Lecture 3:
Linear Classifiers

Reminder: Assignment 1

- http://web.eecs.umich.edu/~justincj/teaching/eecs498/assignment1.html
- Due **Sunday September 15, 11:59pm EST**
- We have written a **homework validation script** to check the format of your .zip file before you submit to Canvas:
- https://github.com/deepvision-class/tools#homework-validation
- This script ensures that your .zip and .ipynb files are properly structured; they do not check correctness
- It is **your responsibility** to make sure your submitted .zip file passes the validation script

Last time: Image Classification

Input: image

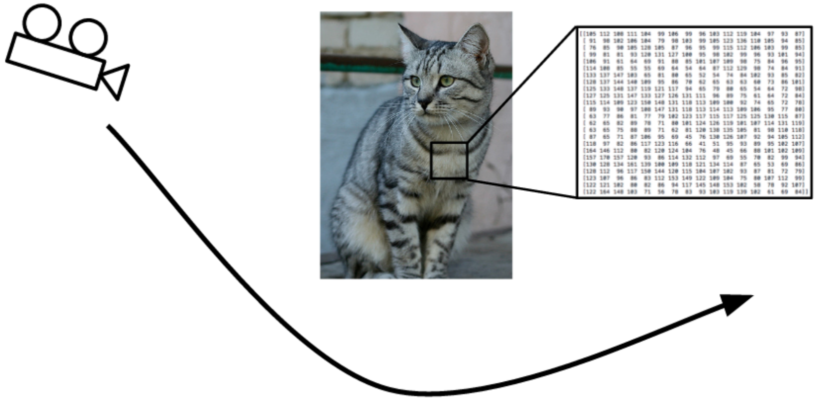
This image by Nikita is licensed under CC-BY 2.0

Output: Assign image to one of a fixed set of categories

cat
bird
deer
dog
truck

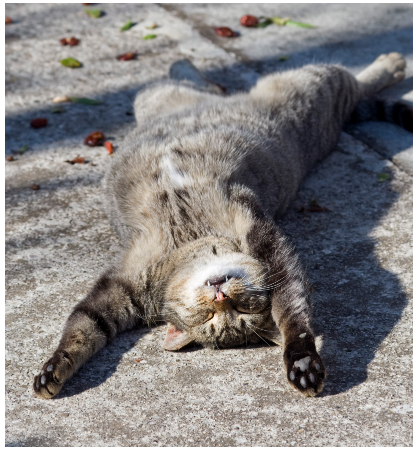
Last Time: Challenges of Recognition

Viewpoint



Illumination

Deformation



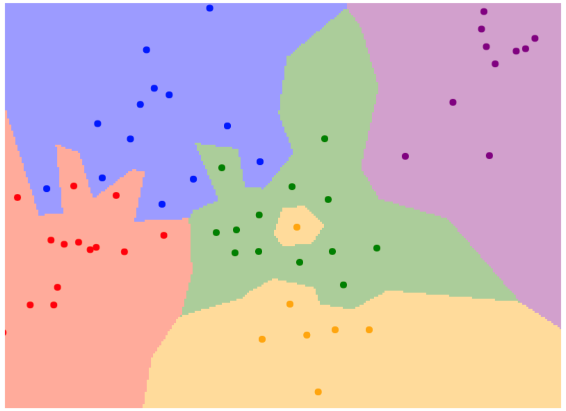
Occlusion

Clutter

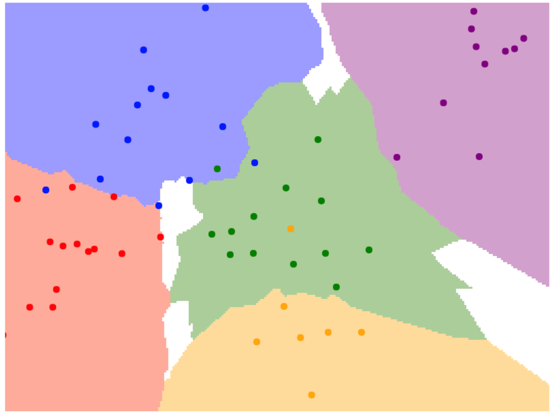
Intraclass Variation

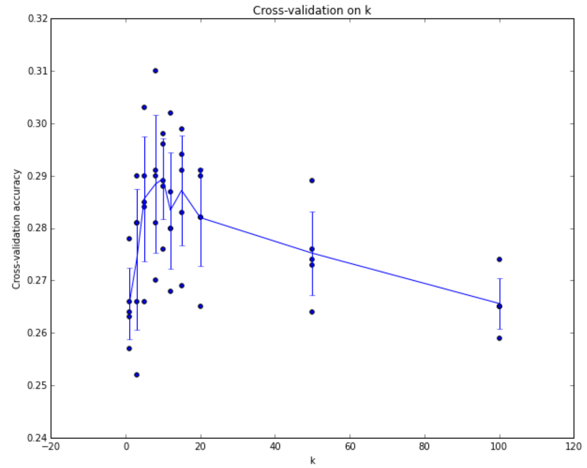
Last time: Data-Drive Approach, kNN

1-NN classifier



5-NN classifier





train	test

train	validation	test

Today: Linear Classifiers

Neural Network

Linear classifiers

This image is CC0 1.0 public domain

Recall CIFAR10

50,000 training images
each image is **32x32x3**

10,000 test images.

Parametric Approach

Image

Array of **32x32x3** numbers
(3072 numbers total)

f(x,W)

10 numbers giving class scores

W
parameters or weights

Parametric Approach: Linear Classifier

$$f(x,W) = Wx$$

Image

Array of **32x32x3** numbers
(3072 numbers total)

f(x,W)

10 numbers giving
class scores

W
parameters
or weights

Parametric Approach: Linear Classifier

$$\boxed{f(x,W)} = \boxed{W}\boxed{x}$$

(3072,)

(10,) (10, 3072)

Image

Array of **32x32x3** numbers
(3072 numbers total)

$\longrightarrow$ f(**x**,**W**) $\longrightarrow$ **10** numbers giving class scores

W
parameters
or weights

Parametric Approach: Linear Classifier

(3072,)

$$\boxed{f(x,W)} = \boxed{W}\boxed{x} + \boxed{b}$$

(10,)

(10,) (10, 3072)

Image

Array of **32x32x3** numbers
(3072 numbers total)

→ f(**x**,**W**) →

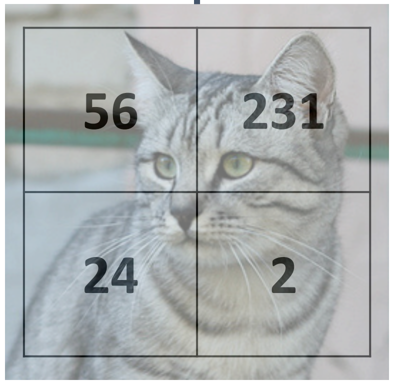
10 numbers giving class scores

W
parameters or weights

Example for 2x2 image, 3 classes (cat/dog/ship)

Stretch pixels into column

$f(x,W) = Wx + b$



Input image
(2, 2)

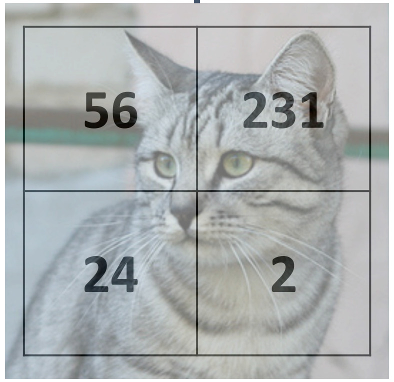
56
231
24
2

(4,)

Example for 2x2 image, 3 classes (cat/dog/ship)

Stretch pixels into column

$$f(x,W) = Wx + b$$



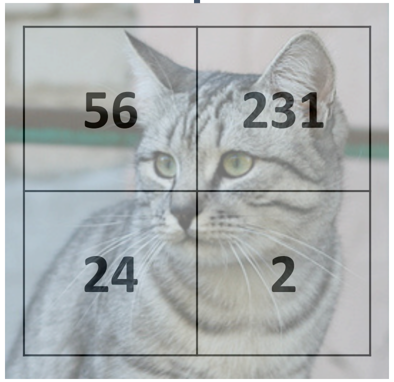
Input image
(2, 2)

$$\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W\ (3,\ 4)} \underbrace{\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}}_{(4,)} + \underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b\ (3,)} = \underbrace{\begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix}}_{(3,)}$$

Linear Classifier: Algebraic Viewpoint

Stretch pixels into column

$$f(x,W) = Wx + b$$



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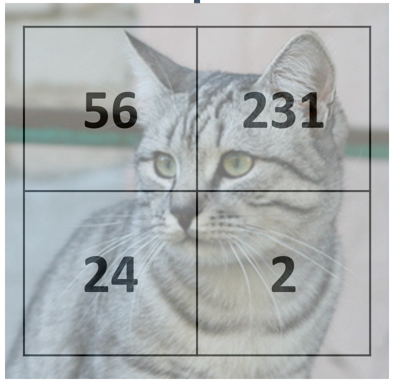
Linear Classifier: Bias Trick

Add extra one to data vector; bias is absorbed into last column of weight matrix

Stretch pixels into column



Input image
(2, 2)

$$\underbrace{\left[\begin{array}{cccc|c} 0.2 & -0.5 & 0.1 & 2.0 & 1.1 \\ 1.5 & 1.3 & 2.1 & 0.0 & 3.2 \\ 0 & 0.25 & 0.2 & -0.3 & -1.2 \end{array}\right]}_{W\ (3,\,5)} \underbrace{\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \\ 1 \end{bmatrix}}_{(5,)} = \underbrace{\begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix}}_{(3,)}$$

Linear Classifier: Predictions are Linear!

$$f(x, W) = Wx \quad \text{(ignore bias)}$$

$$f(cx, W) = W(cx) = c * f(x, W)$$

Linear Classifier: Predictions are Linear!

$$f(x, W) = Wx \quad \text{(ignore bias)}$$

$$f(cx, W) = W(cx) = c * f(x, W)$$

Image

Scores

-96.8
437.8
62.0

0.5 * Image

0.5 * Scores

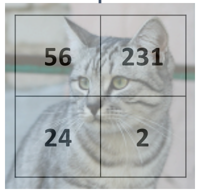
-48.4
218.9
31.0

Interpreting a Linear Classifier

Algebraic Viewpoint

$$f(x,W) = Wx + b$$

Stretch pixels into column



Input image
(2, 2)

$$\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W\ (3,\,4)} \underbrace{\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}}_{(4,)} + \underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b\ (3,)} = \underbrace{\begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix}}_{(3,)}$$

Interpreting a Linear Classifier

Algebraic Viewpoint

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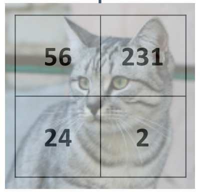

