EECS 230
ENGINEERING ELECTROMAGNETICS

Leland Pierce

Transmission Lines 5

University of Michigan - Fall 2024

Chapter 2 Overview

What is a transmission line?

Why study transmission lines?

model

diffeq

solve diffeq

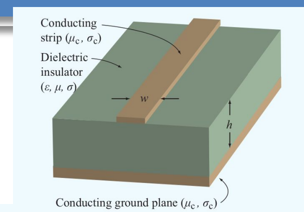
wave propagation

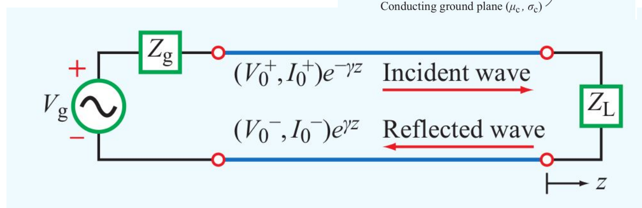
lossless xmission lir

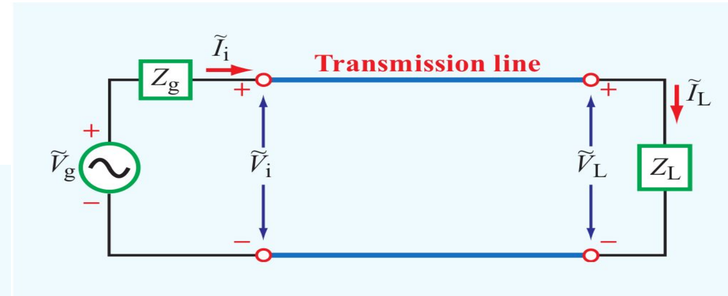
- microstrip lines
- reflections
- standing waves
- impedance
- short, open
- matching
- power flow

smith chart

transients

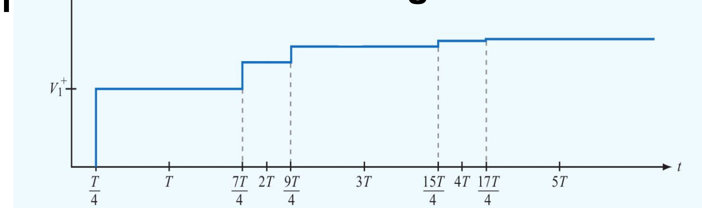


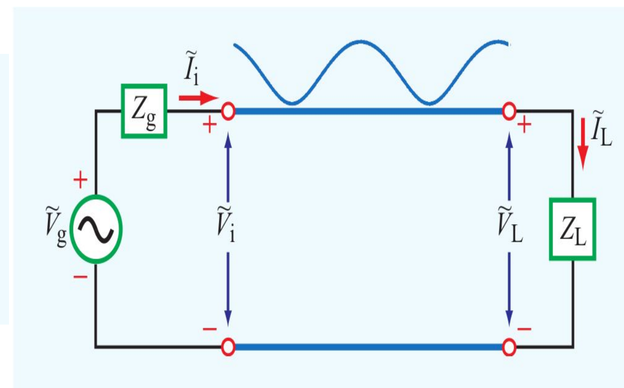




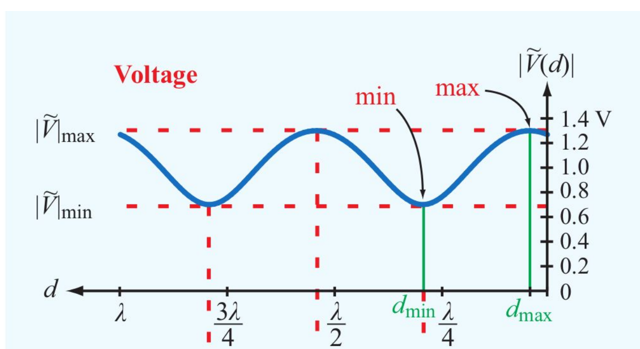
Typical High-Frequency Circuit

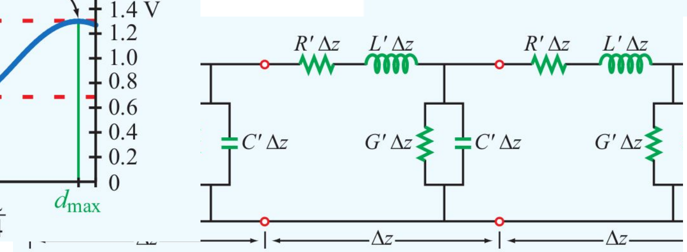
Pulse Voltage vs. time

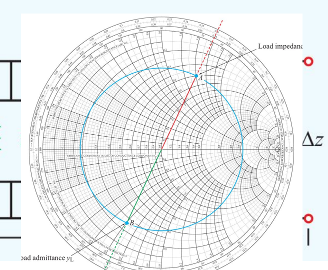




Waves on line: old methods don't work







Today's Lecture Coverage

Review Sections 2-1 through 2-5 of the book:

2-1: What is a transmission line?
Why study transmission lines?
2-2: Lumped-Element Model
2-3: Governing Differential Eqns
2-4: Solve the Differential Equations
Properties of the solution: wave propagation
2-5: Lossless Microstrip Line
2-6: Lossless Transmission Lines
2-7: Lossless Transmission Lines: Wave Impedance

Sections 2-8, 2-9 of the book:

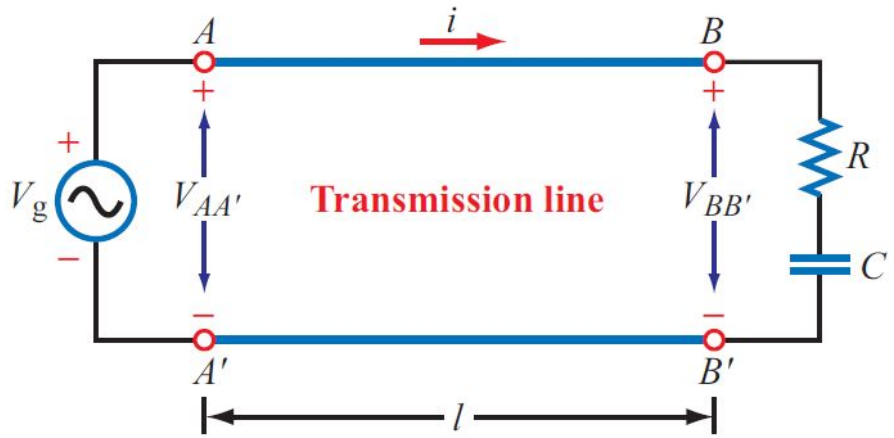
2-8: Lossless Transmission Lines: Special Cases
2-9: Lossless Transmission Lines: Power Flow

Chapter 2 Review

- A transmission line connects a generator to a load.

Chapter 2 Review

Phase Delay due to length of transmission line:



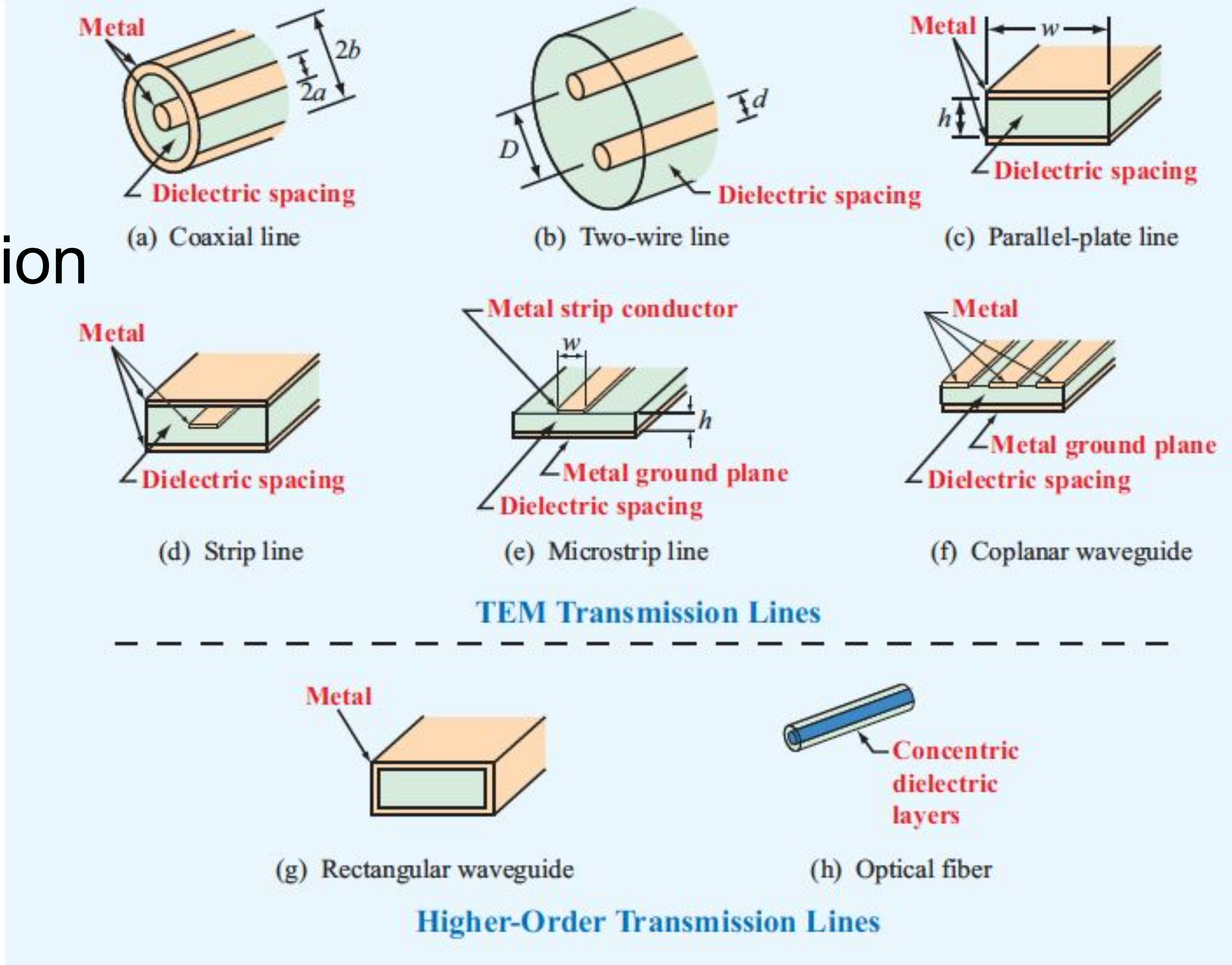
$$\phi_0 = \frac{\omega l}{c} = \frac{2\pi f l}{c} = 2\pi \frac{l}{\lambda} \quad \text{radians.}$$

$l/\lambda \lesssim 0.01$: Can ignore transmission-line effects

$l/\lambda \gtrsim 0.01$: Must deal with phase shift, and other effects...

