Lecture 5:
Neural Networks

Assignment 2

- Use SGD to train linear classifiers and fully-connected networks
- After today, can do full assignment
- If you have a hard time computing derivatives, wait for next lecture on backprop
- Due Friday January 28, 11:59pm ET

Late Enrolls

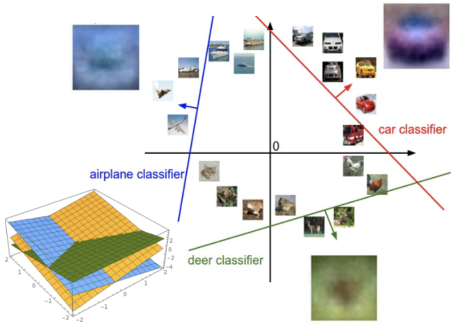
Anyone who enrolled today can have until Friday 2/4 for to turn in A1 and A2 without using late days or penalties

(But please email us / post on Piazza to confirm if you are using this extension)

Where we are:

1. Use **Linear Models** for image classification problems
2. Use **Loss Functions** to express preferences over different choices of weights
3. Use **Regularization** to prevent overfitting to training data
4. Use **Stochastic Gradient Descent** to minimize our loss functions and train the model

$$s = f(x; W) = Wx$$



$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$ Softmax

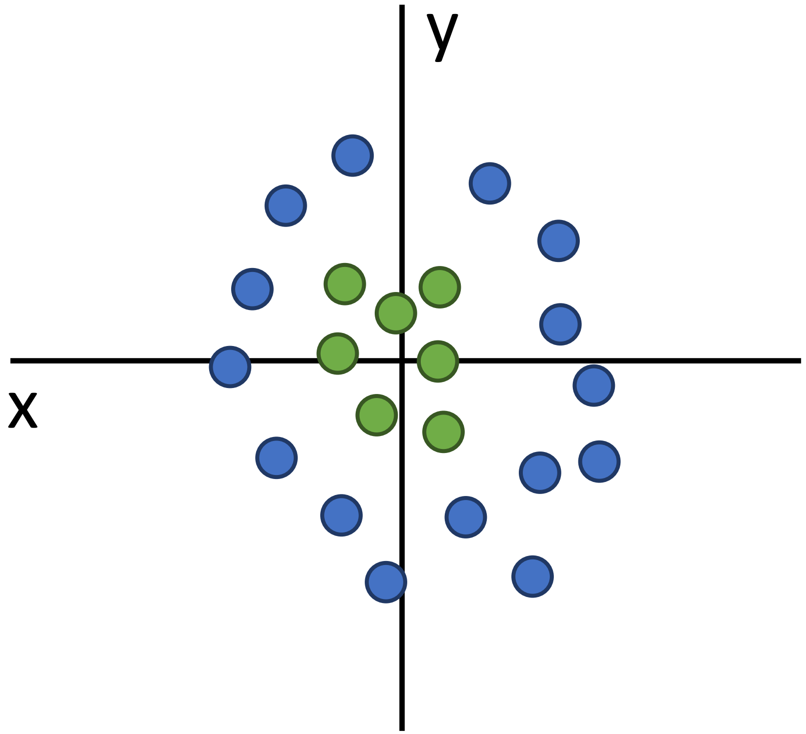
$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ SVM

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$

```
v = 0
for t in range(num_steps):
  dw = compute_gradient(w)
  v = rho * v + dw
  w -= learning_rate * v
```

Problem: Linear Classifiers aren't that powerful

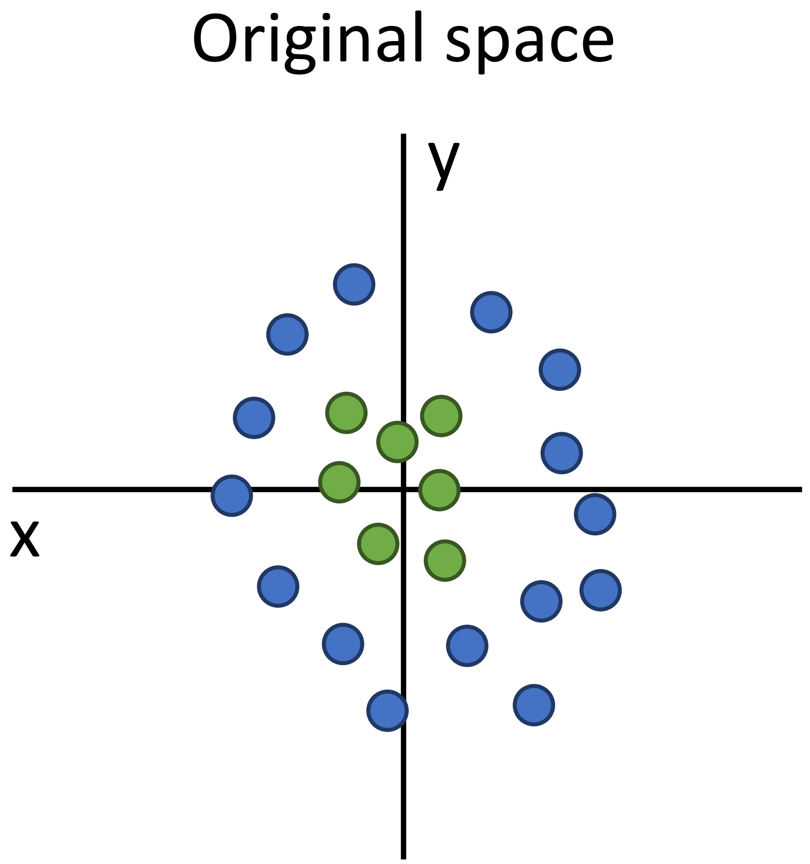
Geometric Viewpoint



Visual Viewpoint

One template per class:
Can't recognize different
modes of a class

One solution: **Feature Transforms**



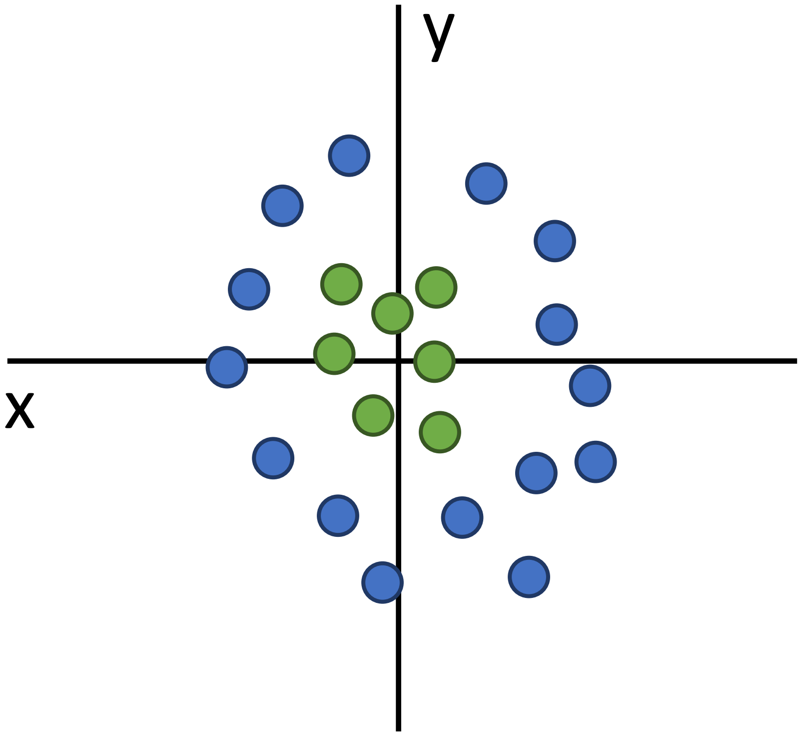
$r = (x^2 + y^2)^{1/2}$

$\theta = \tan^{-1}(y/x)$

Feature transform

One solution: **Feature Transforms**

Original space



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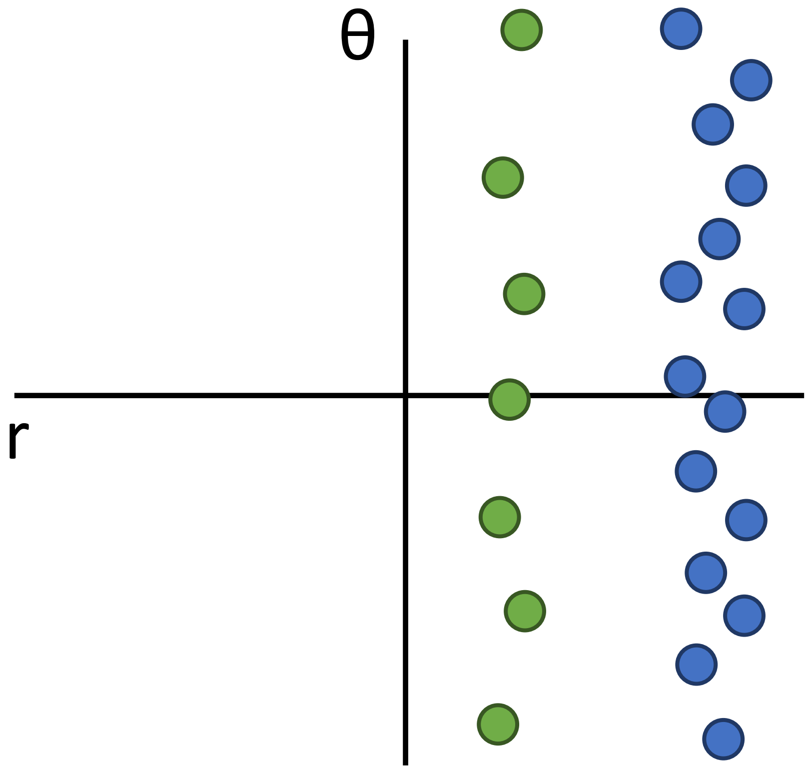
Feature transform

Feature space



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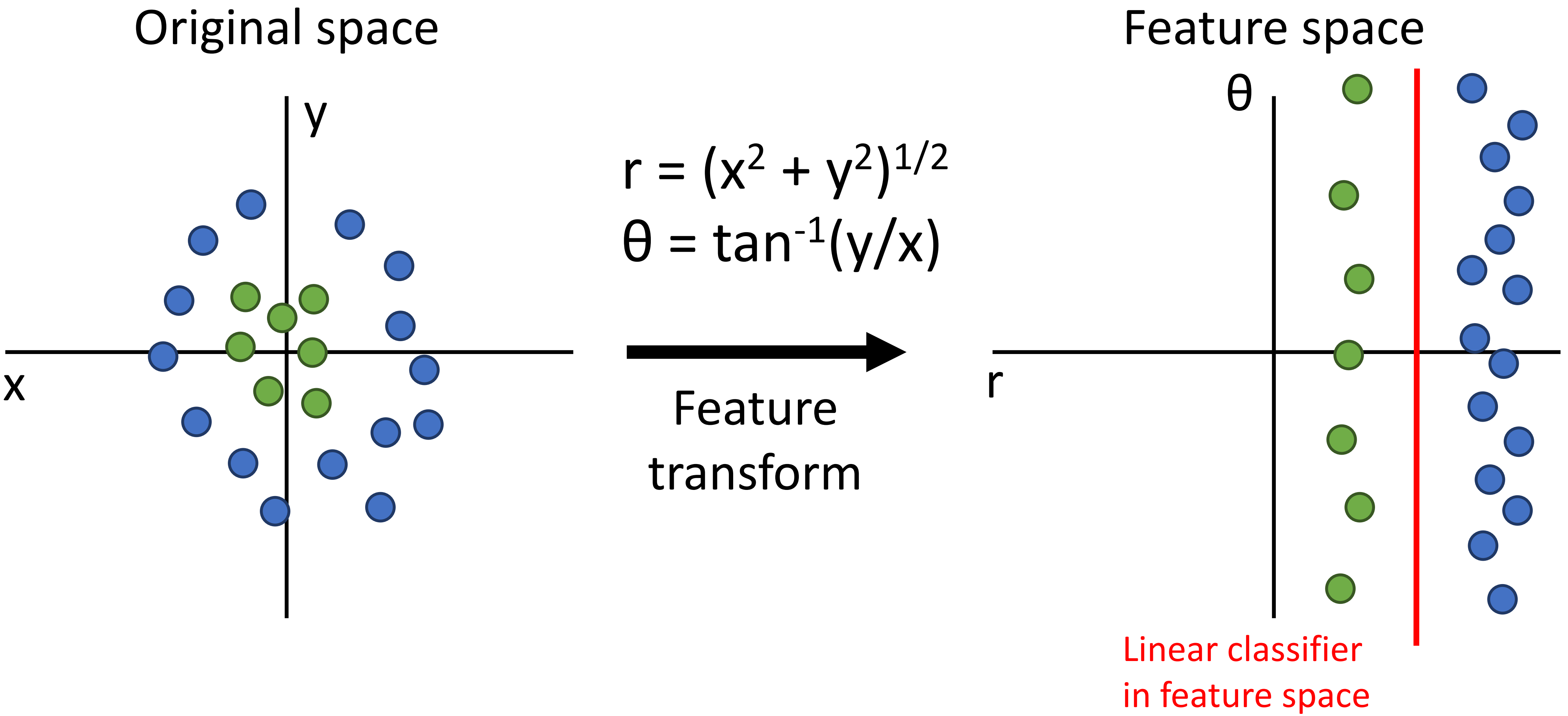

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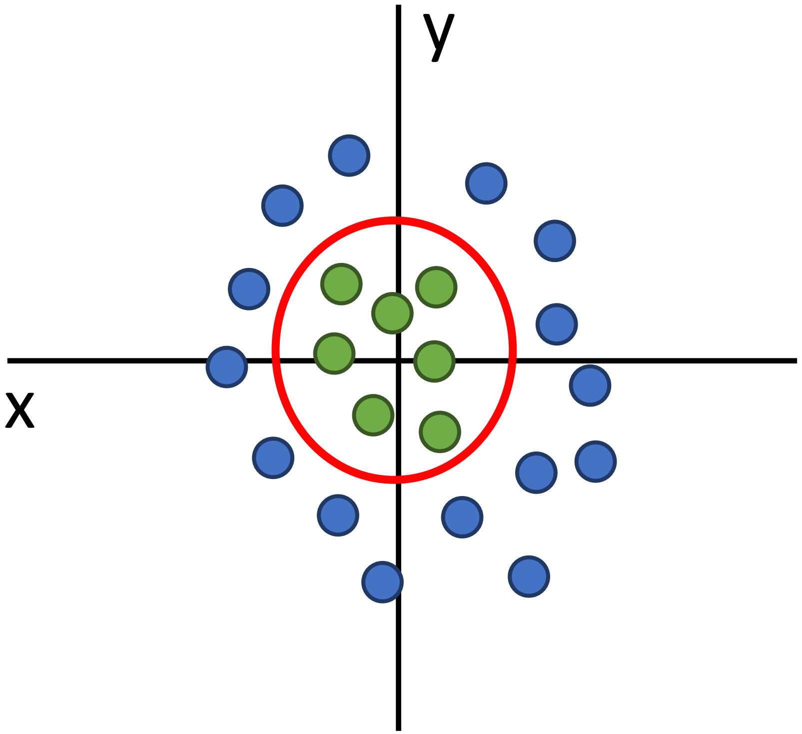

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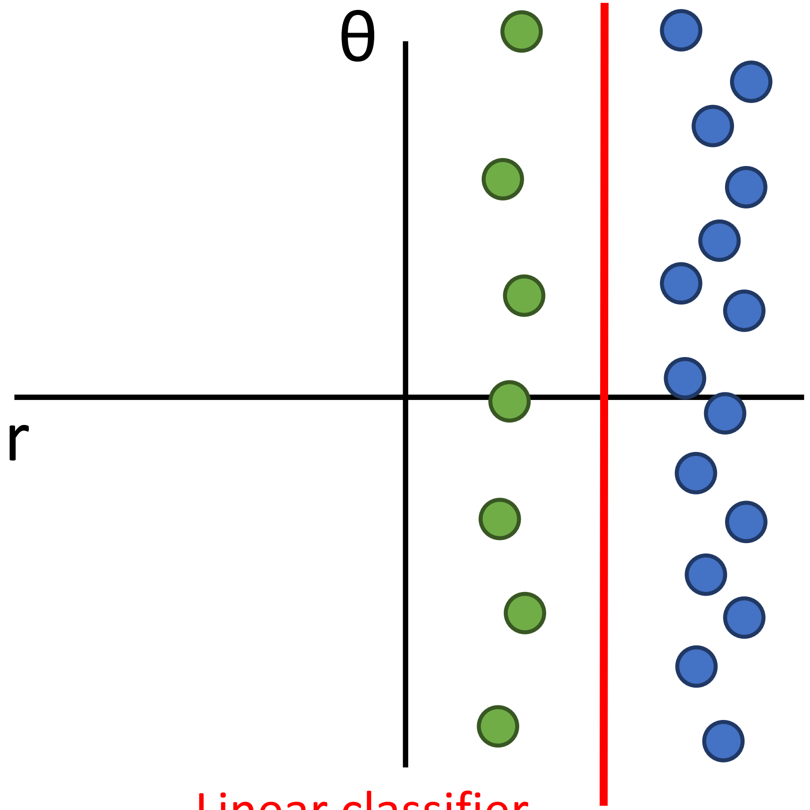
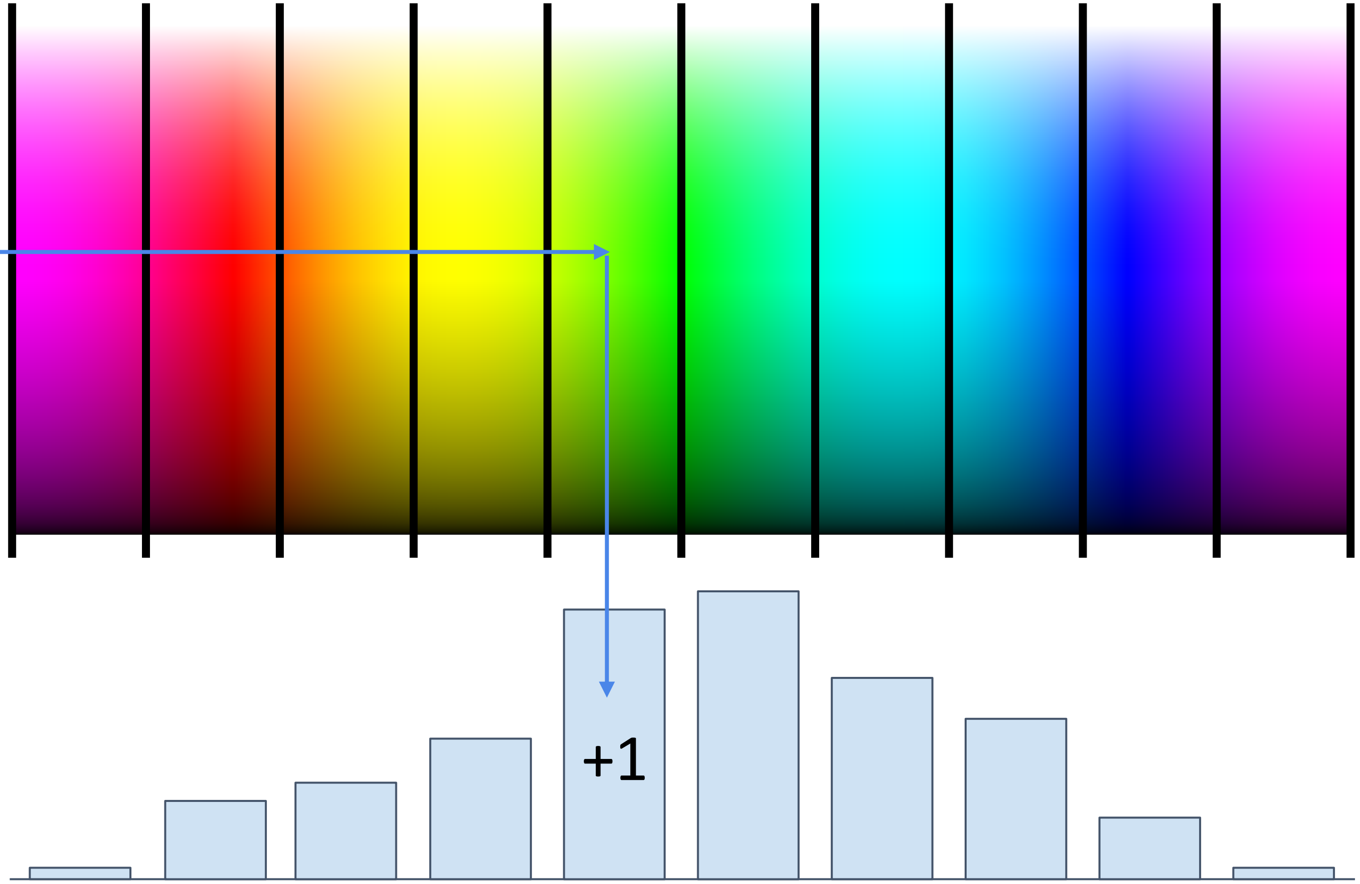


Nonlinear classifier in original space!

Linear classifier in feature space

Image Features: Color Histogram



Ignores texture, spatial positions

Frog image is in the public domain

Image Features: Histogram of Oriented Gradients (HoG)

1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge directions weighted by edge strength

Lowe, "Object recognition from local scale-invariant features", ICCV 1999
Dalal and Triggs, "Histograms of oriented gradients for human detection," CVPR 2005

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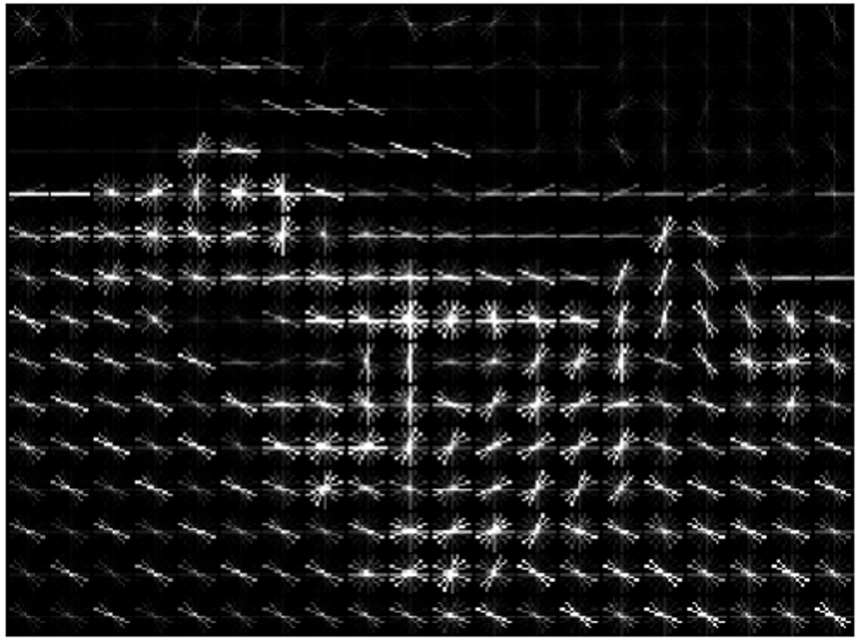

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Example: 320x240 image gets divided into 40x30 bins; 8 directions per bin; feature vector has 30*40*9 = 10,800 numbers

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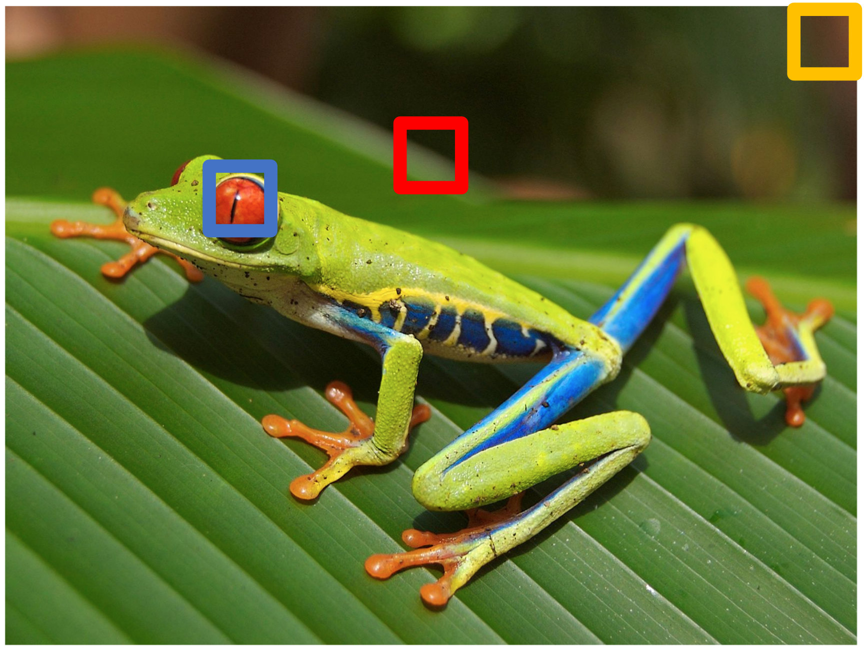
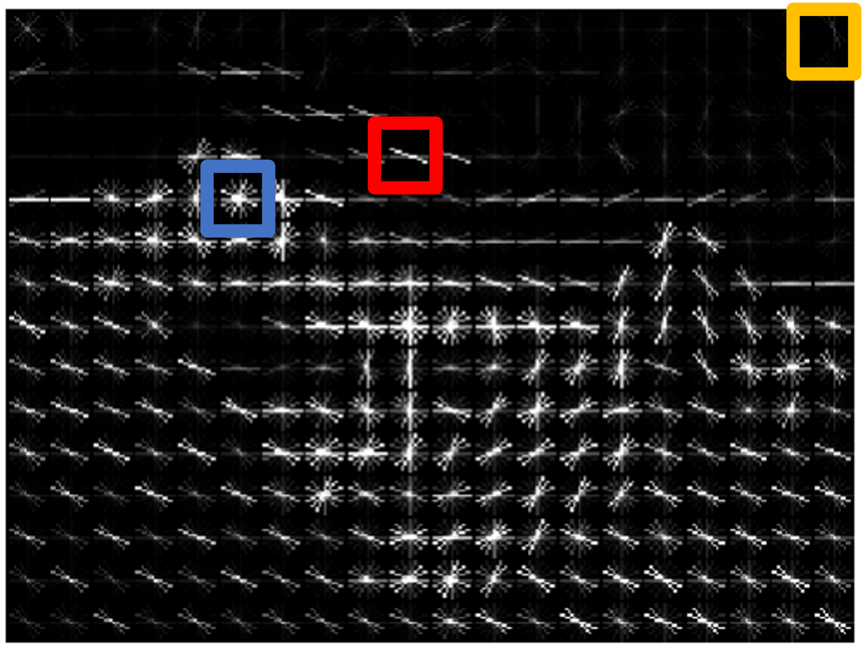


Weak edges

Strong diagonal edges

Edges in all directions



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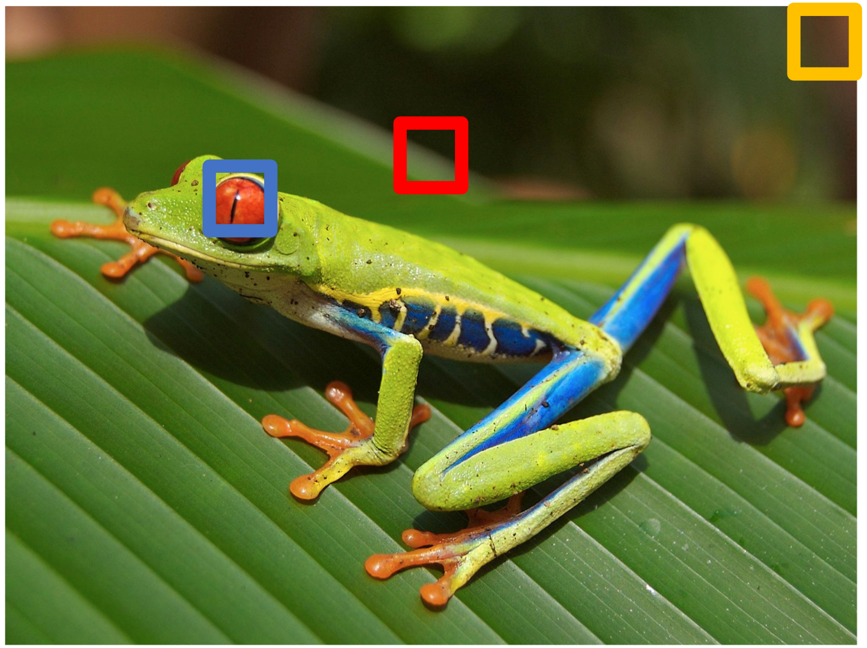

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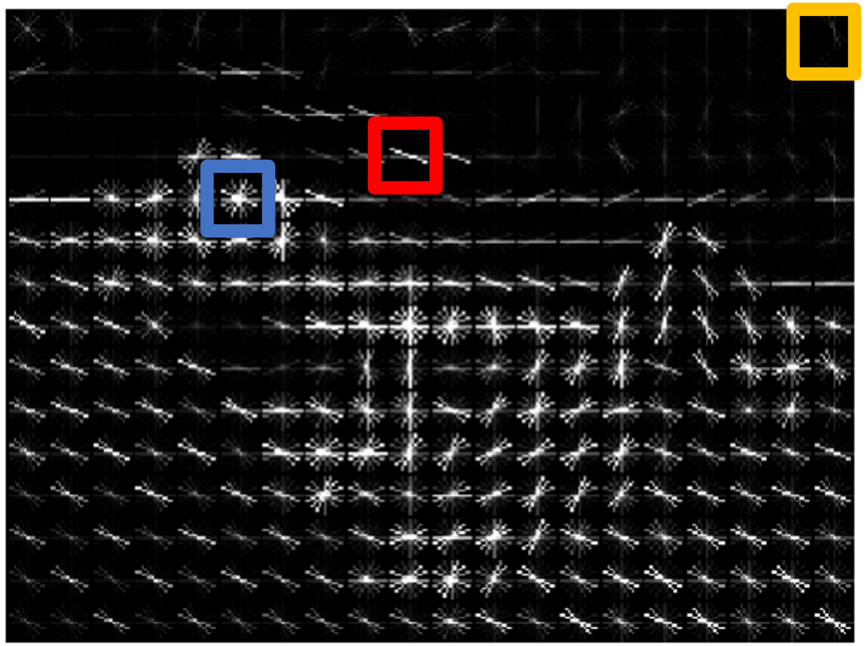

1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge directions weighted by edge strength

Captures texture and position, robust to small image changes

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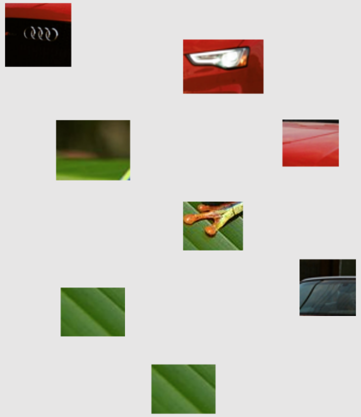
Image Features: Bag of Words (Data-Driven!)

Step 1: Build codebook

Extract random patches



Cluster patches to form “codebook” of “visual words”

Fei-Fei and Perona, “A bayesian hierarchical model for learning natural scene categories”, CVPR 2005

Image Features: Bag of Words (Data-Driven!)