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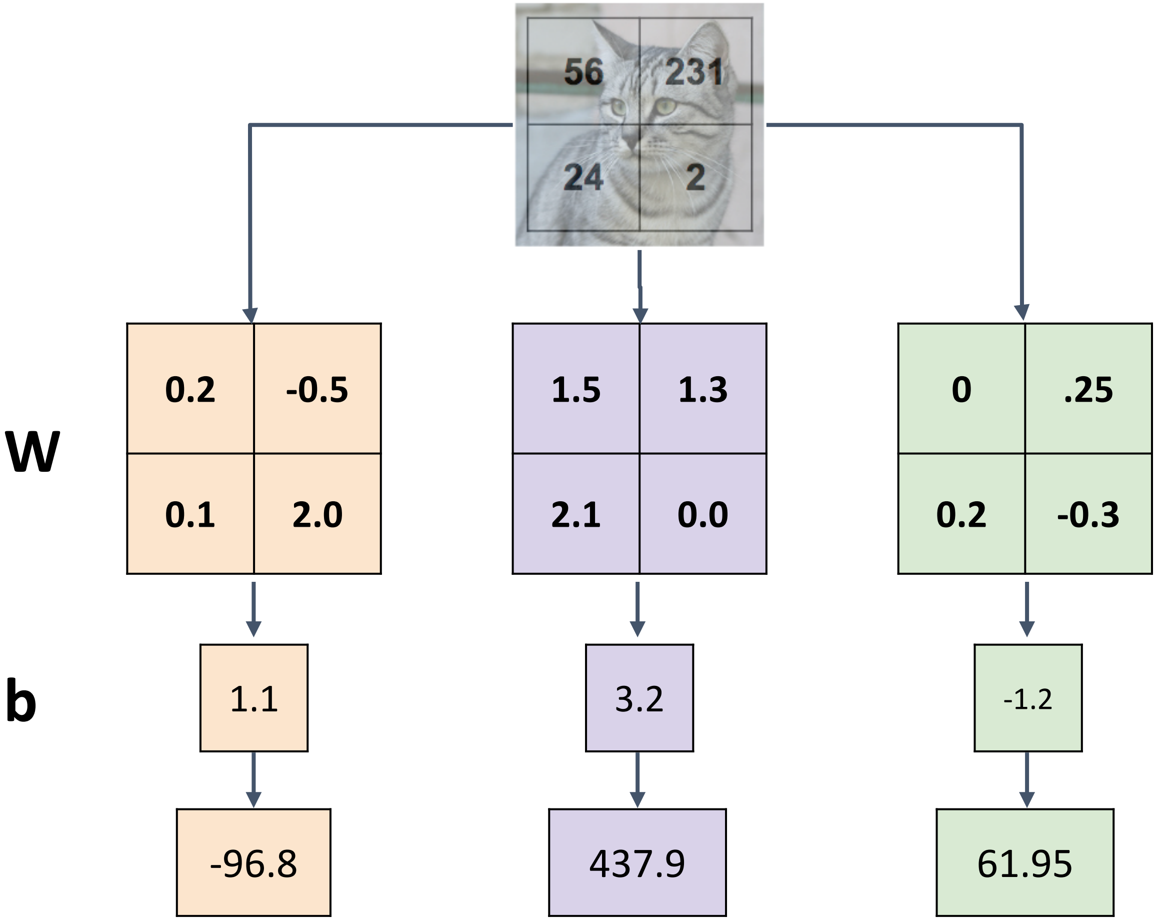
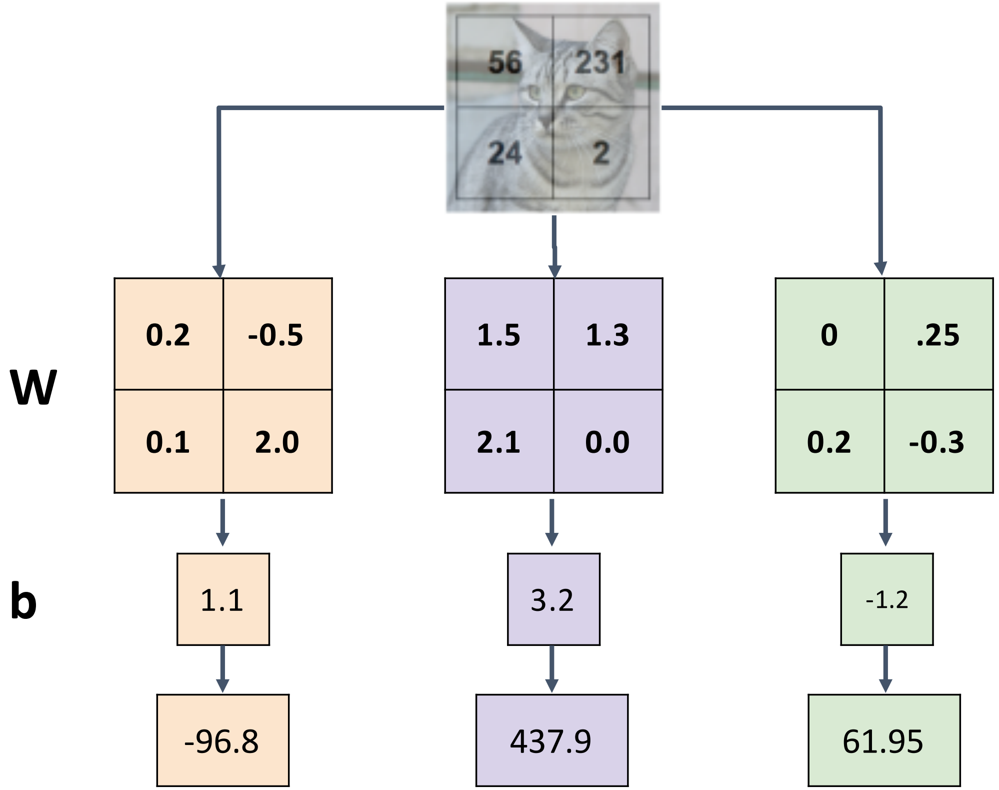
(2, 2)

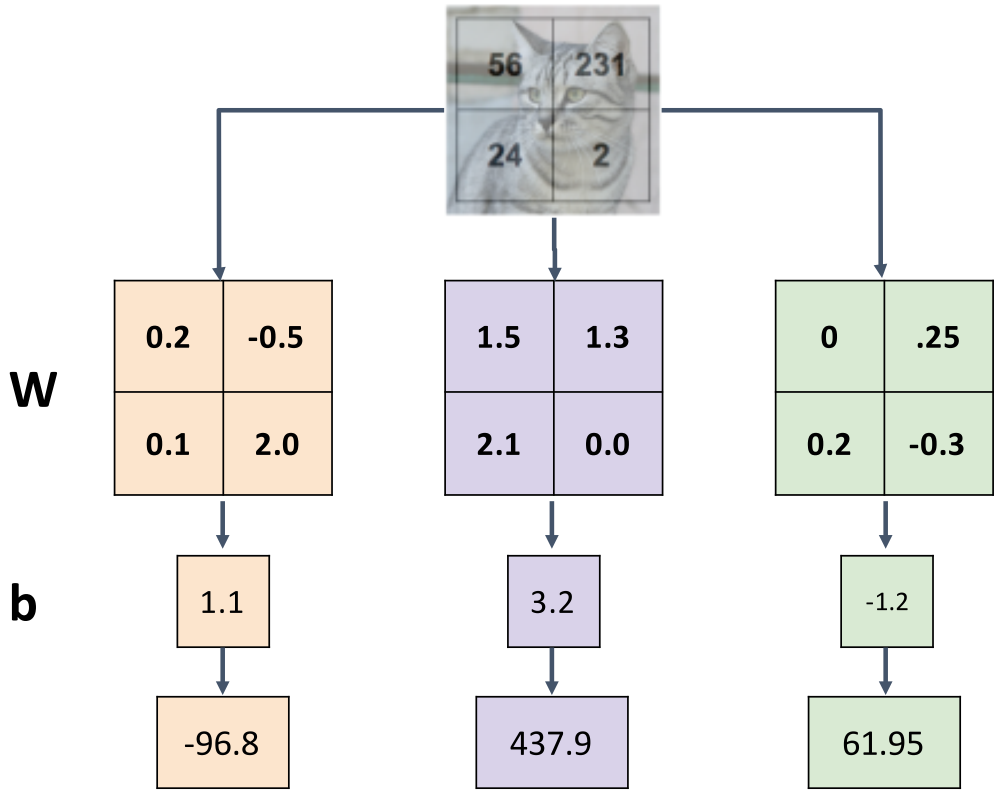
$$\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W\ (3,4)} \underbrace{\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}}_{(4,)} + \underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b\ (3,)} = \underbrace{\begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix}}_{(3,)}$$