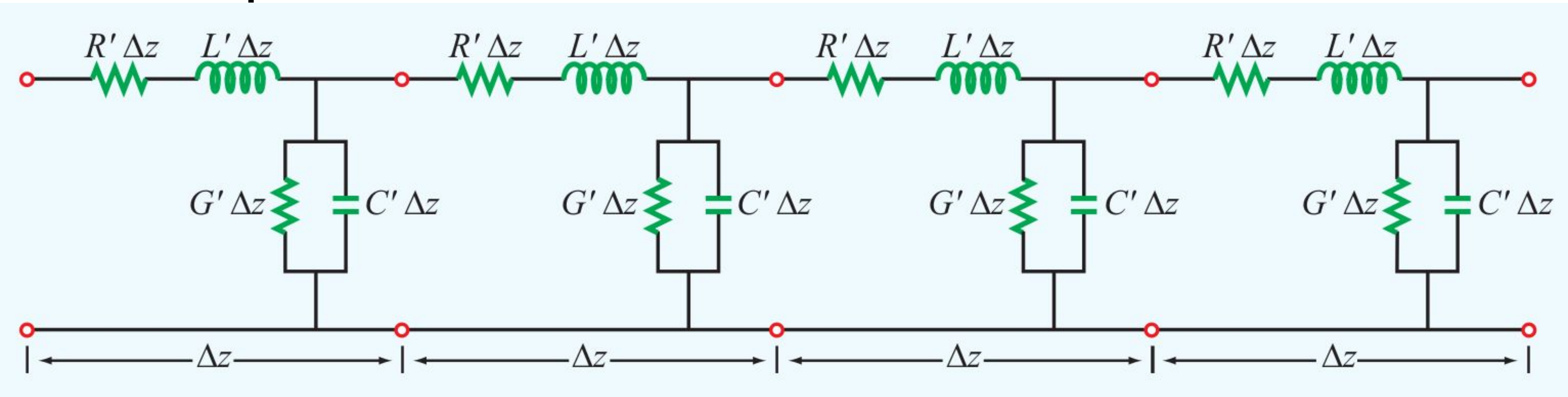
Chapter 2 Review

Different geometries for transmission lines



Chapter 2 Review

Lumped-Element Model:



All parameters are "per unit length":

R': Combined Resistance of BOTH conductors: □/m

L': Combined Inductance of BOTH conductors, H/m

G': Conductance of insulation
between inner and outer conductor, S/m

C': Capacitance
between inner and outer conductors, F/m

Chapter 2 Review

Lumped-Element Values: geometry/materials/freq

Table 2-1 **Transmission-line parameters R', L', G', and C' for three types of lines.**

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\varepsilon w}{h}$	F/m

$$R_s = \sqrt{\pi f \mu_c / \sigma_c}$$

Chapter 2 Review

Transmission-line governing Differential Equations

$$-\frac{d\widetilde{V}(z)}{dz} = (R' + j\omega L')\,\widetilde{I}(z),$$

$$-\frac{d\widetilde{I}(z)}{dz} = (G' + j\omega C')\,\widetilde{V}(z).$$

(telegrapher's equations in phasor form)

Chapter 2 Review

Transmission-line governing Differential Equation for *V*:

$$\frac{d^2\widetilde{V}(z)}{dz^2} - \gamma^2 \widetilde{V}(z) = 0,$$

(wave equation for $\widetilde{V}(z)$)

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}.$$

(propagation constant)

Chapter 2 Review

Complex propagation constant:

$$\gamma = \alpha + j\beta$$

Where:

γ: Units of 1/m

α: Attenuation constant, units of Np/m (>0 in this class)

β: Phase constant, units of rad/m

Np and radians are both "dimensionless", and are the conventional "units" for these parameters.

Chapter 2 Review

Form of the solution: traveling waves, going in both directions:

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad \text{(V)},$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \qquad \text{(A)}.$$

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \qquad (\Omega),$$

Chapter 2 Review

Solution in time-domain

$$v(z,t) = |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \phi^+) + |V_0^-| e^{+\alpha z} \cos(\omega t + \beta z + \phi^-)$$

Remaining unknowns are determined via specification of source and load.

Chapter 2 Review

- The wave equation for a general Transmission Line.

$$\frac{d^2\widetilde{V}(z)}{dz^2} - \gamma^2\,\widetilde{V}(z) = 0, \qquad (2.21)$$

- General solution of the wave equation
    - *It involves both incident and reflected waves*

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad \text{(V)},$$

$$\widetilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \qquad \text{(A)}.$$

Chapter 2 Review

- Useful Relations for lossless Transmission Lines:

$$\alpha = 0 \qquad \text{(lossless line)},$$
$$\beta = \omega\sqrt{L'C'} \qquad \text{(lossless line)}. \qquad (2.45)$$

$$\beta = \omega\sqrt{\mu\varepsilon} \qquad \text{(rad/m)}, \qquad (2.49)$$

$$u_p = \frac{1}{\sqrt{\mu_0\epsilon_0}}\frac{1}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}}$$

$$\lambda = \frac{u_p}{f} = \frac{c}{f}\frac{1}{\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

$$Z_0 = \sqrt{\frac{L'}{C'}} \qquad \text{(lossless line)}, \qquad (2.46)$$

(REAL)

Chapter 2 Review

- Voltage reflection coefficient due to load:

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1}$$

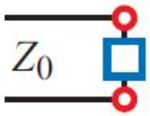
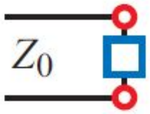
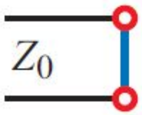
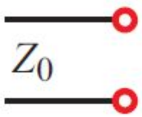
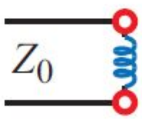
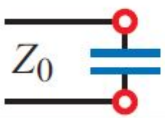
$$= \frac{z_L - 1}{z_L + 1}$$

- Load impedance in terms of Γ:

$$Z_L = \frac{1+\Gamma}{1-\Gamma} Z_0$$

Chapter 2 Review

Reflection Coefficient $\Gamma = |\Gamma| e^{j\theta_r}$

Load	$\lvert\Gamma\rvert$	θ_r
 $Z_L = (r + jx)Z_0$	$\left[\frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$	$\tan^{-1}\left(\frac{x}{r-1}\right) - \tan^{-1}\left(\frac{x}{r+1}\right)$
 Z_0	0 (no reflection)	irrelevant
 (short)	1	$\pm 180°$ (phase opposition)
 (open)	1	0 (in-phase)
 $jX = j\omega L$	1	$\pm 180° - 2\tan^{-1} x$
 $jX = \frac{-j}{\omega C}$	1	$\pm 180° + 2\tan^{-1} x$

$$z_L = Z_L/Z_0 = (R + jX)/Z_0 = r + jx$$

Chapter 2 Review

- Concept of standing wave

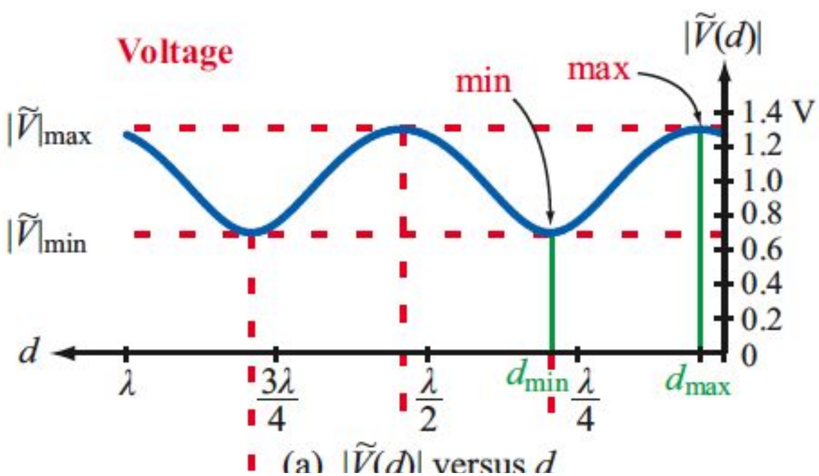
$$\widetilde{V}(z) = V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z}),$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}).$$

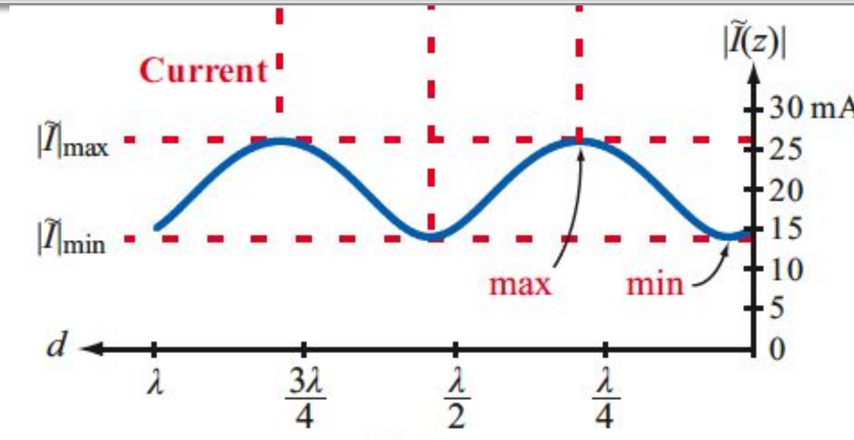
  - Voltage magnitudes at any point on line:

$$|\widetilde{V}(d)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta d - \theta_r)\right]^{1/2}. \quad (2.66)$$

Chapter 2 Review



(a) $|\tilde{V}(d)|$ versus d



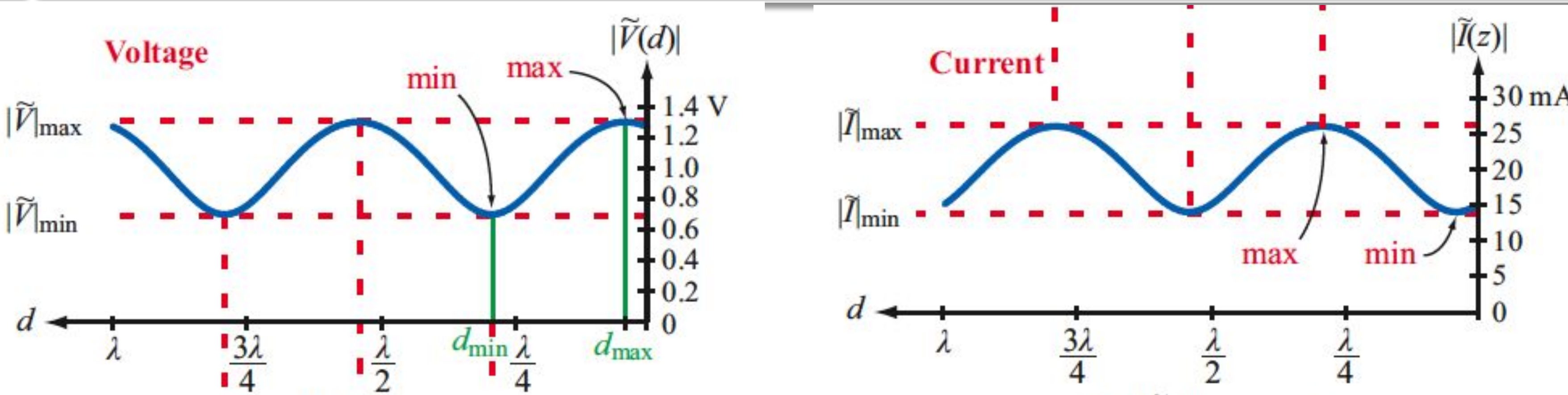
(b) $|\tilde{I}(d)|$ versus d

- Location of minima / maxima

$$d_{\max} = \frac{\theta_r + 2n\pi}{2\beta} = \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2}, \quad \begin{cases} n = 1, 2, \ldots & \text{if } \theta_r < 0, \\ n = 0, 1, 2, \ldots & \text{if } \theta_r \geq 0, \end{cases} \qquad (2.70)$$