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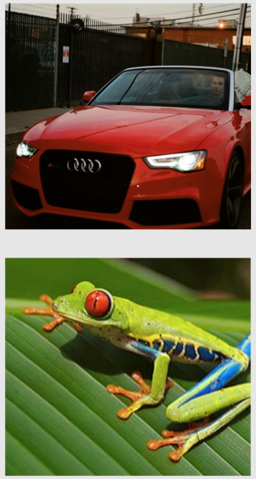
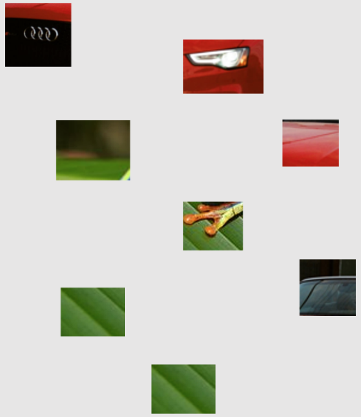


Extract random patches



Cluster patches to form "codebook" of "visual words"

Step 2: Encode images

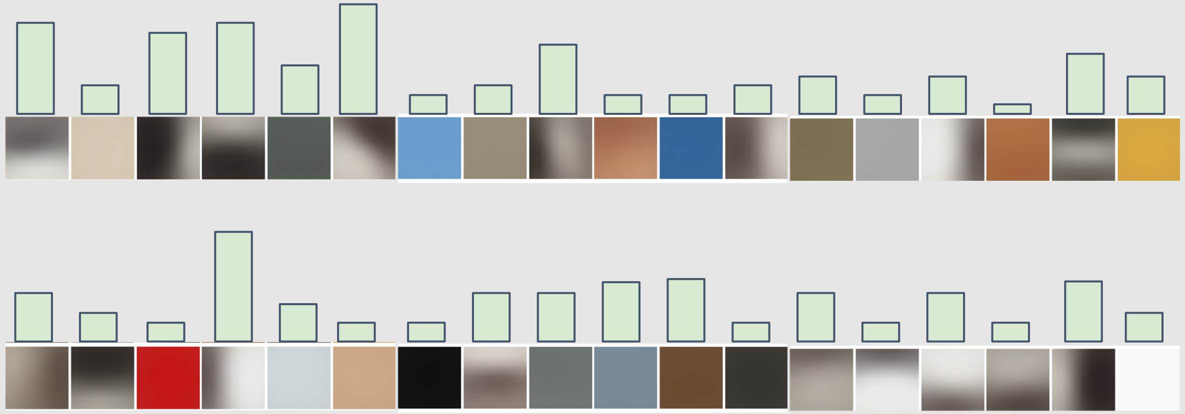


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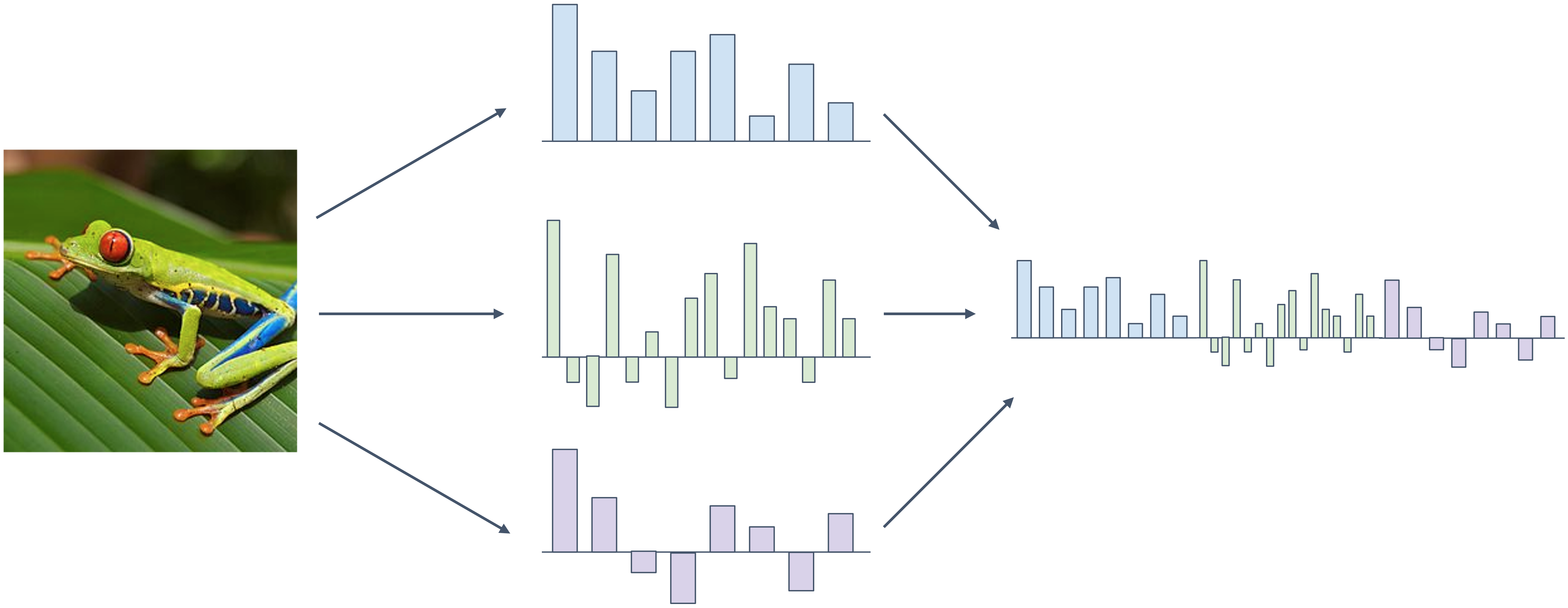
Image Features



Example: Winner of 2011 ImageNet challenge

Low-level feature extraction ≈ 10k patches per image

- SIFT: 128-dim
- color: 96-dim

} reduced to 64-dim with PCA

FV extraction and compression:

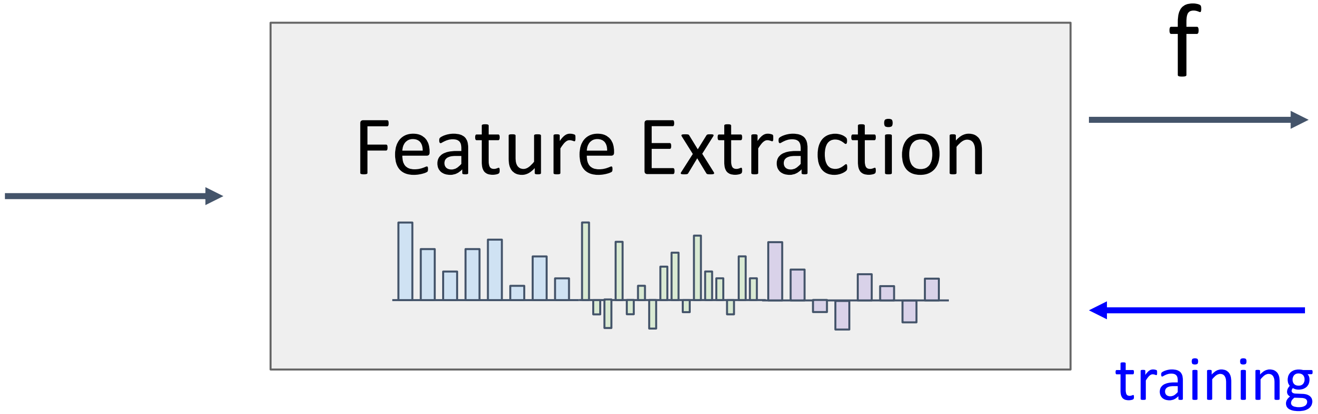
- N=1,024 Gaussians, R=4 regions ⇨ 520K dim x 2
- compression: G=8, b=1 bit per dimension

One-vs-all SVM learning with SGD

Late fusion of SIFT and color systems

F. Perronnin, J. Sánchez, "Compressed Fisher vectors for LSVRC", PASCAL VOC / ImageNet workshop, ICCV, 2011.

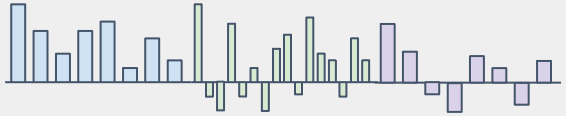
Image Features



10 numbers giving scores for classes

Image Features vs Neural Networks

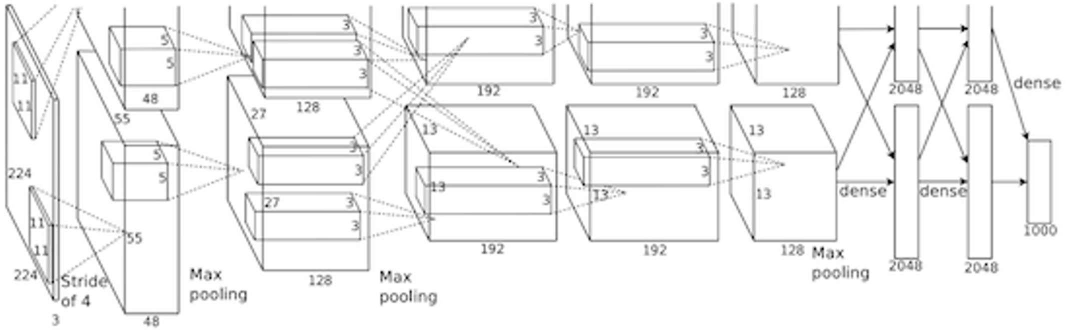
Feature Extraction



f

10 numbers giving scores for classes

training



Krizhevsky, Sutskever, and Hinton, "Imagenet classification with deep convolutional neural networks", NIPS 2012.

10 numbers giving scores for classes

training

Neural Networks

Input: $x \in \mathbb{R}^D$ **Output**: $f(x) \in \mathbb{R}$^C

Before: Linear Classifier: $f(x) = Wx + b$

Learnable parameters: $W \in \mathbb{R}^{D \times C}, b \in \mathbb{R}^C$

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Feature Extraction
Linear Classifier

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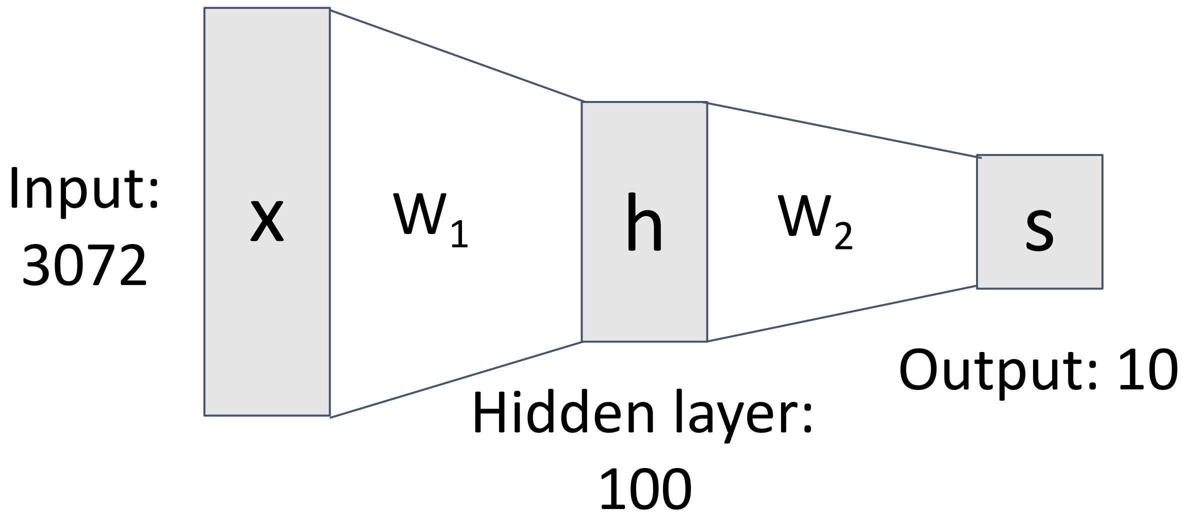
Or Three-Layer Neural Network:
$f(x) = W_3 \max(0, W_2 \max(0, W_1 x + b_1) + b_2) + b_3$

Neural Networks

Before: Linear classifier $f(x) = Wx + b$

Now: 2-layer Neural Network $f(x) = W_2 \max(0, W_1 x + b_1) + b_2$



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Neural Networks

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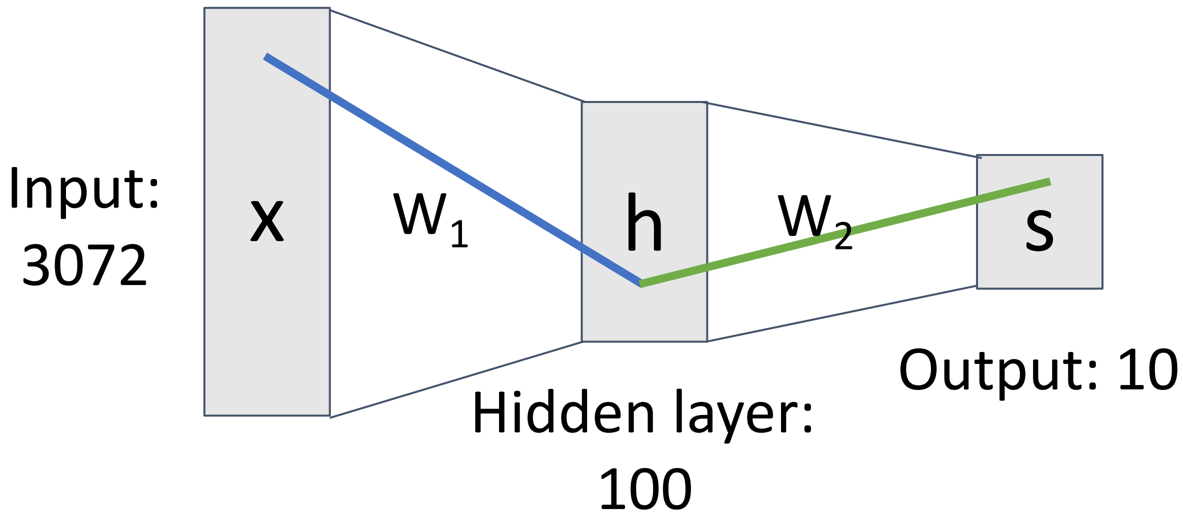
Element (i, j) of W_1 gives the effect on h_i from x_j



Element (i, j) of W_2 gives the effect on s_i from h_j

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

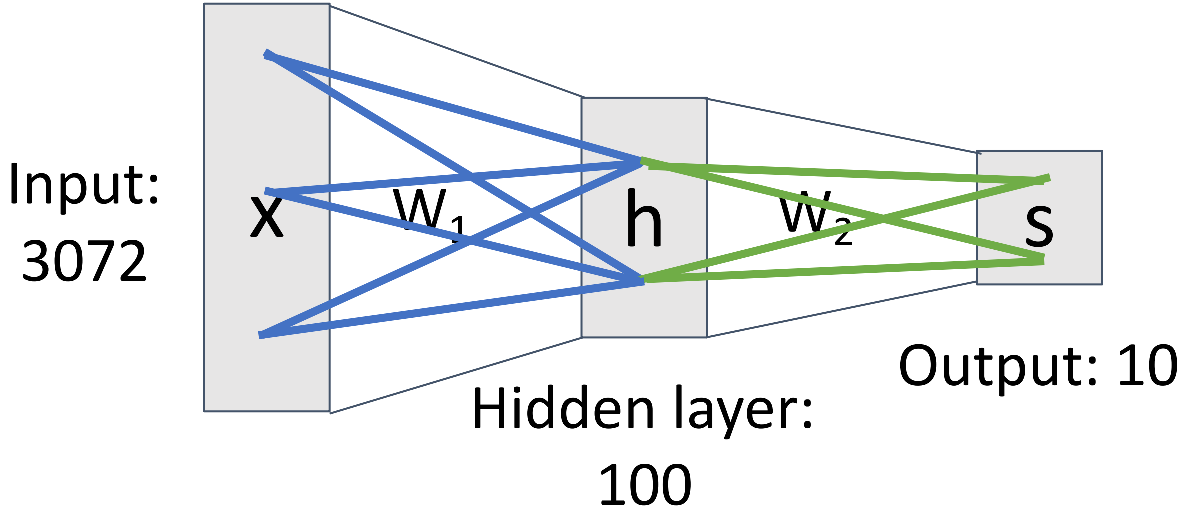
Neural Networks

Before: Linear classifier $f(x) = Wx + b$

Now: 2-layer Neural Network $f(x) = W_2 \max(0, W_1 x + b_1) + b_2$

Element (i, j) of W_1 gives the effect on h_i from x_j

All elements of x affect all elements of h



Element (i, j) of W_2 gives the effect on s_i from h_j

All elements of h affect all elements of s

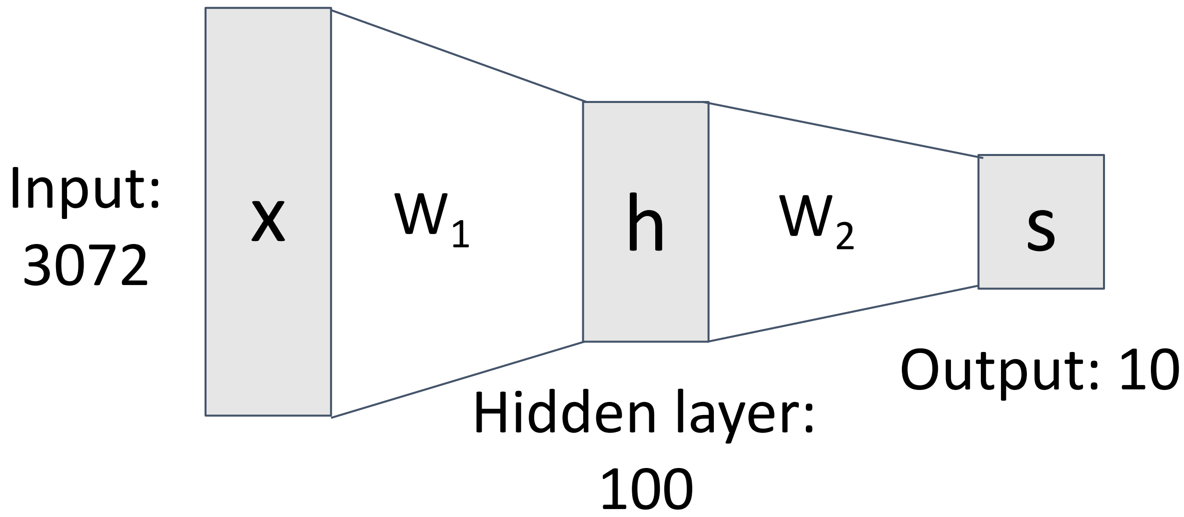
Fully-connected neural network
Also "Multi-Layer Perceptron" (MLP)

Neural Networks

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network

Linear classifier: One template per class



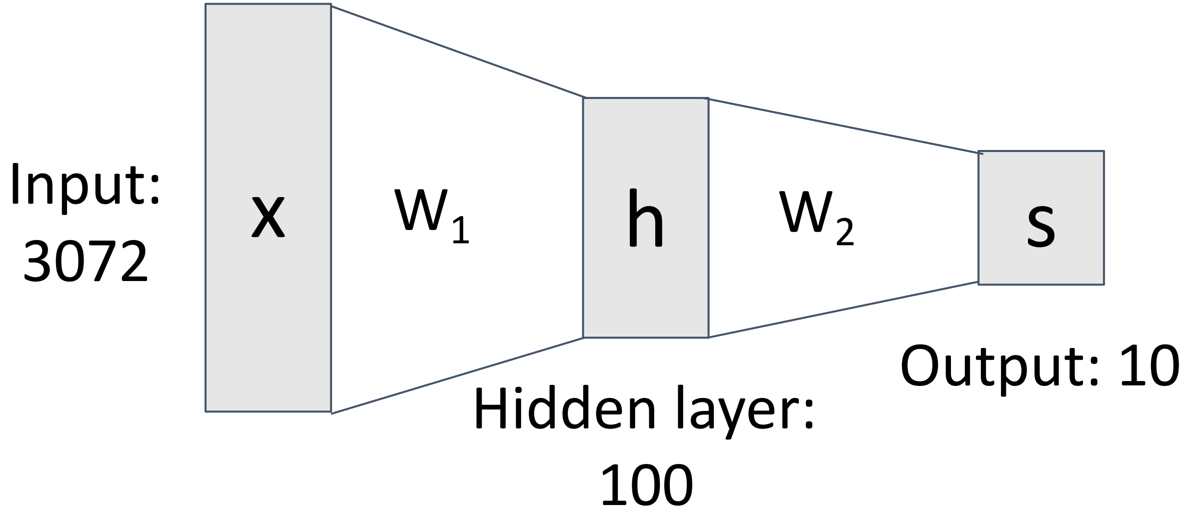
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Neural net: first layer is bank of templates;
Second layer recombines templates

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



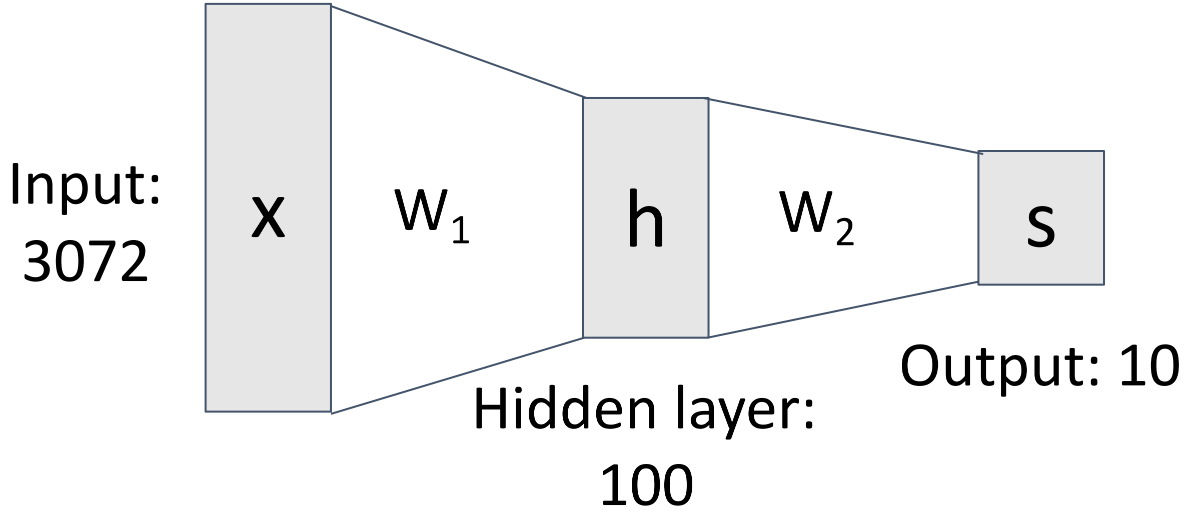
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Can use different templates to cover multiple modes of a class!

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



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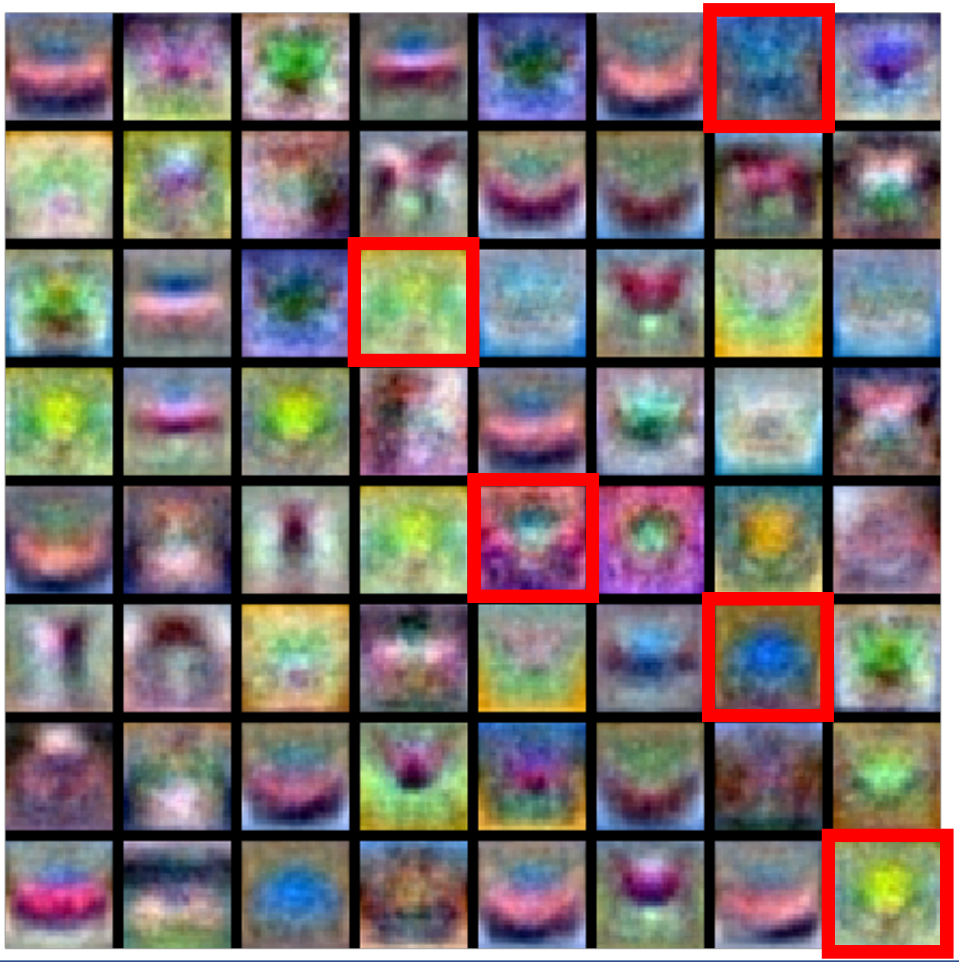
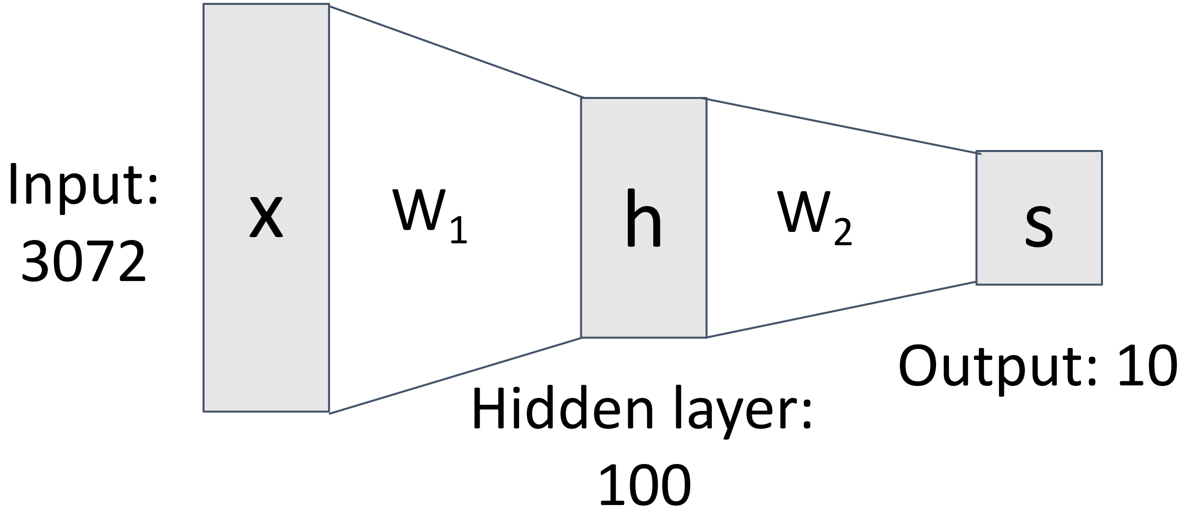
Neural Networks

"Distributed representation":
Most templates not interpretable!



(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



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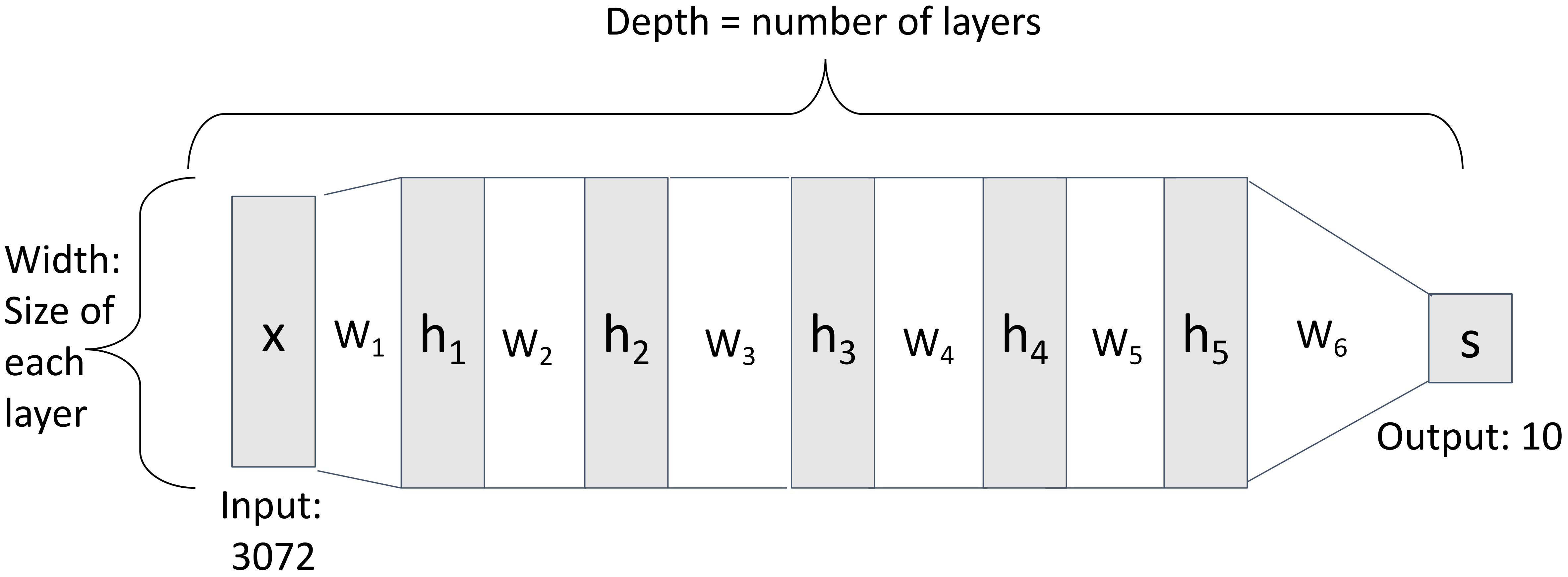
Deep Neural Networks



$$s = W_6 \max(0, W_5 \max(0, W_4 \max(0, W_3 \max(0, W_2 \max(0, W_1 x)))))$$

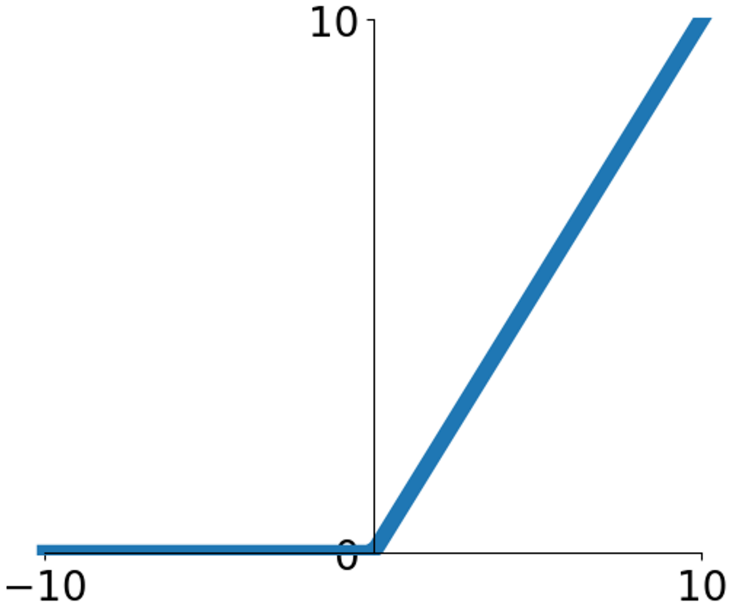
Activation Functions

2-layer Neural Network

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$

The function $ReLU(z) = \max(0, z)$ is called "Rectified Linear Unit"

This is called the **activation function** of the neural network



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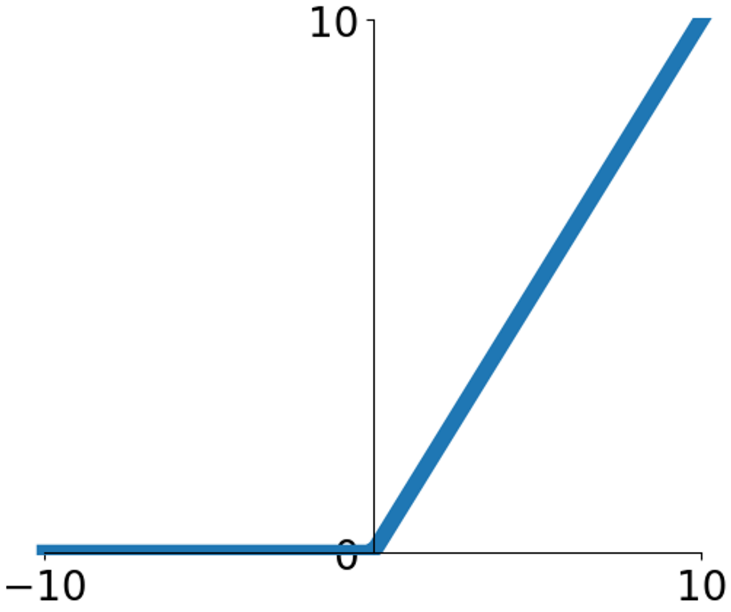

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Q: What happens if we build a neural network with no activation function?

$$f(x) = W_2(W_1 x + b_1) + b_2$$

Activation Functions

2-layer Neural Network

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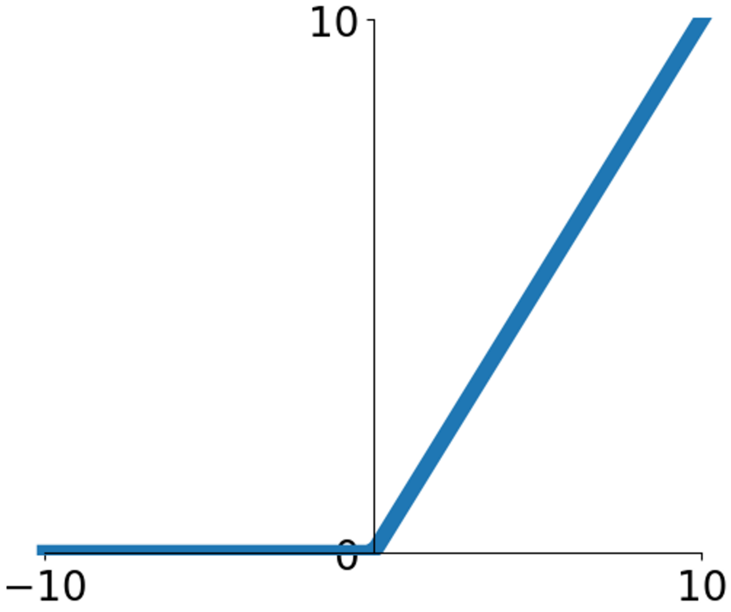


This is called the **activation function** of the neural network

Q: What happens if we build a neural network with no activation function?