Interpreting an Linear Classifier

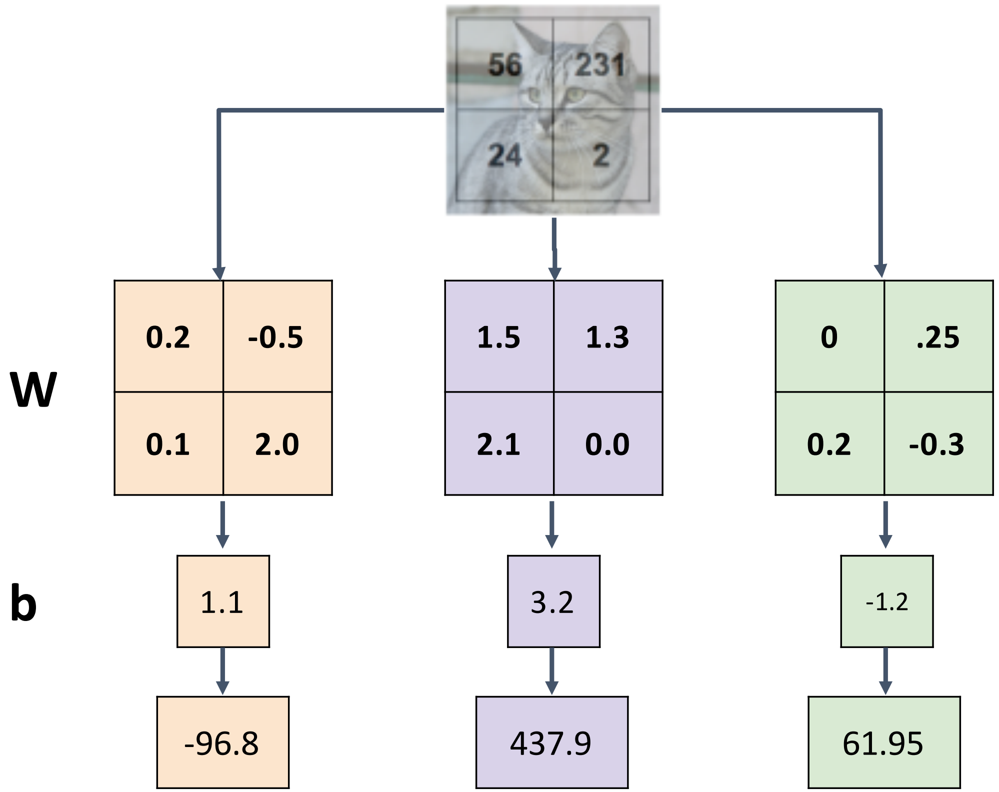


Interpreting an Linear Classifier: Visual Viewpoint



Interpreting an Linear Classifier: Visual Viewpoint

Linear classifier has one "template" per category

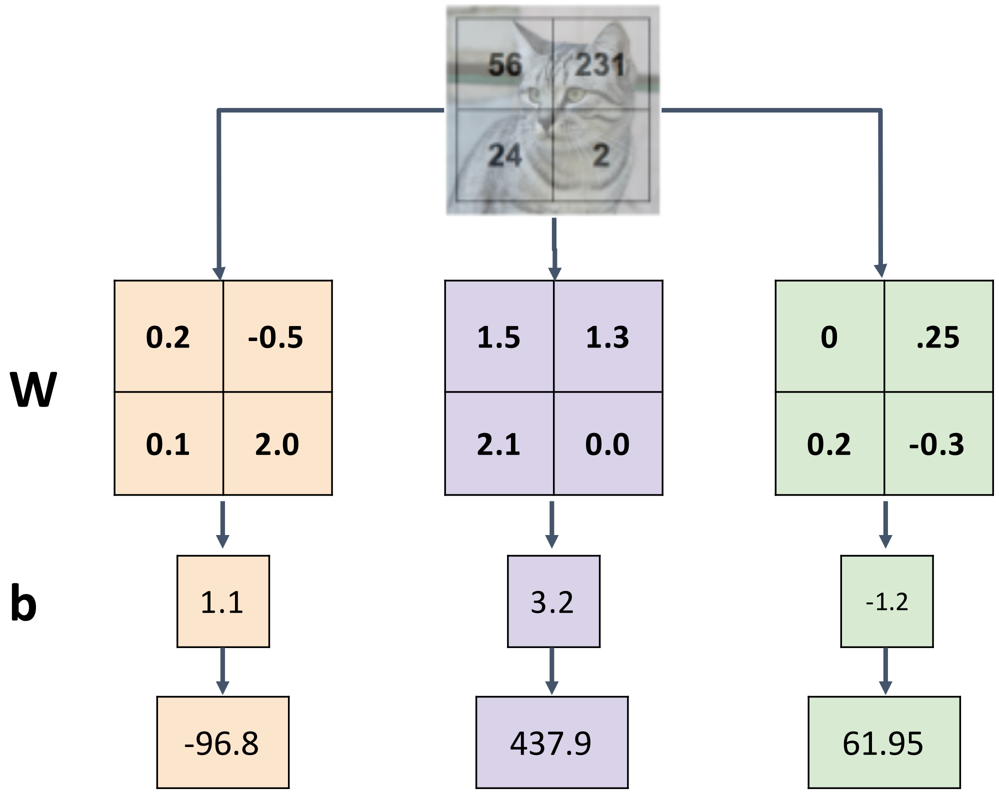


Interpreting an Linear Classifier: Visual Viewpoint

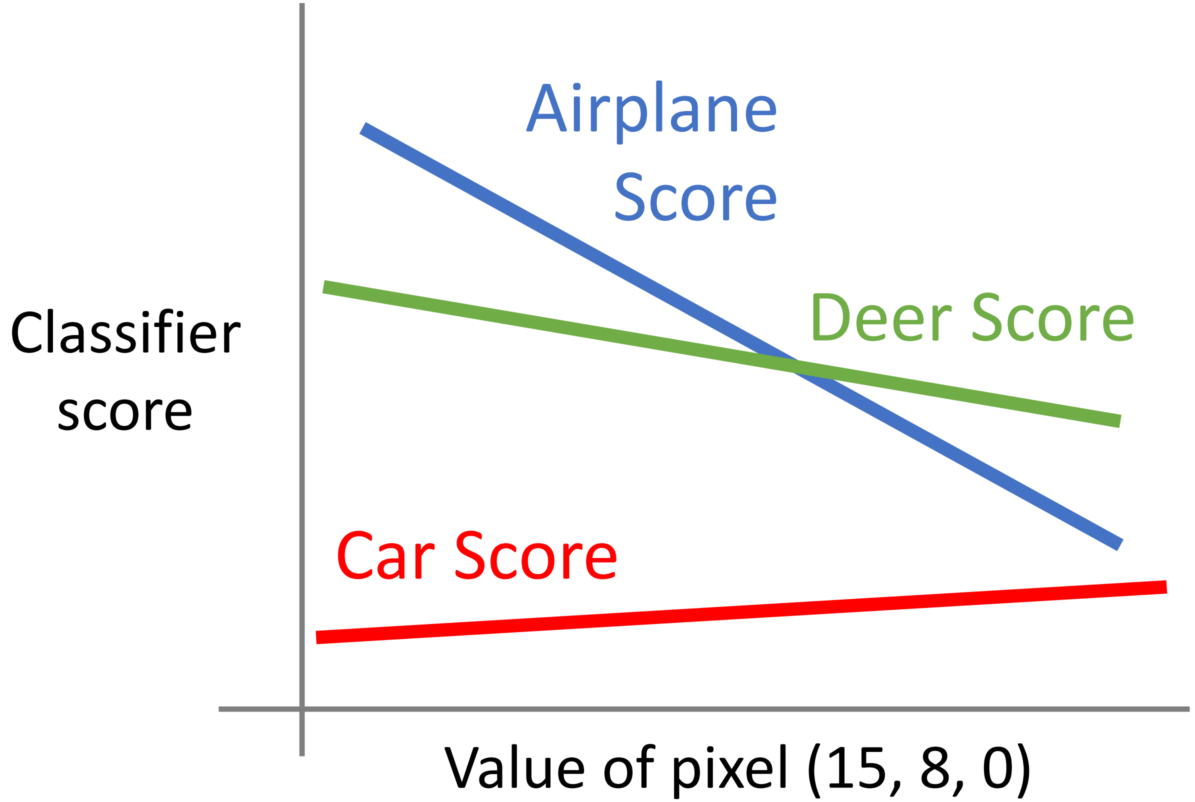
Linear classifier has one "template" per category

A single template cannot capture multiple modes of the data

e.g. horse template has 2 heads!



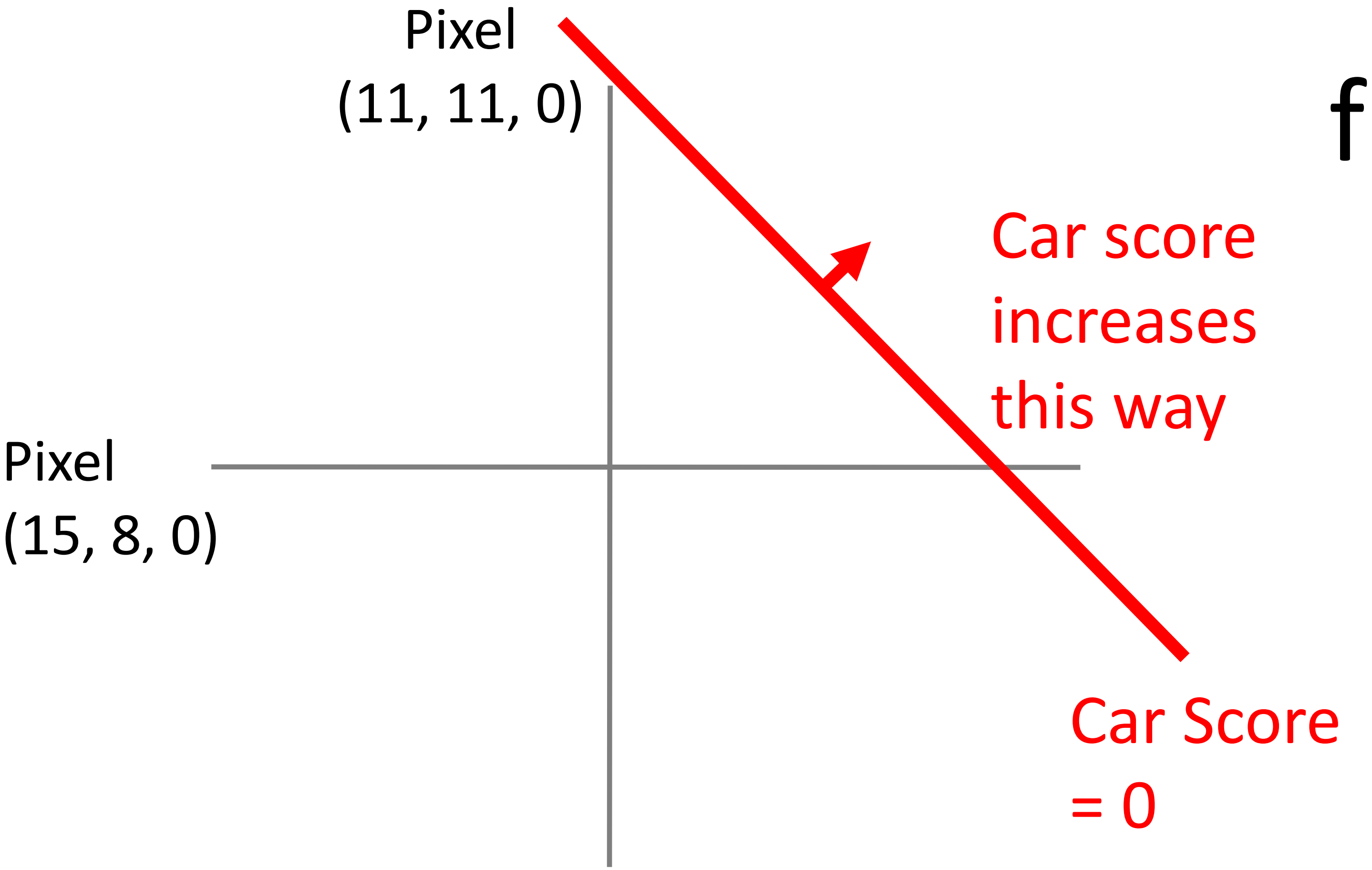
Interpreting a Linear Classifier: Geometric Viewpoint



$$f(x,W) = Wx + b$$

Array of **32x32x3** numbers
(3072 numbers total)

Interpreting a Linear Classifier: <u>Geometric Viewpoint</u>



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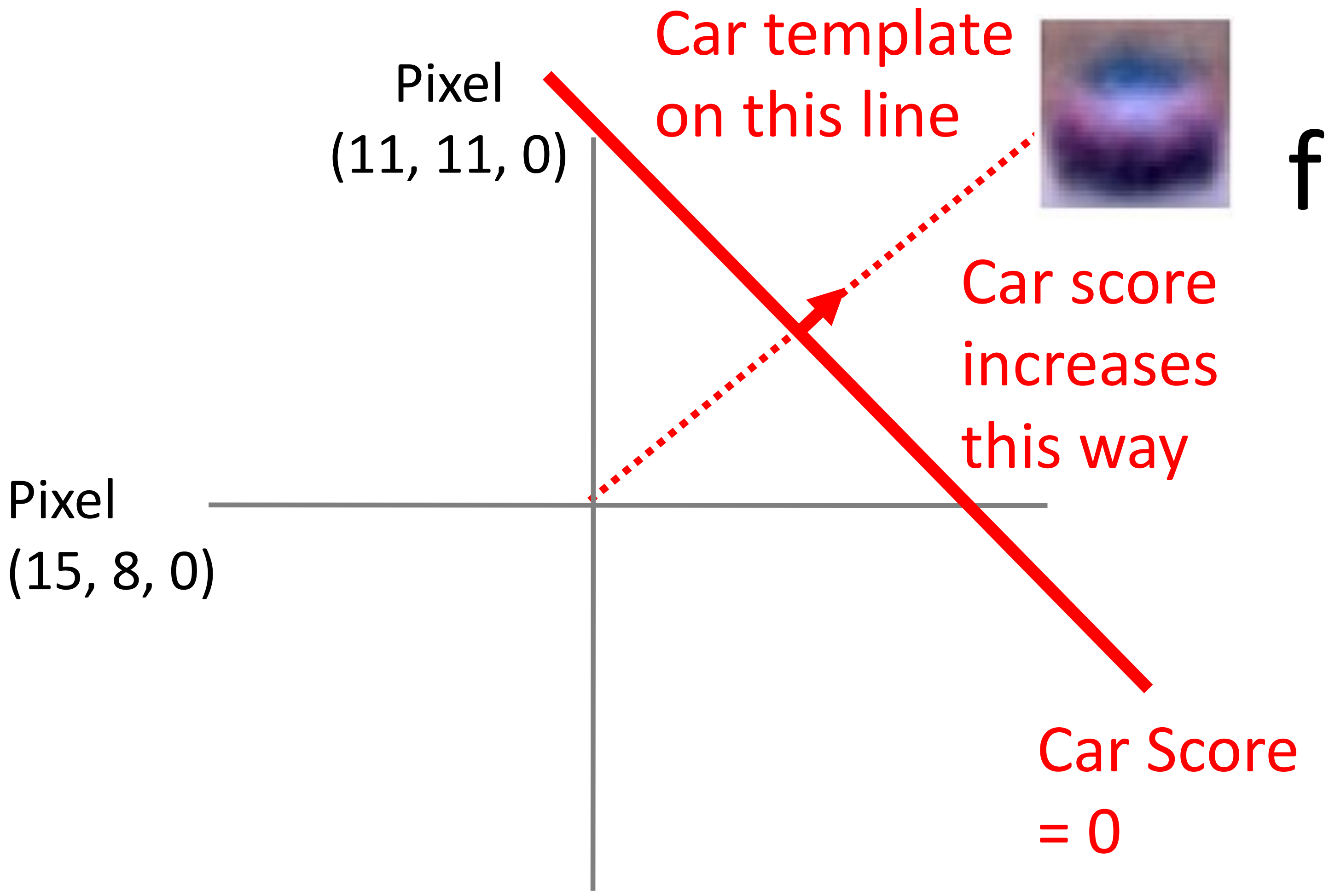
Interpreting a Linear Classifier: Geometric Viewpoint