Value of V_{max}: $|\tilde{V}(d)| = |\tilde{V}|_{\max} = |V_0^+|[1 + |\Gamma|],$

Chapter 2 Review



(a) $|\widetilde{V}(d)|$ versus d

(b) $|\widetilde{I}(d)|$ versus d

- Spatial period of standing wave: $\frac{\lambda}{2}$

- Standing wave ratio S:

$$S = \frac{|\widetilde{V}|_{max}}{|\widetilde{V}|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

Chapter 2 Review

Wave Impedance:

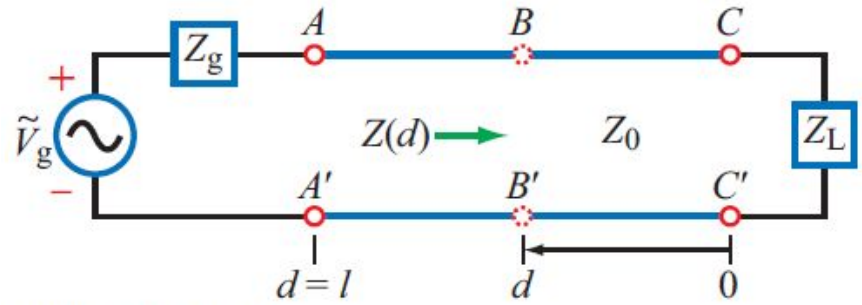
At a distance *d* from the load:

$$Z(d) = \frac{\widetilde{V}(d)}{\widetilde{I}(d)}$$

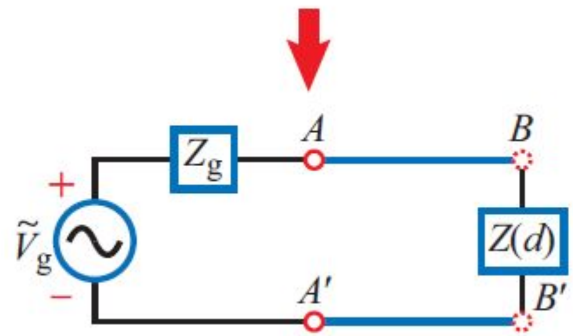
$$= \frac{V_0^+[e^{j\beta d} + \Gamma e^{-j\beta d}]}{V_0^+[e^{j\beta d} - \Gamma e^{-j\beta d}]} Z_0$$

$$= Z_0 \left[\frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}\right]$$

$$= Z_0 \left[\frac{1 + \Gamma_d}{1 - \Gamma_d}\right] \qquad (\Omega),$$



(a) Actual circuit



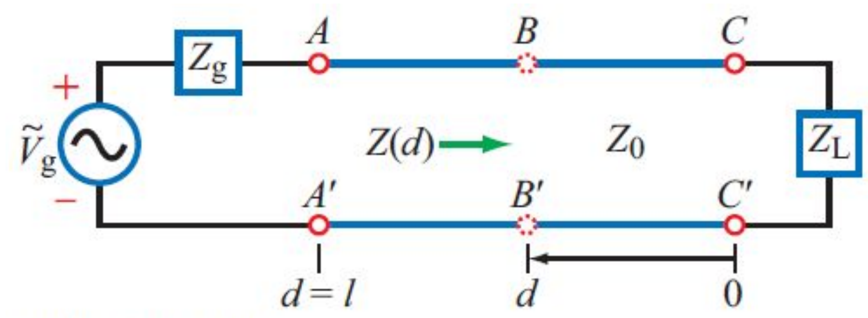
(b) Equivalent circuit

Chapter 2 Review

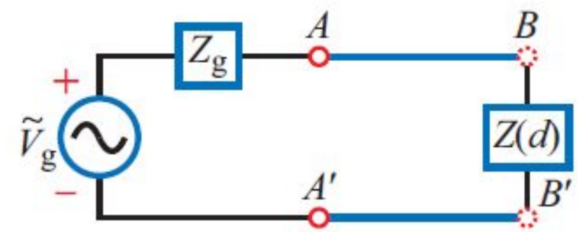
Define the phase-shifted voltage reflection coefficient:

$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_r} e^{-j2\beta d}$$

$$= |\Gamma| e^{j(\theta_r - 2\beta d)}$$



(a) Actual circuit

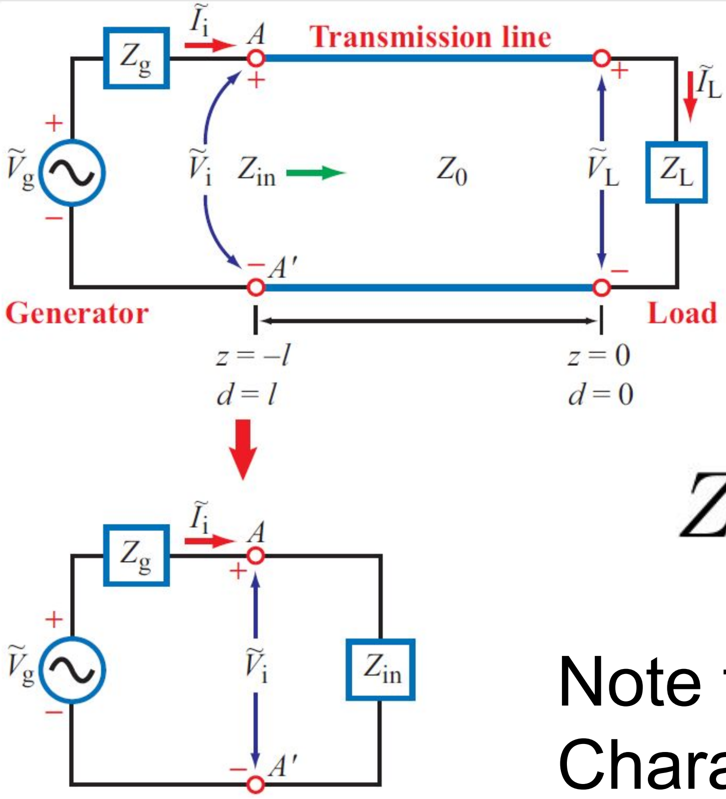


(b) Equivalent circuit

Z(d) is different than Z_0:
Ratio of Total Voltage and Current

Recall: $Z_0 = \dfrac{V_0^+}{I_0^+}$

Chapter 2 Review



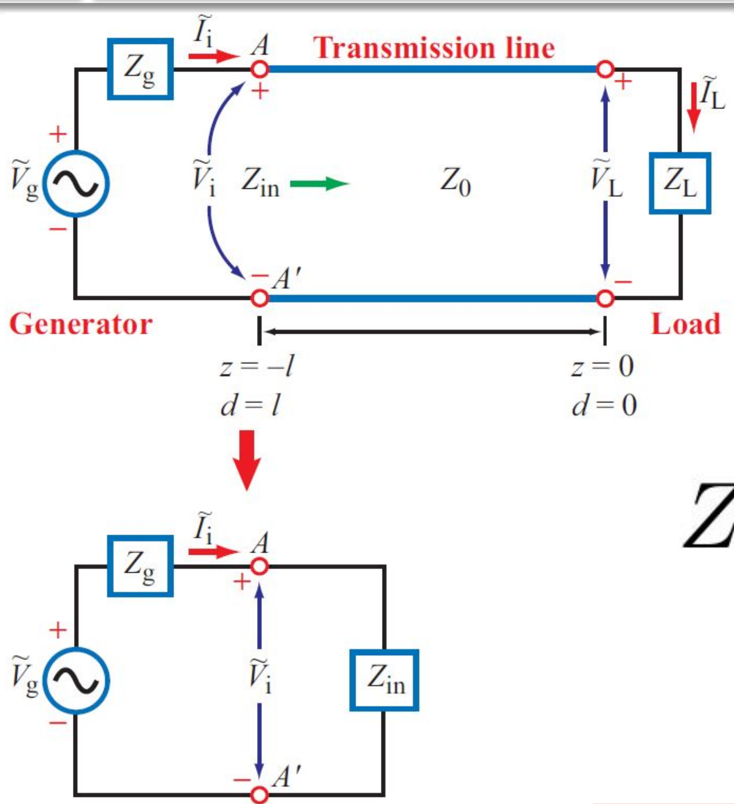
Input impedance:
impedance of the transmission line at *d=l:*

$$Z_{\mathrm{in}} = Z(d=l) = \frac{\widetilde{V}(d=l)}{\widetilde{I}(d=l)}$$

Note that Z_{in} is different from the Characteristic Impedance, and is different from the Load Impedance:

$$Z_{\mathrm{in}} \neq Z_0 \qquad Z_{\mathrm{in}} \neq Z_L$$

Chapter 2 Review

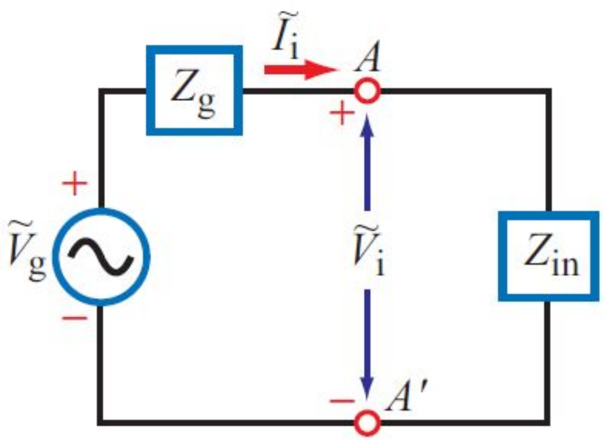


Input impedance:
impedance of the transmission line at *d*=*l*:

$$Z_{in} = Z(d=l) = \frac{\widetilde{V}(d=l)}{\widetilde{I}(d=l)}$$

$$Z_{in} = Z_0 \left[\frac{z_L + j\tan(\beta l)}{1 + j z_L \tan(\beta l)} \right]$$

Chapter 2 Review



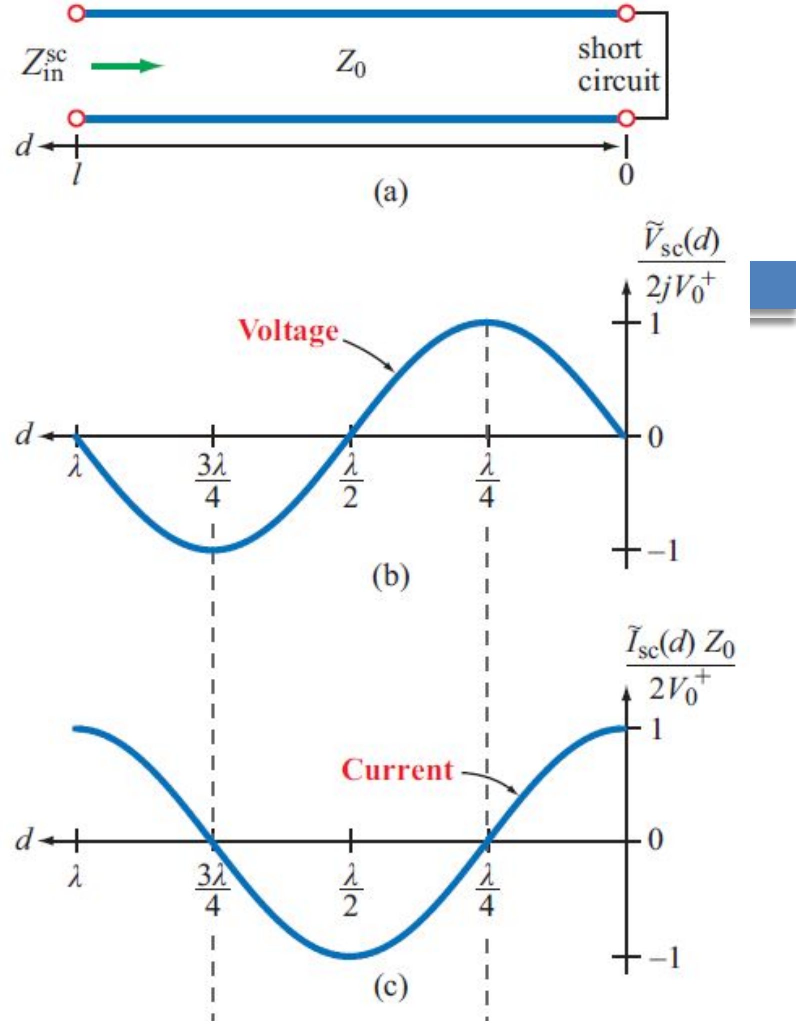
Voltage Amplitude:

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \right) \left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}} \right).$$

We used the boundary conditions (Z_L, Z_g , V_g, l) to solve for the 2 unknowns.

