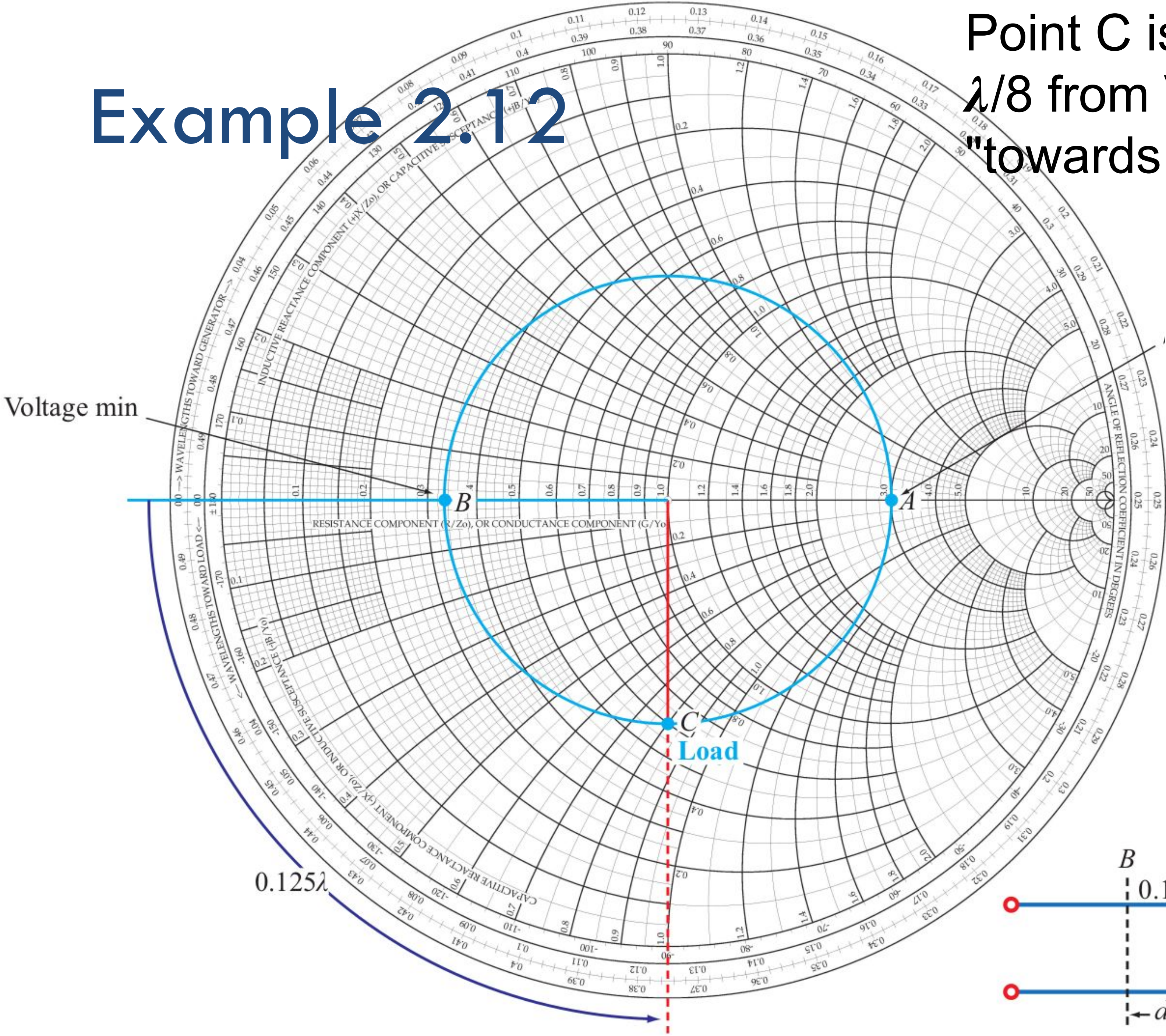


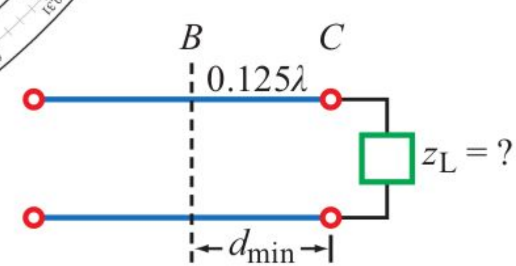


Example 2.12

Point C is the load:
$\lambda/8$ from V_{MIN}
"towards the load"

S=3.0





Example 2.12

Solution:

From the Smith Chart:

$$z_L = 0.6 - j0.8.$$

So:

$$Z_L = 50(0.6 - j0.8) = (30 - j40)\ \Omega.$$

Example

Given: a lossless transmission line with Z_0=50 Ω,

Z_{in} = 50 + j50 Ω, length of 0.2 λ.