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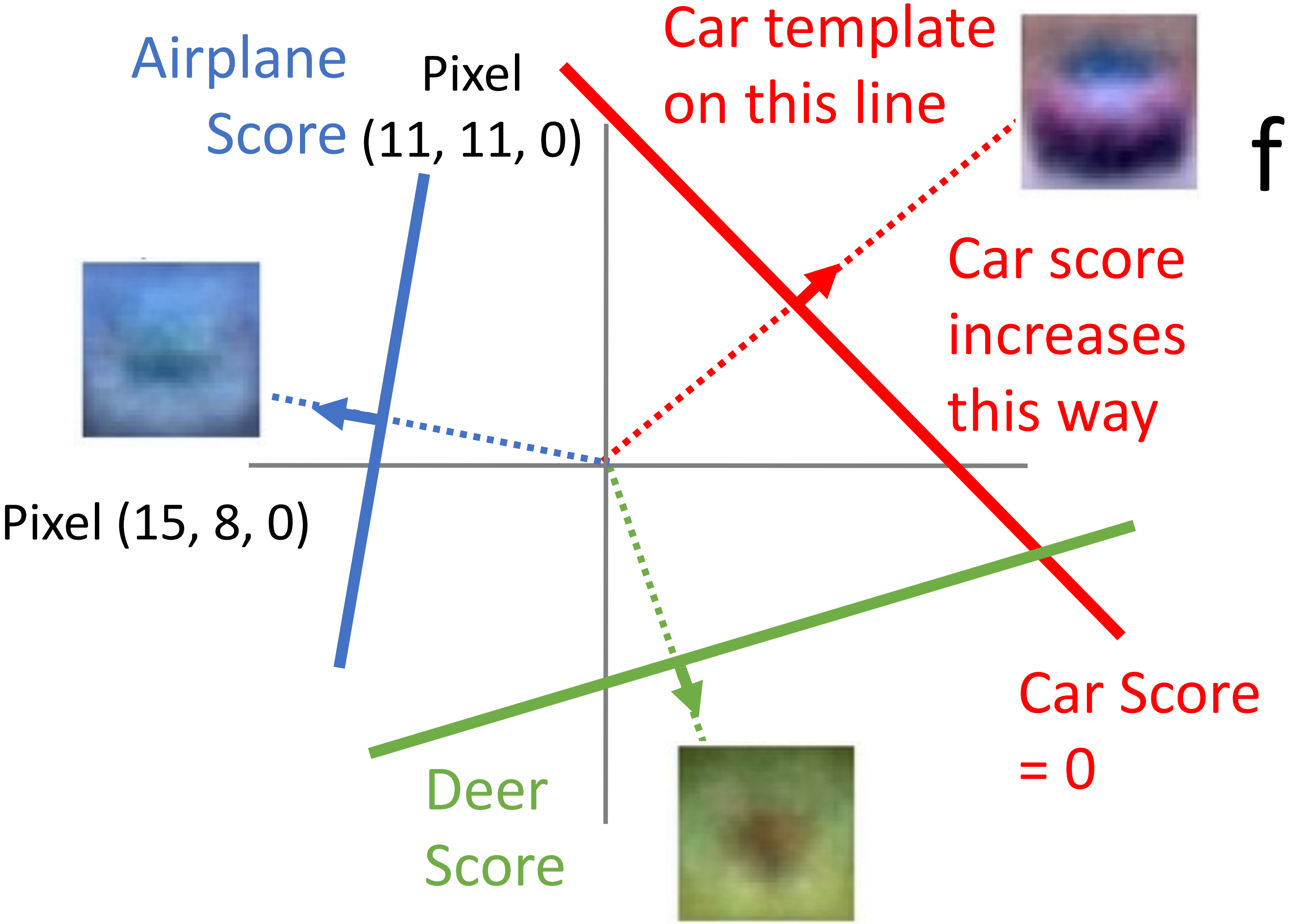
Interpreting a Linear Classifier: Geometric Viewpoint



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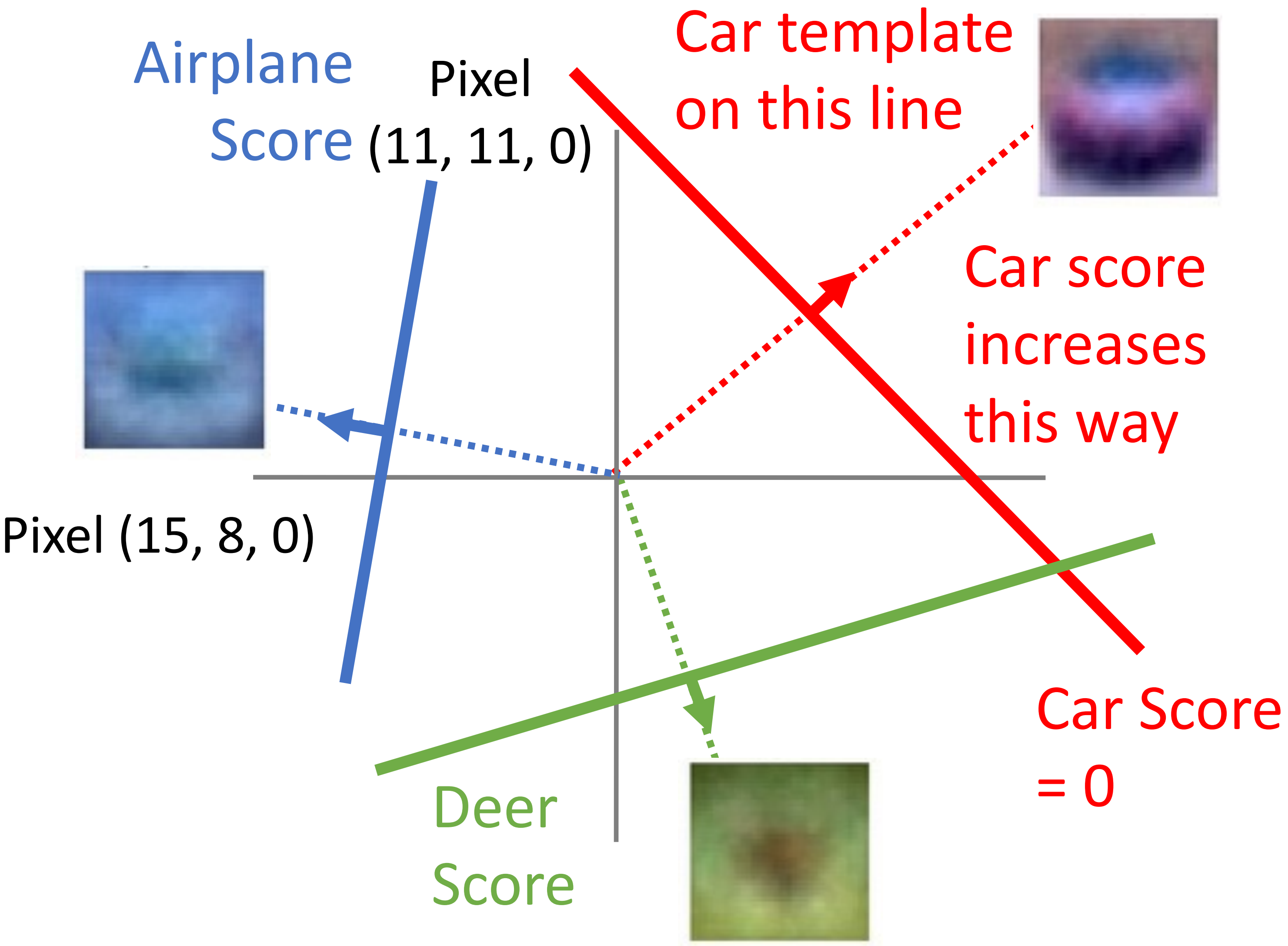
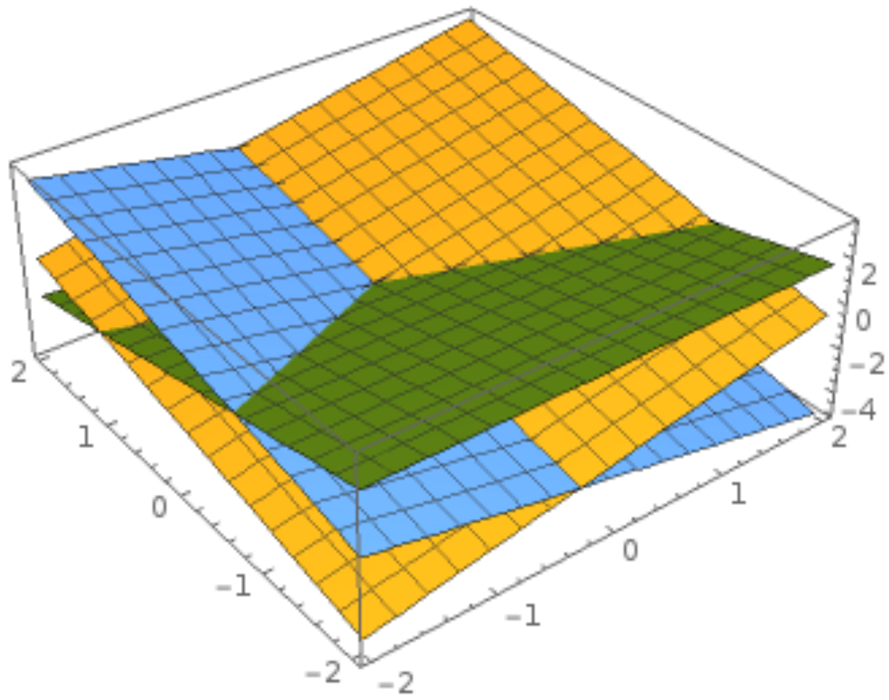


Hyperplanes carving up a high-dimensional space

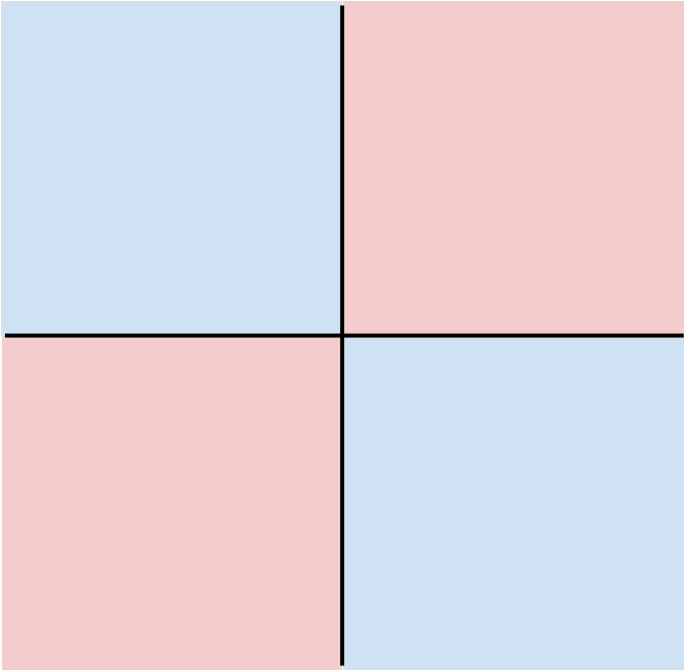


Plot created using [Wolfram Cloud]