This completes the solution of the transmission line differential equation.

2-8 Short-Circuited Line



$$\tilde{V}(d) = V_0^+(e^{j\beta d} + \Gamma e^{-j\beta d})$$

$$\tilde{I}(d) = \frac{V_0^+}{Z_0}(e^{j\beta d} - \Gamma e^{-j\beta d})$$

For the short-circuited line:

$$\Gamma = -1$$

$$\tilde{V}_{sc}(d) = V_0^+[e^{j\beta d} - e^{-j\beta d}]$$

$$= 2jV_0^+ \sin \beta d,$$

$$\tilde{I}_{sc}(d) = \frac{2V_0^+}{Z_0} \cos \beta d,$$

2-8 Short-Circuited Line

For the short-circuited line:

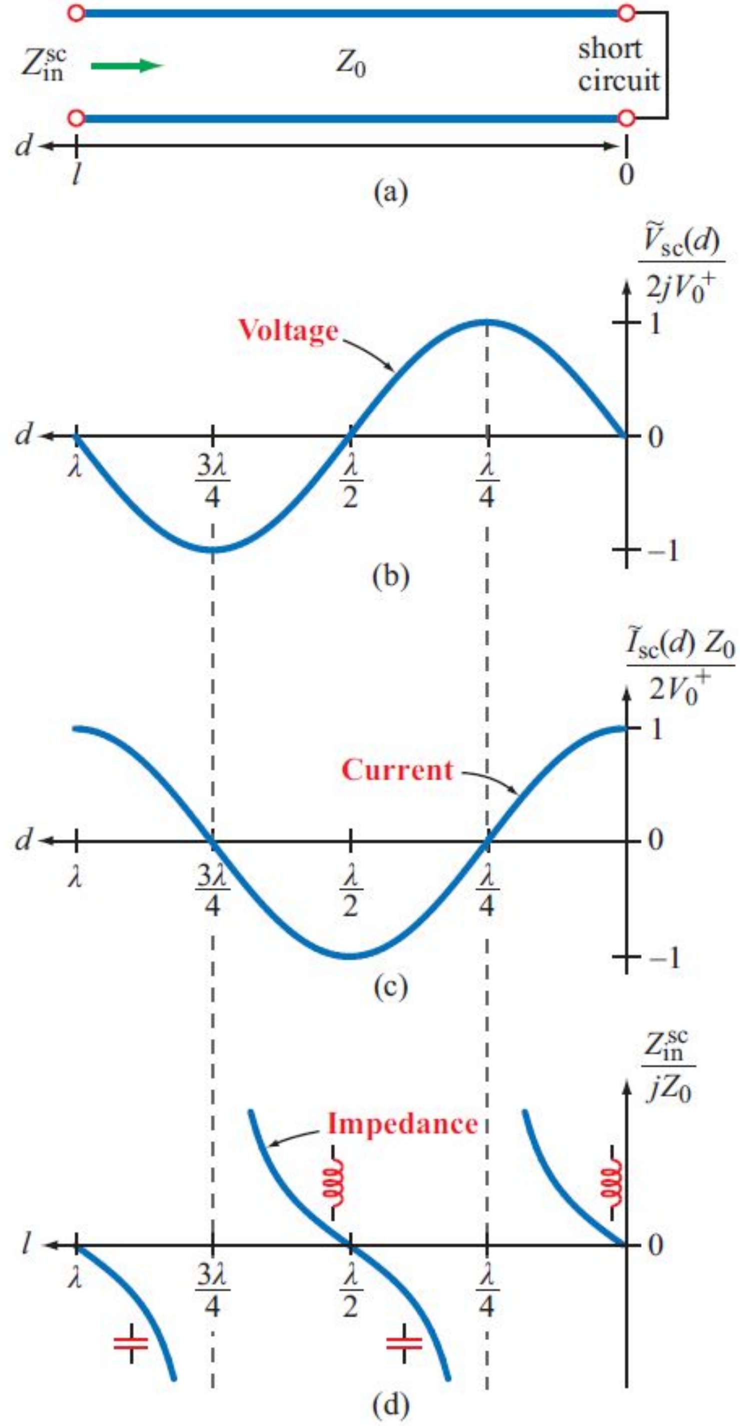
$$\Gamma = -1$$

$$\widetilde{V}_{sc}(d) = 2jV_0^+ \sin\beta d,$$

$$\widetilde{I}_{sc}(d) = \frac{2V_0^+}{Z_0}\cos\beta d,$$

$$Z_{sc}(d) = \frac{\widetilde{V}_{sc}(d)}{\widetilde{I}_{sc}(d)} = jZ_0 \tan\beta d.$$

repeats every $\lambda/2$ (or: π rad)



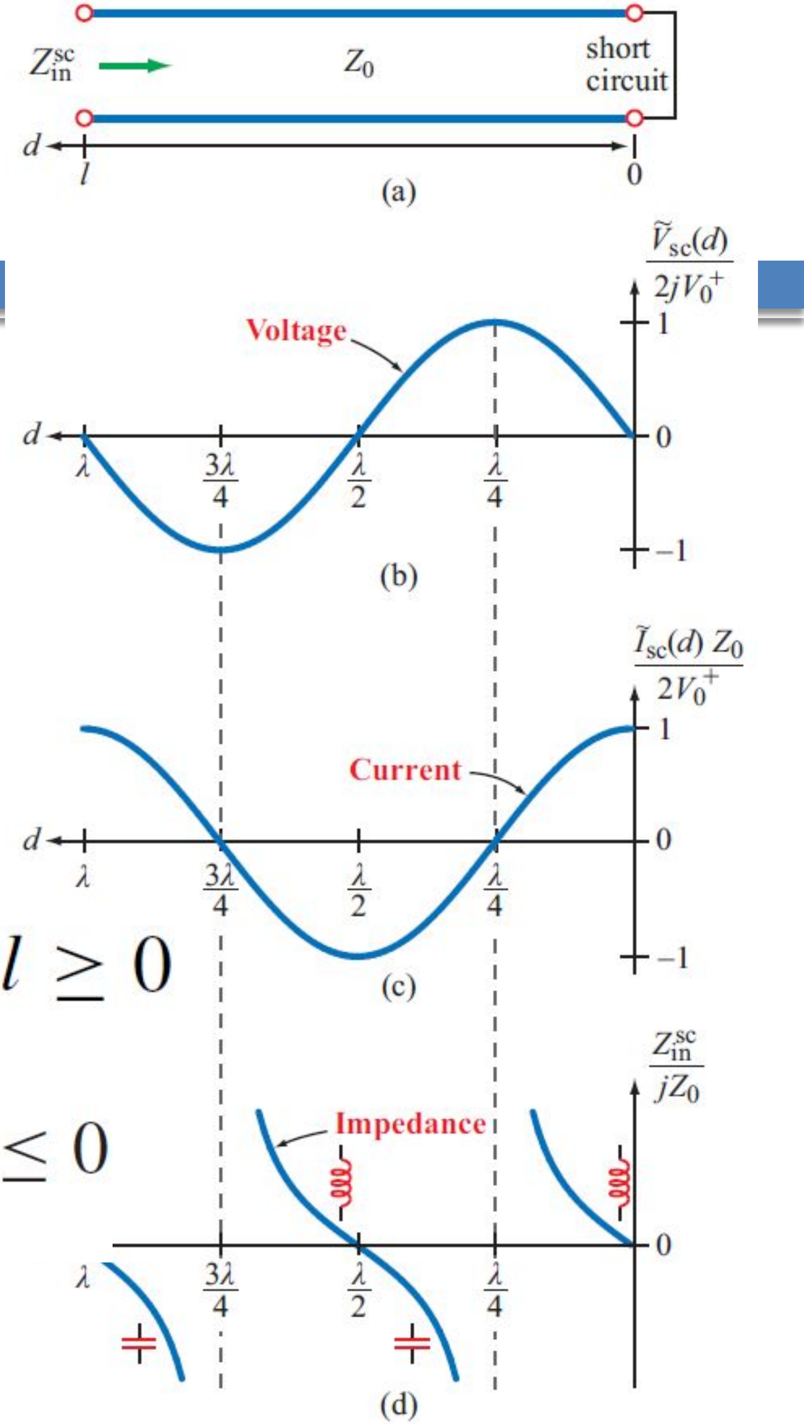
2-8 Short-Circuited Line

At its input, the line appears like an inductor or a capacitor depending on the sign of

$$\tan \beta d$$

$$j\omega L_{\text{eq}} = jZ_0 \tan \beta l, \qquad \text{if } \tan \beta l \geq 0$$

$$\frac{1}{j\omega C_{\text{eq}}} = jZ_0 \tan \beta l, \qquad \text{if } \tan \beta l \leq 0$$

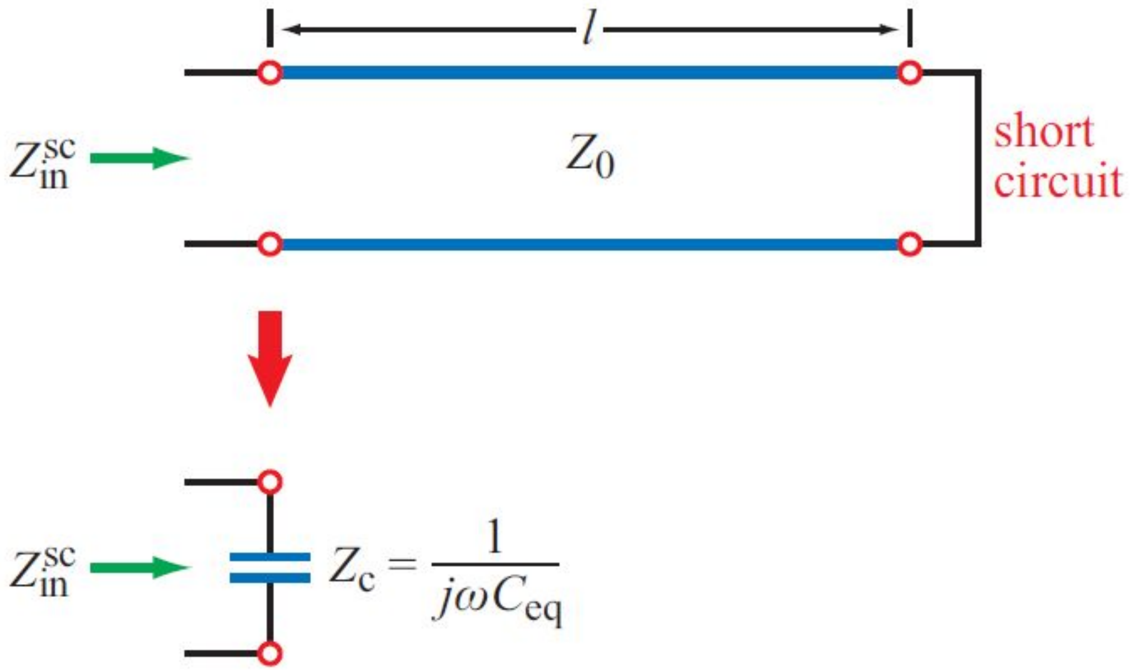


Example 2-8

Given: short-circuited transmission line,

$$Z_0 = 50\Omega,\ f = 2.25 \text{ GHz},\ u_p = 0.75c$$

Find: length, l, so that input impedance is the same as a capacitor with value 4pF.