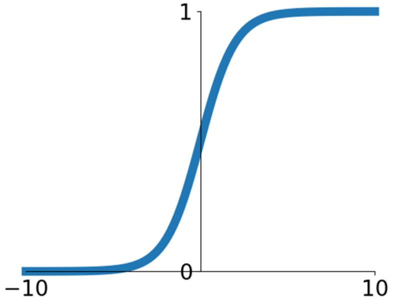
$$f(x) = W_2(W_1 x + b_1) + b_2$$
$$= (W_1 W_2)x + (W_2 b_1 + b_2)$$

A: We end up with a linear classifier!

Activation Functions

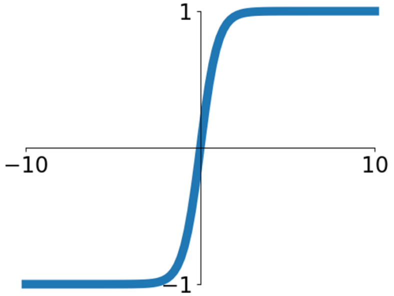
Sigmoid

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



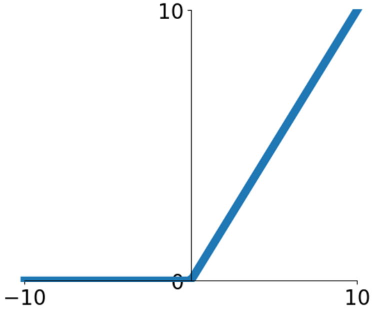
tanh

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$



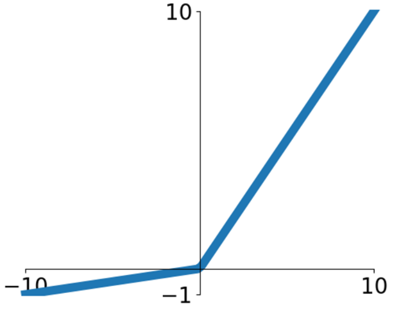
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.2x, x)$$

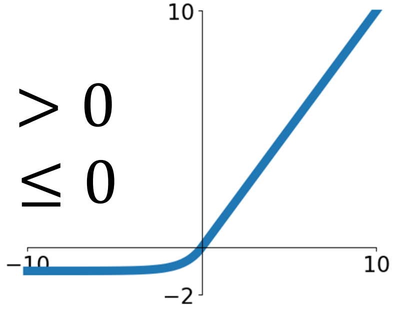


Softplus

$$\log(1 + \exp(x))$$

ELU

$$f(x) = \begin{cases} x, & x > 0 \\ \alpha(\exp(x) - 1), & x \le 0 \end{cases}$$

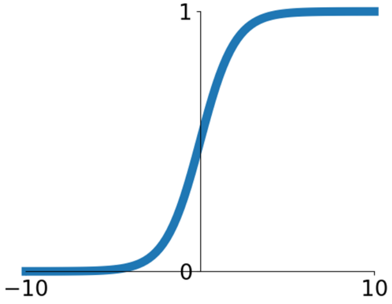


Activation Functions

ReLU is a good default choice for most problems

Sigmoid

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



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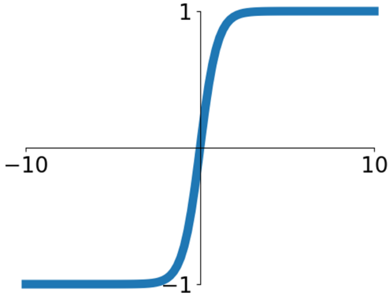
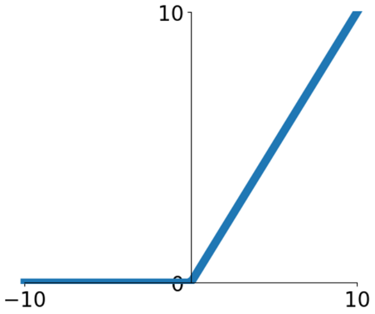


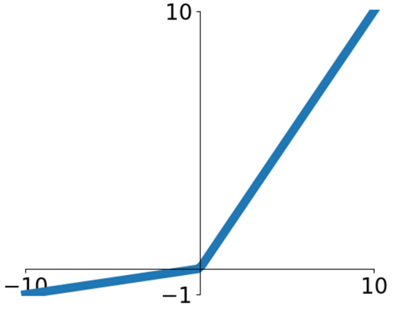
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.2x, x)$$

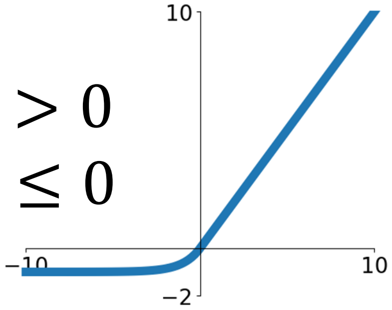


Softplus

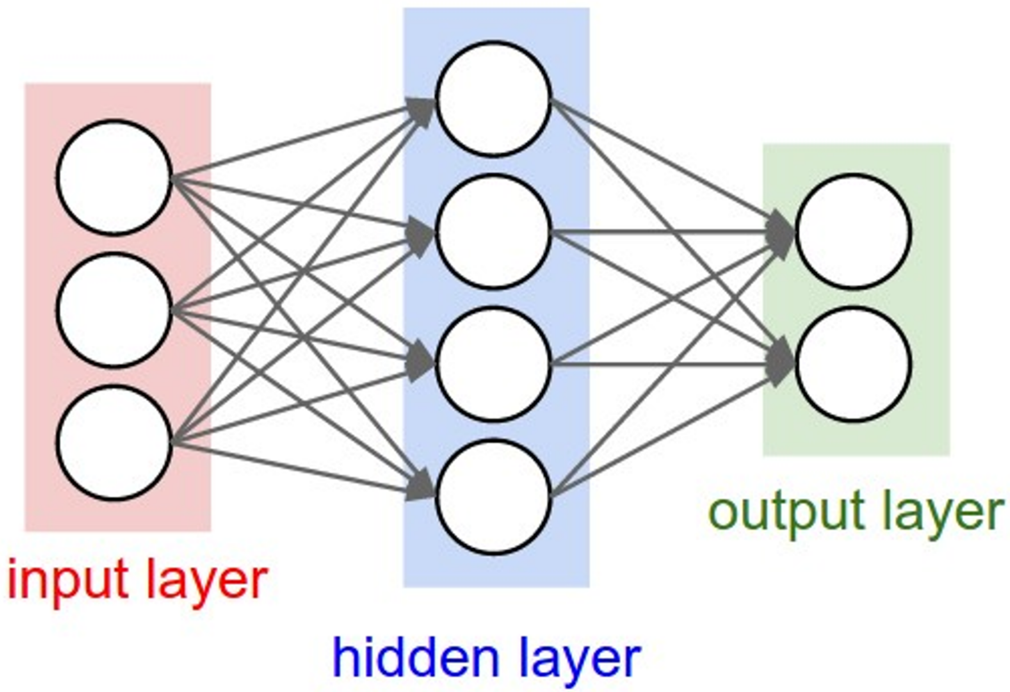
$$\log(1 + \exp(x))$$

ELU

$$f(x) = \begin{cases} x, & x > 0 \\ \alpha(\exp(x) - 1), & x \leq 0 \end{cases}$$



Neural Net in <20 lines!

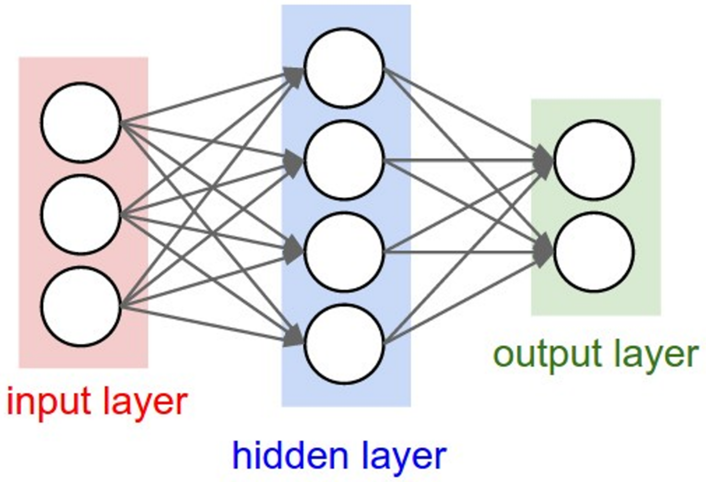



```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```

Neural Net in <20 lines!

Initialize weights and data

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  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
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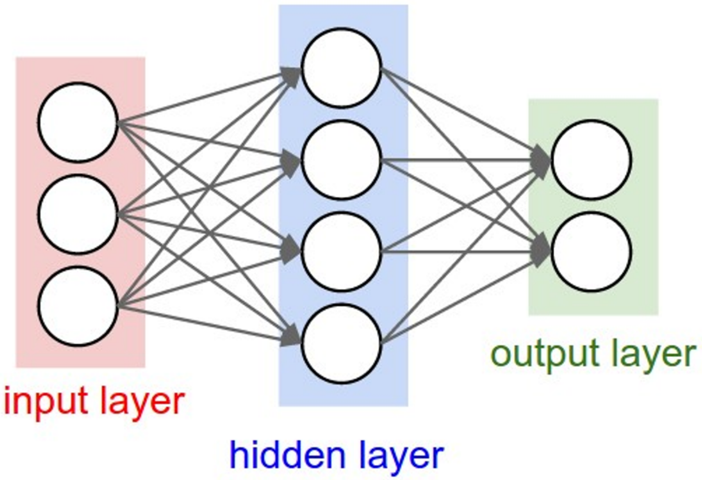
```

Neural Net in <20 lines!



Initialize weights and data

Compute loss (sigmoid activation, L2 loss)

```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
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```
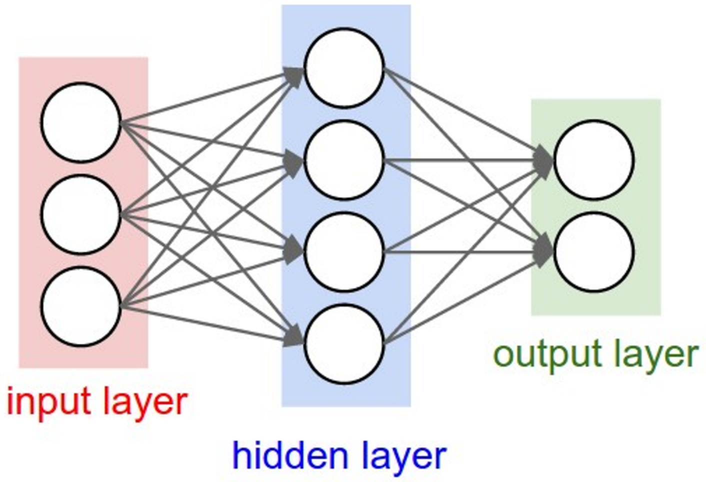

Neural Net in <20 lines!



Initialize weights and data

Compute loss (sigmoid activation, L2 loss)

Compute gradients

```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
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```
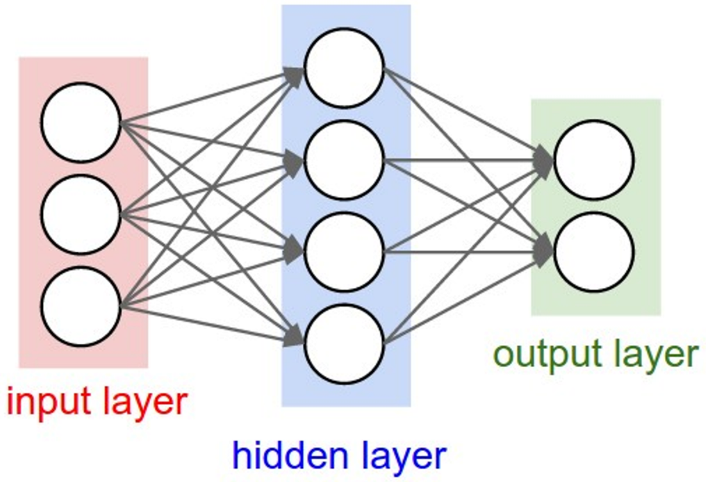

Neural Net in <20 lines!



Initialize weights and data

Compute loss (sigmoid activation, L2 loss)