Hard Cases for a Linear Classifier

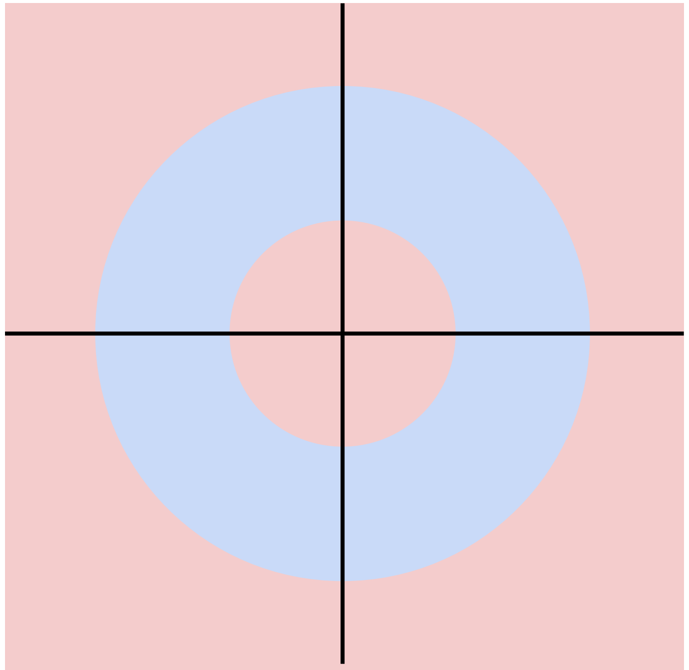
Class 1:
First and third quadrants

Class 2:
Second and fourth quadrants



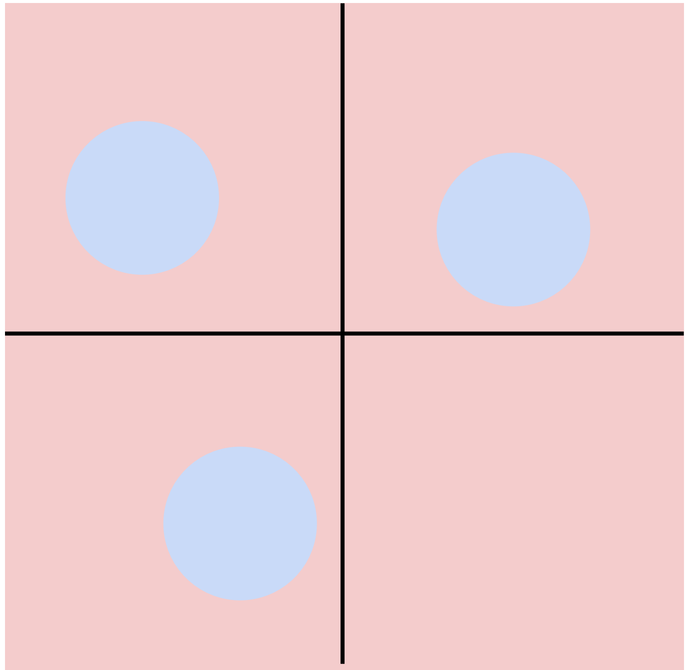
Class 1:
1 <= L2 norm <= 2

Class 2:
Everything else



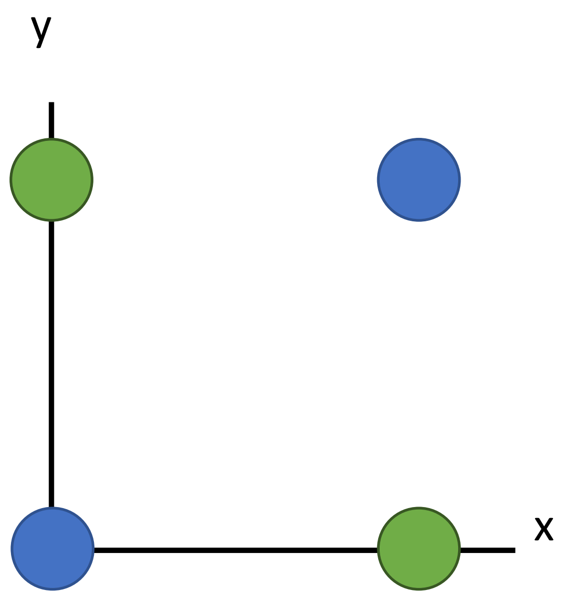
Class 1:
Three modes

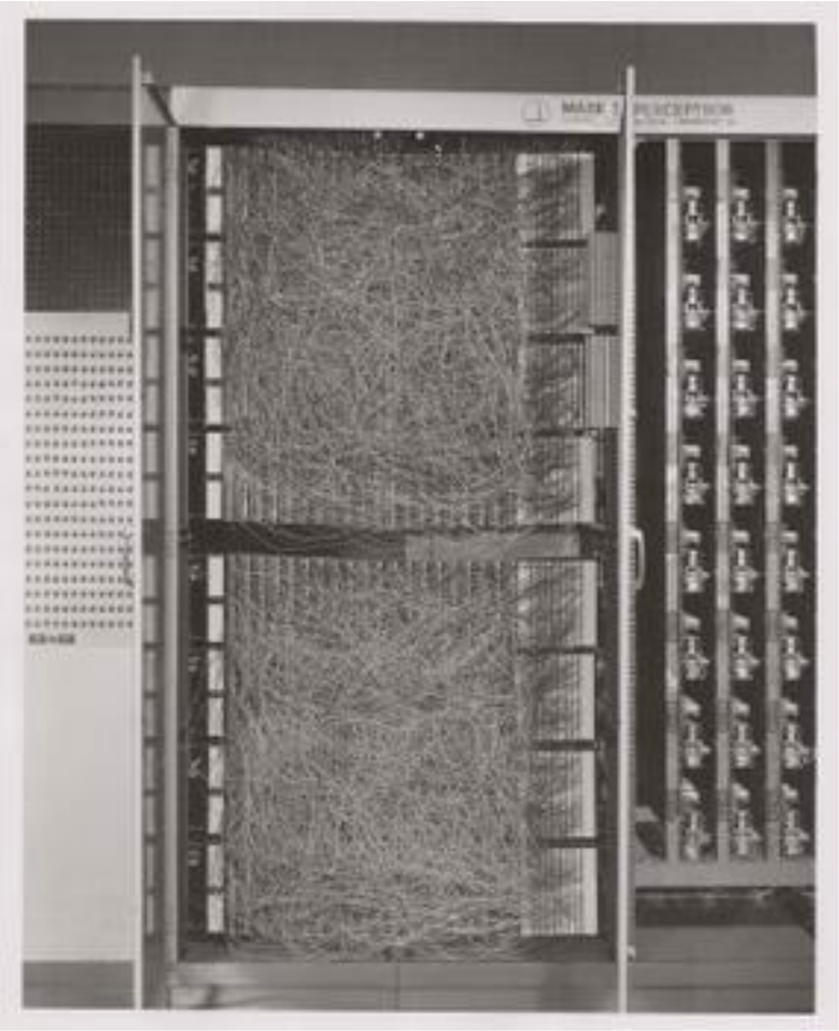
Class 2:
Everything else



Recall: Perceptron couldn't learn XOR

X	Y	F(x,y)
0	0	0
0	1	1
1	0	1
1	1	0

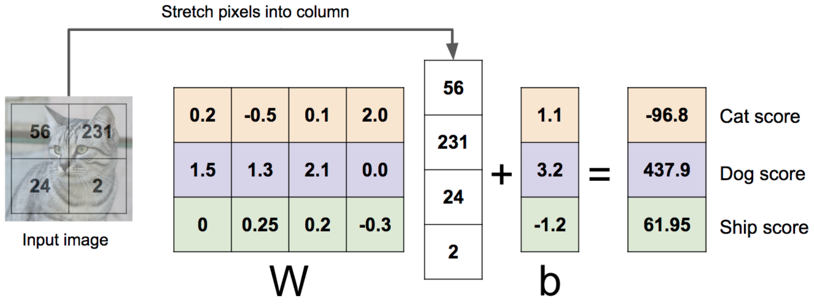




Linear Classifier: Three Viewpoints

Algebraic Viewpoint

$f(x,W) = Wx$

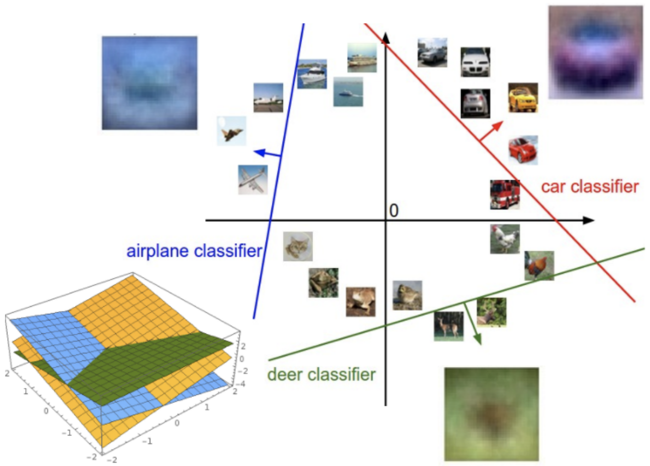


Visual Viewpoint

One template
per class

Geometric Viewpoint

Hyperplanes
cutting up space



So Far: Defined a linear **score function**

$f(x,W) = Wx + b$

airplane	-3.45	-0.51	3.42
automobile	-8.87	**6.04**	4.64
bird	0.09	5.31	2.65
cat	**2.9**	-4.22	5.1
deer	4.48	-4.19	2.64
dog	8.02	3.58	5.55
frog	3.78	4.49	**-4.34**
horse	1.06	-4.37	-1.5
ship	-0.36	-2.09	-4.79
truck	-0.72	-2.93	6.14

Given a W, we can compute class scores for an image x.

But how can we actually choose a good W?

Choosing a good W

$f(x,W) = Wx + b$

airplane	-3.45	-0.51	3.42
automobile	-8.87	**6.04**	4.64
bird	0.09	5.31	2.65
cat	**2.9**	-4.22	5.1
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horse	1.06	-4.37	-1.5
ship	-0.36	-2.09	-4.79
truck	-0.72	-2.93	6.14

TODO:

1. Use a **loss function** to quantify how good a value of W is

2. Find a W that minimizes the loss function (**optimization**)

Loss Function

A **loss function** tells how good our current classifier is

Low loss = good classifier
High loss = bad classifier

(Also called: **objective function**; **cost function**)

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Negative loss function sometimes called **reward function**, **profit function**, **utility function**, **fitness function**, etc

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Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^{N}$$

Where x_i is image and
y_i is (integer) label

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Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^N$$

Where x_i is image and
y_i is (integer) label

Loss for a single example is

$$L_i(f(x_i, W), y_i)$$

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A **loss function** tells how good our current classifier is

Low loss = good classifier
High loss = bad classifier

(Also called: **objective function**; **cost function**)

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Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^N$$

Where x_i is image and
y_i is (integer) label

Loss for a single example is

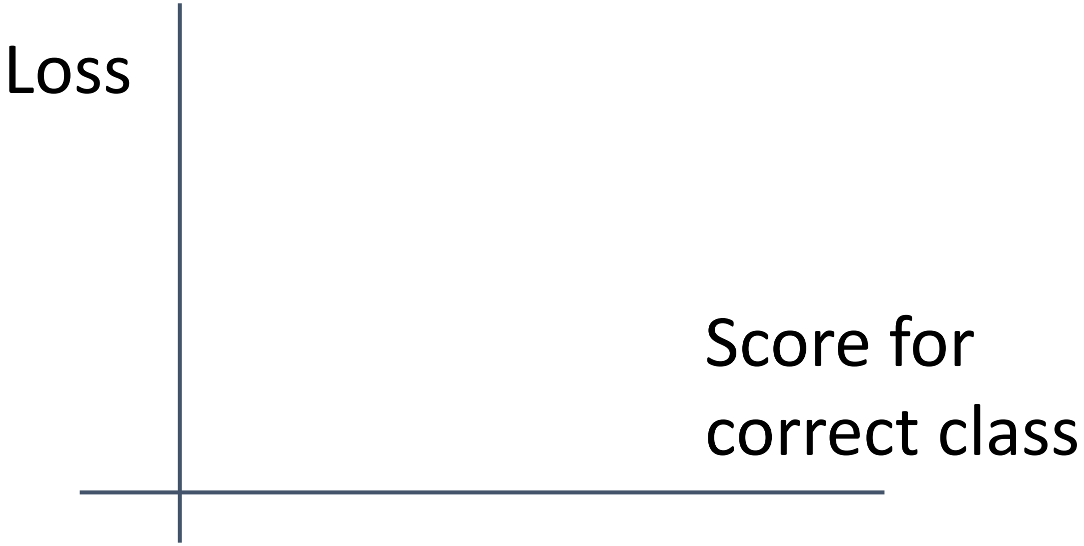
$$L_i(f(x_i, W), y_i)$$

Loss for the dataset is average of per-example losses:

$$L = \frac{1}{N} \sum_i L_i(f(x_i, W), y_i)$$

Multiclass SVM Loss

”The score of the correct class should be higher than all the other scores”



Multiclass SVM Loss

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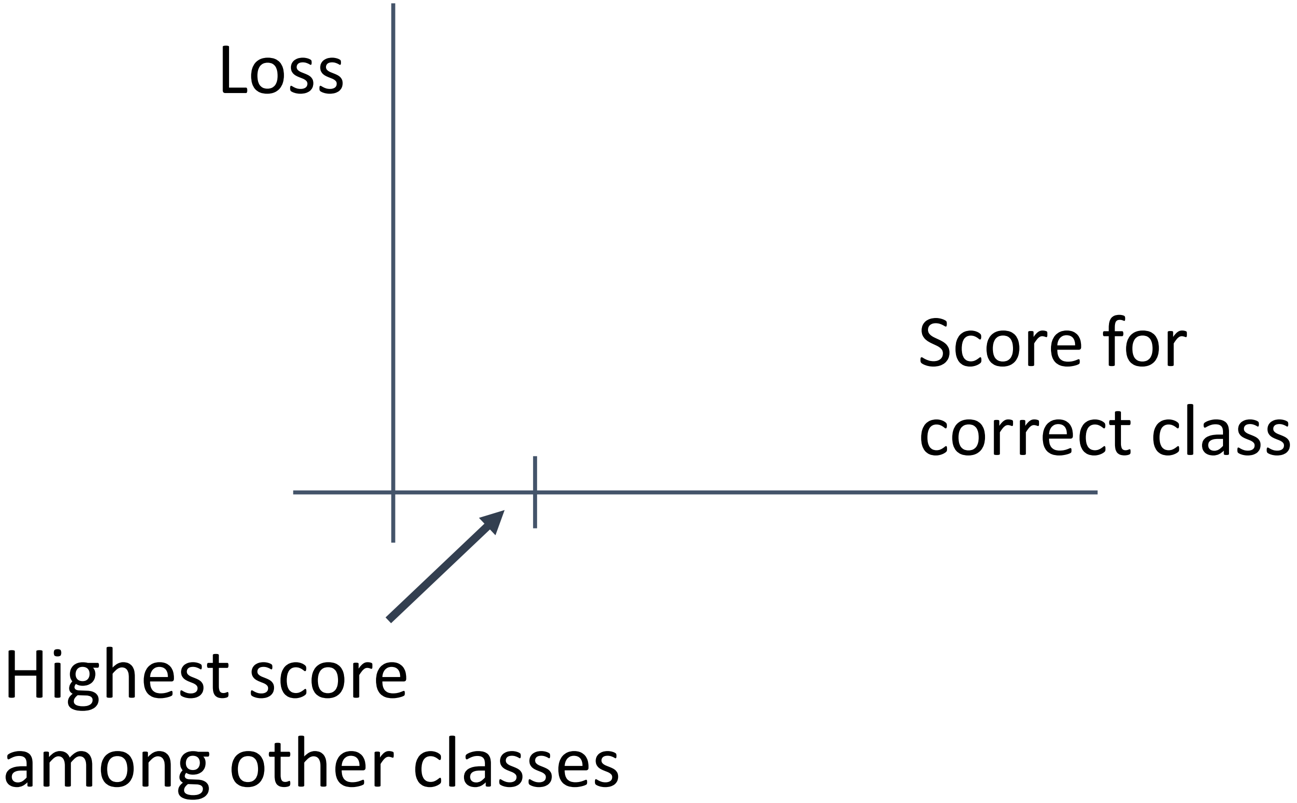