Example 2-8

Solution: Given:

$$u_p = 0.75c = (0.75)3 \times 10^8 \text{ m/sec} = 2.25 \times 10^8 \text{ m/sec}$$

$$Z_0 = 50\,\Omega$$

$$f = 2.25\,\text{GHz} = 2.25 \times 10^9 \text{ Hz}$$

$$C_{\text{eq}} = 4\,\text{pF} = 4 \times 10^{-12} \text{ F}$$

Example 2-8

Solution:

we know that: $$Z_{\text{in}} = jZ_0 \tan\beta l = \frac{1}{j\omega C_{\text{eq}}}$$

so:

$$\tan\beta l = \frac{1}{jZ_0 j\omega C_{\text{eq}}}$$

$$\tan\beta l = -\frac{1}{Z_0\omega C_{\text{eq}}}$$

$$\beta l = \tan^{-1}\left(-\frac{1}{Z_0\omega C_{\text{eq}}}\right)$$

$$l = \frac{1}{\beta}\tan^{-1}\left(-\frac{1}{Z_0\omega C_{\text{eq}}}\right)$$

Example 2-8

Solution:

$$\omega = 2\pi f = (2\pi)2.25 \times 10^9 \text{ Hz} = 14.14 \times 10^9 \text{ rad/sec}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{u_p} = \frac{\omega}{u_p}$$

$$\beta = \frac{14.14 \times 10^9 \text{ rad/sec}}{2.25 \times 10^8 \text{ m/sec}}$$

$$\beta = 62.83 \text{ rad/m}$$

Example 2-8

Solution:

$$l = \frac{1}{\beta}\tan^{-1}\left(-\frac{1}{Z_0\omega C_{\text{eq}}}\right)$$

$$l = \frac{1}{62.83\,\text{rad/m}}\tan^{-1}\left(-\frac{1}{(50\Omega)(14.14\times 10^9\,\text{rad/sec})(4\times 10^{-12}\,\text{F})}\right)$$

$$l = 0.015915\,\text{m/rad}\tan^{-1}\left(-\frac{1}{2.828}\right)$$

$$l = 0.015915\,\text{m/rad}\tan^{-1}(-0.3536)$$

$$l = 0.015915\,\text{m/rad}(-0.34\,\text{rad})$$

But the length can't be negative !

Example 2-8

Solution:

$$l = 0.015915\,\text{m/rad}(-0.34\,\text{rad} + \pi)$$

$$l = 0.015915\,\text{m/rad}(2.8\,\text{rad})$$

$$l = 0.0446\,\text{m}$$

and since it repeats every $\lambda/2$:

$$\boxed{l = 0.0446 + n\lambda/2\,\text{m}}$$

Example 2-8

Solution:

with:

$$\lambda = \frac{u_p}{f} = \frac{2.25 \times 10^8 \text{ m/sec}}{2.25 \times 10^9 \text{ Hz}}$$

$$\lambda = 0.1 \text{ m}$$

so:

$$\boxed{l = 0.0446 + n(0.05) \text{ m}}$$

2-8 Open-Circuited Line

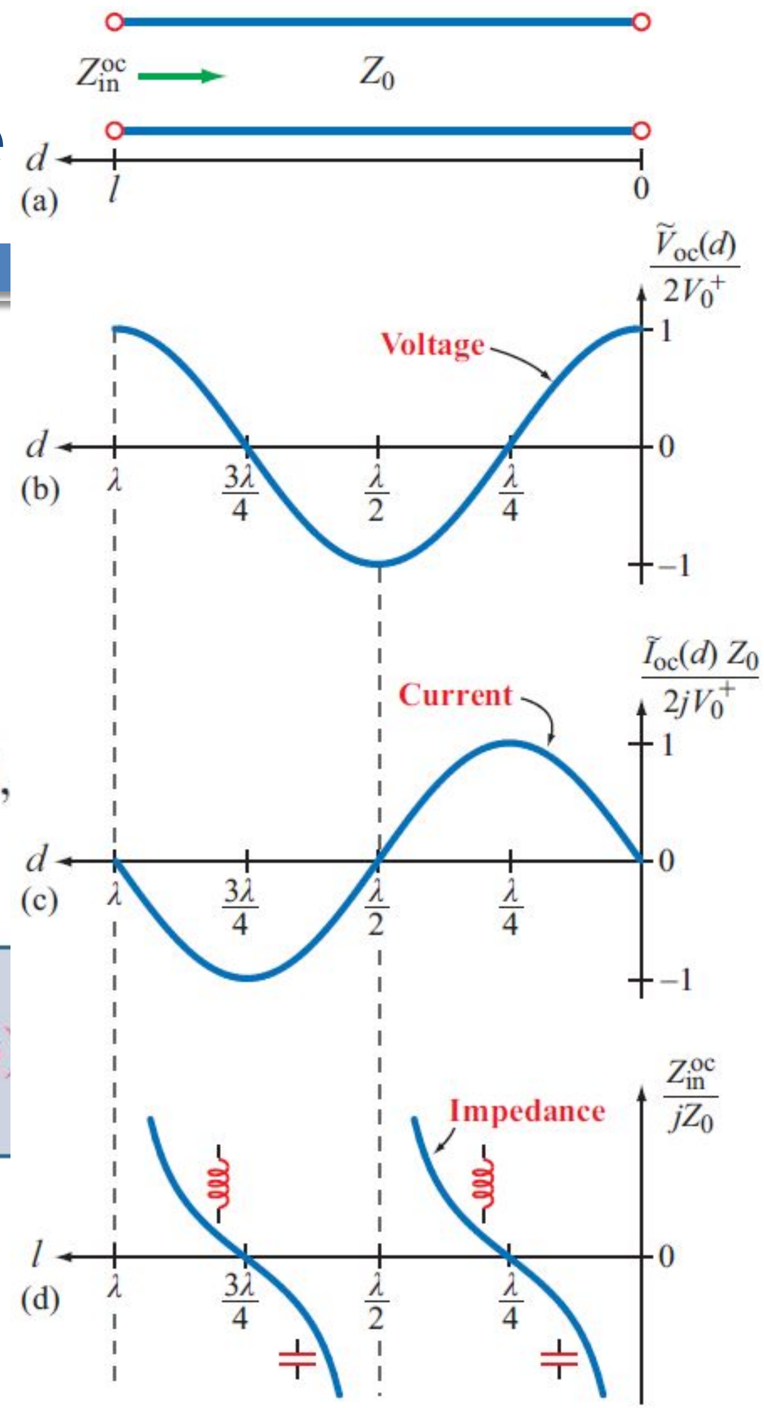
$$\Gamma = 1$$

$$\widetilde{V}_{oc}(d) = V_0^+[e^{j\beta d} + e^{-j\beta d}] = 2V_0^+ \cos\beta d,$$

$$\widetilde{I}_{oc}(d) = \frac{V_0^+}{Z_0}[e^{j\beta d} - e^{-j\beta d}] = \frac{2jV_0^+}{Z_0}\sin\beta d,$$

$$Z_{in}^{oc} = \frac{\widetilde{V}_{oc}(l)}{\widetilde{I}_{oc}(l)} = -jZ_0 \cot\beta l. \qquad (2.93)$$

repeats every $\lambda/2$ (or: π rad)

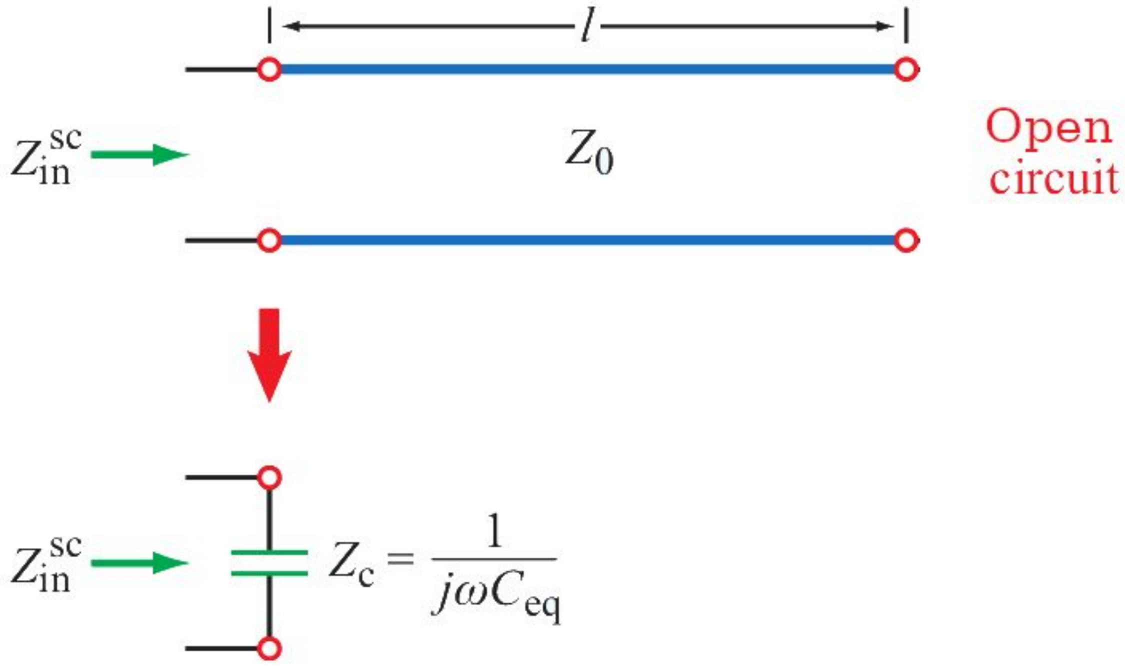


Example 2-8a

Given: open-circuited transmission line,

$$Z_0 = 50\Omega,\ f = 2.25\ \text{GHz},\ u_p = 0.75c$$

Find: length, l, so that input impedance is the same as a capacitor with value 4pF.