Compute gradients

SGD step

```
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
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  w1 -= 1e-4 * dw1
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```
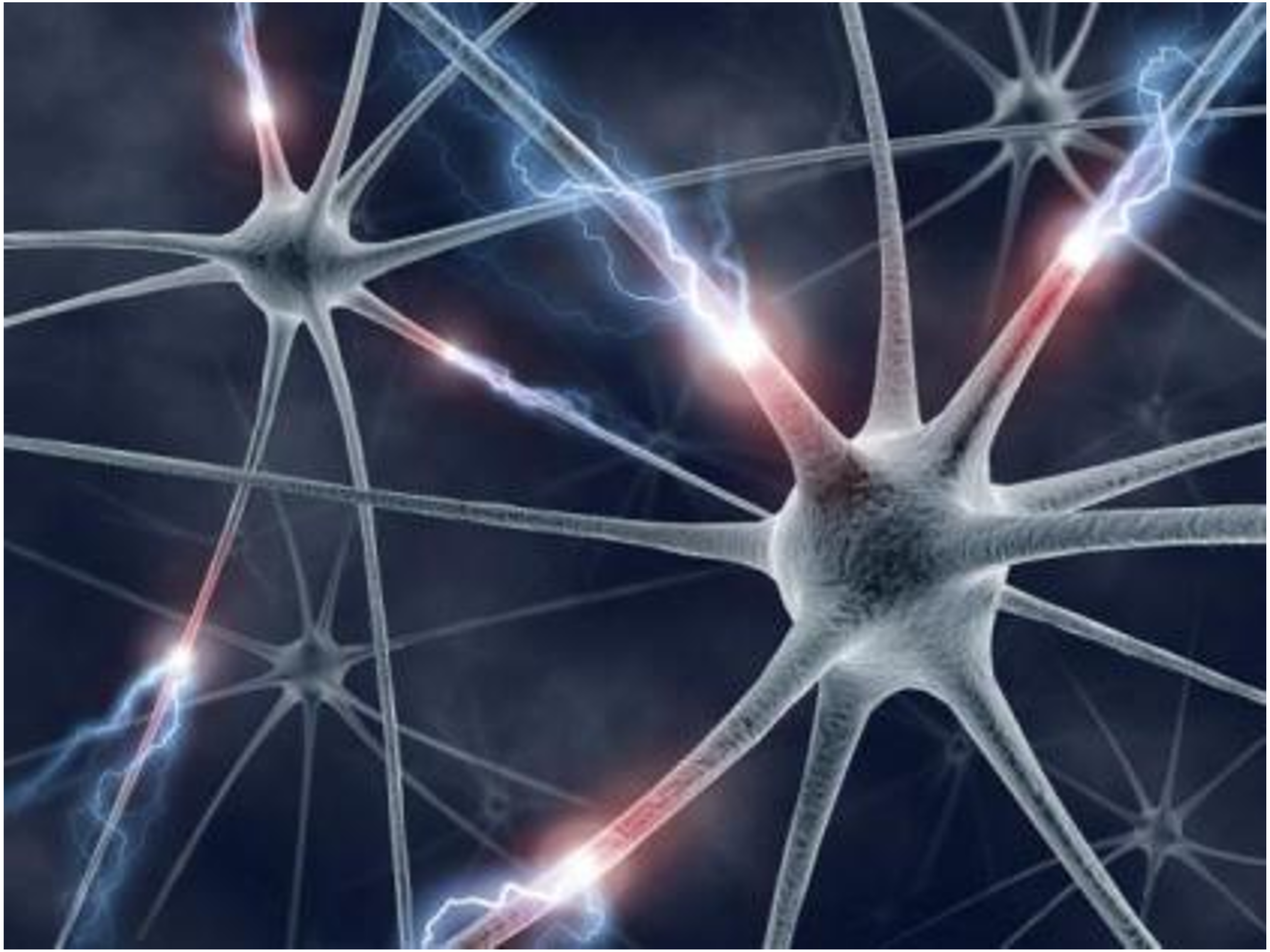
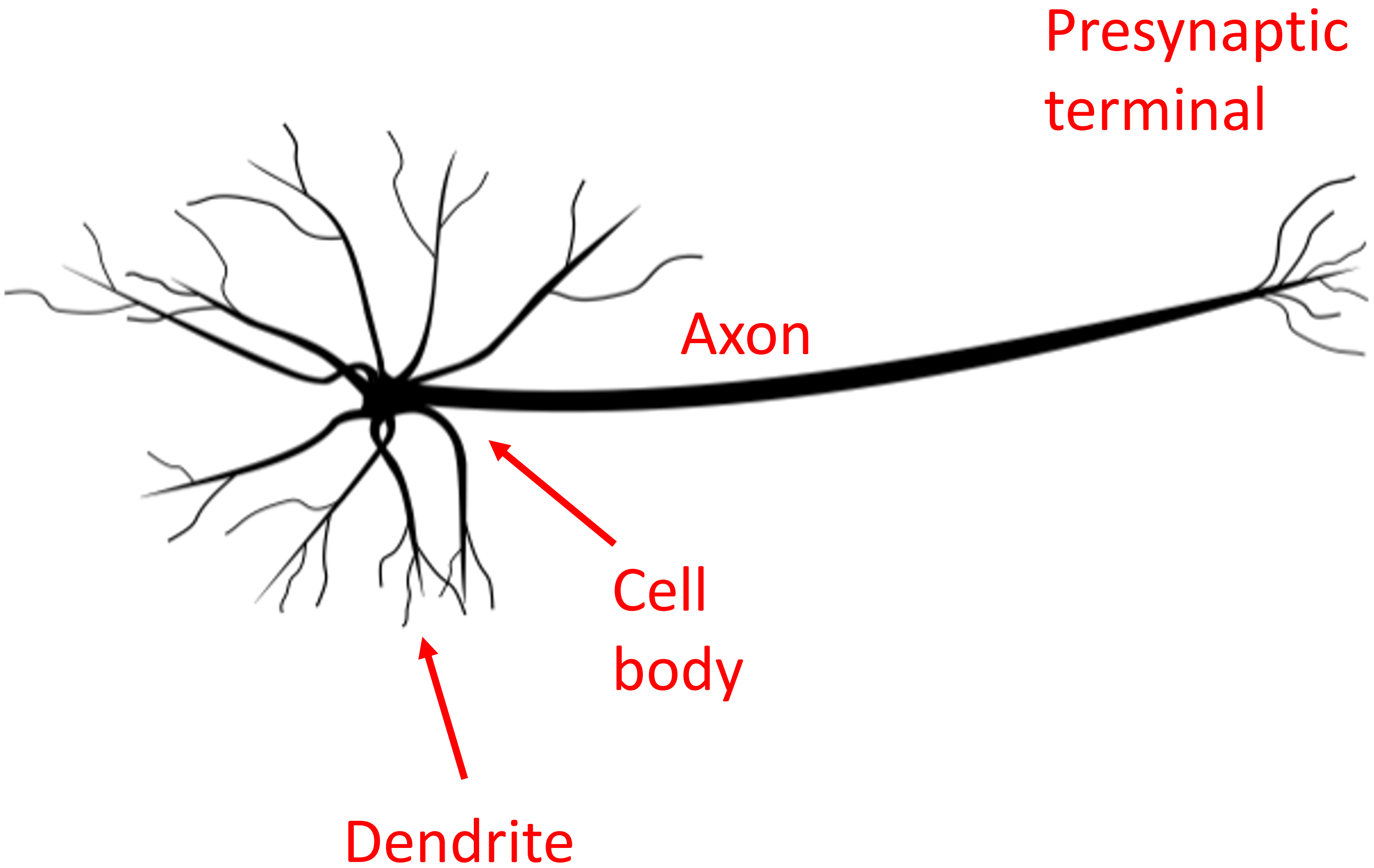
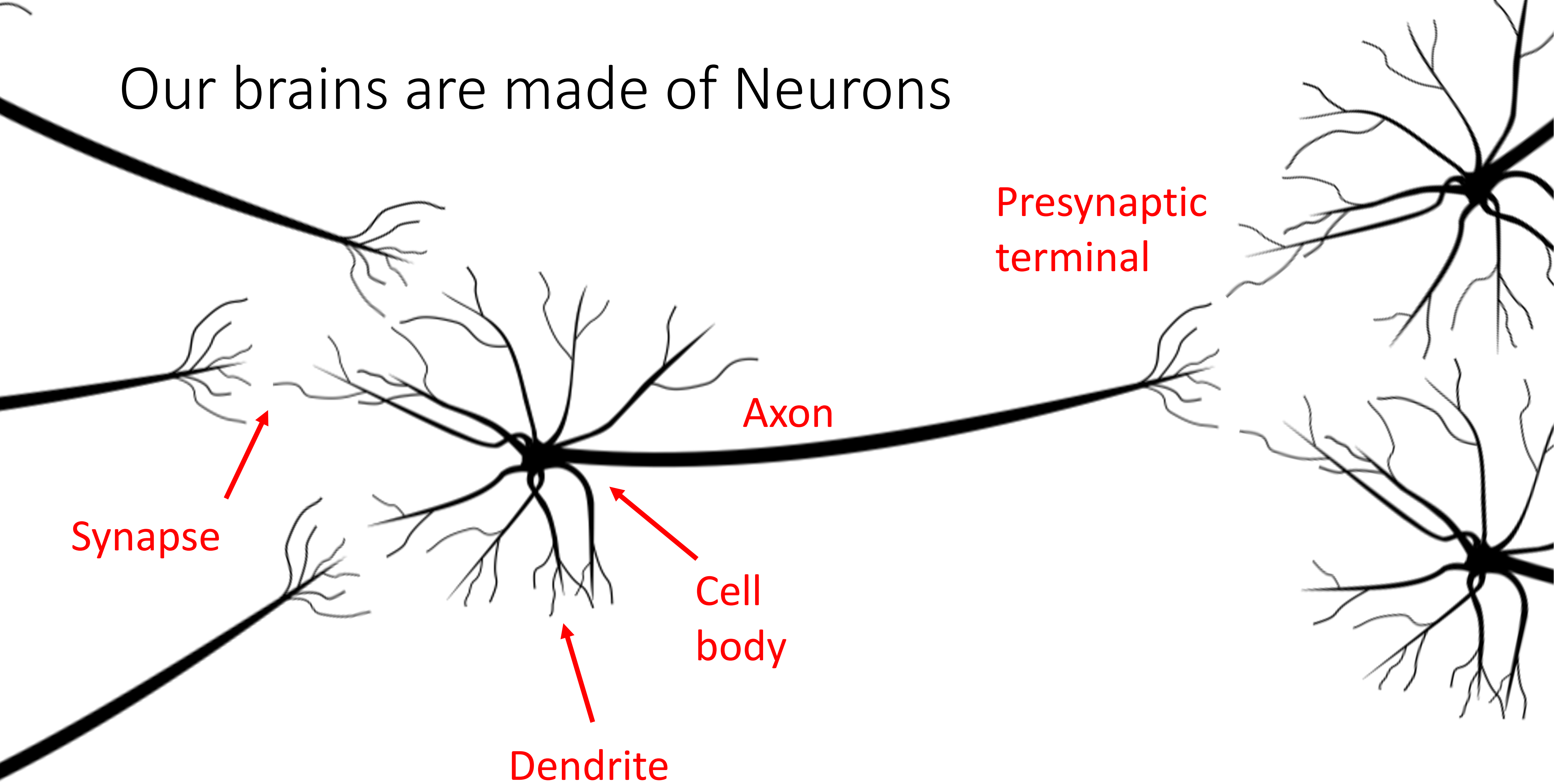



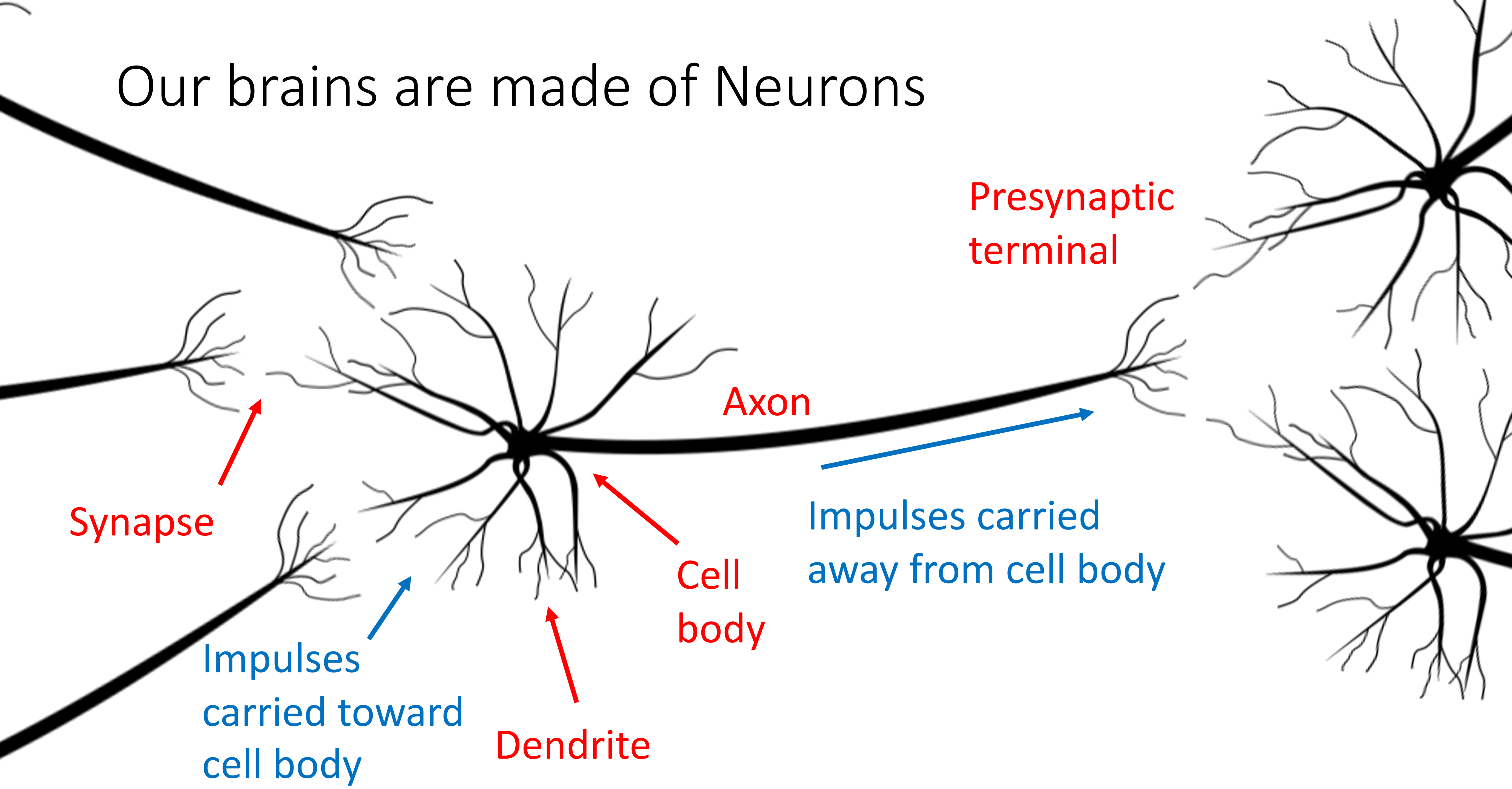
Our brains are made of Neurons



Our brains are made of Neurons



Our brains are made of Neurons



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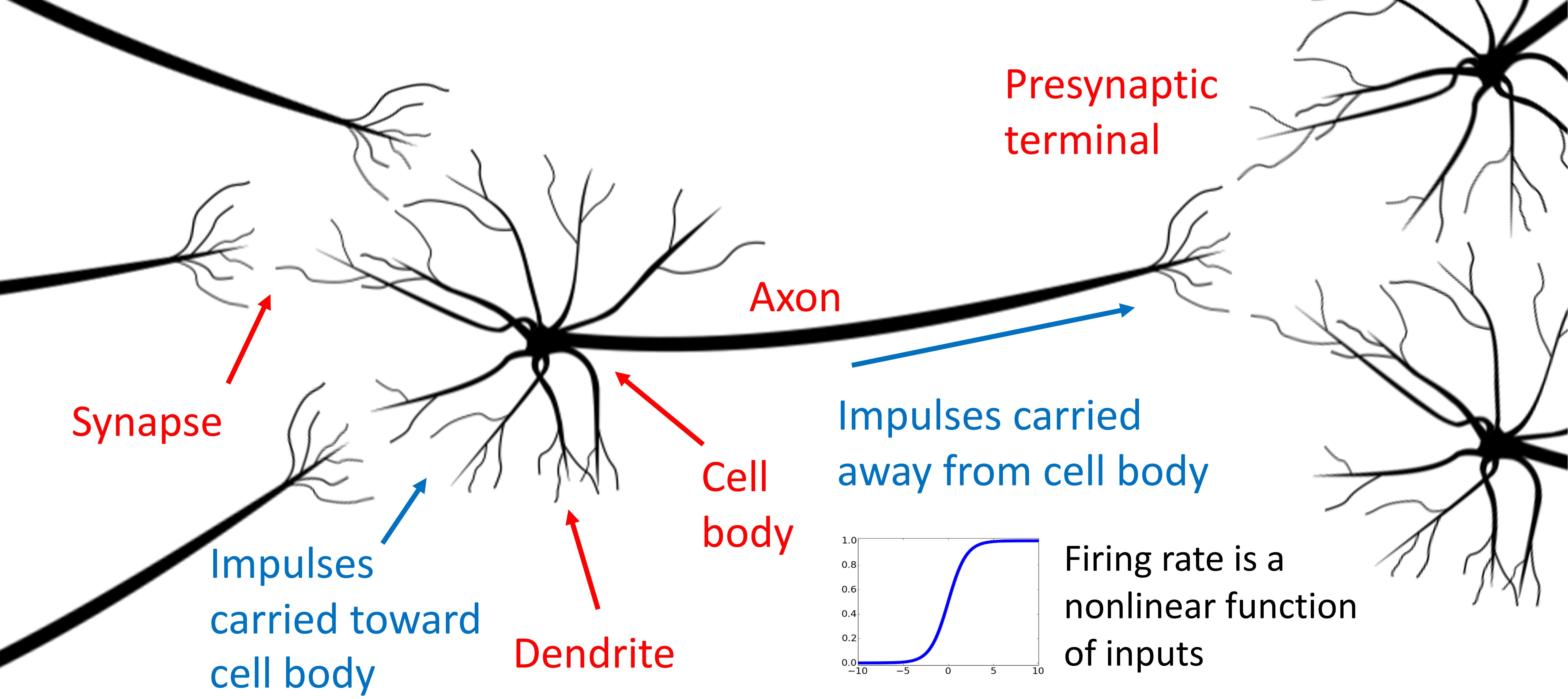
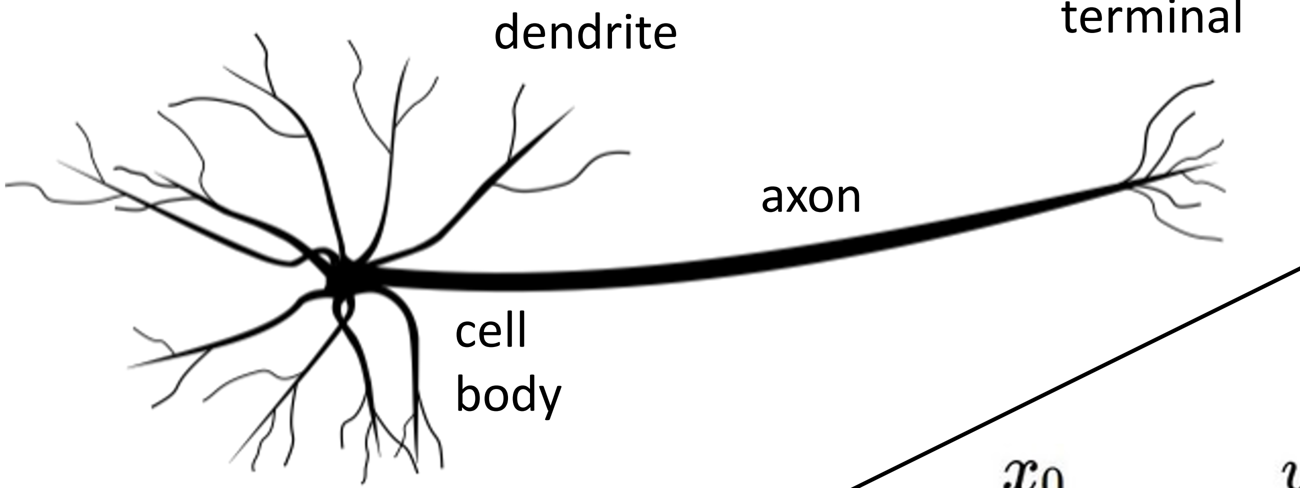
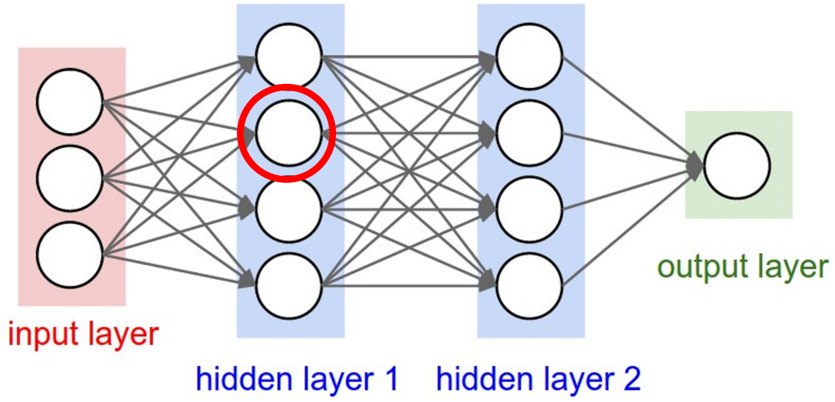


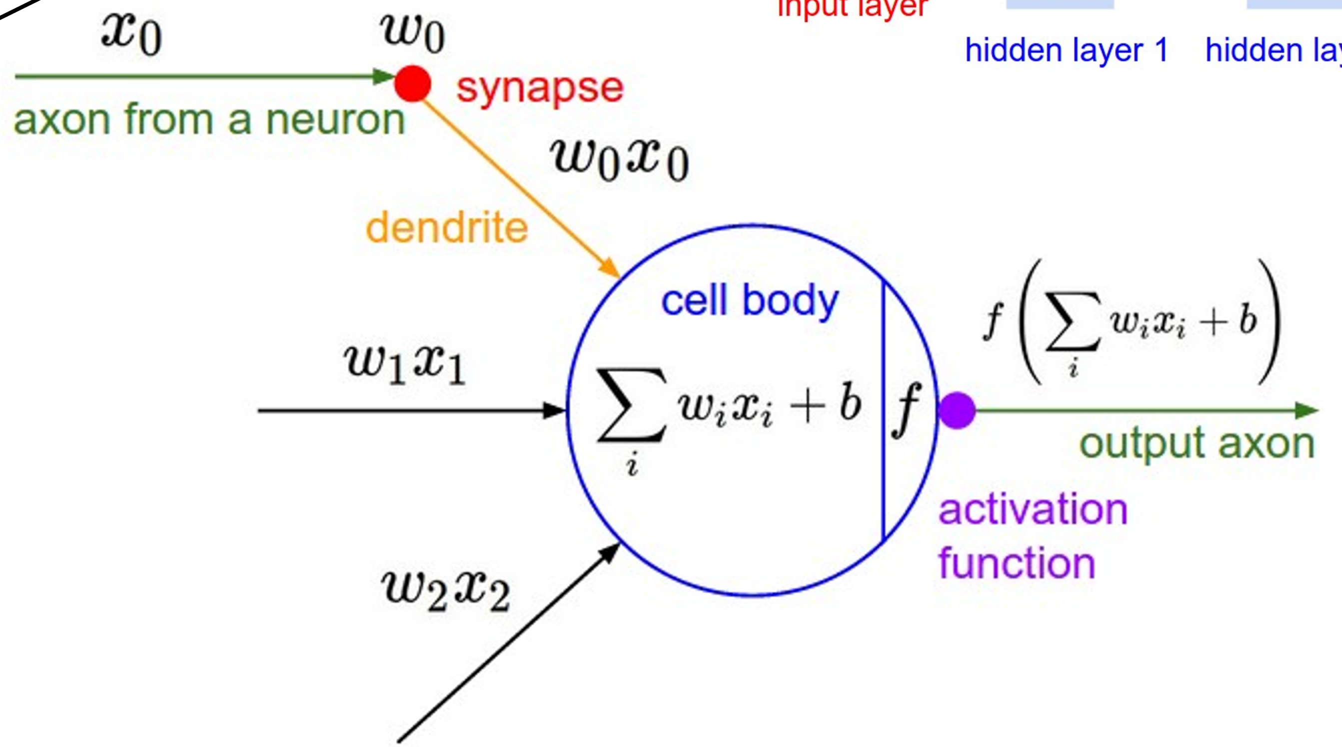
Firing rate is a nonlinear function of inputs

Biological Neuron

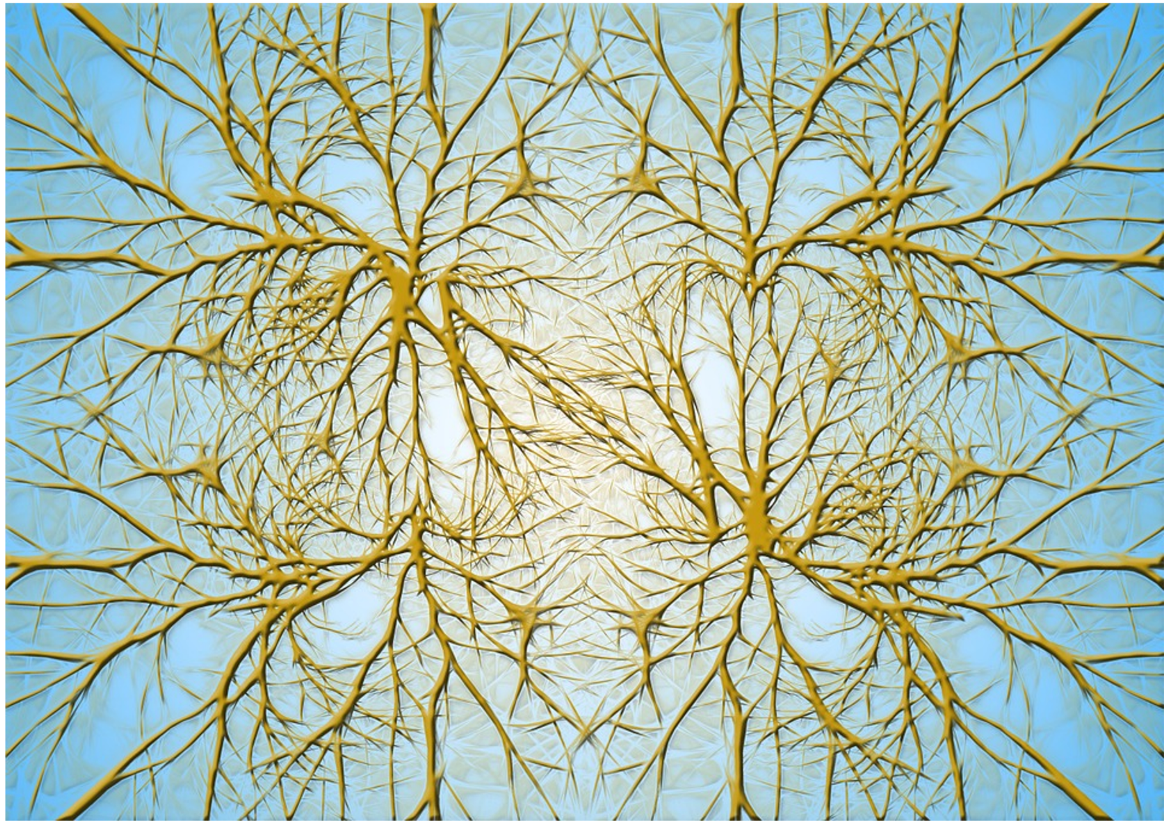


Artificial Neuron





Biological Neurons:
Complex connectivity patterns



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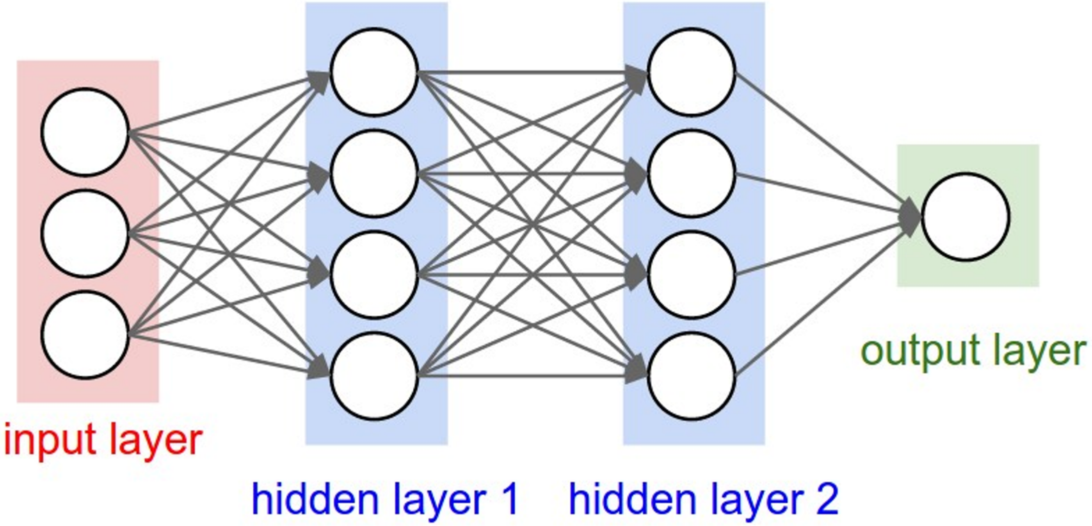
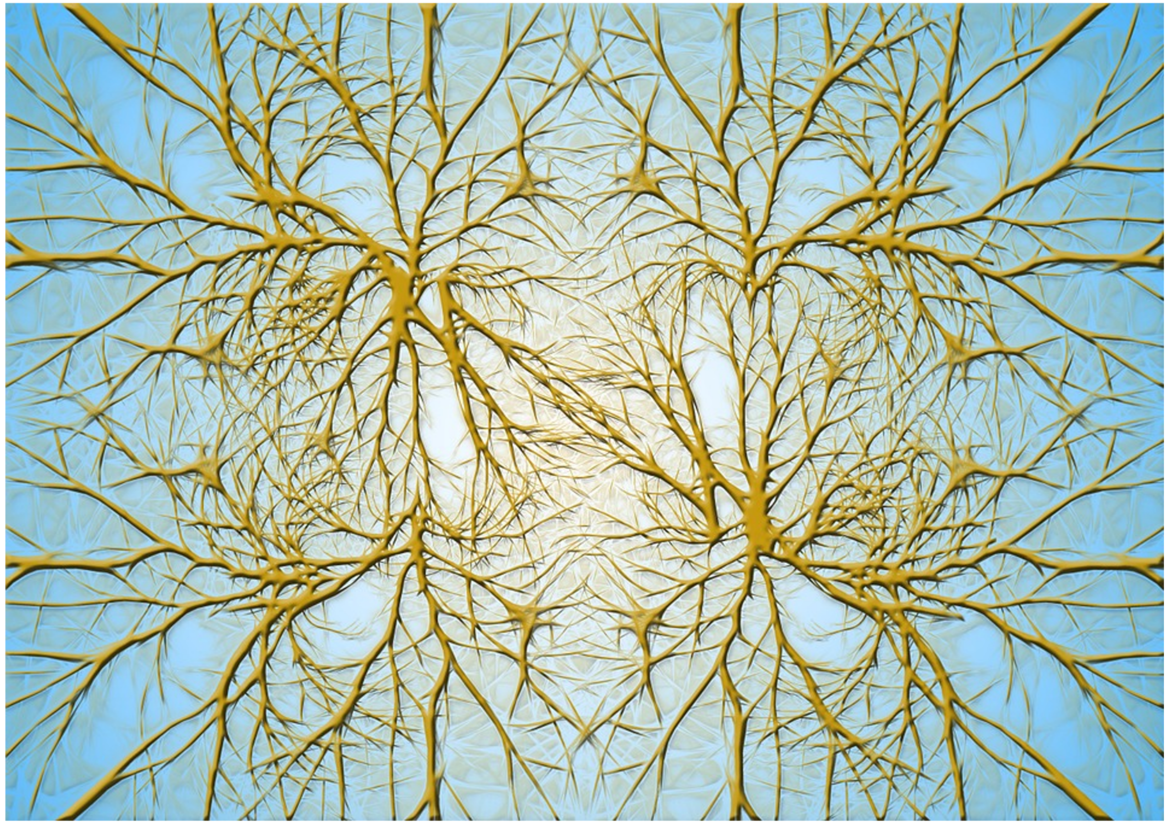
Neurons in a neural network:
Organized into regular layers for computational efficiency

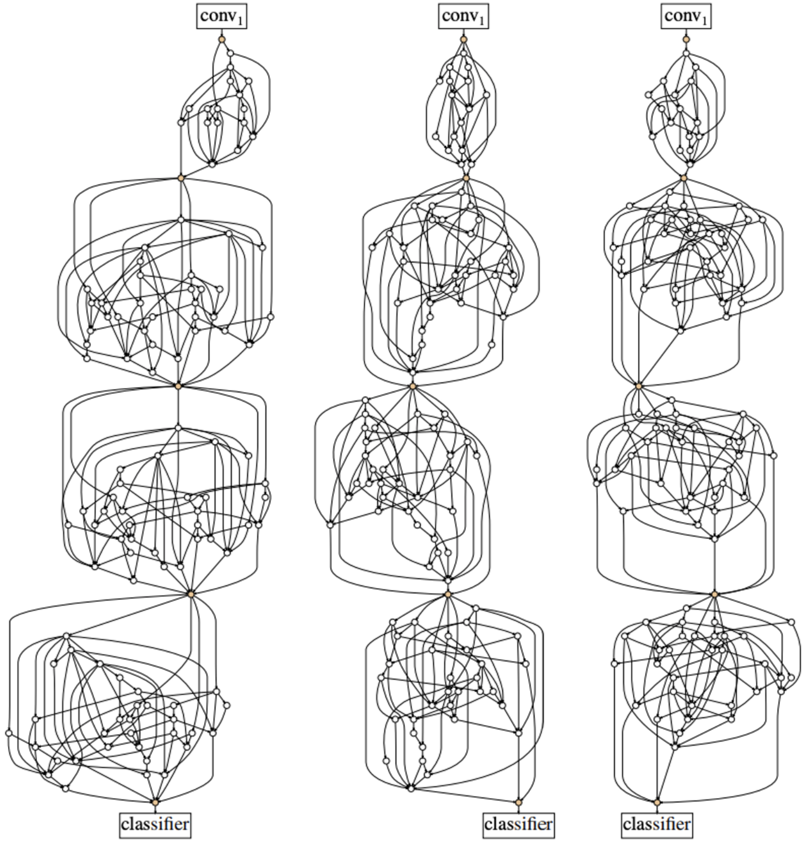


Biological Neurons:
Complex connectivity patterns



This image is CC0 Public Domain

But neural networks with random connections can work too!



Xie et al, "Exploring Randomly Wired Neural Networks for Image Recognition", ICCV 2019

Be very careful with brain analogies!

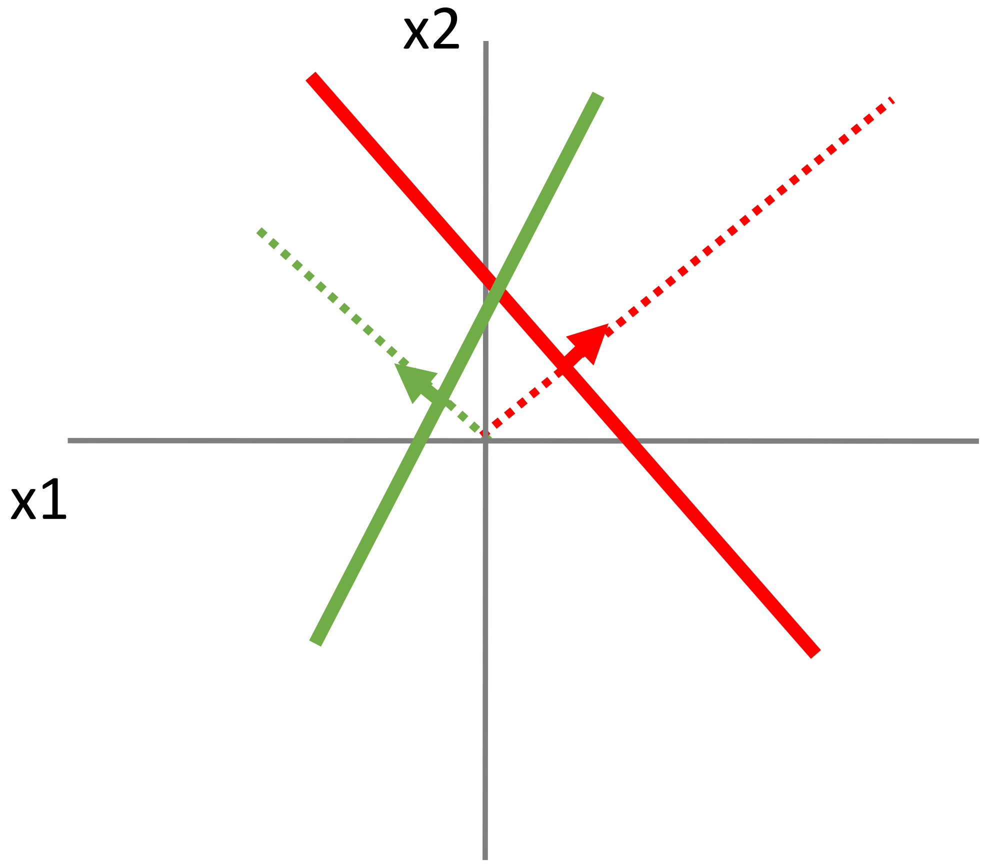
Biological Neurons:

- Many different types
- Dendrites can perform complex non-linear computations
- Synapses are not a single weight but a complex non-linear dynamical system
- Abstracting a neuron by “firing rate” isn’t enough; temporal sequences of activations matter too (spiking neural networks)

[Dendritic Computation. London and Hausser]

Space Warping

Consider a linear transform: $h = Wx$
Where x, h are both 2-dimensional



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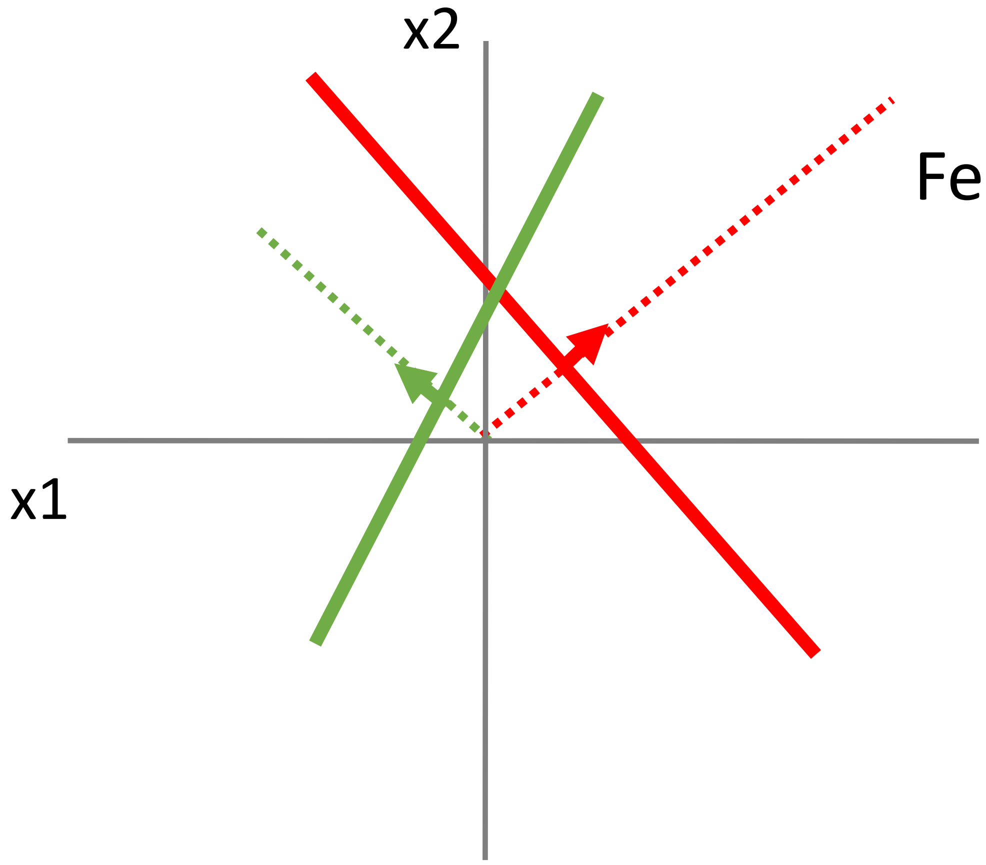
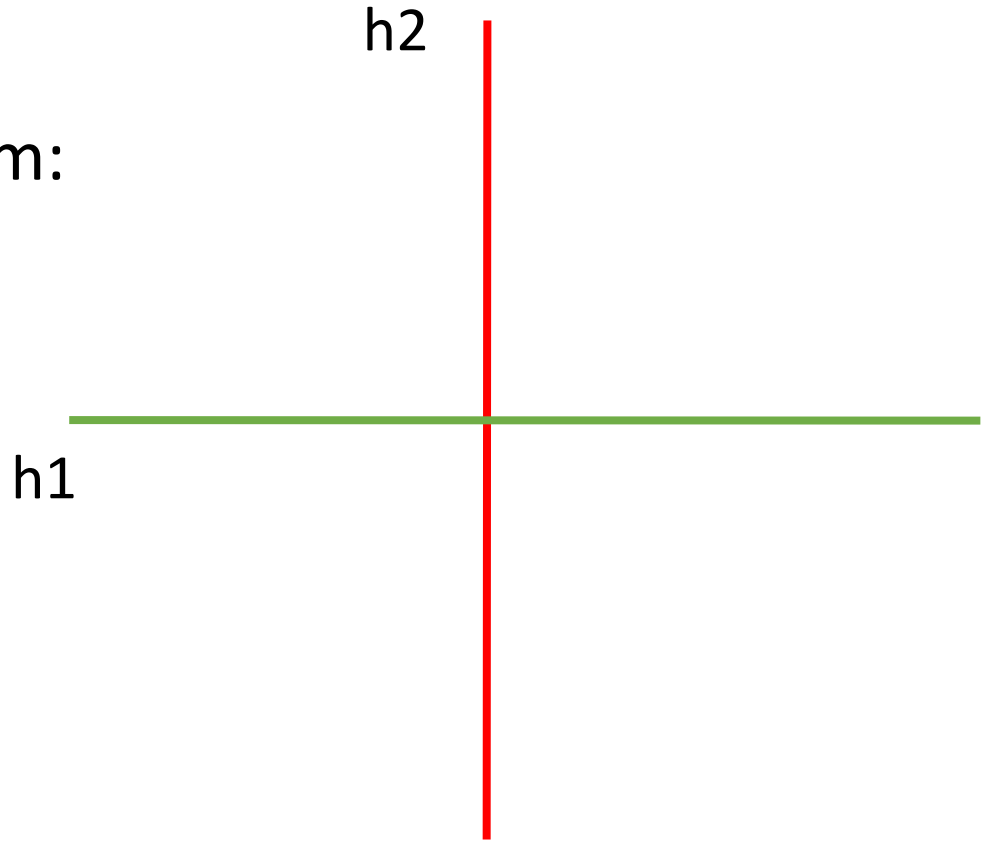


Feature transform:
h = Wx



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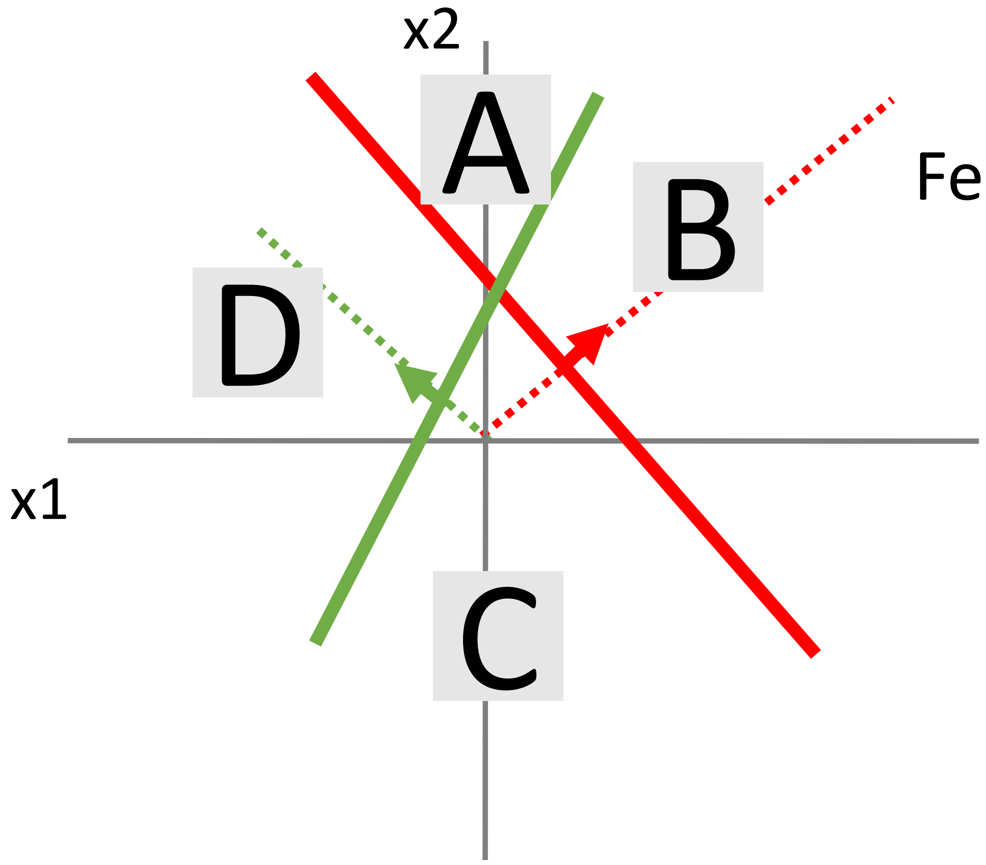

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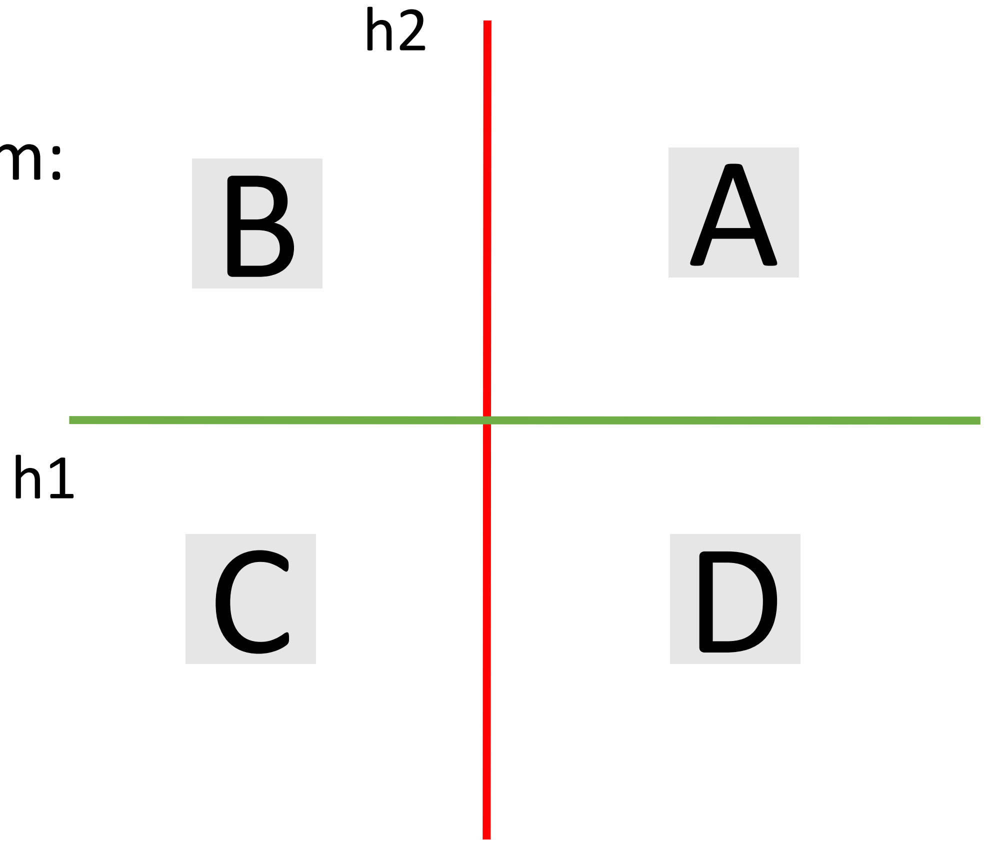
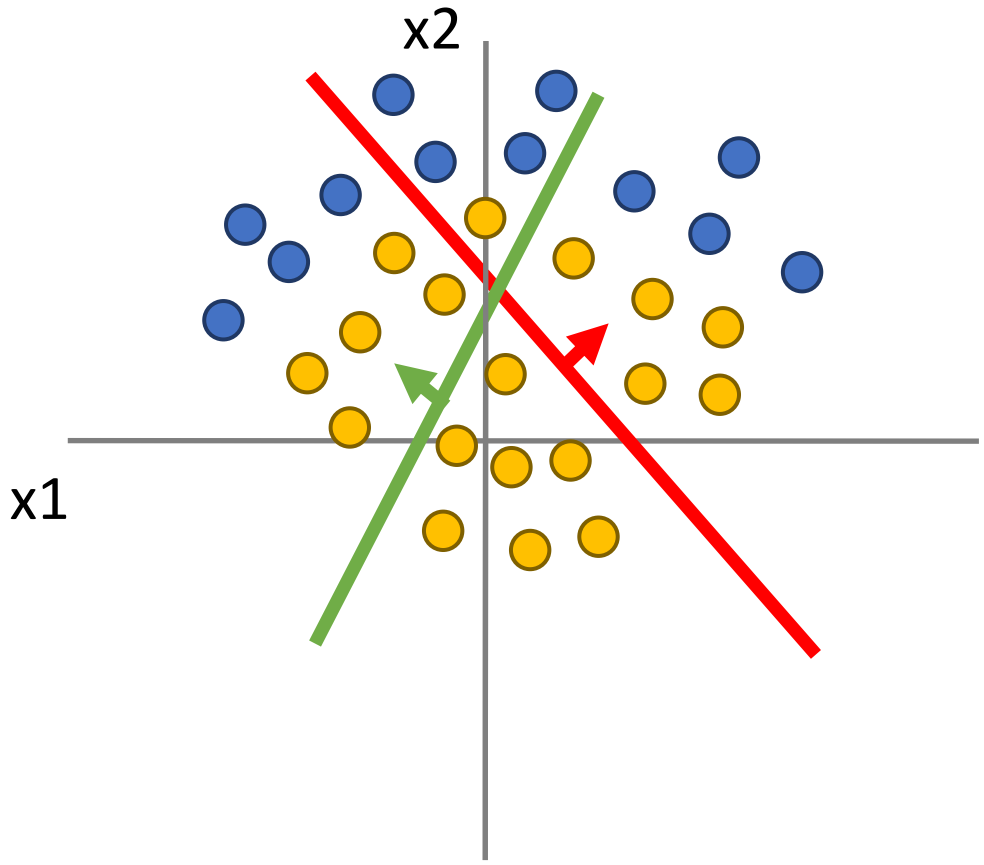


Space Warping

Points not linearly separable in original space



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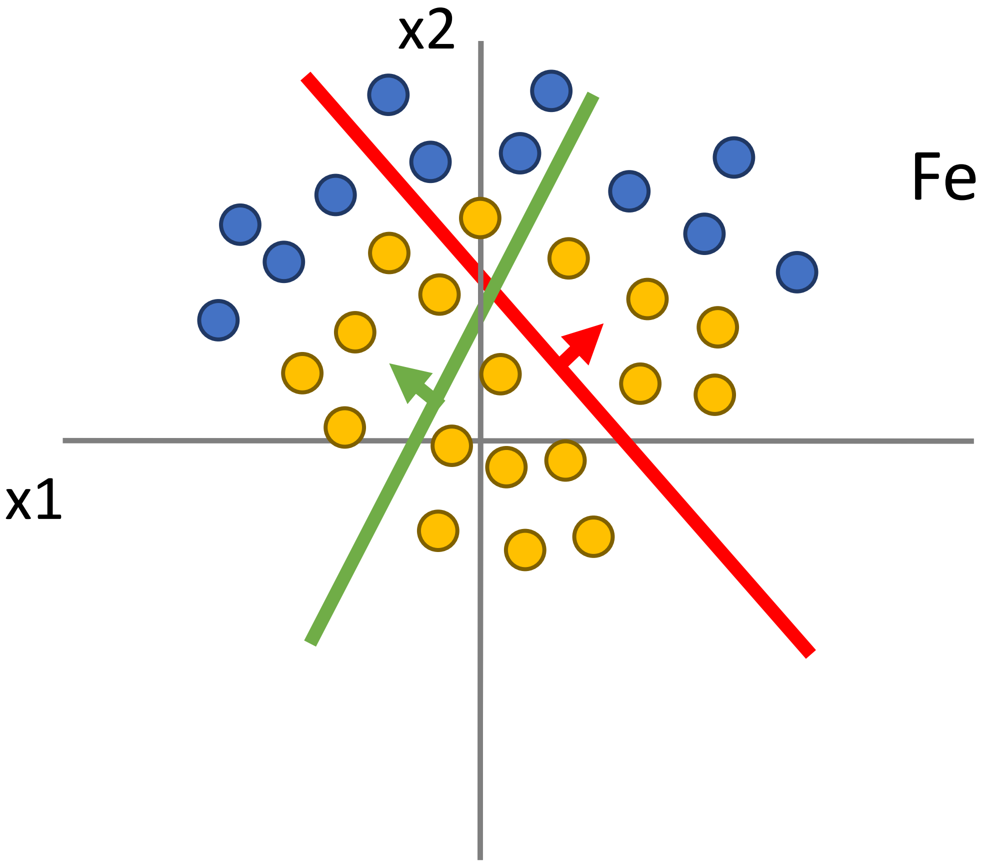
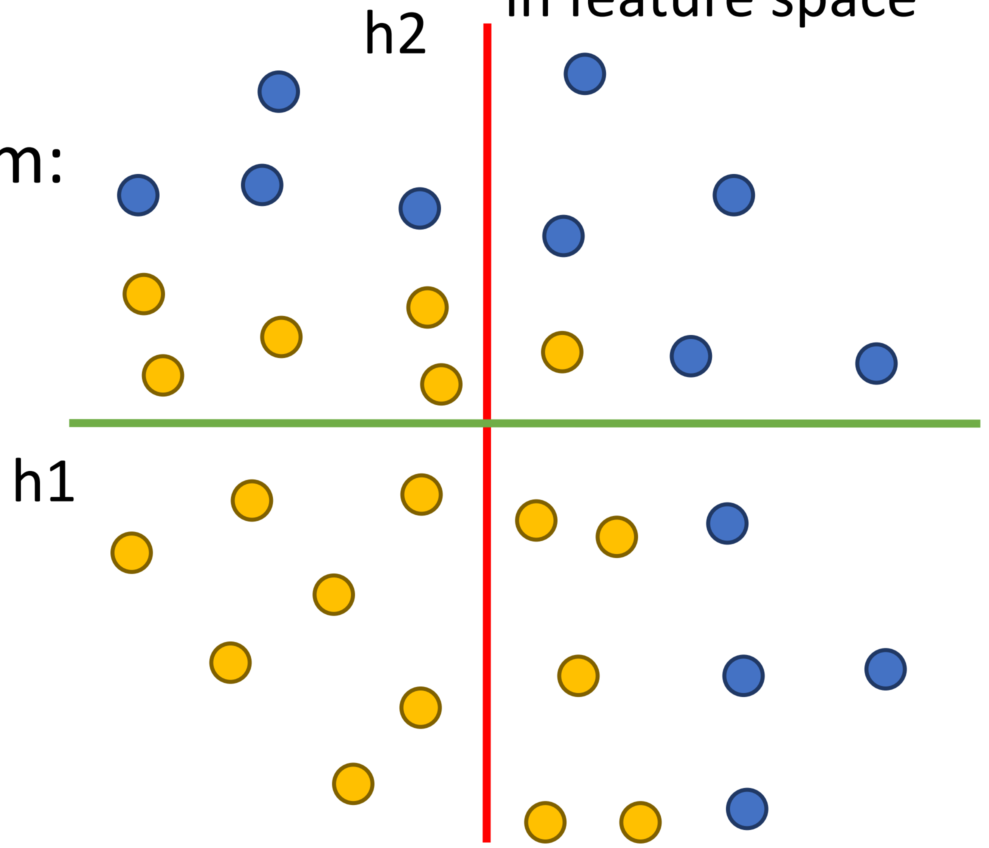


Feature transform:
h = Wx

Not linearly separable in feature space

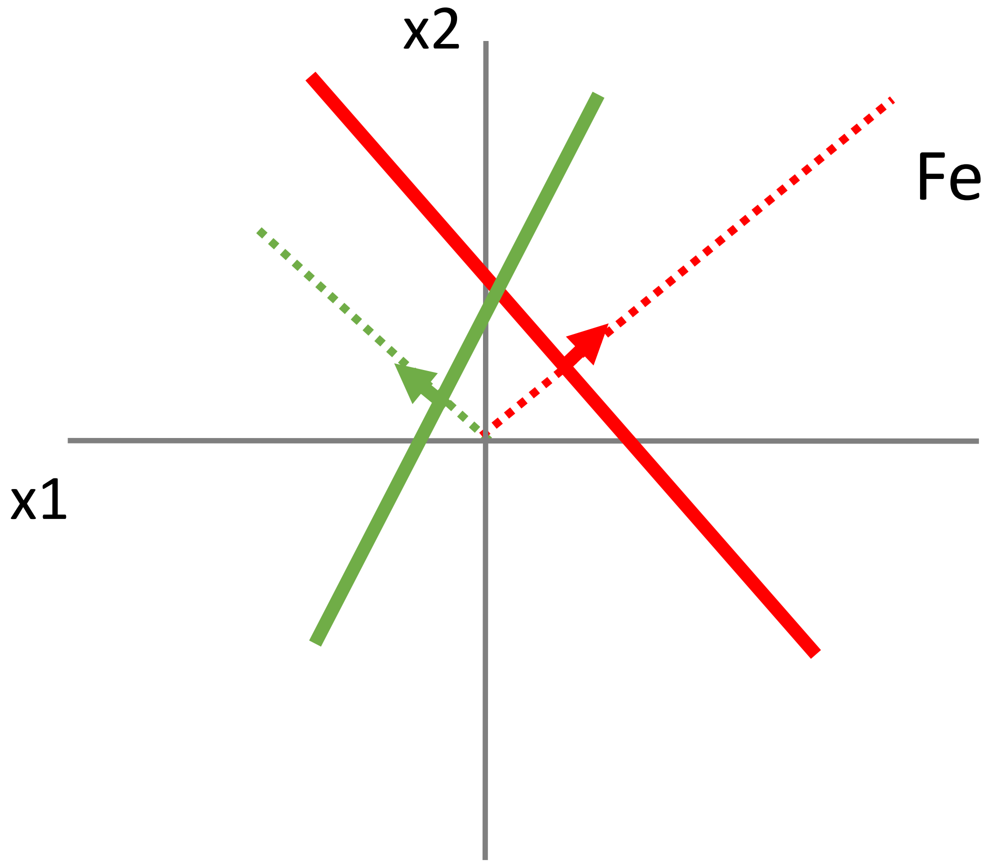


Space Warping

Consider a neural net hidden layer:

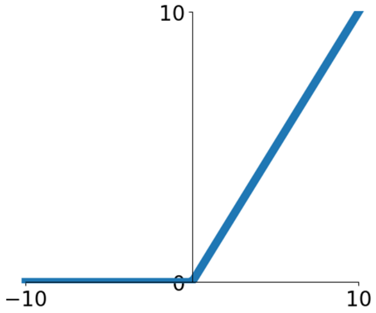
h = ReLU(Wx) = max(0, Wx)

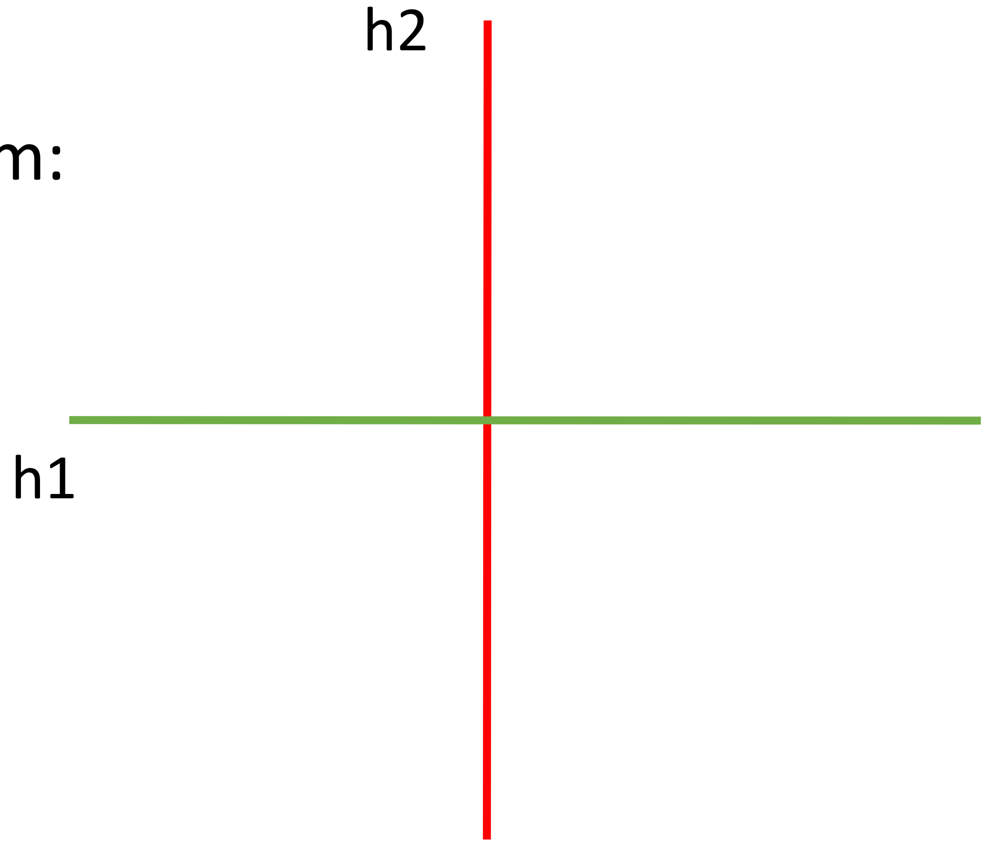
Where x, h are both 2-dimensional



Feature transform:

h = ReLU(Wx)





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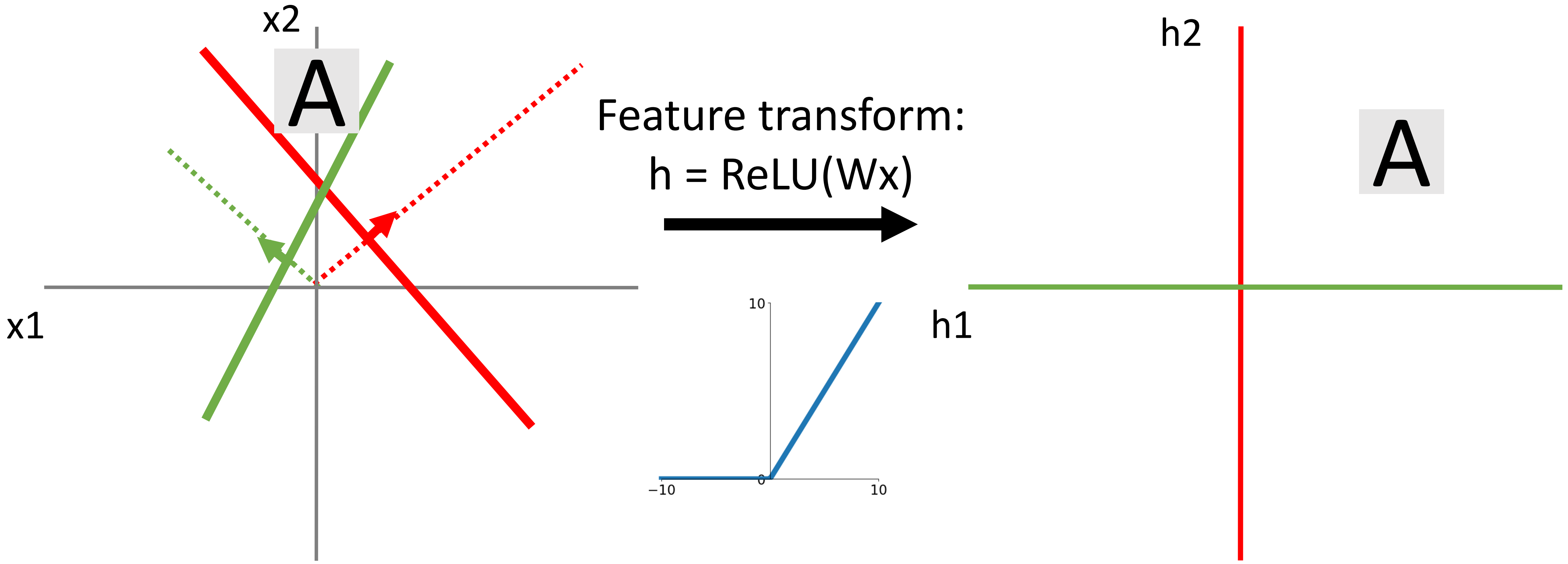

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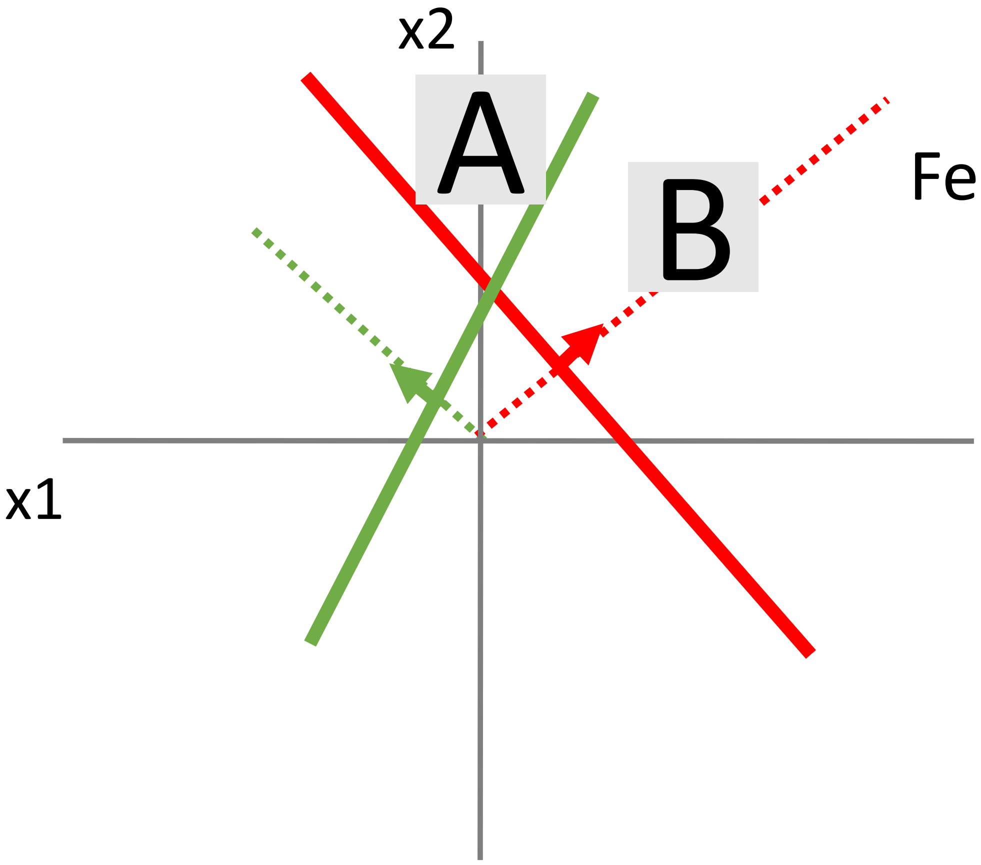