Multiclass SVM Loss

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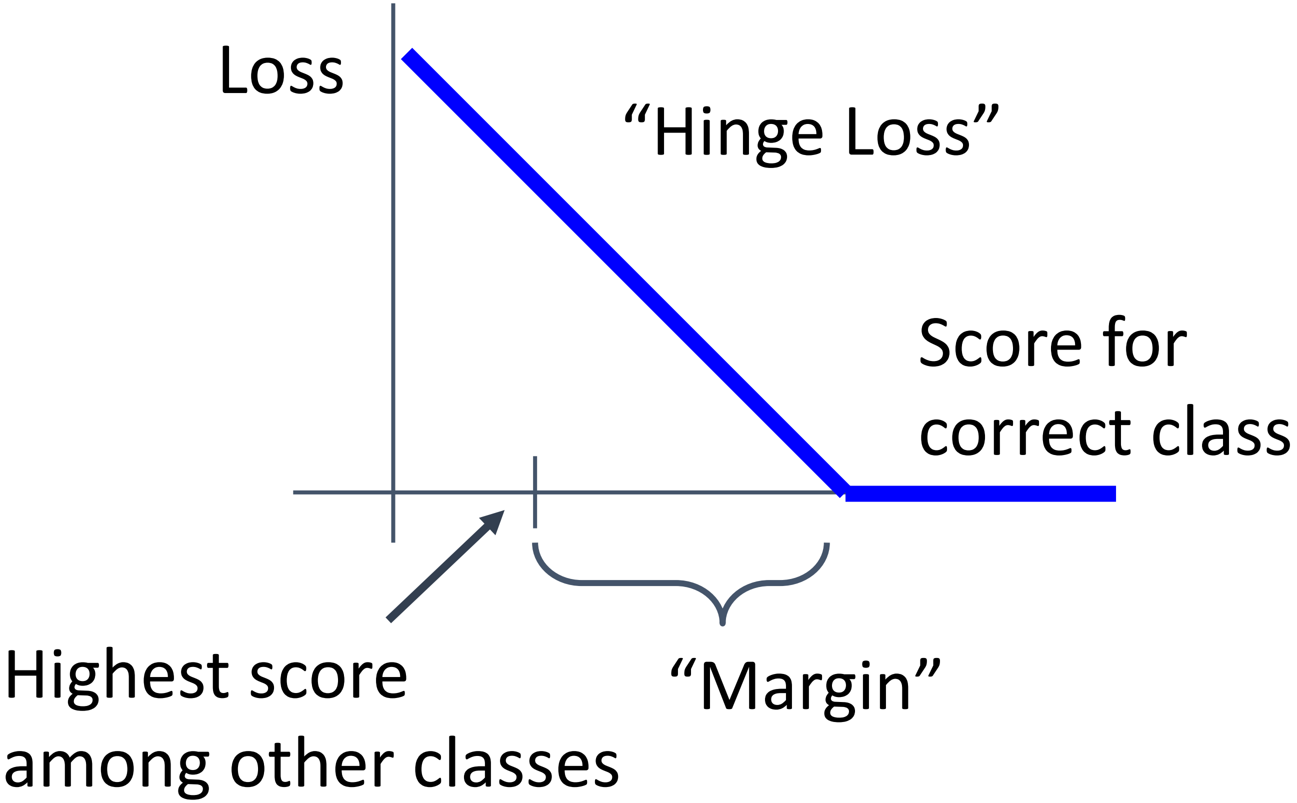
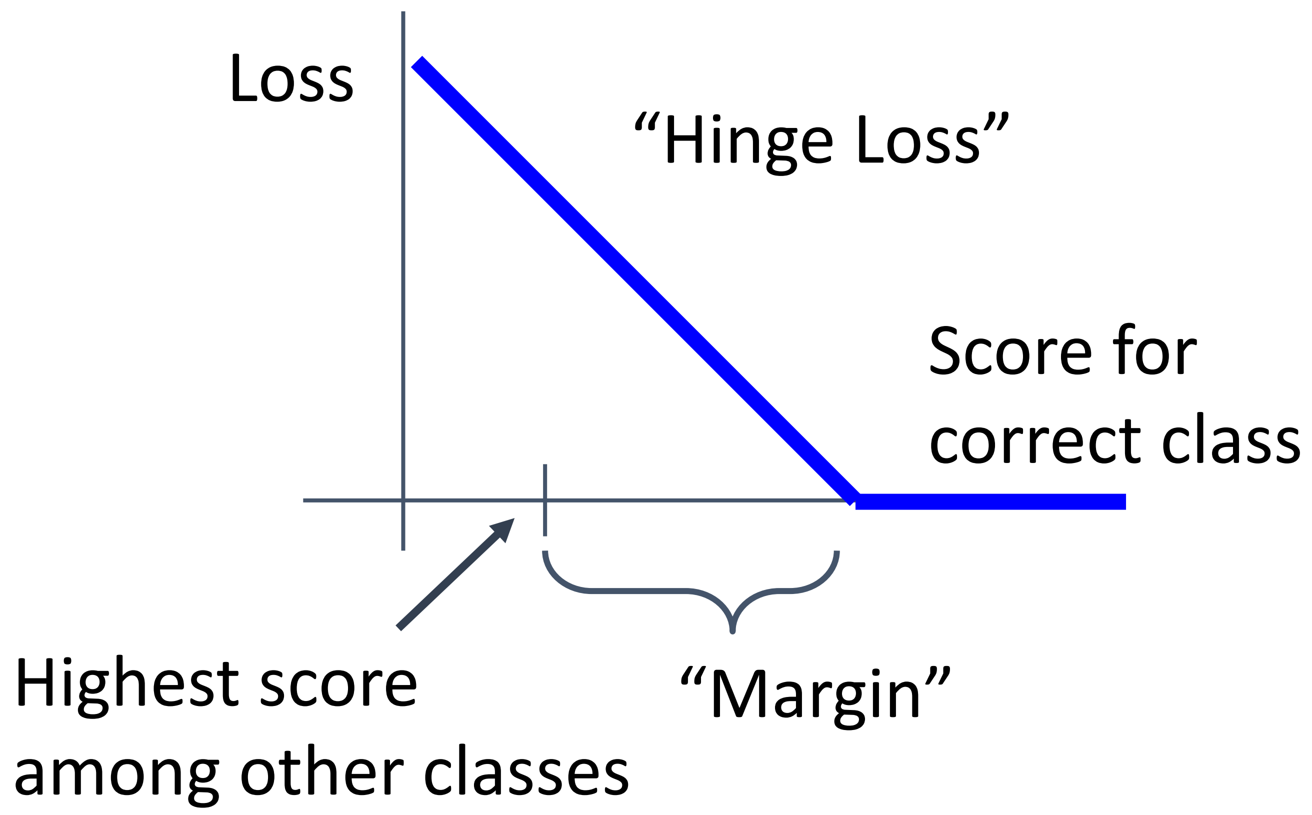


Multiclass SVM Loss

"The score of the correct class should be higher than all the other scores"



Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Multiclass SVM Loss

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

Given an example (x_i, y_i)
(x_i is image, y_i is label)

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Multiclass SVM Loss

  

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9		

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

= max(0, 5.1 - 3.2 + 1)
+ max(0, -1.7 - 3.2 + 1)
= max(0, 2.9) + max(0, -3.9)
= 2.9 + 0
= 2.9

Multiclass SVM Loss

			
cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

= max(0, 1.3 - 4.9 + 1)
+max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

Multiclass SVM Loss

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

= max(0, 2.2 - (-3.1) + 1)
+max(0, 2.5 - (-3.1) + 1)
= max(0, 6.3) + max(0, 6.6)
= 6.3 + 6.6
= 12.9

Multiclass SVM Loss

  

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Loss over the dataset is:

L = (2.9 + 0.0 + 12.9) / 3
= 5.27

Multiclass SVM Loss

  

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q: What happens to the loss if the scores for the car image change a bit?

Multiclass SVM Loss



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

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Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q2: What are the min and max possible loss?

Multiclass SVM Loss

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q3: If all the scores were random, what loss would we expect?

Multiclass SVM Loss



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q4: What would happen if the sum were over all classes? (including $i = y_i$)

Multiclass SVM Loss



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q5: What if the loss used a mean instead of a sum?

Multiclass SVM Loss

			
cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q6: What if we used this loss instead?

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)^2$$

Multiclass SVM Loss

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

Q: Suppose we found some W with L = 0. Is it unique?

Multiclass SVM Loss

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

Q: Suppose we found some W with L = 0. Is it unique?

No! 2W is also has L = 0!

Multiclass SVM Loss

$$f(x, W) = Wx$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)^2$$

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Original W:

= max(0, 1.3 - 4.9 + 1)
+max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

Using 2W instead:

= max(0, 2.6 - 9.8 + 1)
+max(0, 4.0 - 9.8 + 1)
= max(0, -6.2) + max(0, -4.8)
= 0 + 0
= 0

Multiclass SVM Loss

$$f(x, W) = Wx$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)^2$$

  

cat	**3.2**	1.3	2.2
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frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

How should we choose between W and 2W if they both perform the same on the training data?

Regularization: Beyond Training Error

$$\underbrace{L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i)}$$

Data loss: Model predictions should match training data

Regularization: Beyond Training Error

$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i)}_{} + \underbrace{\lambda R(W)}_{}$$

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

Regularization: Beyond Training Error

$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i)}_{\text{Data loss}} + \underbrace{\lambda R(W)}_{\text{Regularization}}$$

λ = regularization strength (hyperparameter)

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

Regularization: Beyond Training Error

$$\underbrace{L(W) = \frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)}_{} + \underbrace{\lambda R(W)}_{}$$

λ = regularization strength (hyperparameter)

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

Simple examples

<u>L2 regularization:</u> $R(W) = \sum_k \sum_l W_{k,l}^2$

L1 regularization: $R(W) = \sum_k \sum_l |W_{k,l}|$

Elastic net (L1 + L2): $R(W) = \sum_k \sum_l \beta W_{k,l}^2 + |W_{k,l}|$

More complex:

Dropout

Batch normalization

Cutout, Mixup, Stochastic depth, etc…

Regularization: Beyond Training Error

$$L(W) = \underbrace{\frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)}_{\text{Data loss}} + \underbrace{\lambda R(W)}_{\text{Regularization}}$$

λ = regularization strength (hyperparameter)

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

Purpose of Regularization:

- Express preferences in among models beyond ”minimize training error”
- Avoid **overfitting**: Prefer simple models that generalize better
- Improve optimization by adding curvature