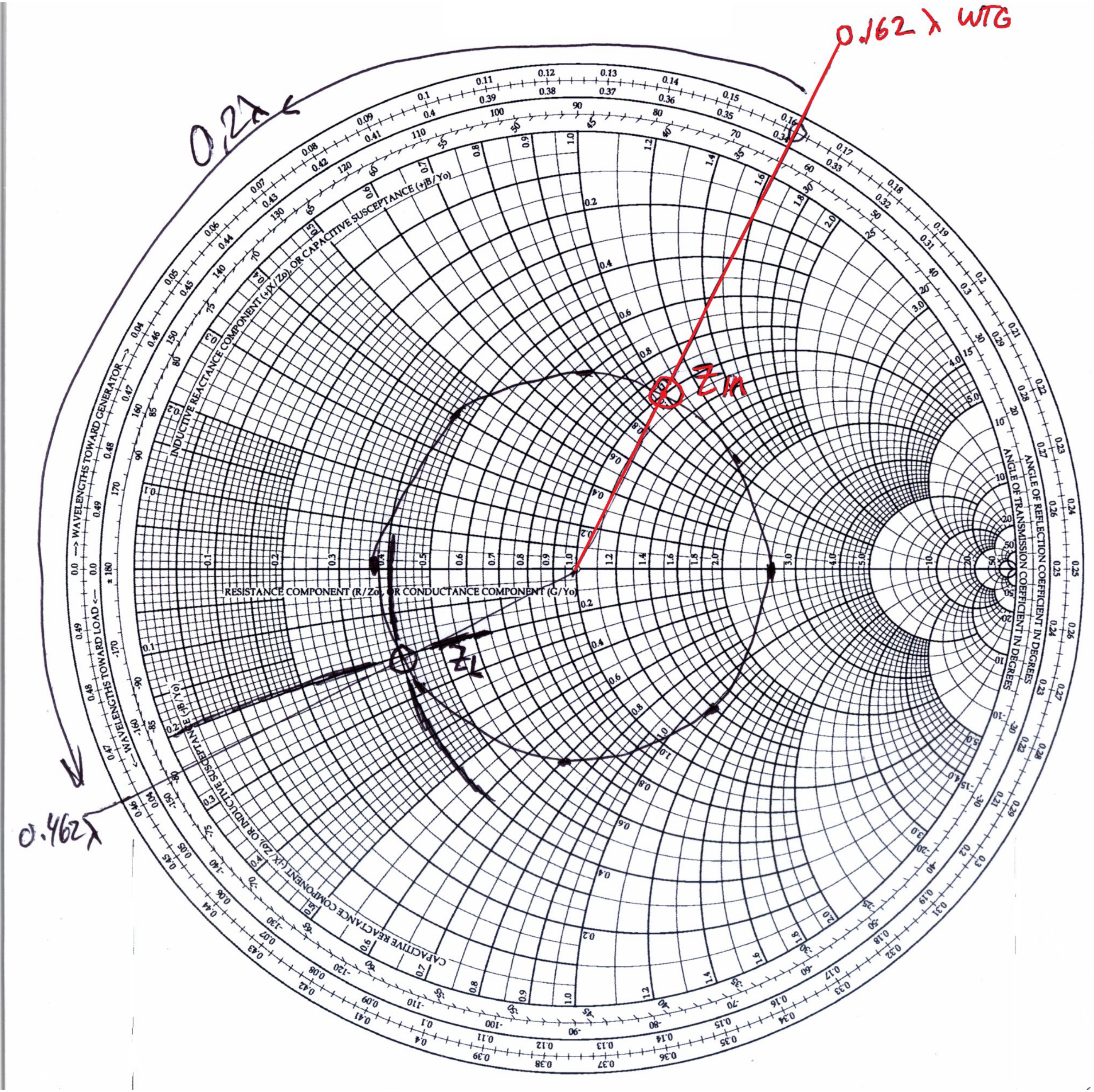
Find: Z_L using the Smith Chart

Solution:

step1: $z_{in} = Z_{in} / Z_0 = 1 + j1$. Plot on Smith Chart

step2: draw radial line to determine value for WTG:
0.162λ

Ex



0.162 λ WTG
0.21λ
Zin
ZL
0.462λ
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES