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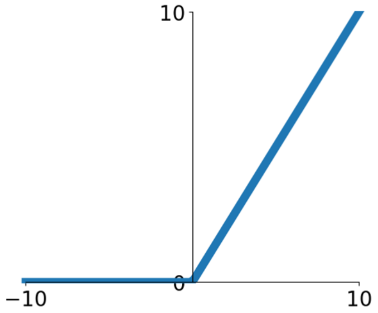

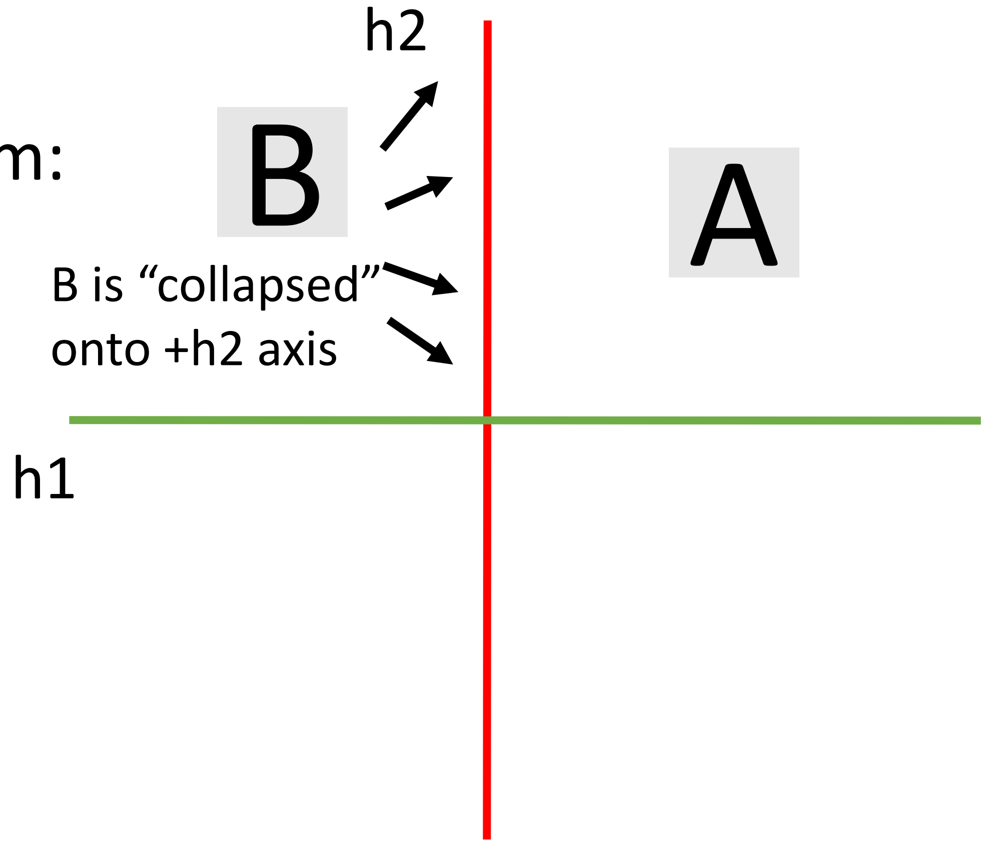

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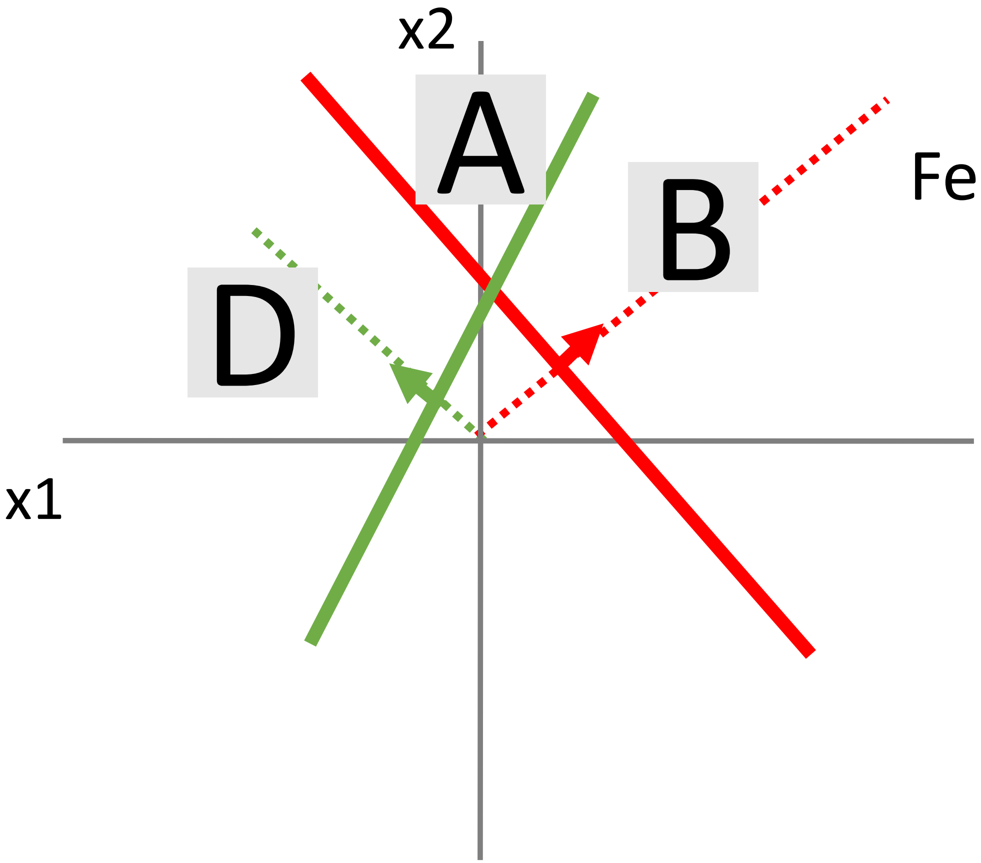

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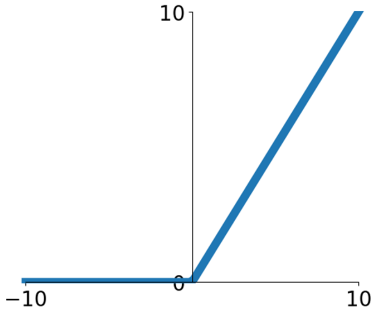

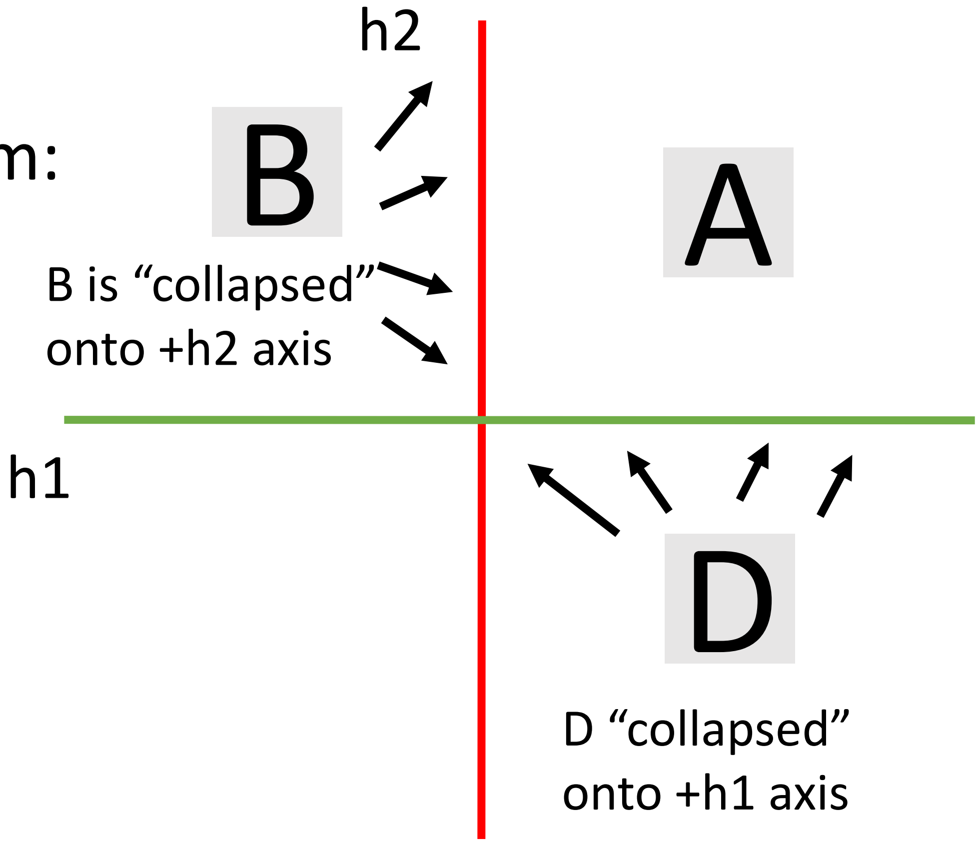

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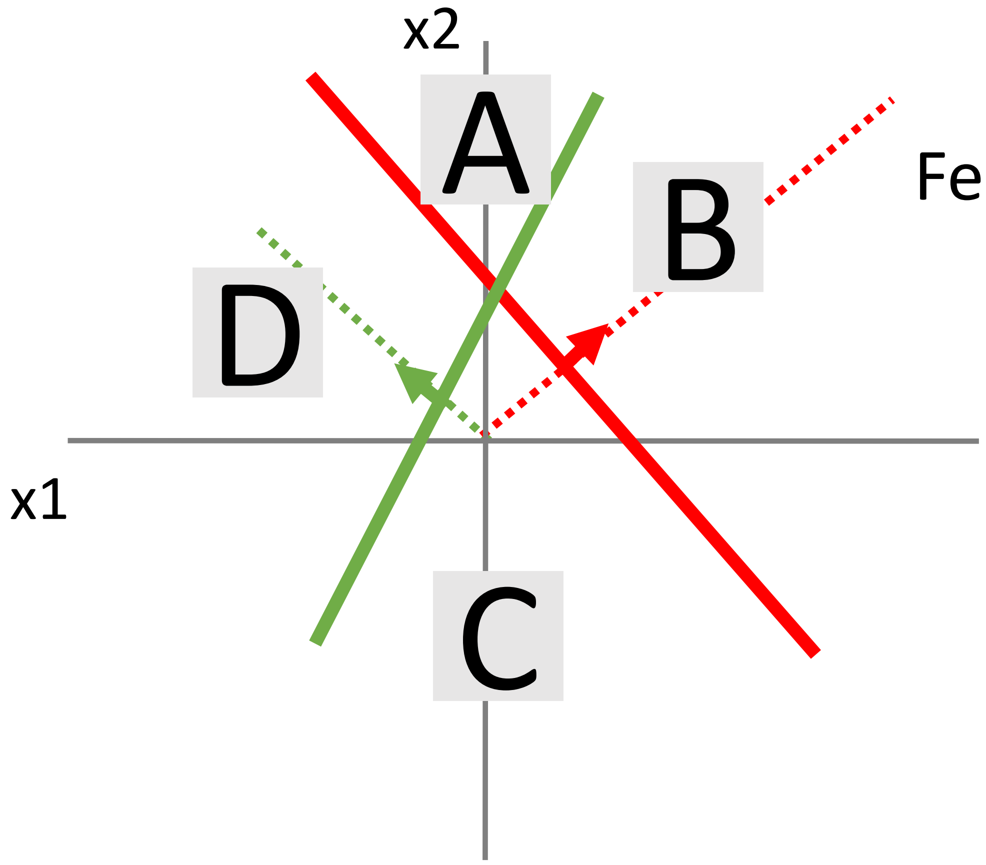

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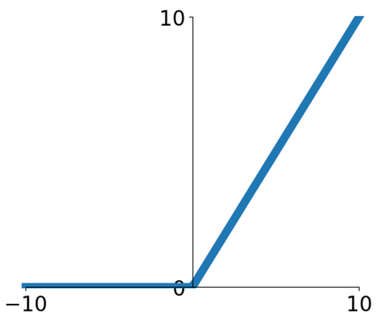
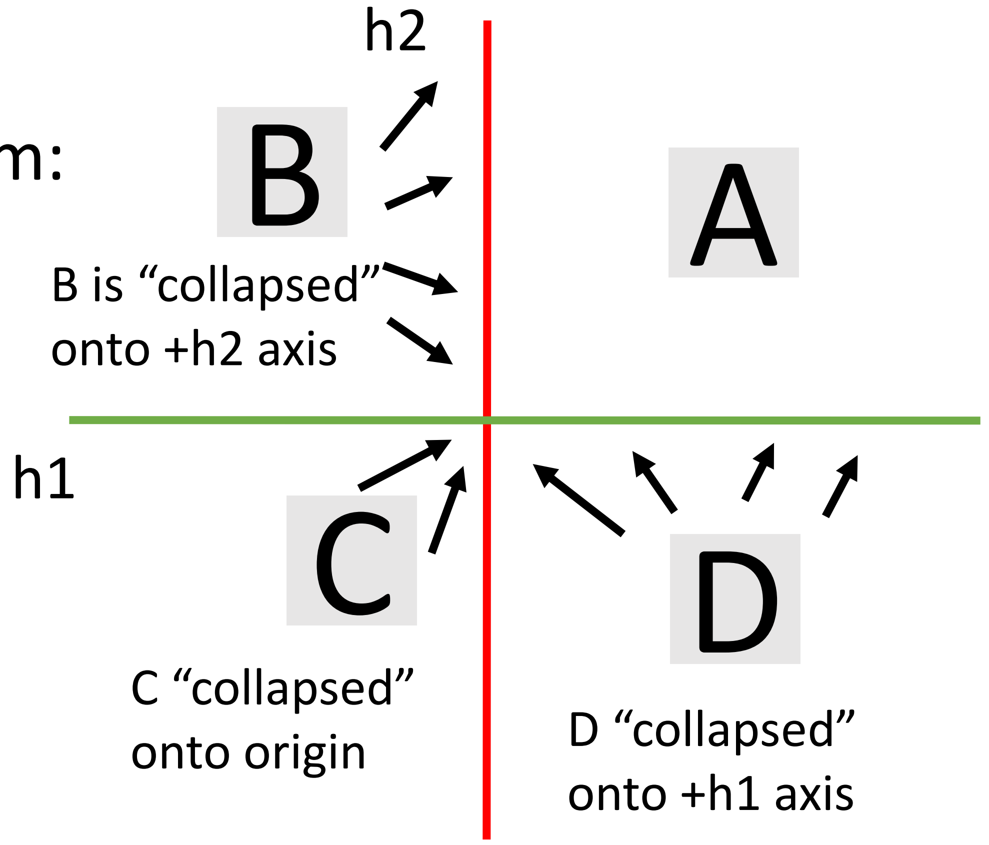
$h = \text{ReLU}(Wx)$





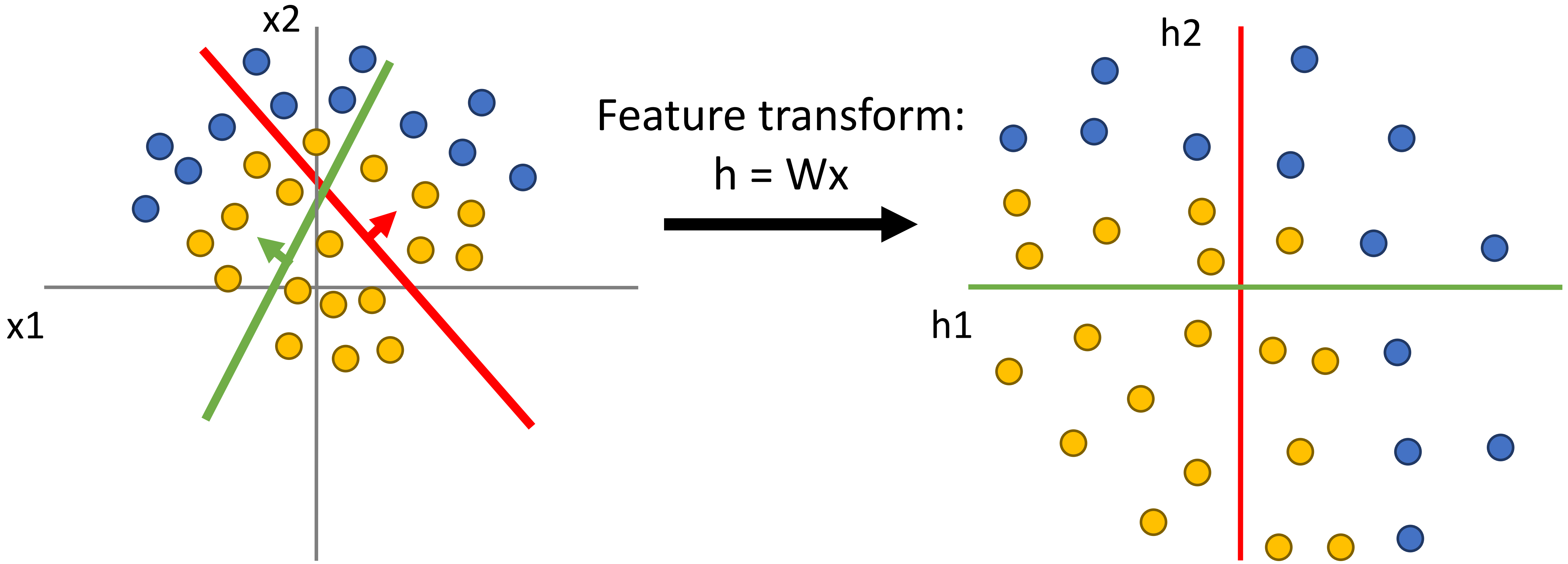
Space Warping

Points not linearly separable in original space

Consider a neural net hidden layer:

$h = \text{ReLU}(Wx) = \max(0, Wx)$

Where x, h are both 2-dimensional



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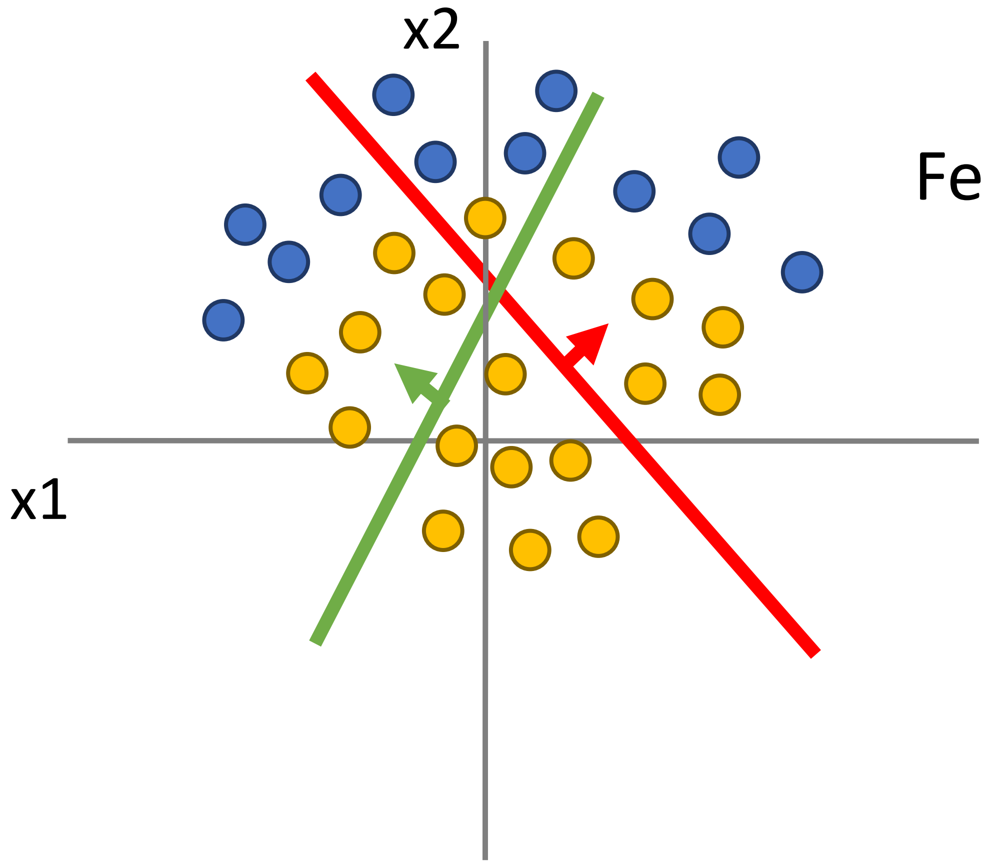

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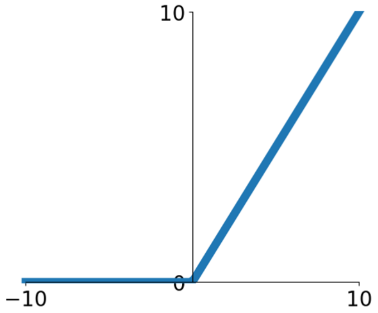
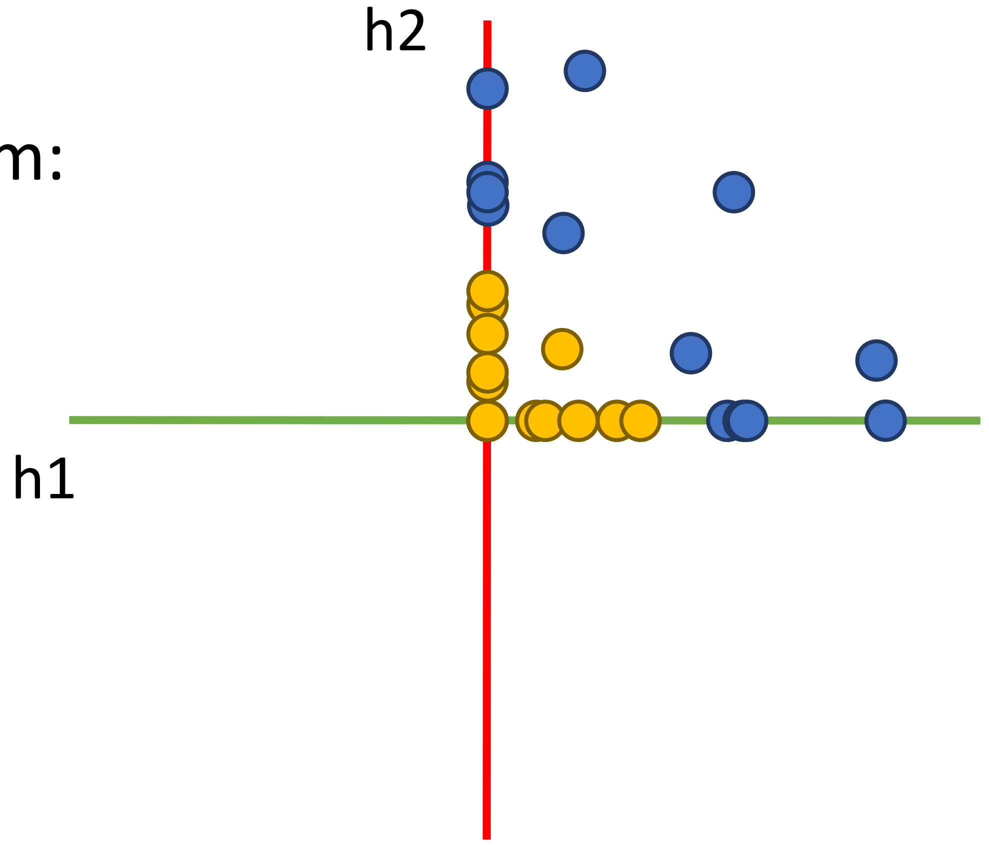
h = ReLU(Wx) = max(0, Wx)

Where x, h are both 2-dimensional

Feature transform:

h = ReLU(Wx)





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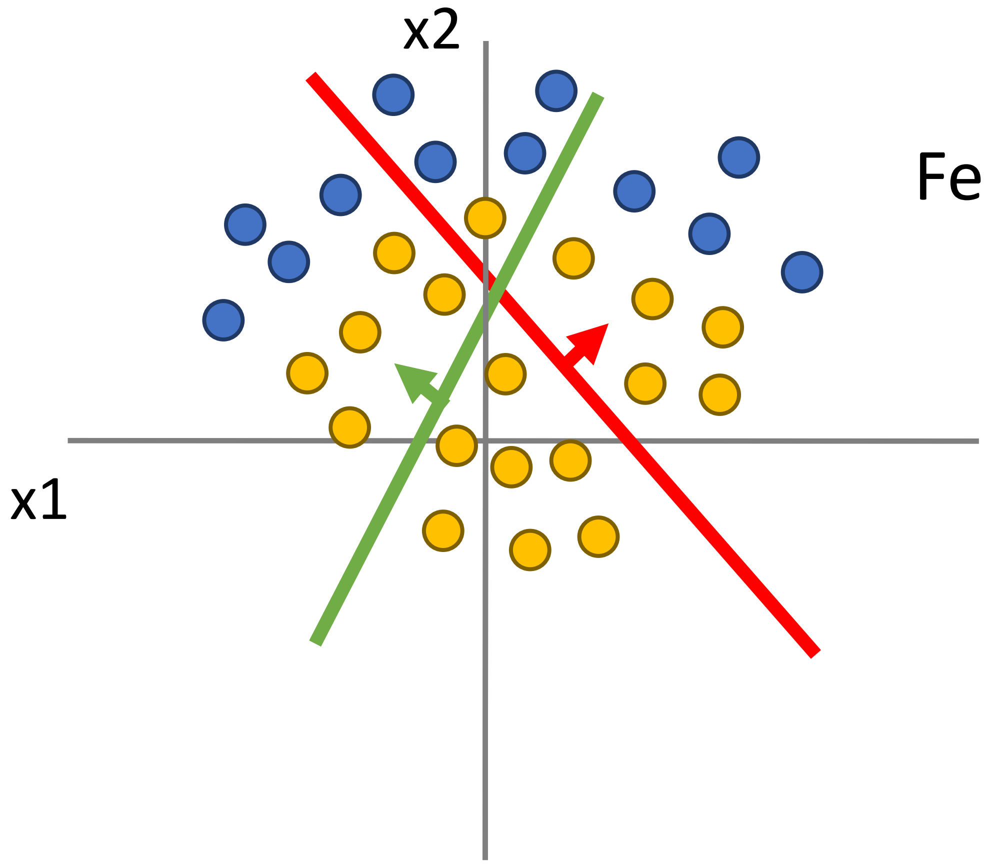


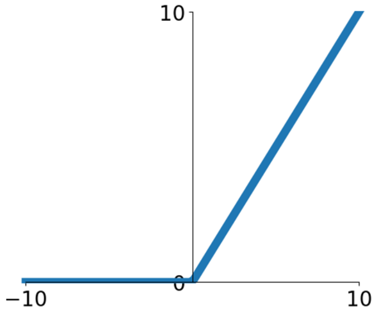
Consider a neural net hidden layer:

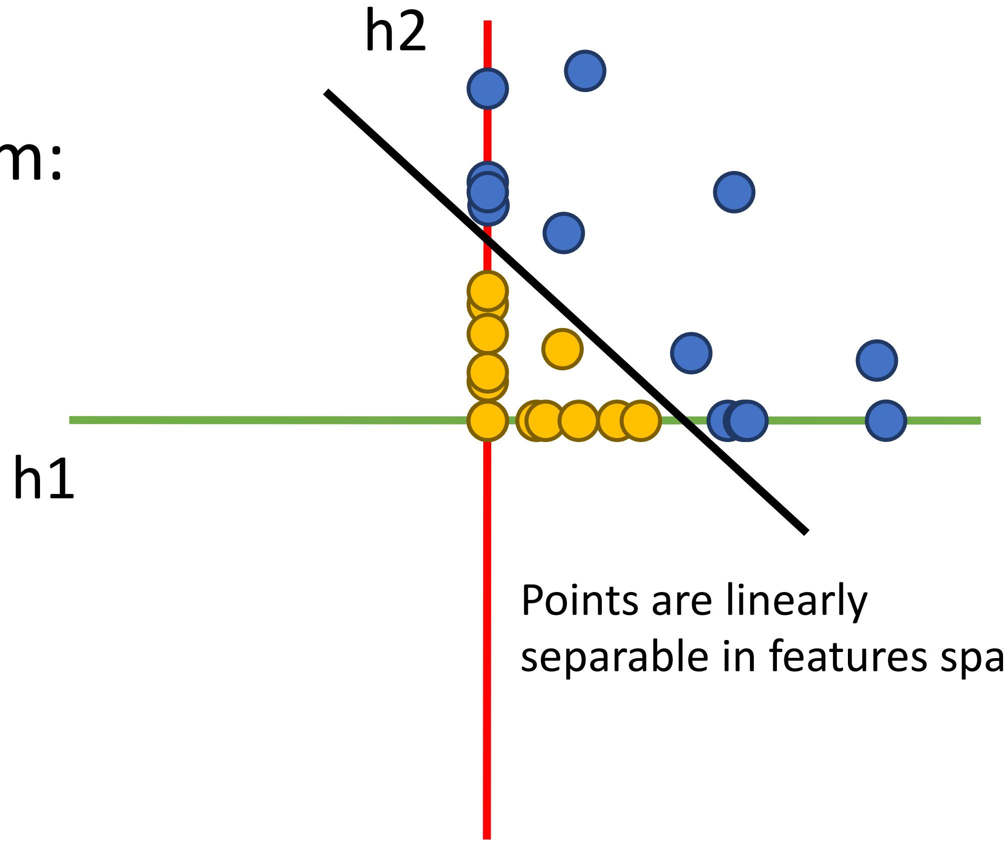
$h = \text{ReLU}(Wx) = \max(0, Wx)$

Where x, h are both 2-dimensional

Feature transform:

$h = \text{ReLU}(Wx)$





Points are linearly separable in features space!

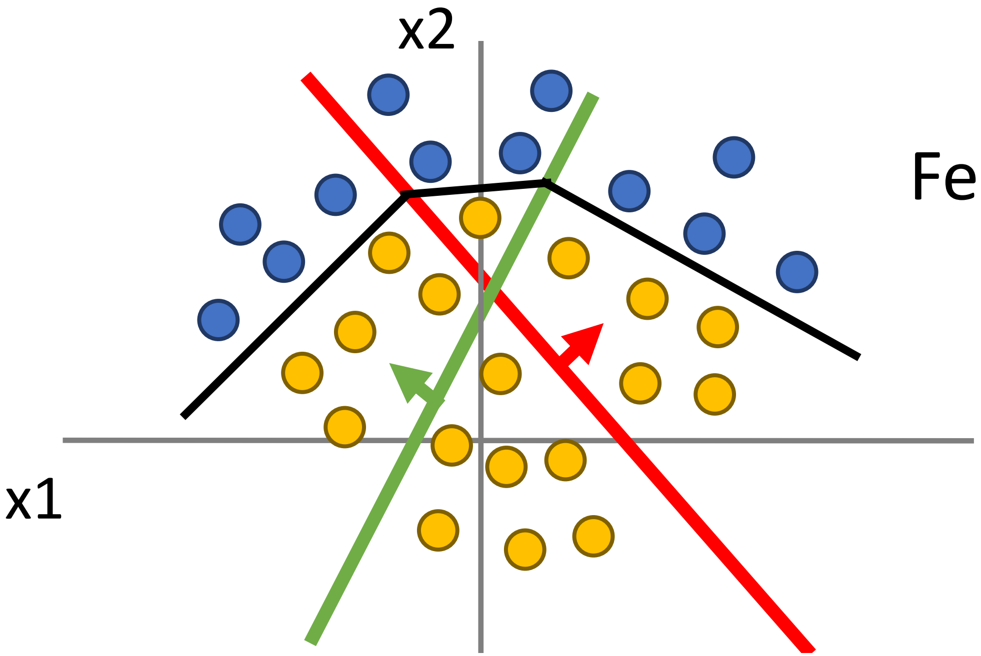
Space Warping

Consider a neural net hidden layer:

$h = \text{ReLU}(Wx) = \max(0, Wx)$

Where x, h are both 2-dimensional

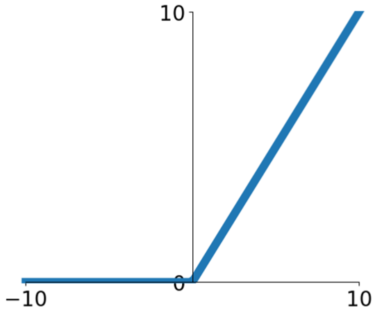
Points not linearly separable in original space

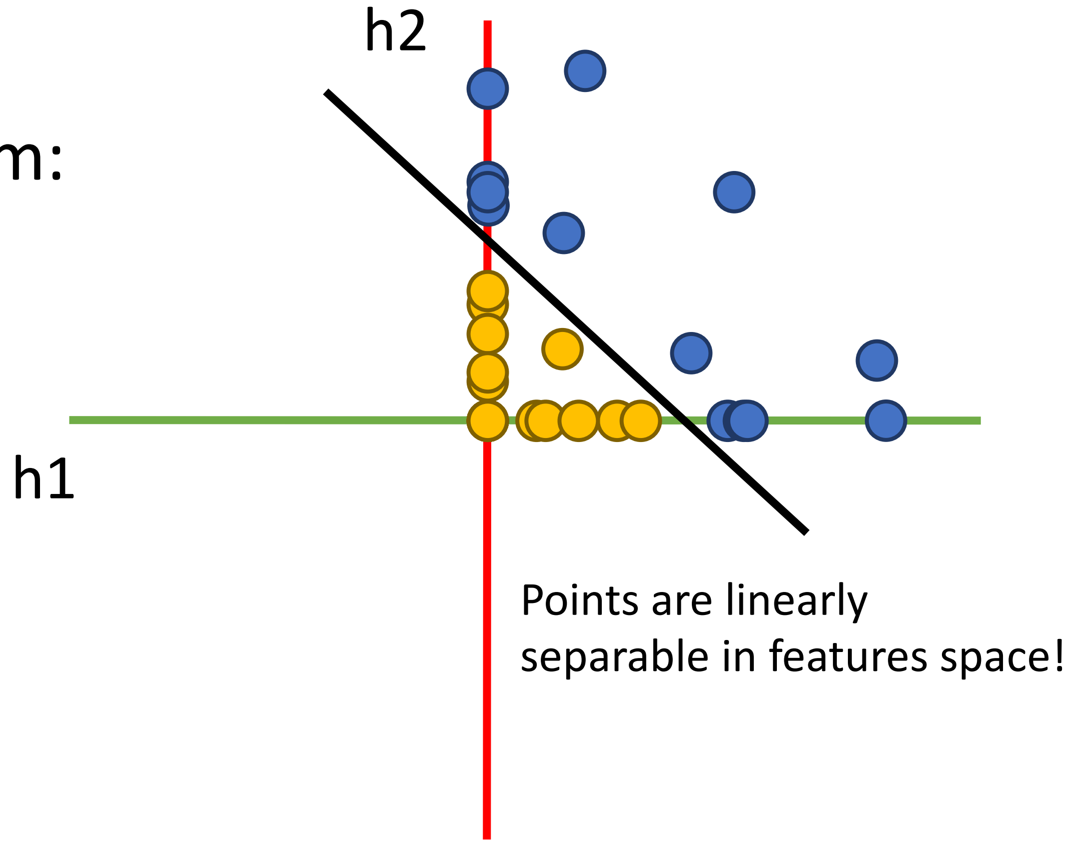


Linear classifier in feature space gives nonlinear classifier in original space

Feature transform:

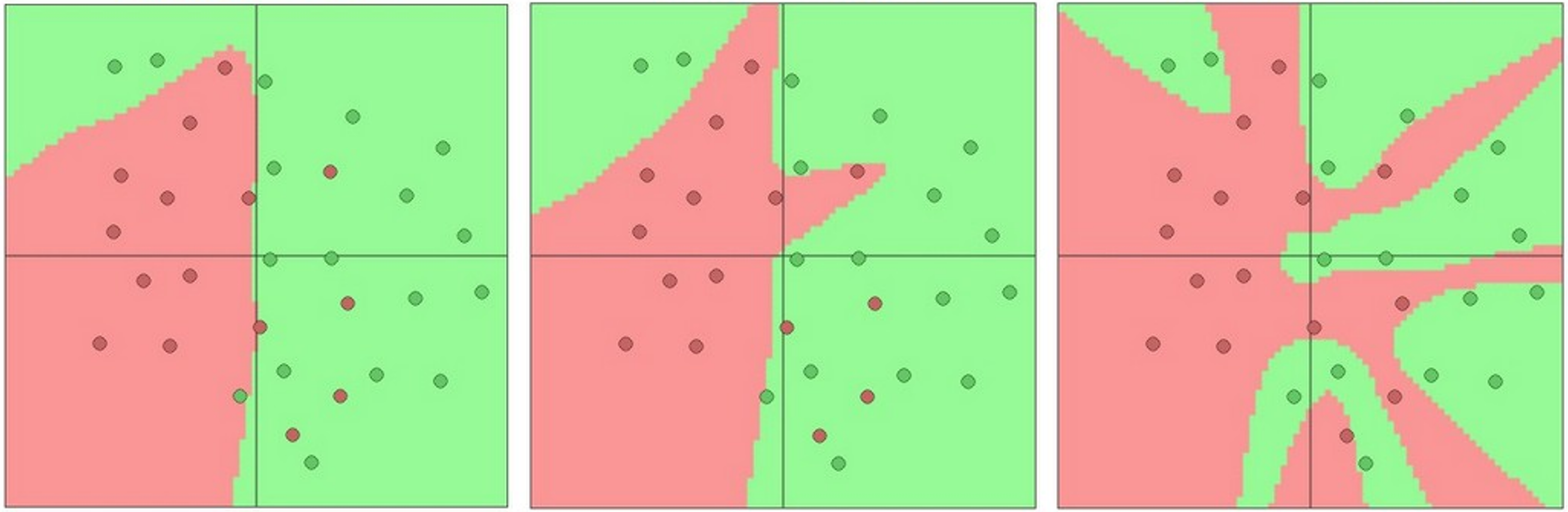
$h = \text{ReLU}(Wx)$





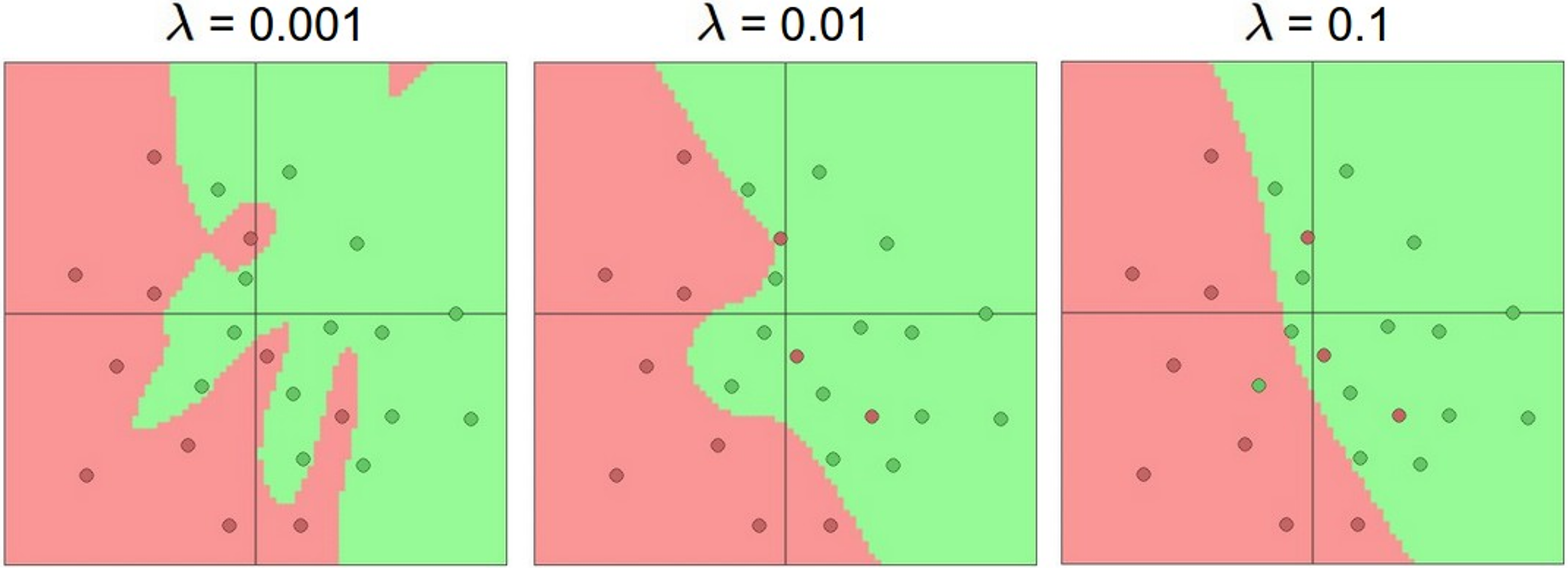
Setting the number of layers and their sizes

3 hidden units | 6 hidden units | 20 hidden units



More hidden units = more capacity

Don’t regularize with size; instead use stronger L2



(Web demo with ConvNetJS:
http://cs.stanford.edu/people/karpathy/convnetjs/demo/classify2d.html)

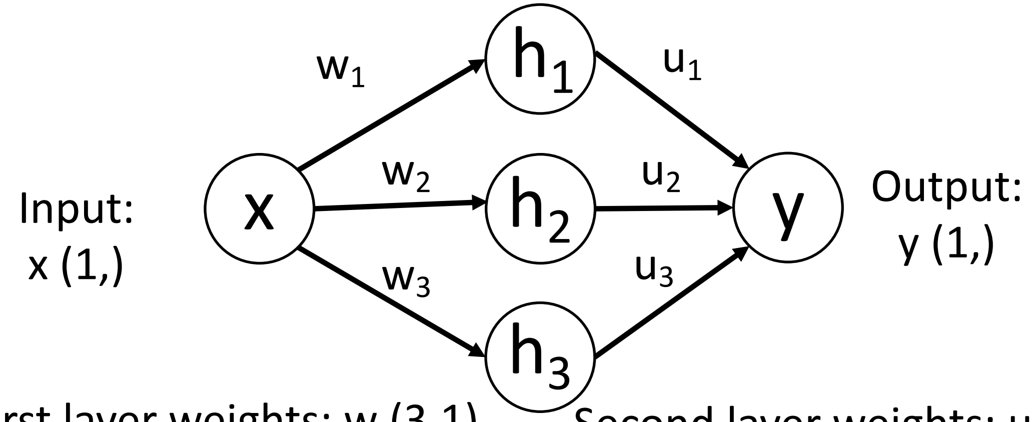
Universal Approximation

A neural network with one hidden layer can approximate any function f: R^N -> R^M with arbitrary precision*

*Many technical conditions: Only holds on compact subsets of R^N; function must be continuous; need to define “arbitrary precision”; etc

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

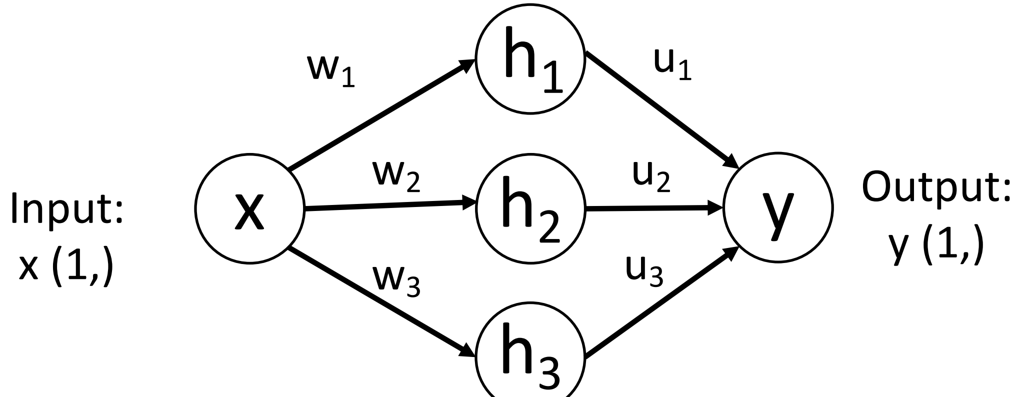


First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



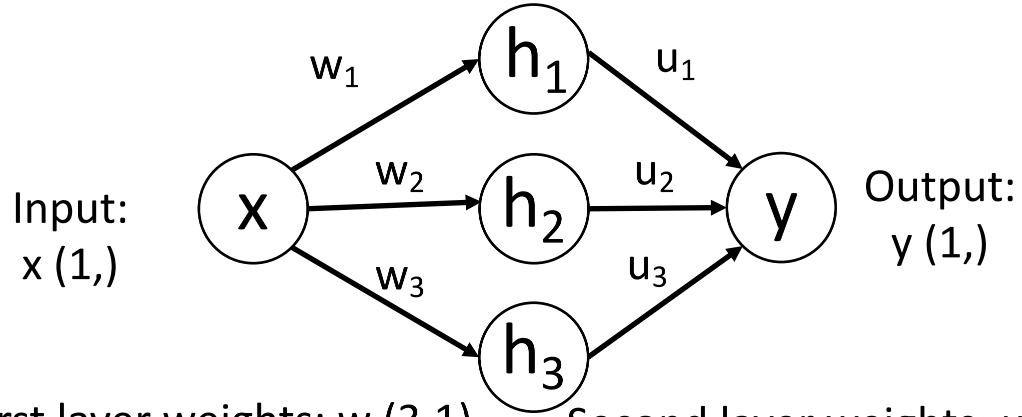
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



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h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
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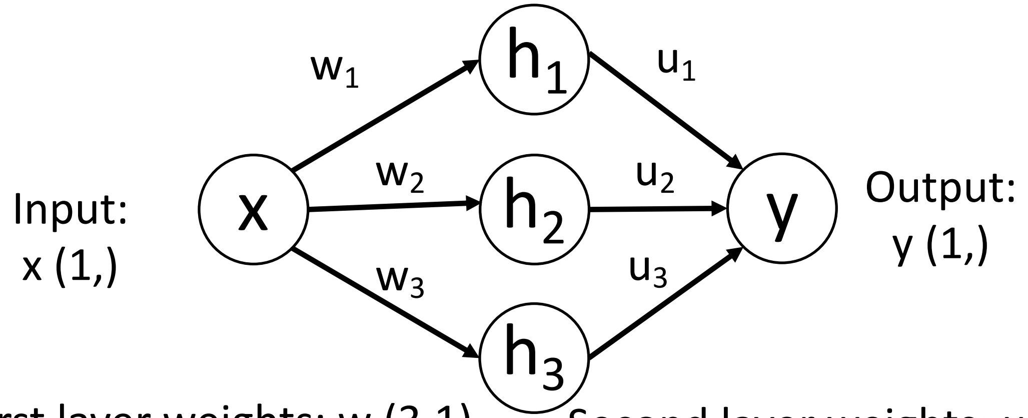