Regularization: Expressing Preferences

$$x = [1, 1, 1, 1]$$

$$w_1 = [1, 0, 0, 0]$$

$$w_2 = [0.25, 0.25, 0.25, 0.25]$$

$$w_1^T x = w_2^T x = 1$$

L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

Regularization: Expressing Preferences

$$x = [1, 1, 1, 1]$$

$$w_1 = [1, 0, 0, 0]$$

$$w_2 = \boxed{[0.25, 0.25, 0.25, 0.25]}$$

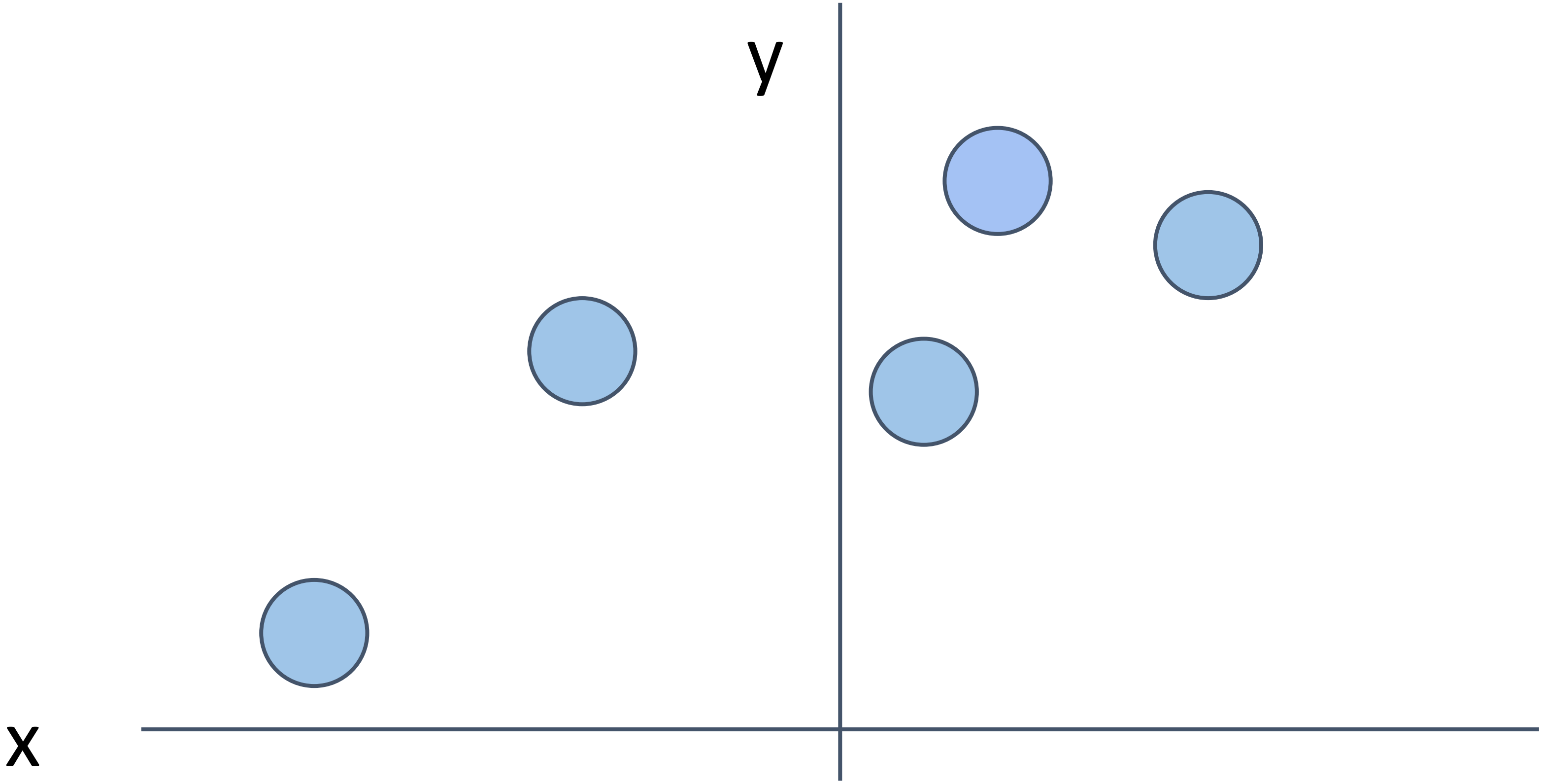
$$w_1^T x = w_2^T x = 1$$

L2 Regularization

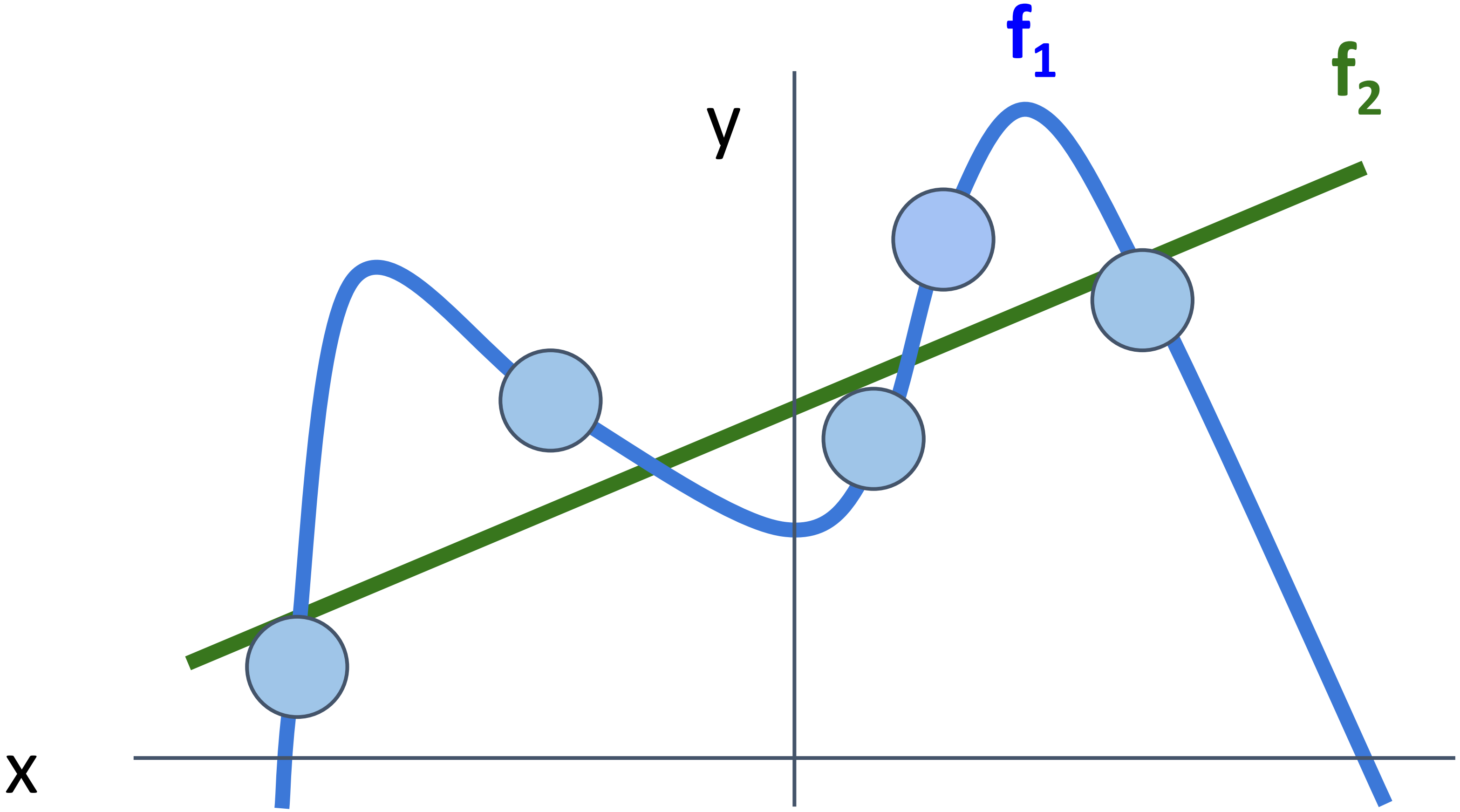
$$R(W) = \sum_k \sum_l W_{k,l}^2$$

L2 regularization likes to "spread out" the weights

Regularization: Prefer Simpler Models

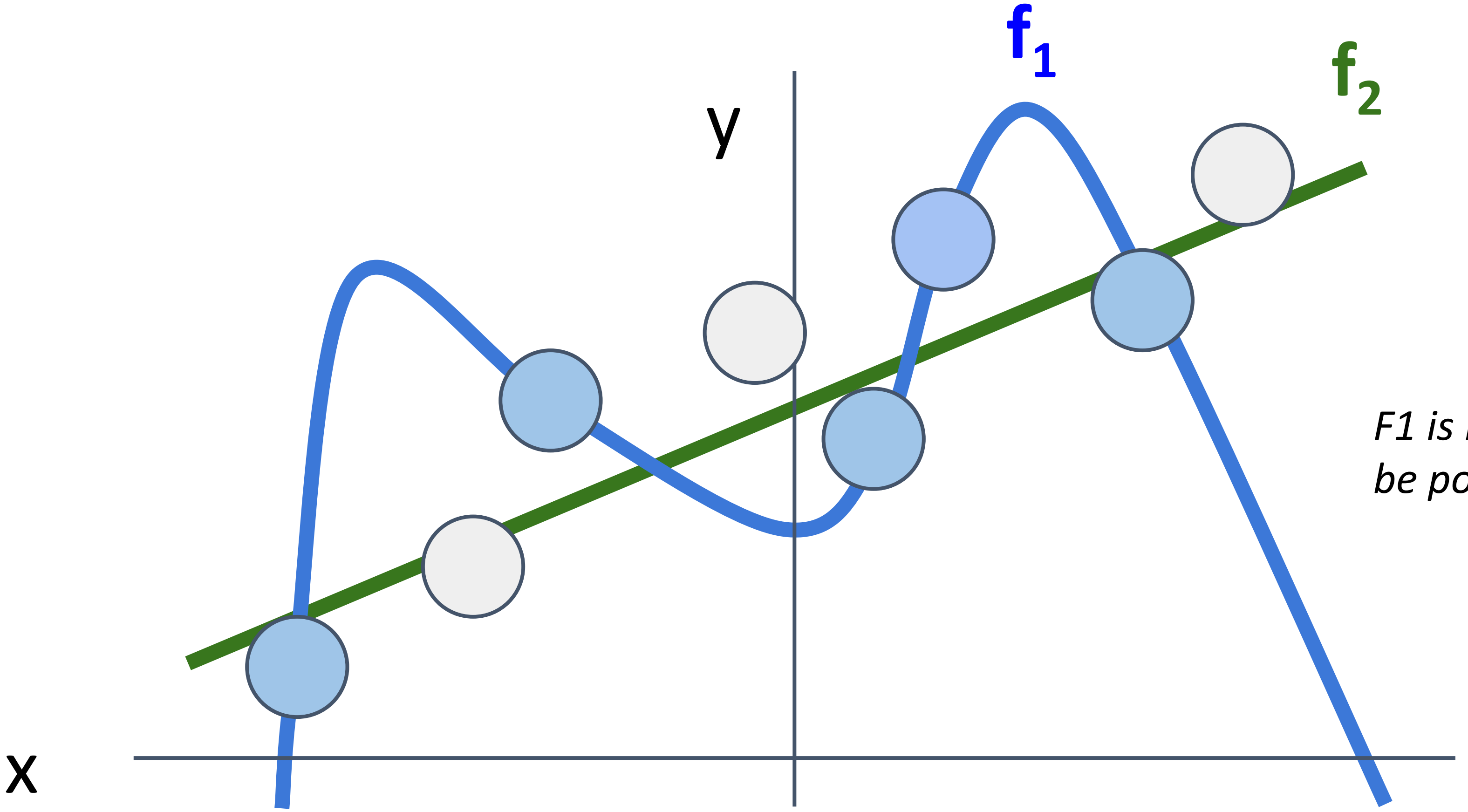


Regularization: Prefer Simpler Models



The model f_1 fits the training data perfectly
The model f_2 has training error, but is simpler