Example 2-8a

Solution: Given:

$$u_p = 0.75c = (0.75)3 \times 10^8 \text{ m/sec} = 2.25 \times 10^8 \text{ m/sec}$$

$$Z_0 = 50\,\Omega$$

$$f = 2.25\,\text{GHz} = 2.25 \times 10^9\ \text{Hz}$$

$$C_{\text{eq}} = 4\,\text{pF} = 4 \times 10^{-12}\ \text{F}$$

Example 2-8a

Solution:

we know that: $$Z_{\text{in}} = -jZ_0 \cot \beta l = \frac{1}{j\omega C_{\text{eq}}}$$

so:

$$\cot \beta l = -\frac{1}{jZ_0 j\omega C_{\text{eq}}}$$

$$\cot \beta l = \frac{1}{Z_0 \omega C_{\text{eq}}}$$

$$\tan \beta l = Z_0 \omega C_{\text{eq}}$$

$$\beta l = \tan^{-1}\left(Z_0 \omega C_{\text{eq}}\right)$$

$$\boxed{l = \frac{1}{\beta} \tan^{-1}\left(Z_0 \omega C_{\text{eq}}\right)}$$

Example 2-8a

Solution:

$$\omega = 2\pi f = (2\pi)2.25 \times 10^9 \text{ Hz} = 14.14 \times 10^9 \text{ rad/sec}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{u_p} = \frac{\omega}{u_p}$$

$$\beta = \frac{14.14 \times 10^9 \text{ rad/sec}}{2.25 \times 10^8 \text{ m/sec}}$$

$$\beta = 62.83 \text{ rad/m}$$

Example 2-8a

Solution:

$$l = \frac{1}{\beta}\tan^{-1}\left(Z_0\omega C_{\text{eq}}\right)$$

$$l = \frac{1}{62.83\,\text{rad/m}}\tan^{-1}\left((50\Omega)(14.14\times10^9\,\text{rad/sec})(4\times10^{-12}\,\text{F})\right)$$

$$l = 0.015915\,\text{m/rad}\tan^{-1}(2.828)$$

$$l = 0.015915\,\text{m/rad}(1.231\,\text{rad})$$

$$l = 0.0196\,\text{m}$$

and since it repeats every $\lambda/2$:

$$l = 0.0196 + n\lambda/2\,\text{m}$$

Example 2-8a

Solution:

with:

$$\lambda = \frac{u_p}{f} = \frac{2.25 \times 10^8 \text{ m/sec}}{2.25 \times 10^9 \text{ Hz}}$$

$$\lambda = 0.1 \text{ m}$$

so:

$$\boxed{l = 0.0196 + n(0.05) \text{ m}}$$

2-8 Short-Circuit/Open-Circuit Method

Starting with:

$$Z_{\text{in}}^{\text{SC}} = jZ_0 \tan \beta l$$
$$Z_{\text{in}}^{\text{OC}} = -jZ_0 \cot \beta l$$

multiply the 2 equations together:

$$Z_{\text{in}}^{\text{SC}} Z_{\text{in}}^{\text{OC}} = [jZ_0 \tan \beta l]\,[-jZ_0 \cot \beta l]$$
$$Z_{\text{in}}^{\text{SC}} Z_{\text{in}}^{\text{OC}} = Z_0^2$$

so:

$$\boxed{Z_0 = \sqrt{Z_{\text{in}}^{\text{SC}} Z_{\text{in}}^{\text{OC}}}}$$

2-8 Short-Circuit/Open-Circuit Method

- Measure Z_{in} twice:
  - when terminated in a short
  - when terminated in an open

Use both: get Z_0:

$$Z_0 = \sqrt{Z_{\text{in}}^{\text{SC}} Z_{\text{in}}^{\text{OC}}}$$

Example 2-9a

Given: lossless transmission line

$$Z_{in}^{sc} = j40.42\ \Omega\ , \quad Z_{in}^{oc} = -j121.24\ \Omega$$

Find: Z_0

Solution:

$$Z_0 = \sqrt{Z_{in}^{SC} Z_{in}^{OC}}$$

$$Z_0 = \sqrt{(j40.42\,\Omega)(-j121.24\,\Omega)}$$

$$Z_0 = \sqrt{4900\,\Omega^2}$$

$$\boxed{Z_0 = 70\,\Omega}$$

2-8 Special Case: $\lambda/2$

Recall the input impedance of a lossless line:

$$Z_{\text{in}} = Z_0 \left(\frac{z_L + j\tan(\beta l)}{1 + j z_L \tan(\beta l)} \right)$$

If the length is constrained to be a multiple of $\lambda/2$:

$$\beta l = (2\pi/\lambda)\,(n\lambda/2) = n\pi$$

so:

$$\tan \beta l = \tan n\pi = 0$$

Hence:

$$Z_{in} = Z_0\, z_L = Z_0\, Z_L / \; Z_0$$

$$\boxed{Z_{in} = Z_L}$$

2-8 Special Case: $\lambda/2$

Recall the input impedance of a lossless line:

$$Z_{\text{in}} = Z_0 \left(\frac{z_L + j\tan(\beta l)}{1 + j z_L \tan(\beta l)} \right)$$

If

So for a line of length $n\lambda/2$, $Z_{in} = Z_L$!

s

$$\tan \beta l = \tan n\pi = 0$$

Hence:

$$Z_{in} = Z_0\, z_L = Z_0\, Z_L / Z_0$$

$$\boxed{Z_{in} = Z_L}$$

2-8 Special Case: $\lambda/4$

$$Z_{\text{in}} = Z_0 \left(\frac{z_L + j\tan(\beta l)}{1 + j z_L \tan(\beta l)} \right)$$

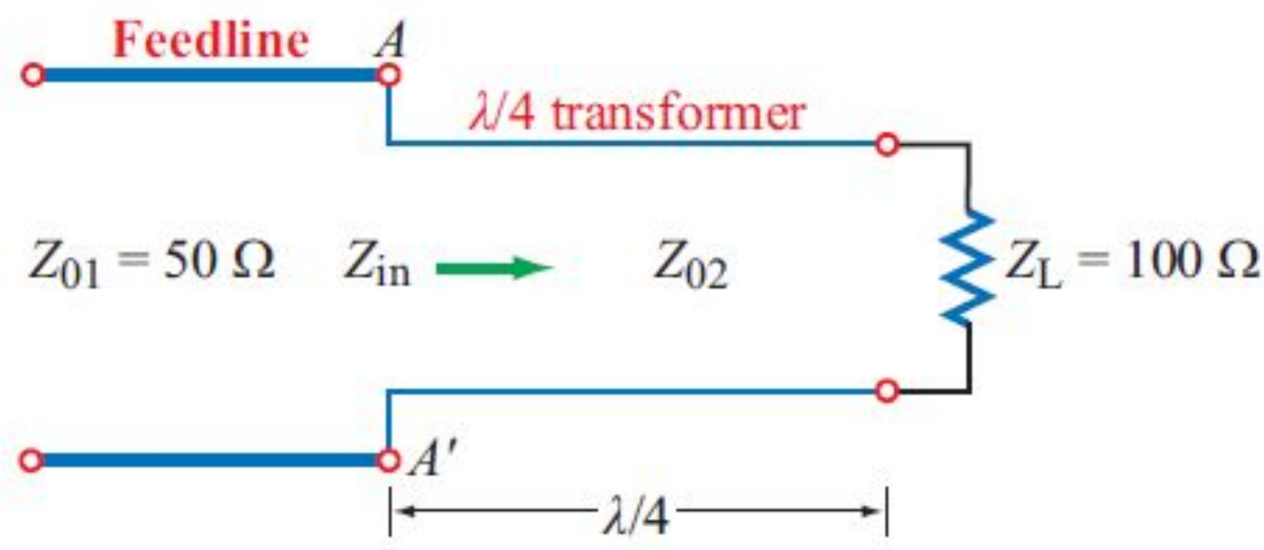
For $l = \lambda/4$, $\beta l = (2\pi/\lambda)(\lambda/4) = \pi/2$

So, as $\beta l \to \pi/2$, $\tan(\beta l) \to \infty$

$$\lim_{\beta l \to \pi/2} Z_{\text{in}} = Z_0 \left(\frac{j\tan(\beta l)}{j z_L \tan(\beta l)} \right) = \frac{Z_0^2}{Z_L}$$

$$Z_{\text{in}} = \frac{Z_0^2}{Z_{\text{L}}}, \qquad \text{for } l = \lambda/4 + n\lambda/2.$$

Example 2-10



Given: lossless transmission line above.

$$Z_{01} = 50\Omega,\ Z_L = 100\Omega$$

Find: Z_{02} to make $Z_{in} = 50\Omega$ (Matched Line)

Example 2-10

Solution: We know:

$$Z_{in} = \frac{Z_{02}^2}{Z_L}$$

so:

$$Z_{02} = \sqrt{Z_{in} Z_L}$$

Plug in known values:

$$Z_{02} = \sqrt{50 \times 100}$$

$$\boxed{Z_{02} = 70.7\ \Omega.}$$

2-9 Instantaneous Power Flow

From EECS 215:

Power anywhere on the line, at any time *t* is:

$$
\begin{aligned}
P(d,t) &= v(d,t)\, i(d,t) \\
&= |V_0^+|[\cos(\omega t + \beta d + \phi^+) \\
&\qquad + |\Gamma|\cos(\omega t - \beta d + \phi^+ + \theta_r)] \\
&\quad \times \frac{|V_0^+|}{Z_0}[\cos(\omega t + \beta d + \phi^+) \\
&\qquad - |\Gamma|\cos(\omega t - \beta d + \phi^+ + \theta_r)]
\end{aligned}
$$

2-9 Instantaneous Power Flow

From EECS 215:

This looks like:

$$P = C\,(a+b)(a-b)$$

$$P = C\,(a^2 + ab - ab - b^2)$$

$$P = C\,(a^2 - b^2)$$

2-9 Instantaneous Power Flow

$$P(d,t) = \upsilon(d,t)\, i(d,t)$$

$$= \frac{|V_0^+|^2}{Z_0}[\cos^2(\omega t + \beta d + \phi^+) - |\Gamma|^2 \cos^2(\omega t - \beta d + \phi^+ + \theta_r)]$$

2-9 Instantaneous Power Flow

$$P(d,t) = P^{\mathrm{i}}(d,t) + P^{\mathrm{r}}(d,t)$$

The 2 terms are the Incident and Reflected power:

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{Z_0}\cos^2(\omega t + \beta d + \phi^+) \qquad \text{(W)},$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{Z_0}\cos^2(\omega t - \beta d + \phi^+ + \theta_{\mathrm{r}})$$

2-9 Instantaneous Power Flow

Using the trigonometric identity

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x),$$

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t + 2\beta d + 2\phi^+)],$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_{\mathrm{r}})].$$