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

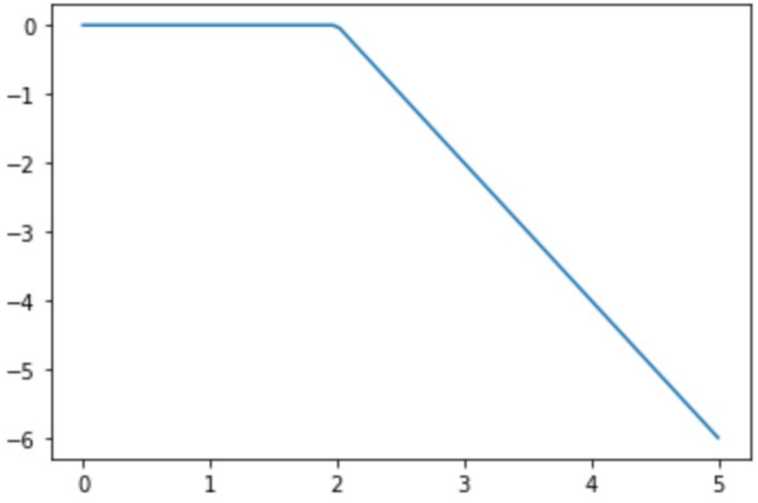
Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

Output is a sum of shifted, scaled ReLUs:

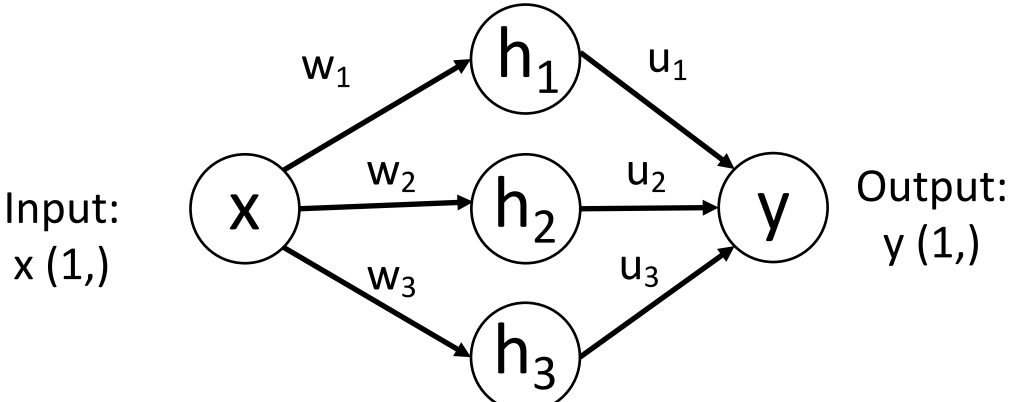


h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
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Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
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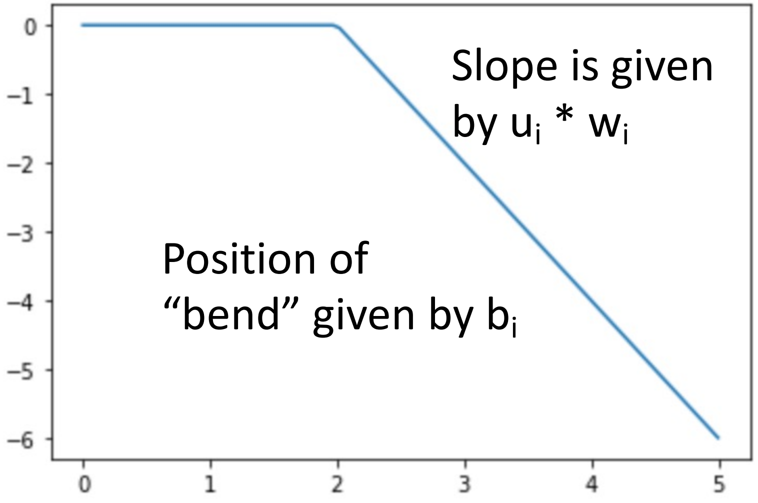
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

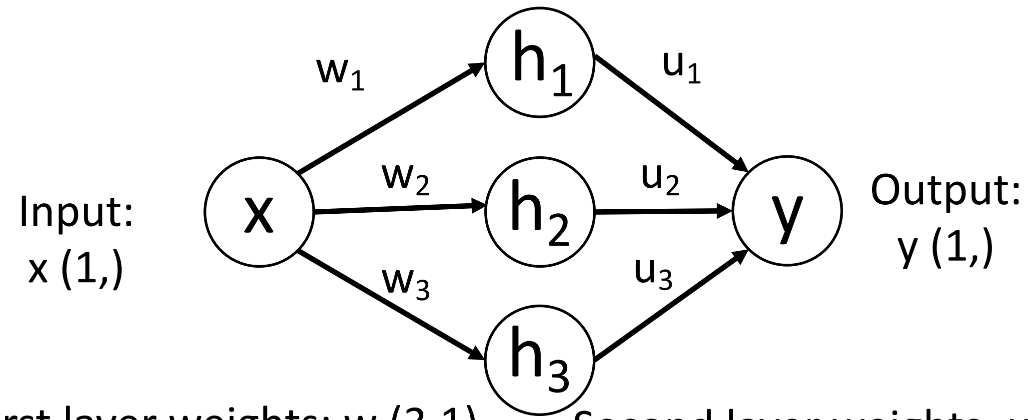
Output is a sum of shifted, scaled ReLUs:

Flip left / right based on sign of w_i



Universal Approximation

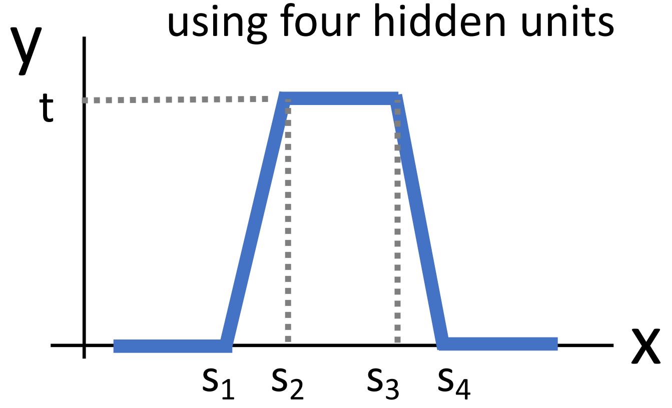
Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

We can build a "bump function" using four hidden units



h1 = max(0, w1 * x + b1)
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h3 = max(0, w3 * x + b3)
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Universal Approximation

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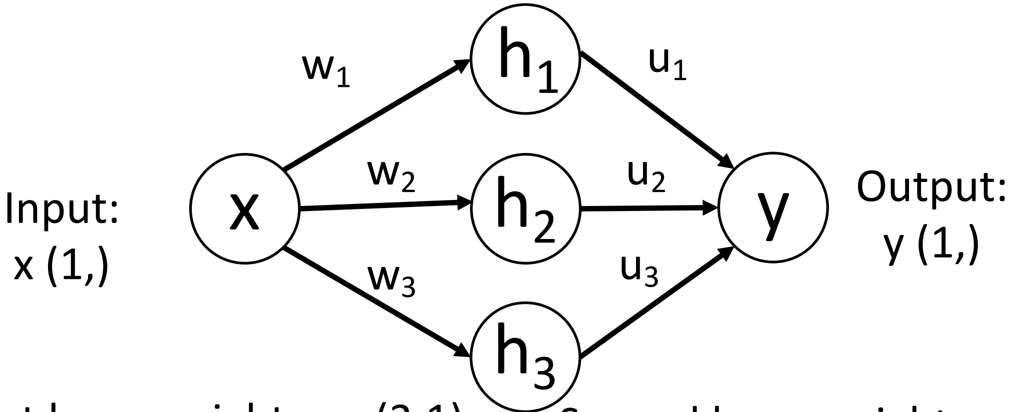

First layer weights: w (3,1)
First layer bias: b (3,)

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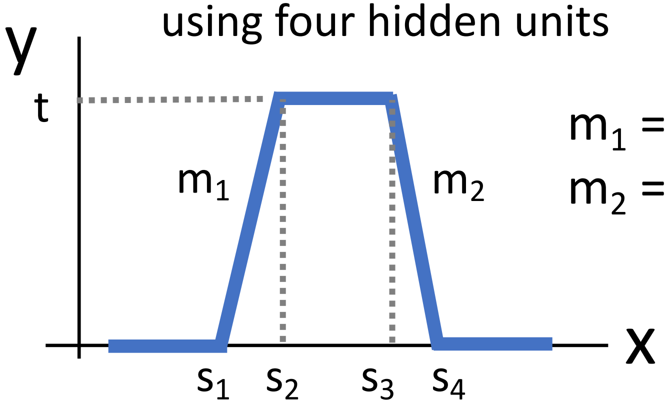
First layer bias: p (1,)

We can build a “bump function” using four hidden units



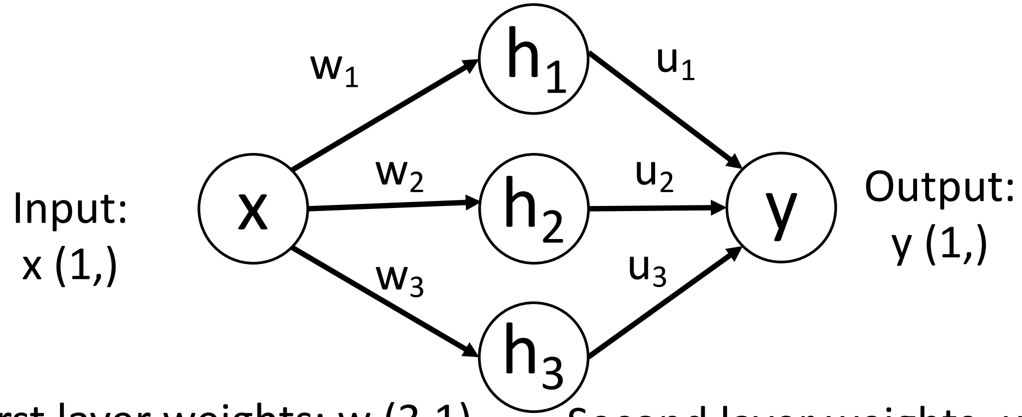
$m_1 = t / (s_2 – s_1)$
$m_2 = t / (s_4 – s_3)$

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

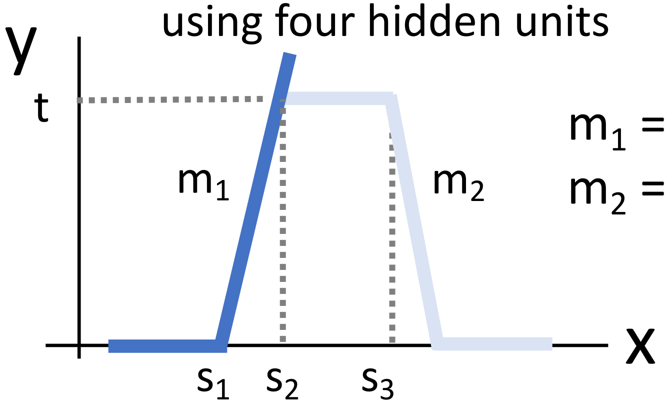
Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
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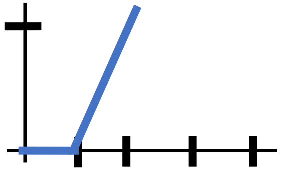
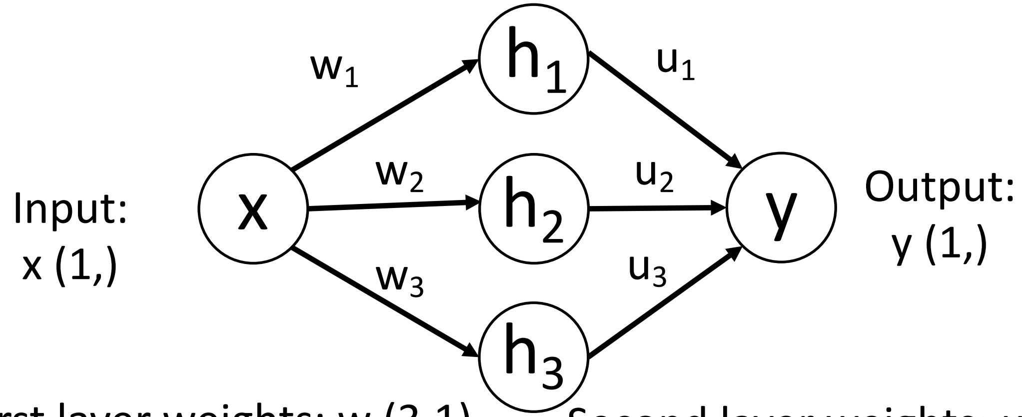
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p



$m_1 * \max(0, x - s_1)$

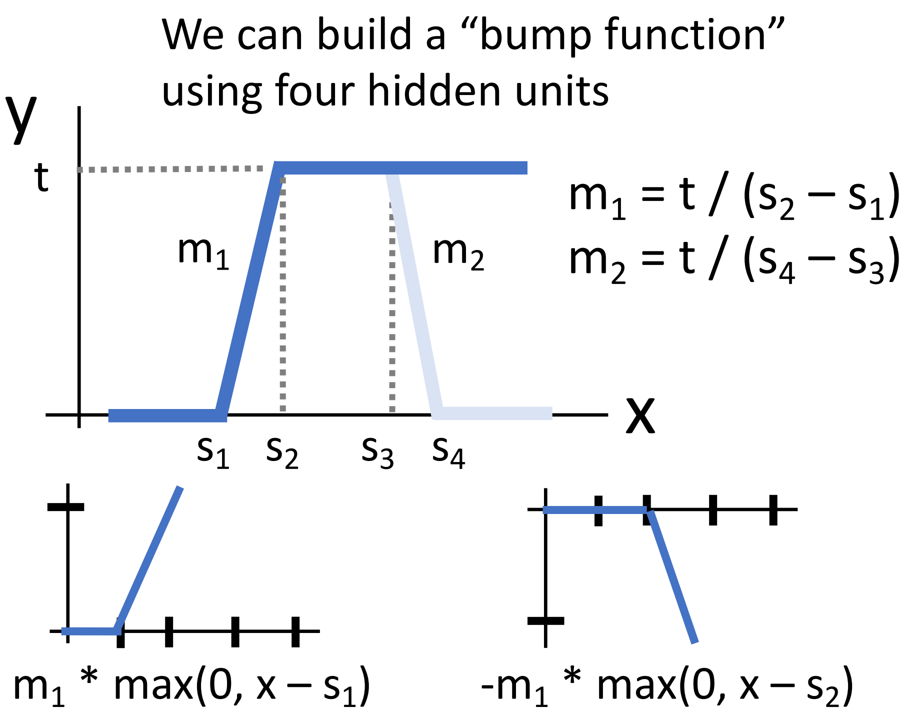
Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)



h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
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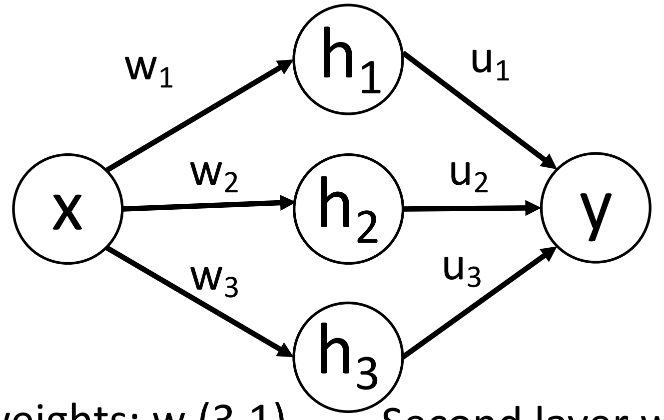
+ p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

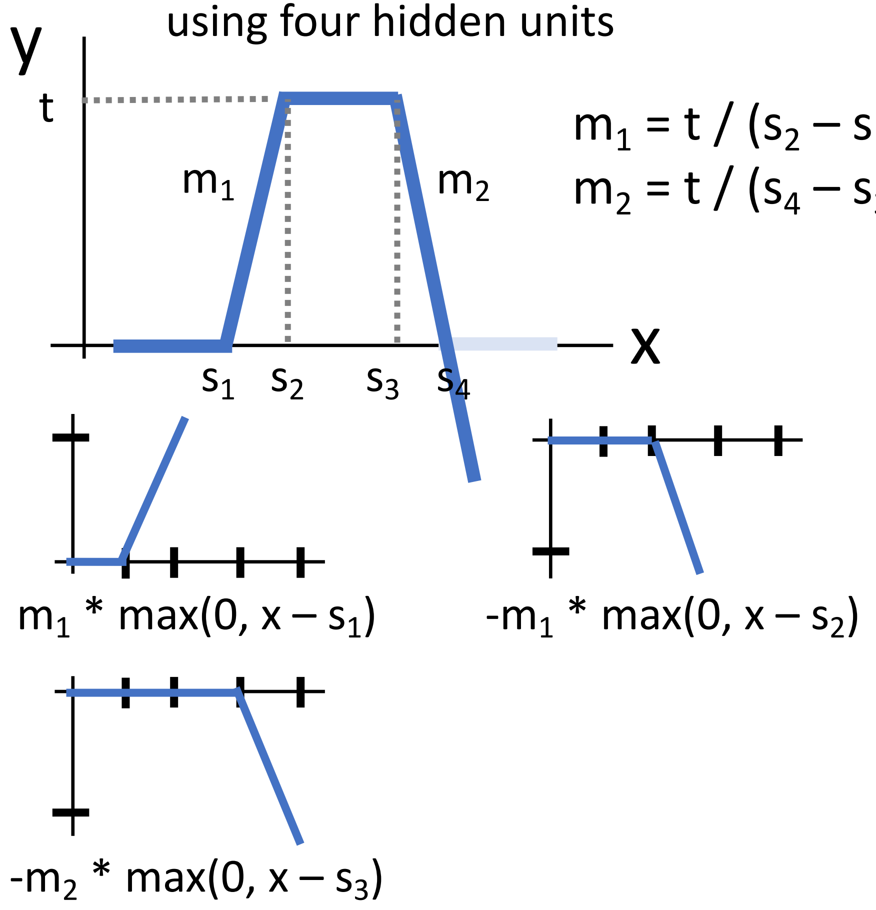
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

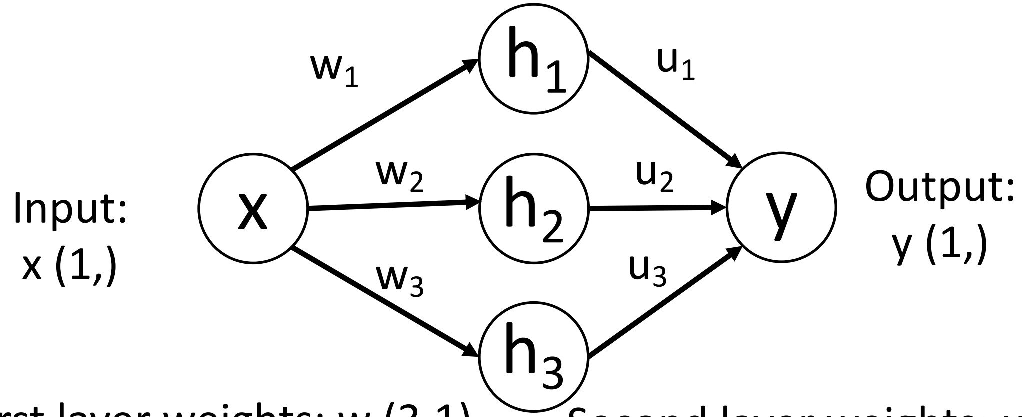
We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
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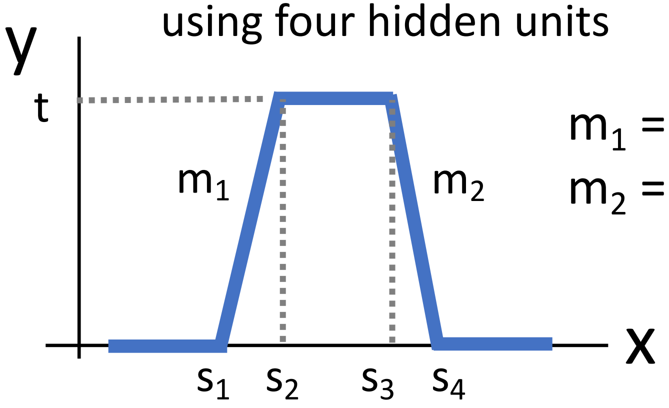
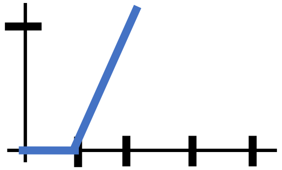
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

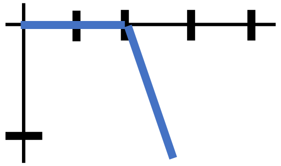
We can build a "bump function" using four hidden units



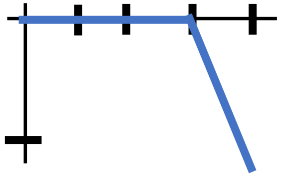
$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$



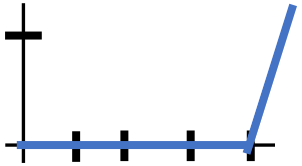
$m_1 * \max(0, x - s_1)$



$-m_1 * \max(0, x - s_2)$



$-m_2 * \max(0, x - s_3)$

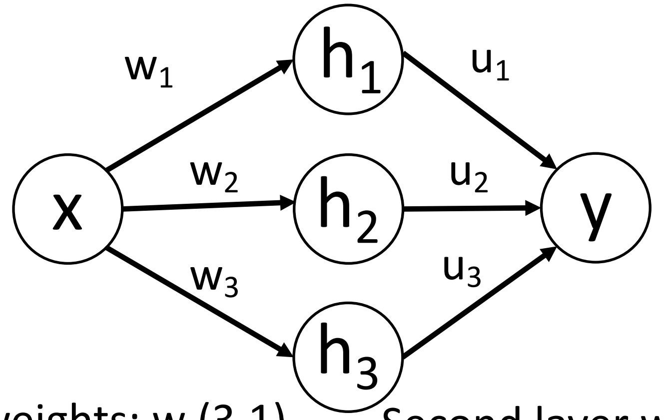


$m_2 * \max(0, x - s_4)$

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

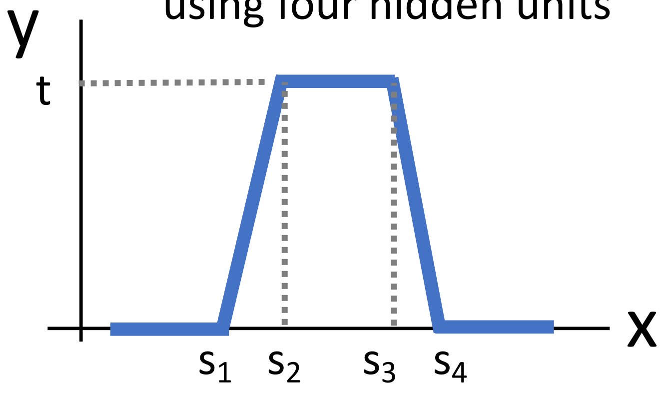
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