Example

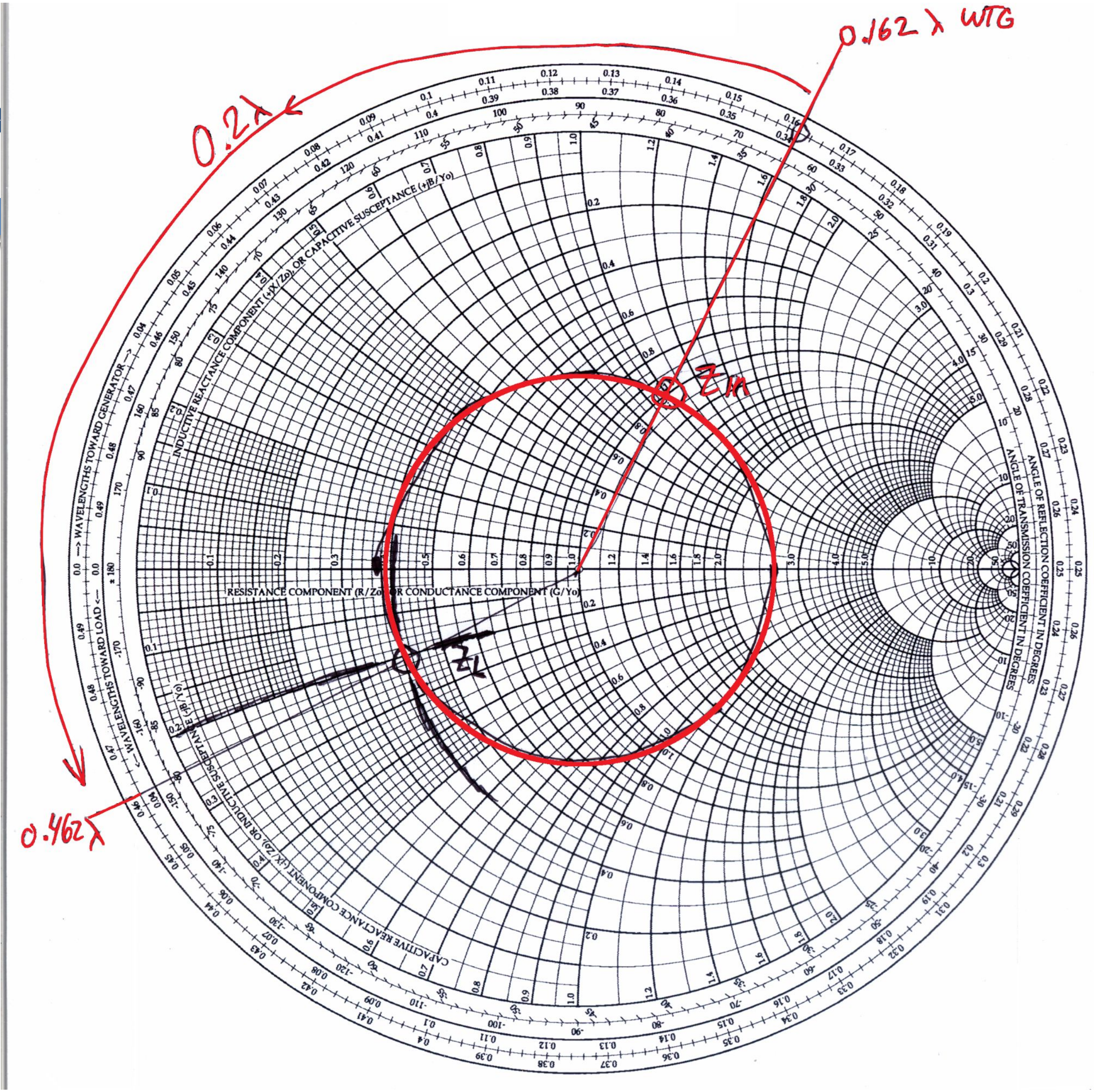
Solution:

Since we know this point was reached by rotating clockwise from z_L by 0.2λ, rotate counter-clockwise from z_{in} to find z_L

step3: draw VSWR circle through z_{in}

step4: rotate 0.2λ CCW: from: 0.162λ to: 0.462λ

Ex



0.162 λ WTG
0.2λ
Zin
ZL
0.462λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES

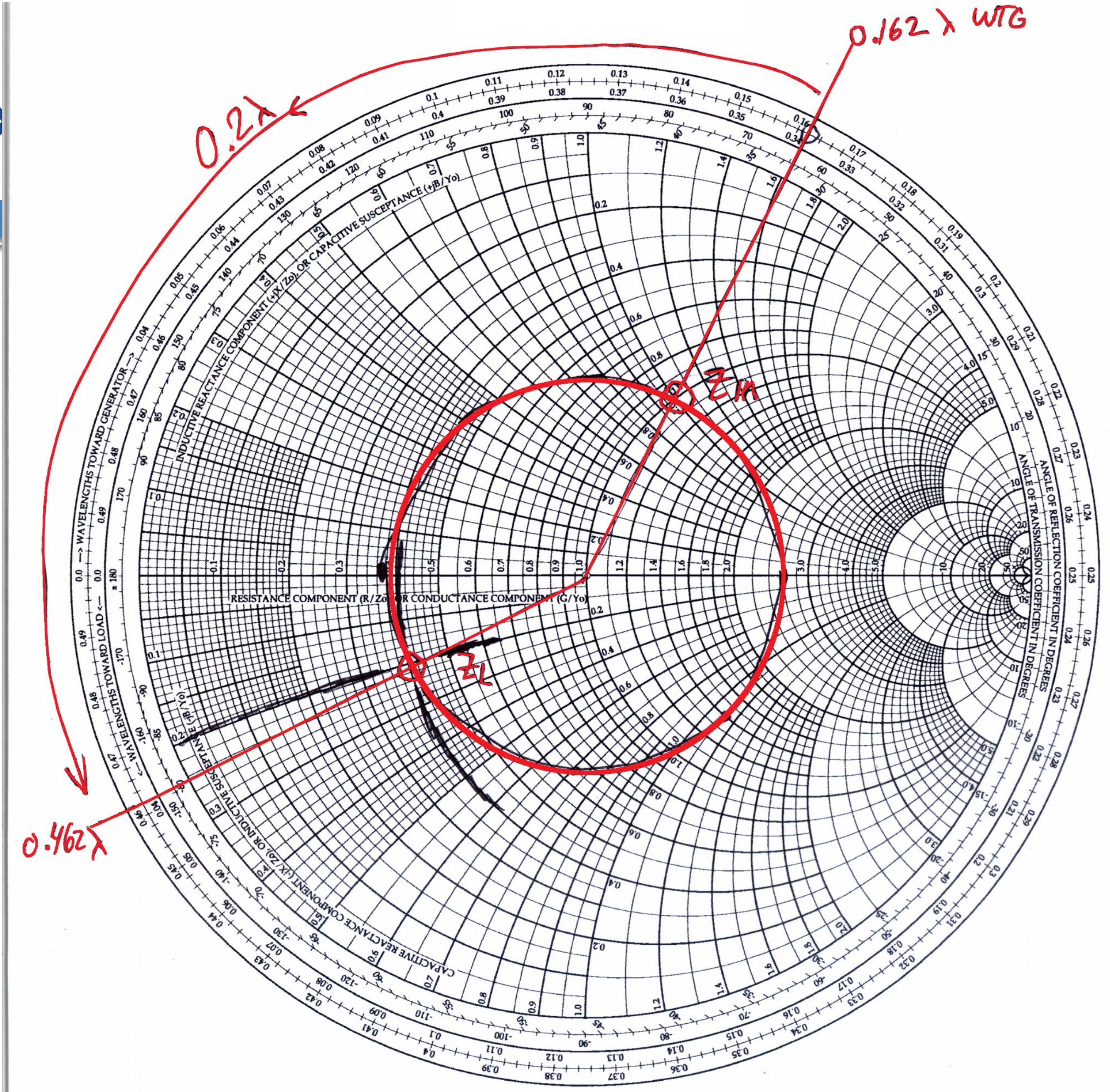
Example

Solution:

step5: draw radial line at this new value for WTG

step6: intersect with VSWR circle to get z_L: 0.4-j0.2

Ex



0.162 λ WTG
0.2λ
Zin
ZL
0.462λ
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES

Example

Solution:

step7: un-normalize to get $Z_L = z_L\ Z_0 = (0.4\text{-}j0.2)(50\Omega)$

$$Z_L = 20 - j10\ \Omega$$

Homework

Homework 8 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Section 2-11:

The Smith Chart: Impedance Matching