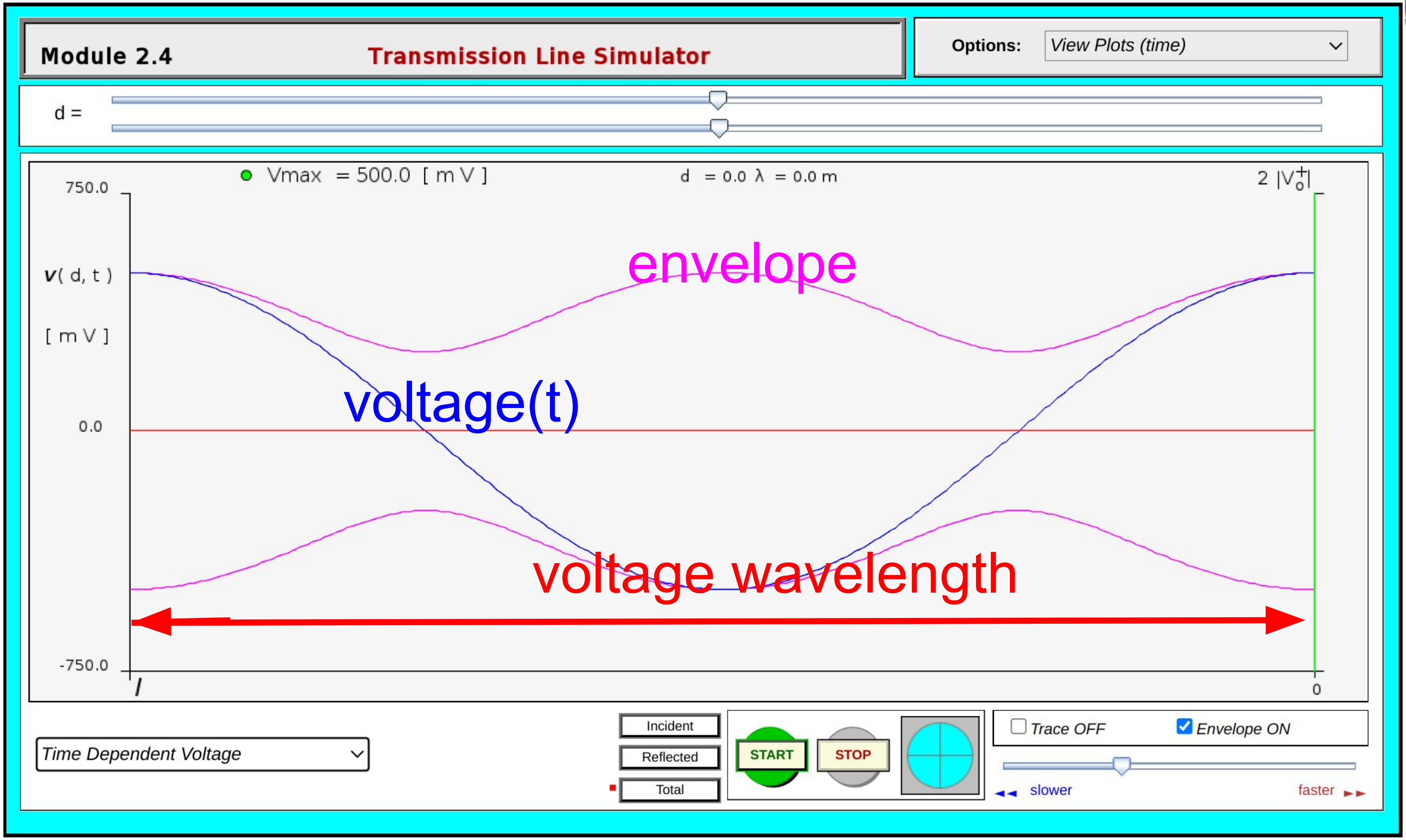
2-9 Instantaneous Power Flow

P oscillates at **twice** the frequency of V and I:

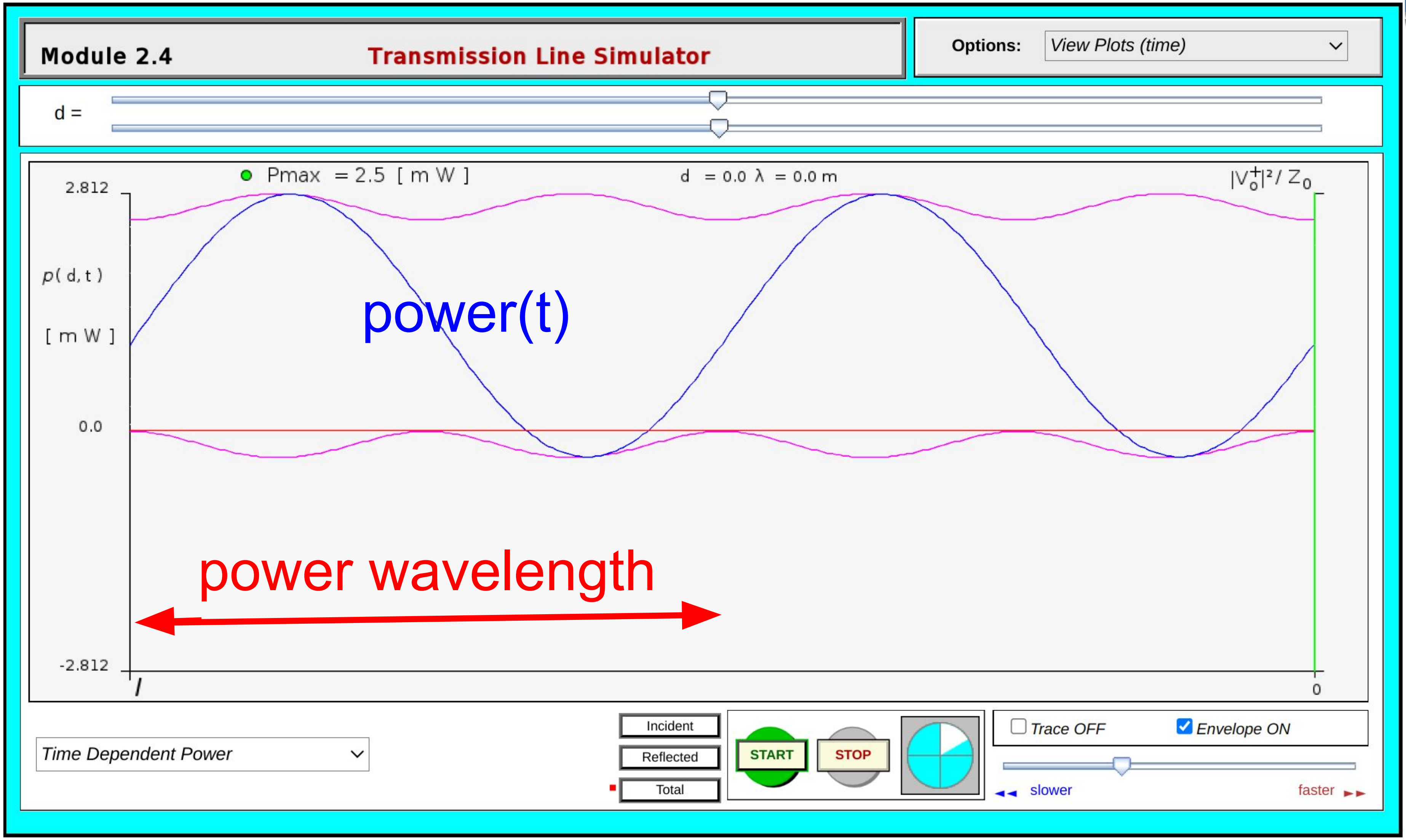
$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t + 2\beta d + 2\phi^+)],$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_{\mathrm{r}})].$$

2-9 Instantaneous Voltage

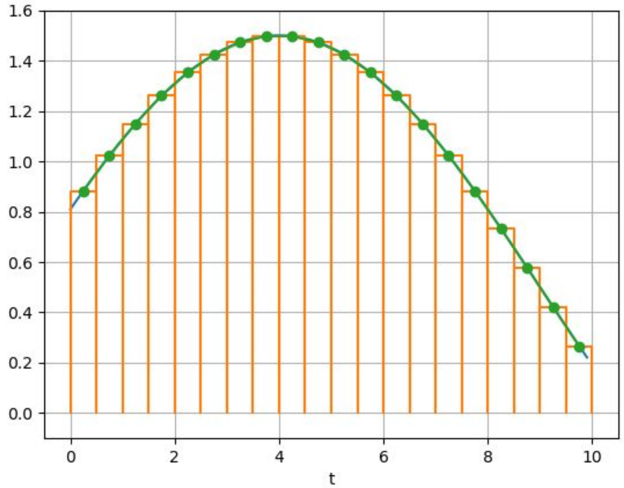


2-9 Instantaneous Power



Average a Signal

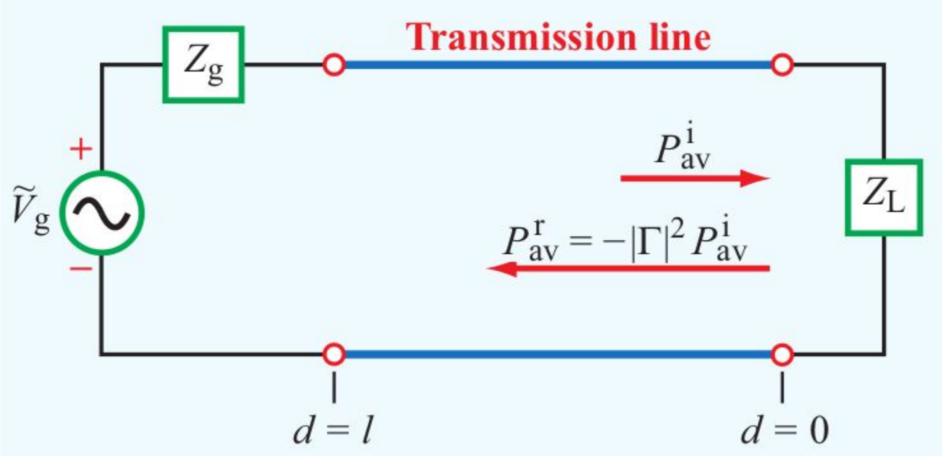
To find the average of a signal over a time period:
1. Divide up time period into small bins
2. for each bin get the center value
3. add up all the areas: center value X dt
4. divide by the total time
5. take the limit as dt goes to zero



$$\text{average} = \frac{\sum_{0}^{N} y(i)\, dt}{T}$$

$$\text{average} = \frac{1}{T}\int_{0}^{T} y(t) dt$$

2-9 Average Power Flow

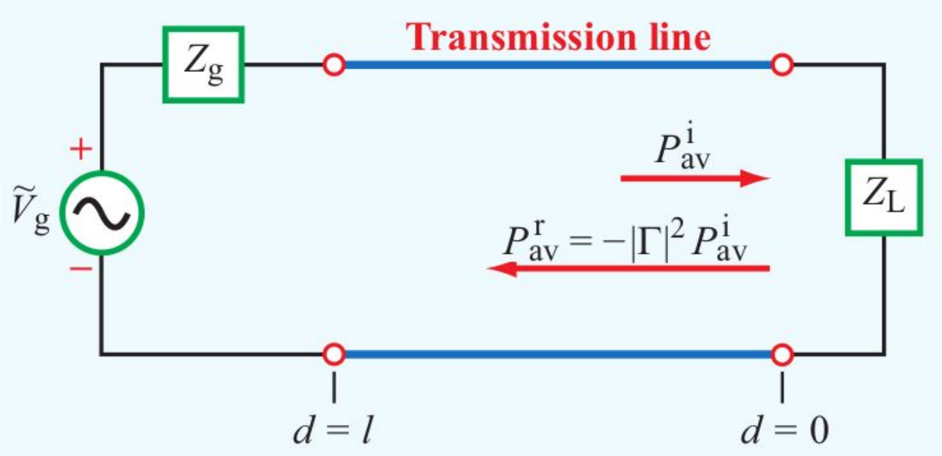


For the incident power, average over one period:

$$P^i_{avg} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \frac{|V_0^+|^2}{2Z_0} \left[1 + \cos(2\omega t + 2\beta d + 2\phi^+)\right] \mathrm{d}t$$

$$\omega = \frac{2\pi}{T}, \qquad hence \quad T = \frac{2\pi}{\omega}$$

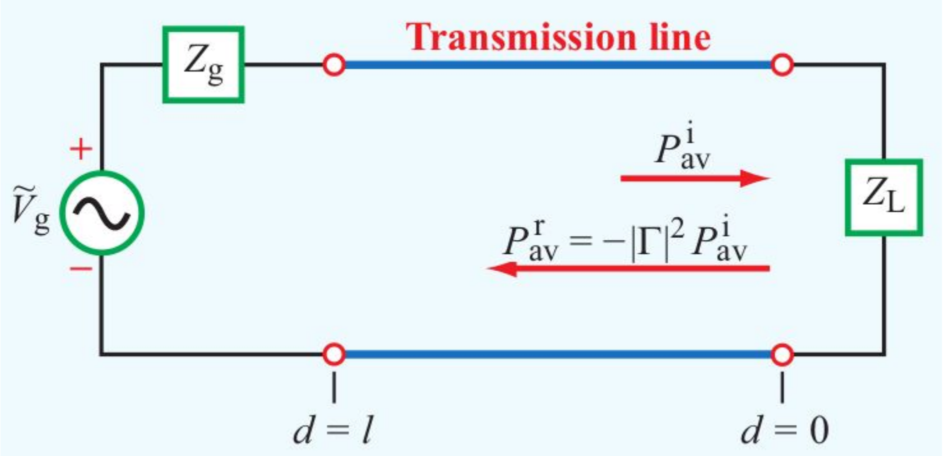
2-9 Average Power Flow



$$P^i_{avg} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \frac{|V_0^+|^2}{2Z_0} \left[1 + \cos(2\omega t + 2\beta d + 2\phi^+)\right] \mathrm{d}t$$

$$P^i_{avg} = \frac{\omega}{2\pi} \frac{|V_0^+|^2}{2Z_0} \left[t + \frac{1}{2\omega} \sin(2\omega t + 2\beta d + 2\phi^+)\right]_0^{\frac{2\pi}{\omega}}$$