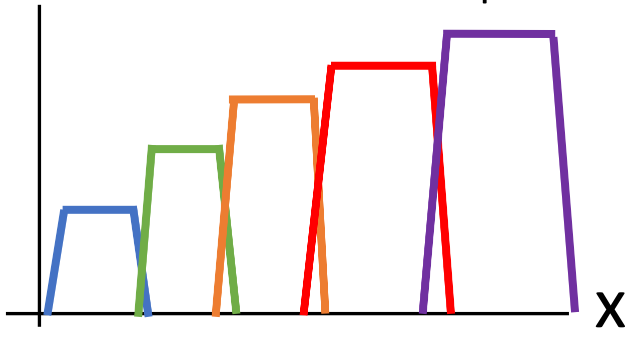
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

We can build a "bump function" using four hidden units



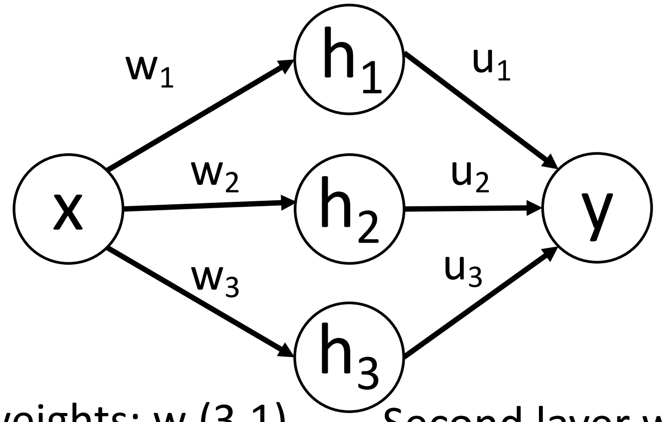
With 4K hidden units we can build a sum of K bumps



Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

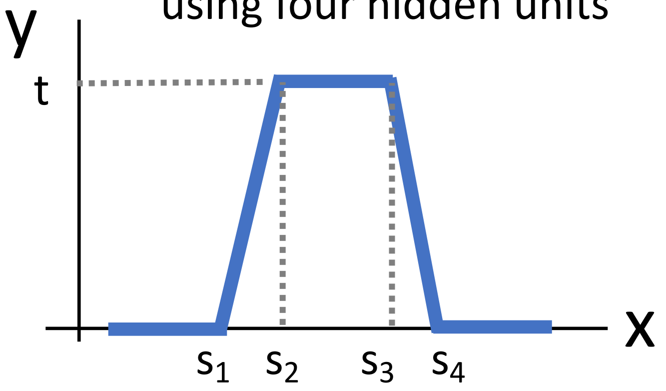
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

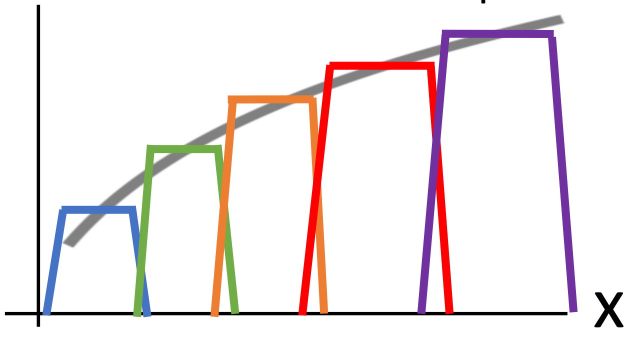
$$h1 = \max(0, w1 * x + b1)$$
$$h2 = \max(0, w2 * x + b2)$$
$$h3 = \max(0, w3 * x + b3)$$
$$y = u1 * h1 + u2 * h2 + u3 * h3 + p$$

$$y = u1 * \max(0, w1 * x + b1) + u2 * \max(0, w2 * x + b2) + u3 * \max(0, w3 * x + b3) + p$$

We can build a "bump function" using four hidden units



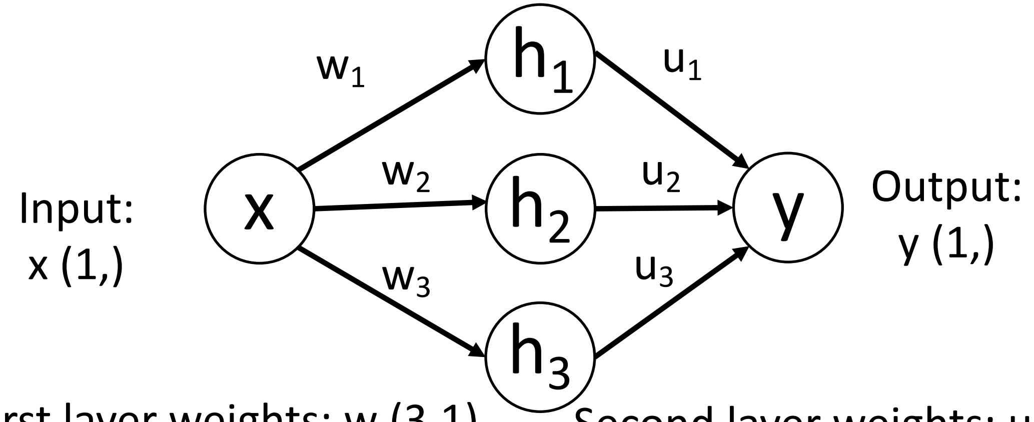
With 4K hidden units we can build a sum of K bumps



Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



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h1 = max(0, w1 * x + b1)
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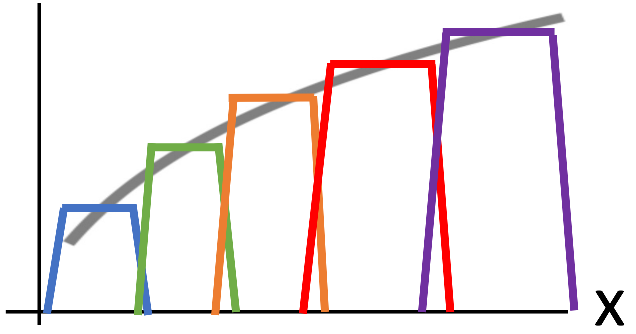
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

What about…
- Gaps between bumps?
- Other nonlinearities?
- Higher-dimensional functions?

See [Nielsen, Chapter 4]

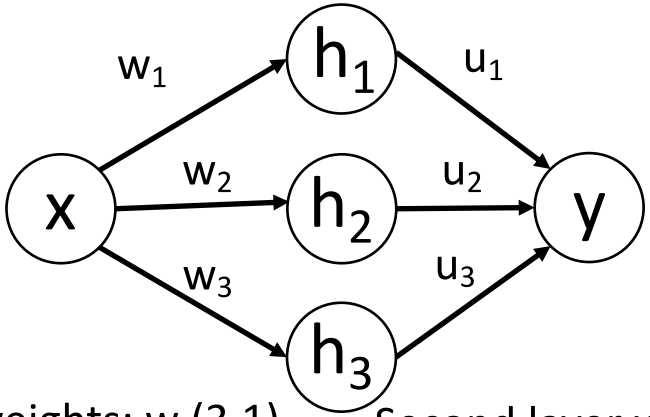


Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)

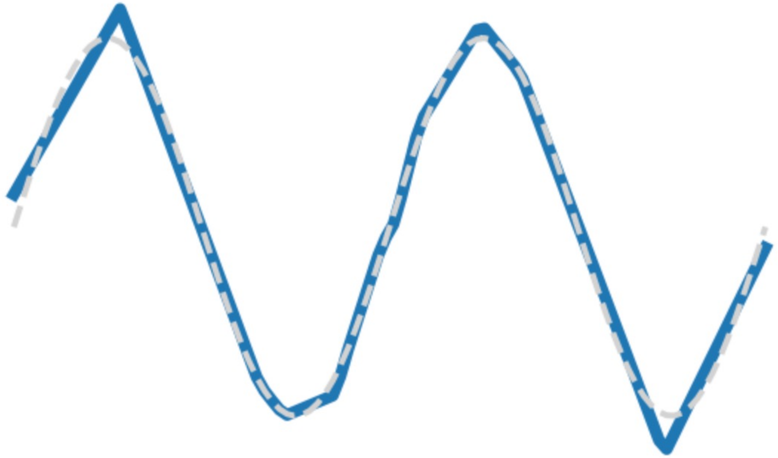


Output:
y (1,)

First layer weights: w (3,1)
First layer bias: b (3,)

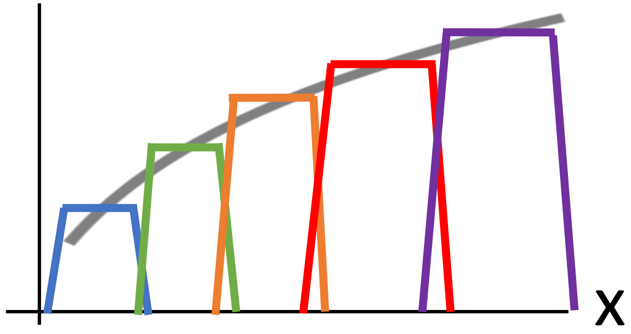
Second layer weights: u (1,3)
First layer bias: p (1,)

Reality check: Networks don't really learn bumps!



h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

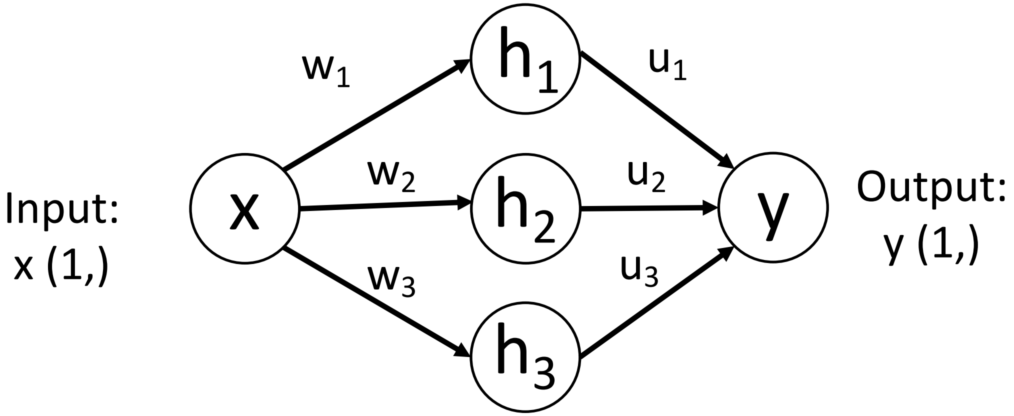
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p



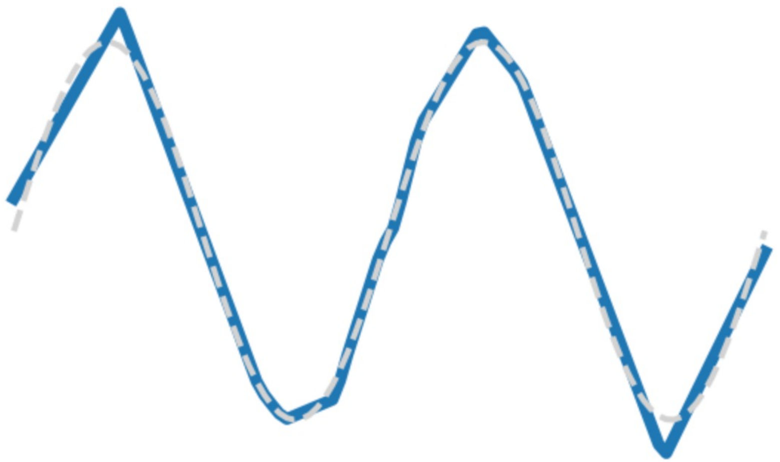
Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



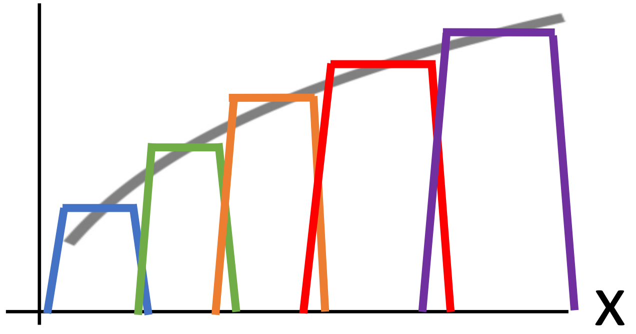
Reality check: Networks don't really learn bumps!



Universal approximation tells us:
- Neural nets can represent any function

Universal approximation DOES NOT tell us:
- Whether we can actually learn any function with SGD
- How much data we need to learn a function

Remember: kNN is also a universal approximator!



Approximate functions with bumps!

Extra topic
(Won’t be on HW / Exam)

Convex Functions

A function $f : X \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0, 1]$,

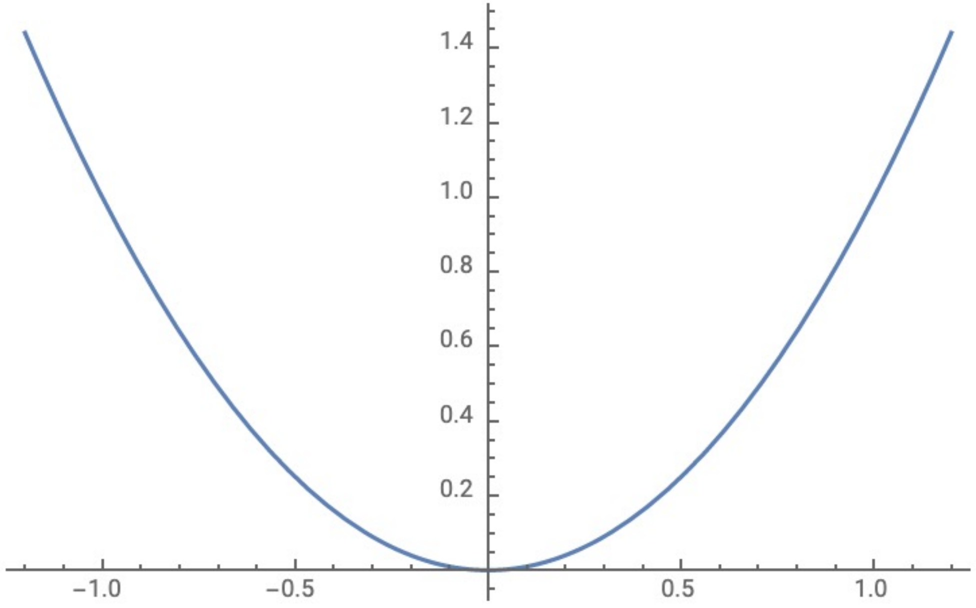
$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Convex Functions

A function $f : X \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Example: $f(x) = x^2$ is convex:

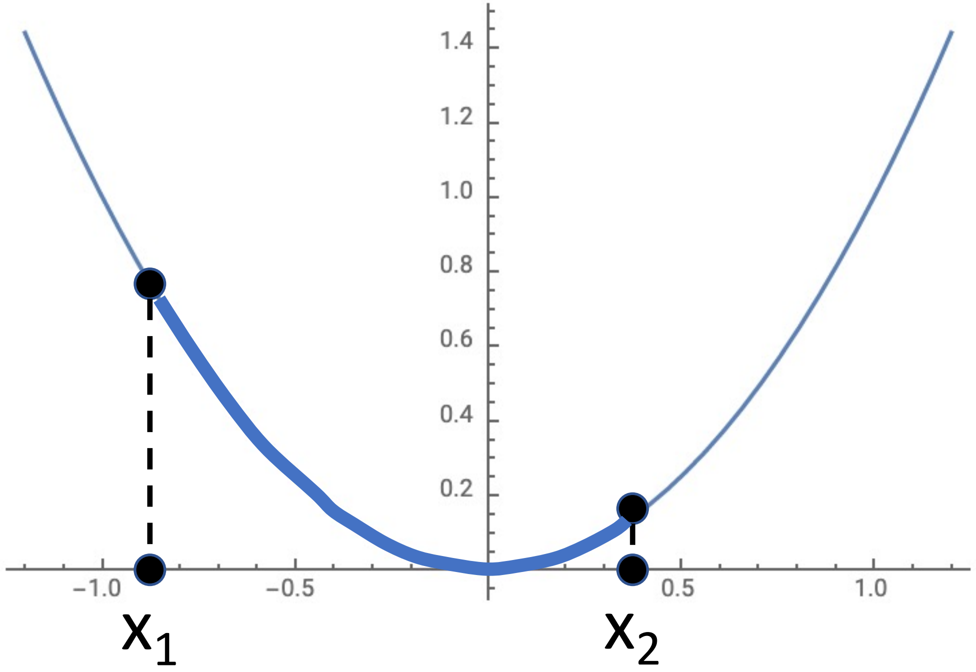


Convex Functions

A function $f : X \subseteq \mathbb{R}^N \to \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Example: $f(x) = x^2$ is convex:

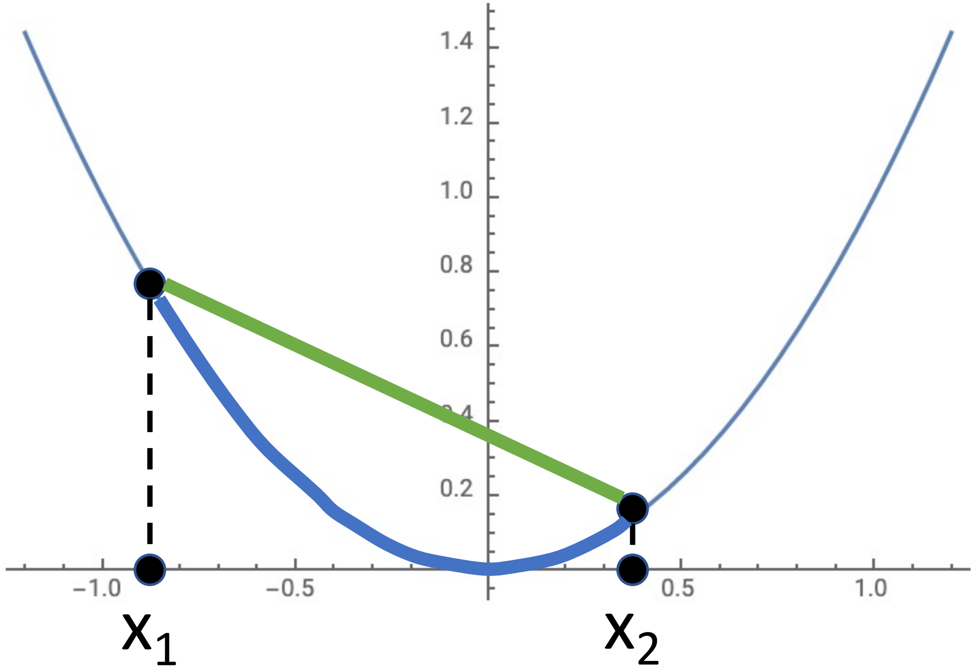


Convex Functions

A function $f : X \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Example: $f(x) = x^2$ is convex:



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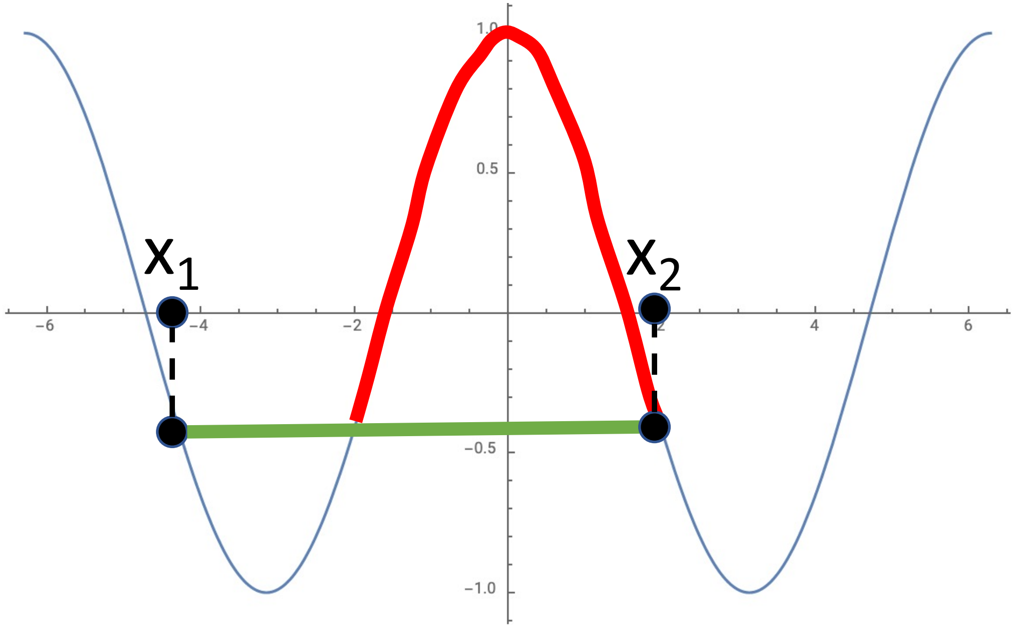
Example: $f(x) = \cos(x)$
is not convex:



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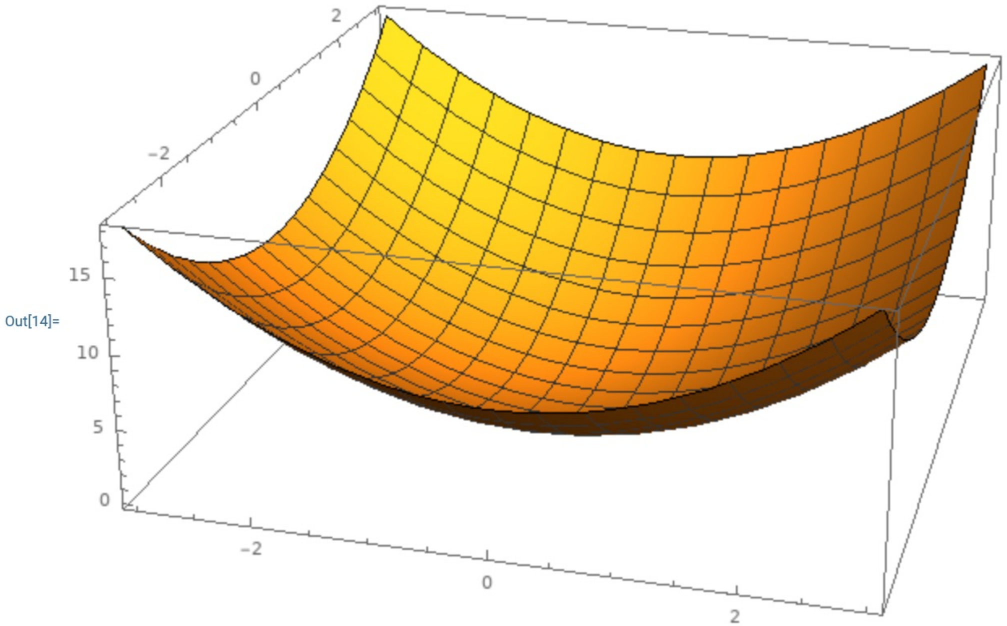
Intuition: A convex function is a (multidimensional) bowl



*Many technical details! See e.g. IOE 661 / MATH 663

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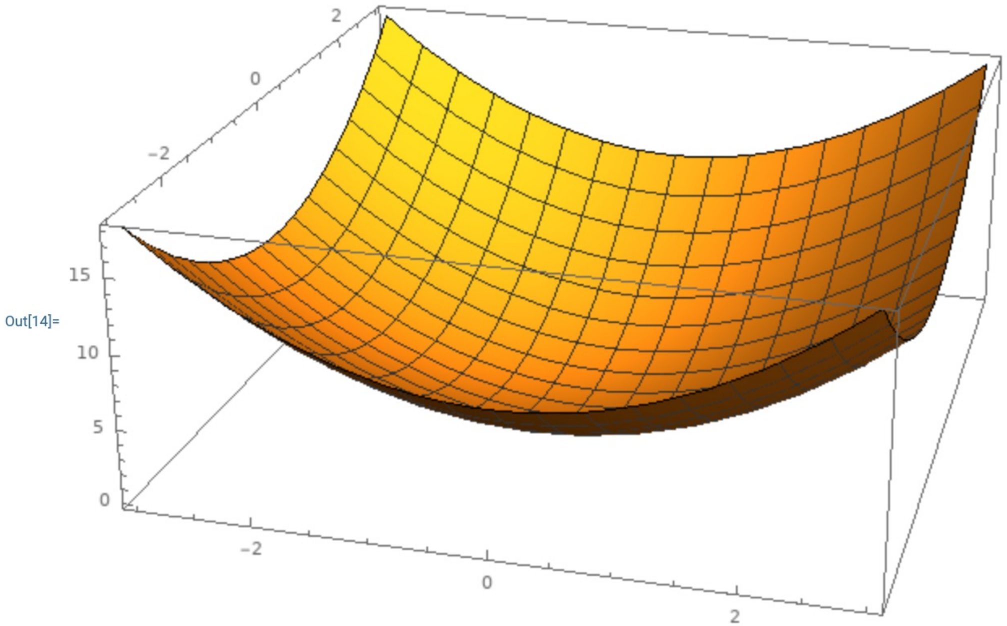
Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum***



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Linear classifiers optimize a **convex function!**

$$s = f(x; W) = Wx$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$ Softmax

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ SVM

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$

R(W) = L2 or L1 regularization

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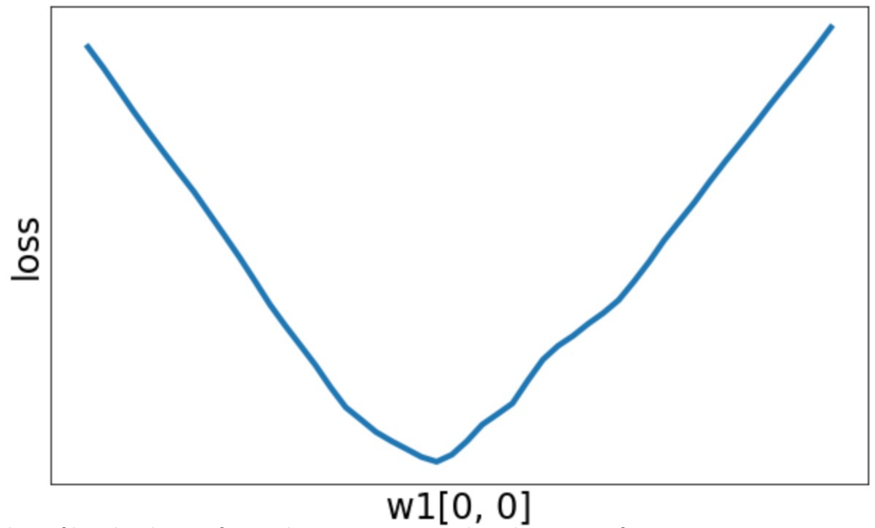
Neural net losses sometimes look convex-ish:



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

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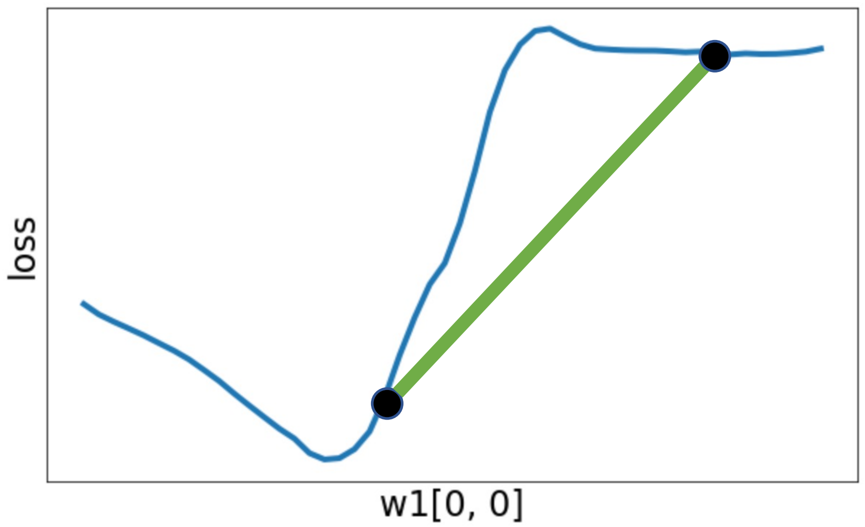
But often clearly nonconvex:



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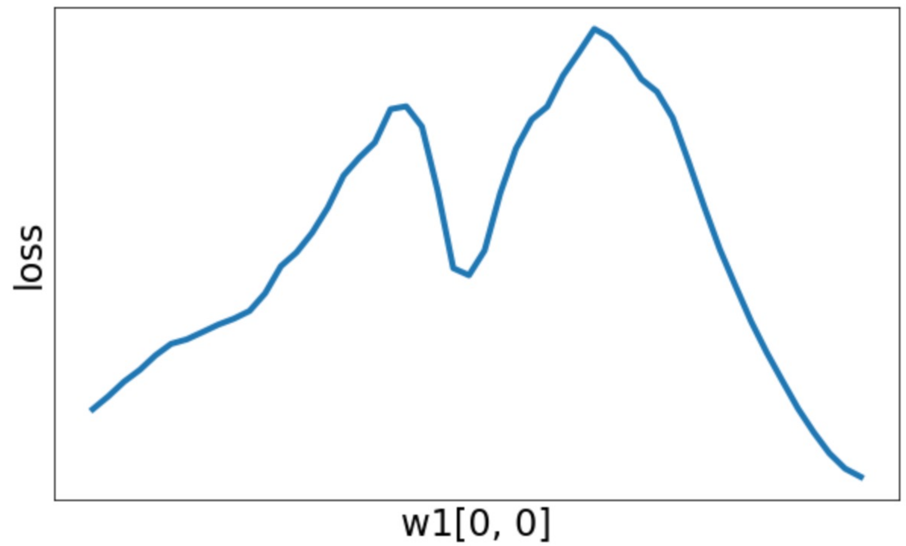
With local minima:



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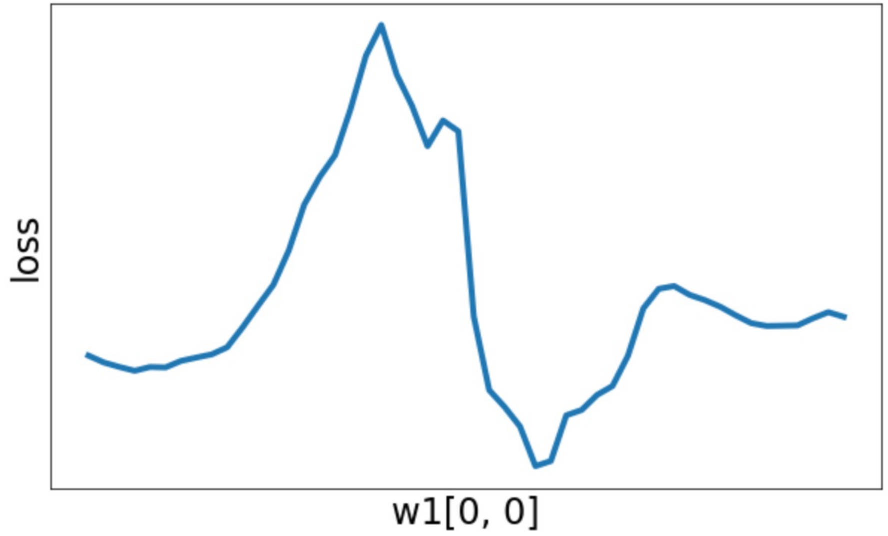
Can get very wild!



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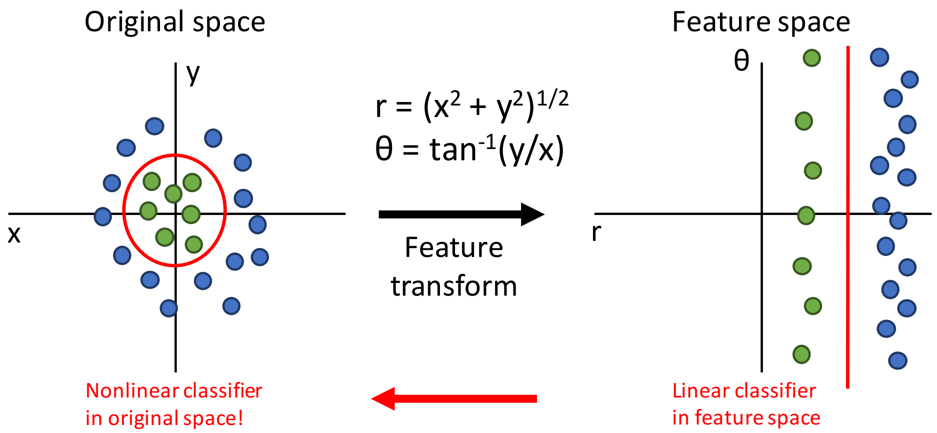
Most neural networks need **nonconvex optimization**

- Few or no guarantees about convergence
- Empirically it seems to work anyway
- Active area of research

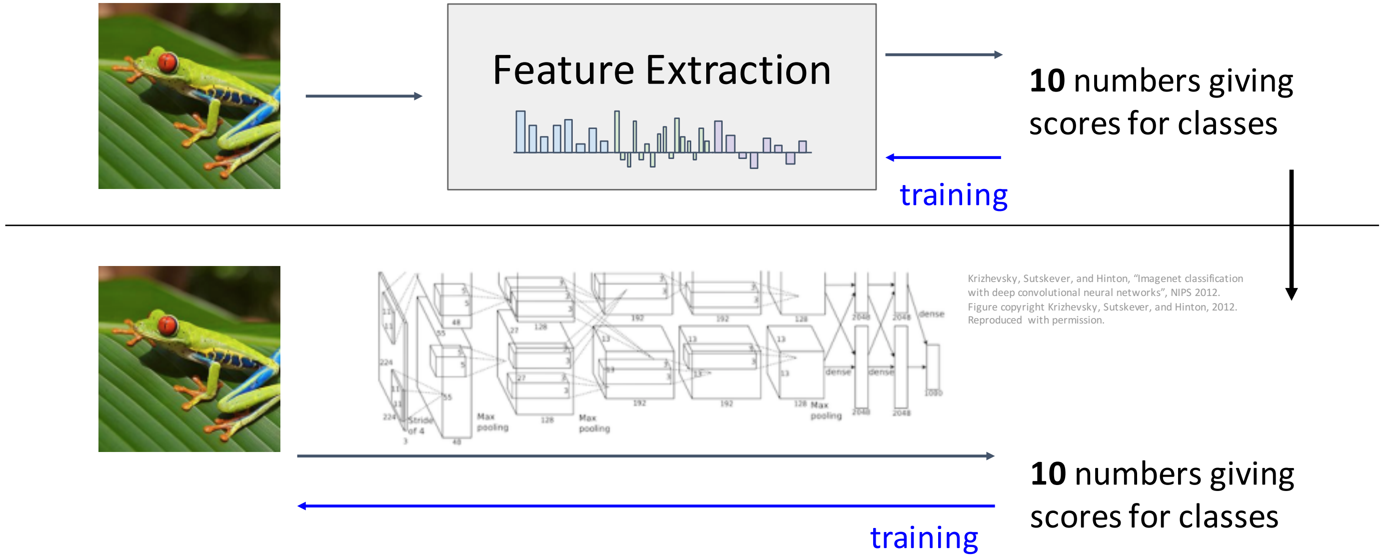
*Many technical details! See e.g. IOE 661 / MATH 663

Summary

Feature transform + Linear classifier allows nonlinear decision boundaries



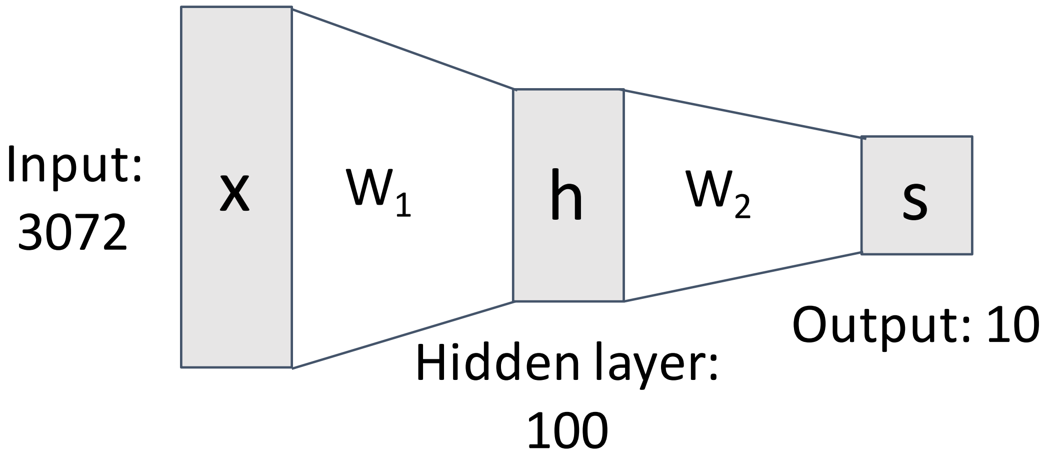
Neural Networks as learnable feature transforms



Summary

From linear classifiers to
fully-connected networks

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$



Linear classifier: One template per class

Neural networks: Many reusable templates

Summary

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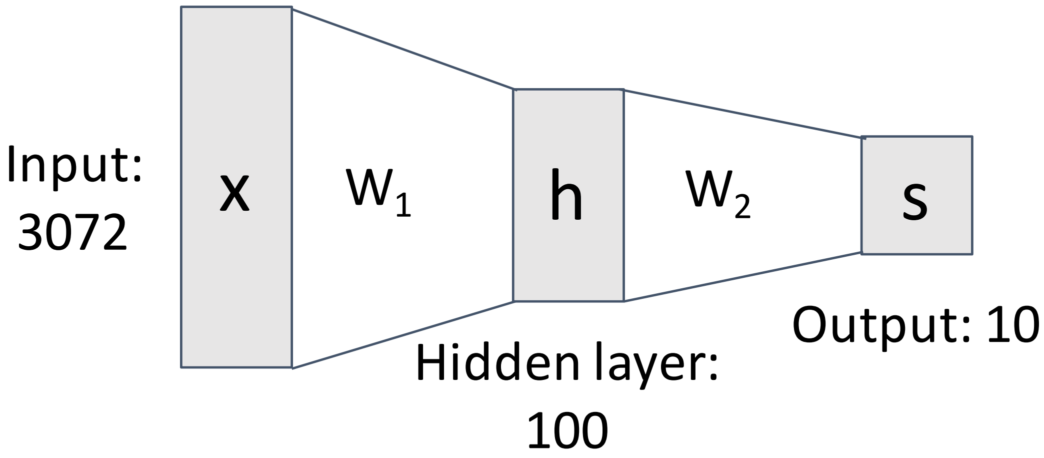
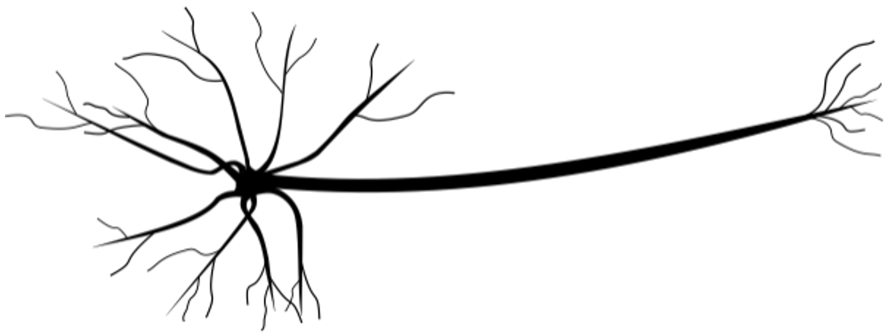


Neural networks loosely inspired by biological neurons but be careful with analogies



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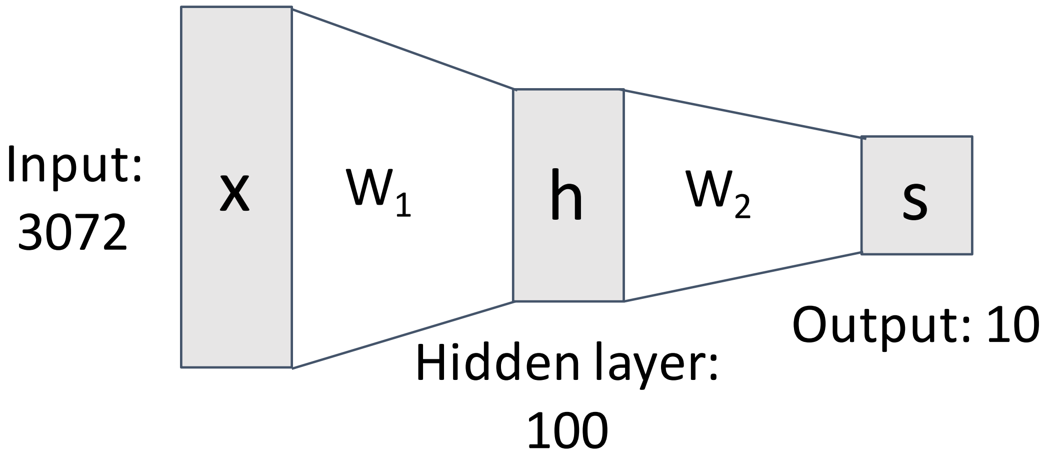
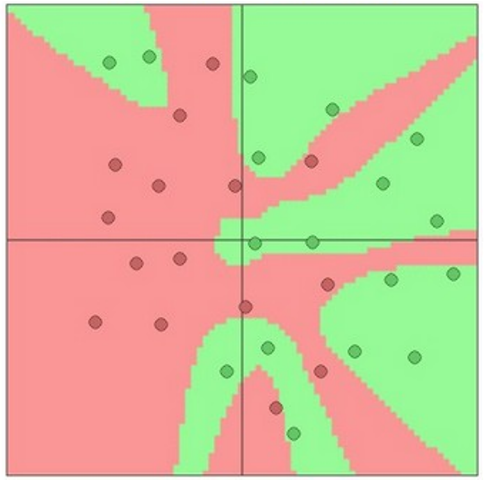
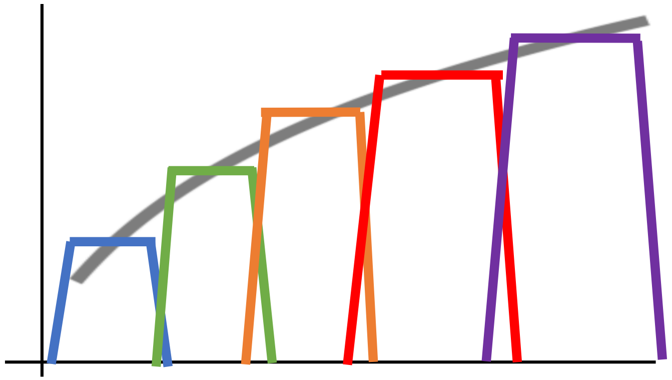


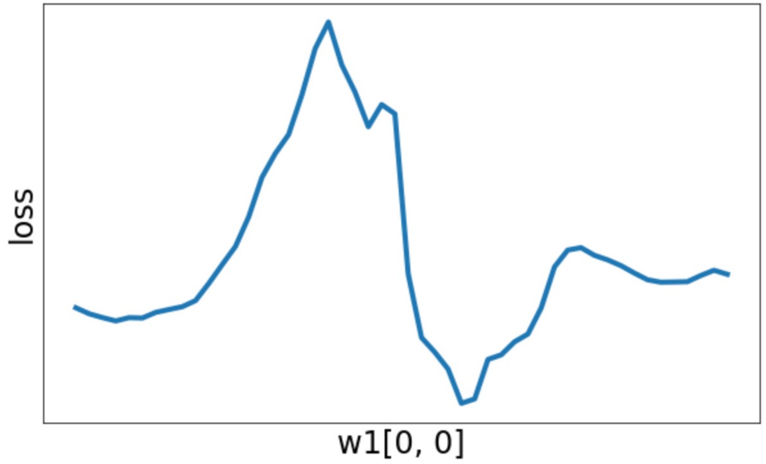
Space Warping



Universal Approximation



Nonconvex



Problem: How to compute gradients?

$s = W_2 \max(0, W_1 x + b_1) + b_2$ Nonlinear score function

$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$ Per-element data loss

$R(W) = \sum_k W_k^2$ L2 Regularization

$L(W_1, W_2, b_1, b_2) = \frac{1}{N} \sum_{i=1}^{N} L_i + \lambda R(W_1) + \lambda R(W_2)$ Total loss

If we can compute $\frac{\partial L}{\partial W_1}, \frac{\partial L}{\partial W_2}, \frac{\partial L}{\partial b_1}, \frac{\partial L}{\partial b_2}$ then we can optimize with SGD

Next time:
Backpropagation