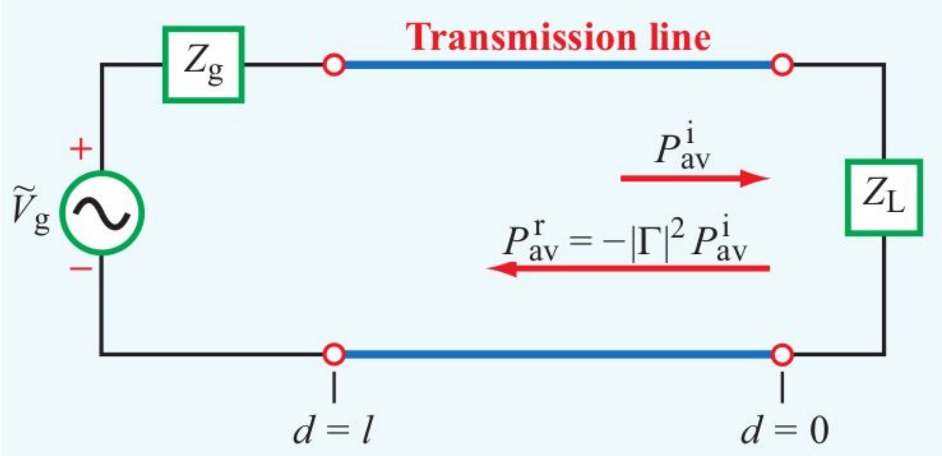
2-9 Average Power Flow



$$P^i_{avg} = \frac{\omega}{2\pi}\frac{|V_0^+|^2}{2Z_0}\left[t + \frac{1}{2\omega}\sin(2\omega t + 2\beta d + 2\phi^+)\right]_0^{\frac{2\pi}{\omega}}$$

$$P^i_{avg} = \frac{\omega}{2\pi}\frac{|V_0^+|^2}{2Z_0}\left\{\left[\frac{2\pi}{\omega} + \frac{1}{2\omega}\sin(4\pi + 2\beta d + 2\phi^+)\right] - \left[0 + \frac{1}{2\omega}\sin(0 + 2\beta d + 2\phi^+)\right]\right\}$$

2-9 Average Power Flow

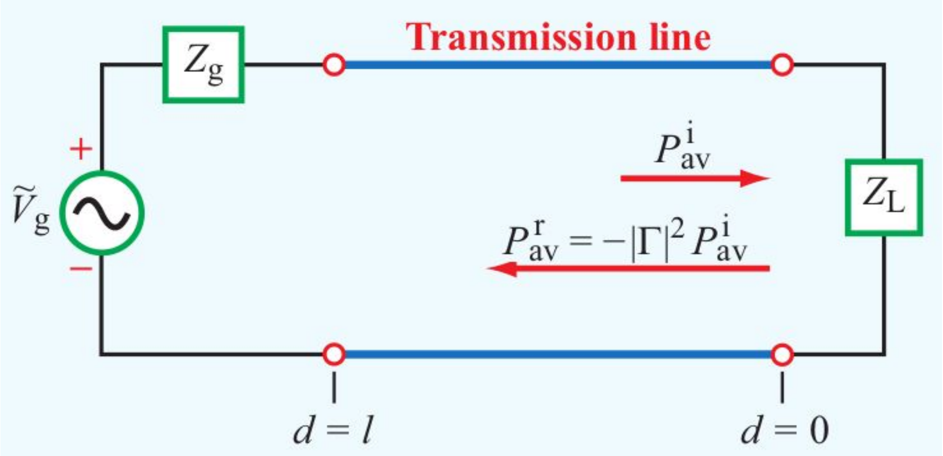


$$P^i_{avg} = \frac{\omega}{2\pi}\frac{|V_0^+|^2}{2Z_0}\left\{\left[\frac{2\pi}{\omega} + \frac{1}{2\omega}\sin(4\pi + 2\beta d + 2\phi^+)\right] - \left[0 + \frac{1}{2\omega}\sin(0 + 2\beta d + 2\phi^+)\right]\right\}$$

$$P^i_{avg} = \frac{\omega}{2\pi}\frac{|V_0^+|^2}{2Z_0}\left[\frac{2\pi}{\omega}\right]$$

$$P^i_{avg} = \frac{|V_0^+|^2}{2Z_0}$$

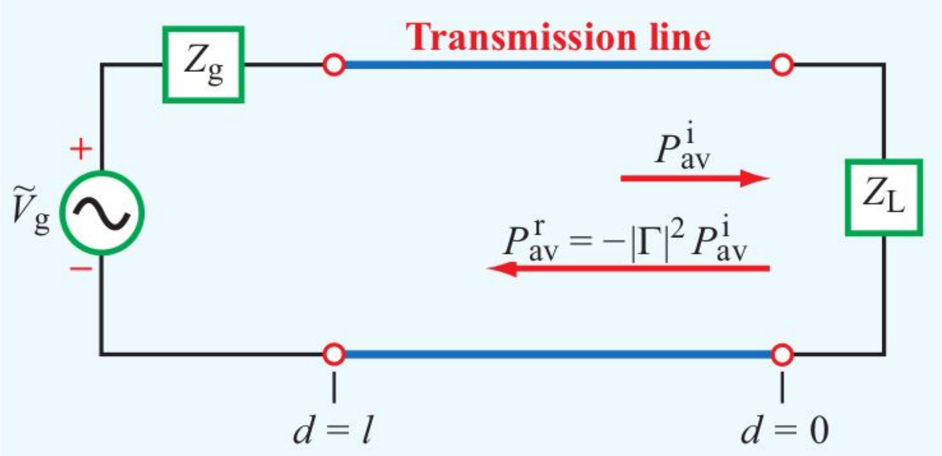
2-9 Average Power Flow



Similarly for the reflected power:

$$P^r_{avg} = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}$$

2-9 Average Power Flow



$$P^i_{avg} = \frac{|V_0^+|^2}{2Z_0}$$

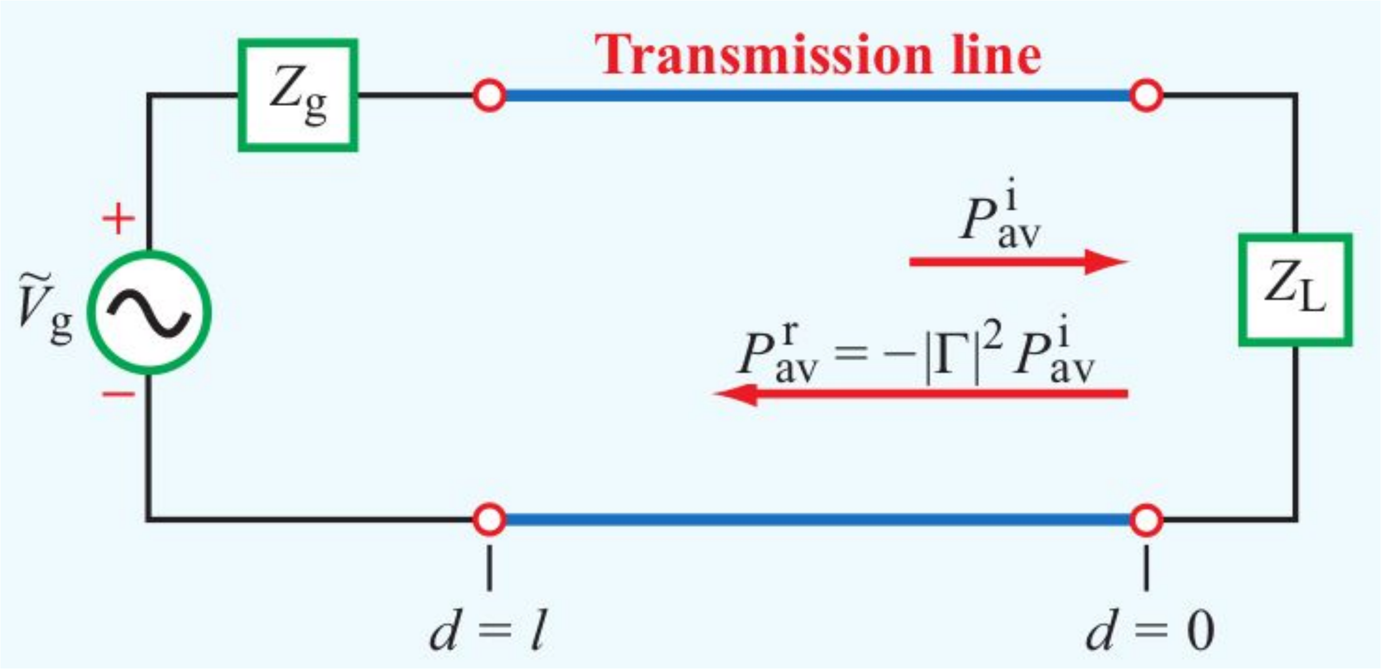
$$P^r_{avg} = -|\Gamma|^2 P^i_{avg}$$

$$P_{av} = P^i_{av} + P^r_{av} = \frac{|V_0^+|^2}{2Z_0}[1 - |\Gamma|^2]$$

The average power absorbed by the load.

Exercise 2-17

Given: Z_0 = 50Ω, Z_L = 100 + j 50 Ω
Find: fraction of the avg incident power that is reflected from the load



Exercise 2-17

Solution: the "fraction of the avg incident power that is reflected from the load" is just: $|\Gamma|^2$, so:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma = \frac{100 + j50 - 50}{100 + j50 + 50}$$

$$\Gamma = \frac{50 + j50}{150 + j50}$$

$$\Gamma = \frac{\sqrt{50^2 + 50^2} e^{j \tan^{-1}(50/50)}}{\sqrt{150^2 + 50^2} e^{j \tan^{-1}(50/150)}}$$

Exercise 2-17

Solution:

$$\Gamma = \frac{\sqrt{50^2 + 50^2} e^{j \tan^{-1}(50/50)}}{\sqrt{150^2 + 50^2} e^{j \tan^{-1}(50/150)}}$$

$$\Gamma = \frac{70.7 e^{j45^\circ}}{158.1 e^{j18.4^\circ}}$$

$$\Gamma = 0.447 e^{j26.6^\circ}$$

hence:

$$|\Gamma|^2 = 0.447^2$$

$$\boxed{|\Gamma|^2 = 0.2}$$

Homework

Homework 7 is due tomorrow at midnight.

submit to gradescope via the canvas site.

Next Time

Section 2-10:

The Smith Chart

(A graphical method to solve transmission line problems.)