Regularization: Prefer Simpler Models

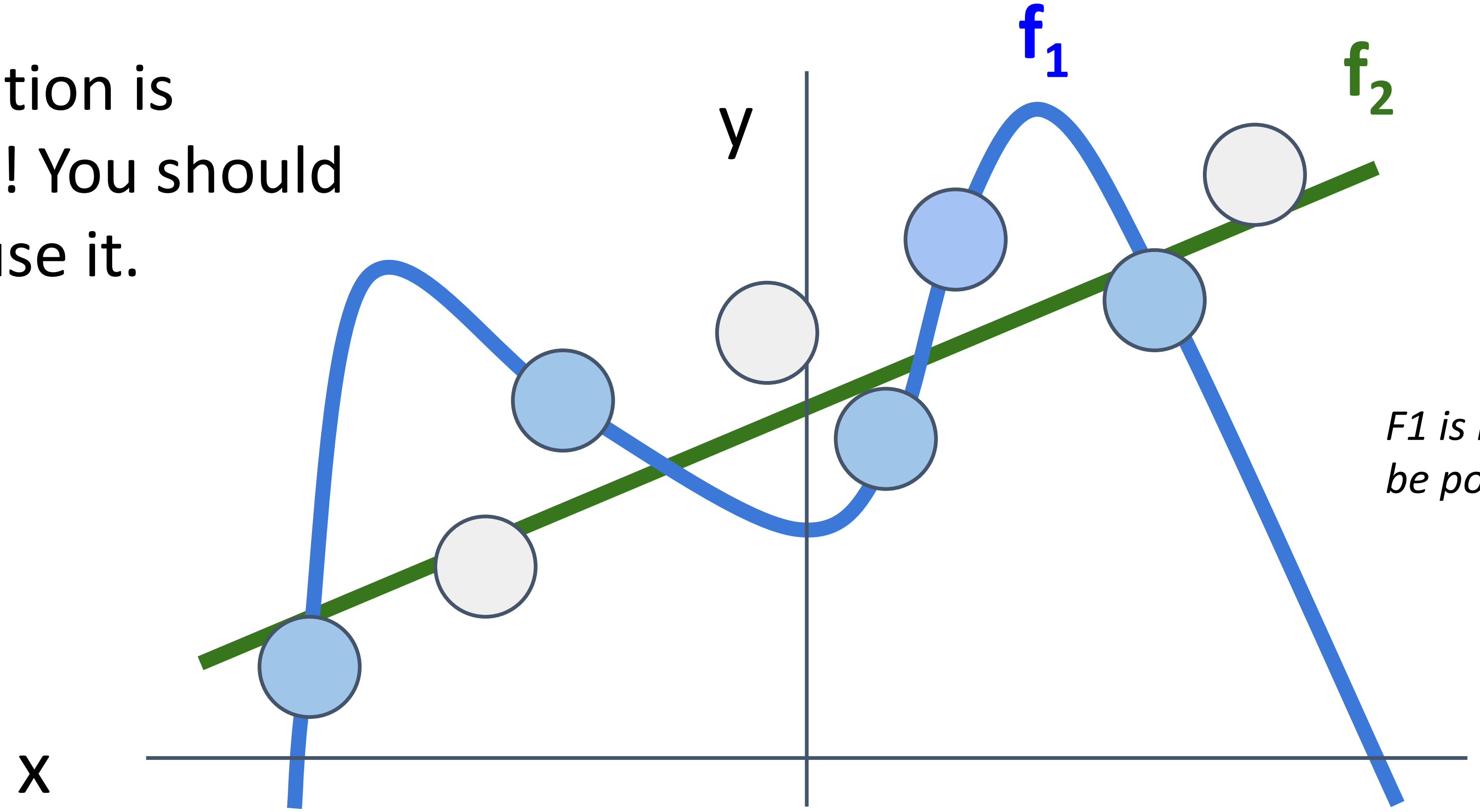


F1 is not a linear model; could be polynomial regression, etc

Regularization pushes against fitting the data *too* well so we don't fit noise in the data

Regularization: Prefer Simpler Models

Regularization is important! You should (usually) use it.



F1 is not a linear model; could be polynomial regression, etc

Regularization pushes against fitting the data *too* well so we don't fit noise in the data

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

cat	**3.2**
car	5.1
frog	-1.7

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

Softmax function

cat	**3.2**
car	5.1
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Unnormalized log-probabilities / logits

Cross-Entropy Loss (Multinomial Logistic Regression)

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Softmax function

Probabilities must be >= 0

cat	**3.2**	exp →	**24.5**
car	5.1		164.0
frog	-1.7		0.18

Unnormalized log-probabilities / logits

unnormalized probabilities

Cross-Entropy Loss (Multinomial Logistic Regression)

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$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax function

Probabilities must be >= 0

Probabilities must sum to 1

		exp		normalize	
cat	**3.2**	$\rightarrow$	**24.5**	$\rightarrow$	**0.13**
car	5.1		164.0		0.87
frog	-1.7		0.18		0.00

Unnormalized log-probabilities / logits

unnormalized probabilities

probabilities

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$ Softmax function

Probabilities must be >= 0

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

	Unnormalized log-probabilities / logits		unnormalized probabilities		probabilities
cat	**3.2**	exp →	**24.5**	normalize →	**0.13**
car	5.1		164.0		0.87
frog	-1.7		0.18		0.00

$L_i = -\log(0.13)$
$= \mathbf{2.04}$

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$ Softmax function

$$L_i = -\log P(Y = y_i | X = x_i)$$

	Unnormalized log-probabilities / logits		unnormalized probabilities		probabilities
cat	**3.2**	exp →	**24.5**	normalize →	**0.13**
car	5.1		164.0		0.87
frog	-1.7		0.18		0.00

Probabilities must be >= 0

Probabilities must sum to 1

$L_i = -\log(0.13)$
$= \mathbf{2.04}$

Maximum Likelihood Estimation
Choose weights to maximize the likelihood of the observed data
(See EECS 445 or EECS 545)

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$ Softmax function

Probabilities must be >= 0

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

	Unnormalized log-probabilities / logits	exp →	unnormalized probabilities	normalize →	probabilities	→ Compare ←	Correct probs
cat	**3.2**		**24.5**		**0.13**		**1.00**
car	5.1		164.0		0.87		0.00
frog	-1.7		0.18		0.00		0.00

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$ Softmax function

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Probabilities must be >= 0

Probabilities must sum to 1

	Unnormalized log-probabilities / logits	exp →	unnormalized probabilities	normalize →	probabilities	→ Compare ←	Correct probs
cat	**3.2**		**24.5**		**0.13**		**1.00**
car	5.1		164.0		0.87		0.00
frog	-1.7		0.18		0.00		0.00

Kullback–Leibler divergence

$$D_{KL}(P \| Q) = \sum_y P(y) \log \frac{P(y)}{Q(y)}$$

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$ Softmax function

Probabilities must be >= 0

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

	Unnormalized log-probabilities / logits	$\xrightarrow{\text{exp}}$	unnormalized probabilities	$\xrightarrow{\text{normalize}}$	probabilities		Correct probs
cat	**3.2**		**24.5**		**0.13**	→ Compare ←	**1.00**
car	5.1		164.0		0.87		0.00
frog	-1.7		0.18		0.00		0.00

Cross Entropy

$$H(P, Q) =$$
$$H(p) + D_{KL}(P \| Q)$$

Cross-Entropy Loss (Multinomial Logistic Regression)

cat	**3.2**
car	5.1
frog	-1.7

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

Softmax function

Maximize probability of correct class

$$L_i = -\log P(Y = y_i | X = x_i)$$

Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

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Maximize probability of correct class

$$L_i = -\log P(Y = y_i | X = x_i)$$

Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

Q: What is the min / max possible loss L_i?

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$ Softmax function

cat	**3.2**
car	5.1
frog	-1.7

Maximize probability of correct class

$$L_i = -\log P(Y = y_i | X = x_i)$$

Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

Q: What is the min / max possible loss L_i?

A: Min 0, max +infinity

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

cat	**3.2**
car	5.1
frog	-1.7

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

Softmax function

Maximize probability of correct class

$$L_i = -\log P(Y = y_i | X = x_i)$$

Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

Q: If all scores are small random values, what is the loss?

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax function

cat	**3.2**
car	5.1
frog	-1.7

Maximize probability of correct class

$$L_i = -\log P(Y = y_i | X = x_i)$$

Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

Q: If all scores are small random values, what is the loss?

A: -log(C)
log(10) ≈ 2.3

Cross-Entropy vs SVM Loss

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $\boxed{y_i = 0}$

Q: What is cross-entropy loss?
What is SVM loss?

Cross-Entropy vs SVM Loss

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $\boxed{y_i = 0}$

Q: What is cross-entropy loss?
What is SVM loss?

A: Cross-entropy loss > 0
SVM loss = 0

Cross-Entropy vs SVM Loss

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $\boxed{y_i = 0}$

Q: What happens to each loss if I slightly change the scores of the last datapoint?

Cross-Entropy vs SVM Loss

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $\boxed{y_i = 0}$

Q: What happens to each loss if I slightly change the scores of the last datapoint?

A: Cross-entropy loss will change; SVM loss will stay the same

Cross-Entropy vs SVM Loss

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $\boxed{y_i = 0}$

Q: What happens to each loss if I double the score of the correct class from 10 to 20?

Cross-Entropy vs SVM Loss

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $\boxed{y_i = 0}$

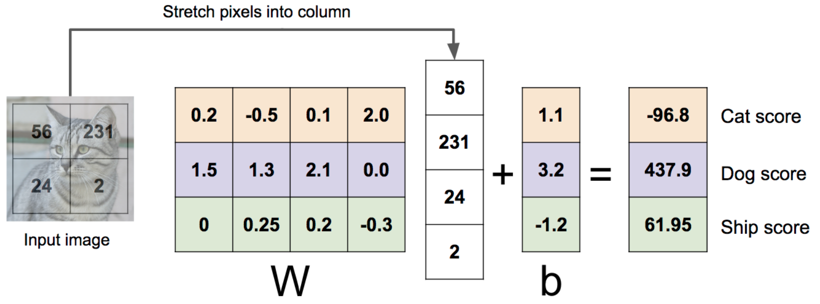
Q: What happens to each loss if I double the score of the correct class from 10 to 20?

A: Cross-entropy loss will decrease, SVM loss still 0

Recap: Three ways to think about linear classifiers

Algebraic Viewpoint

$f(x,W) = Wx$

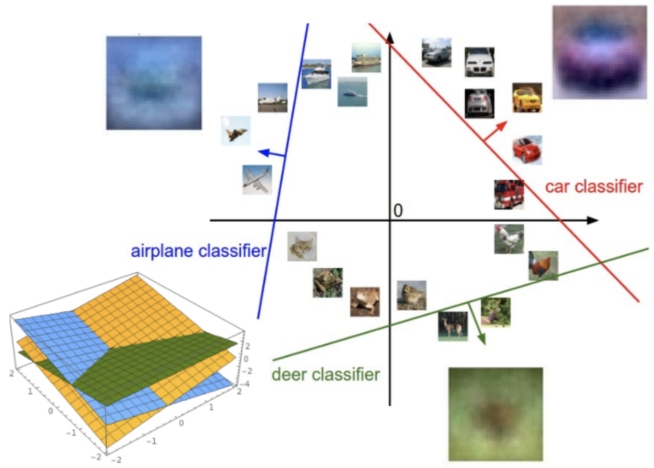


Visual Viewpoint

One template
per class

Geometric Viewpoint

Hyperplanes
cutting up space



Recap: Loss Functions quantify preferences

- We have some dataset of (x, y)
- We have a **score function:**
- We have a **loss function**:

$$s = f(x; W) = Wx$$

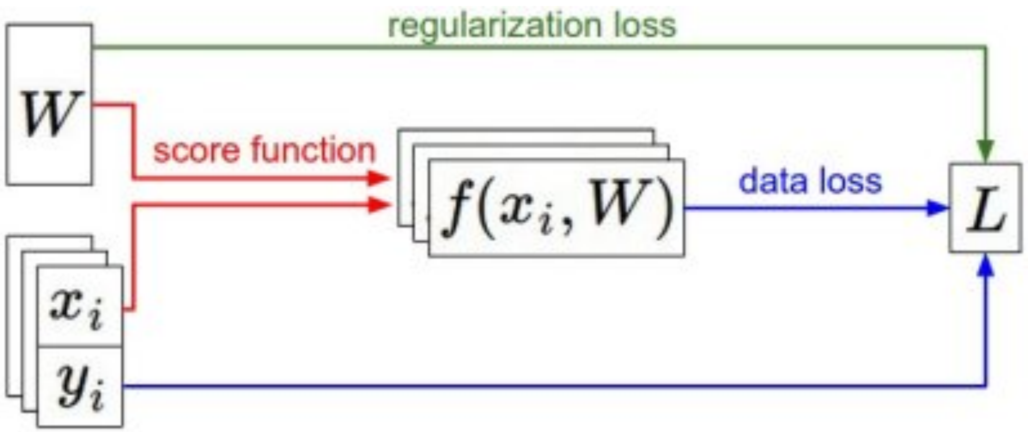
Linear classifier

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$ Softmax

SVM

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$ Full loss



Recap: Loss Functions quantify preferences

- We have some dataset of (x, y)
- We have a **score function:**
- We have a **loss function**:

Q: How do we find the best W?

$$s = f(x; W) = Wx$$

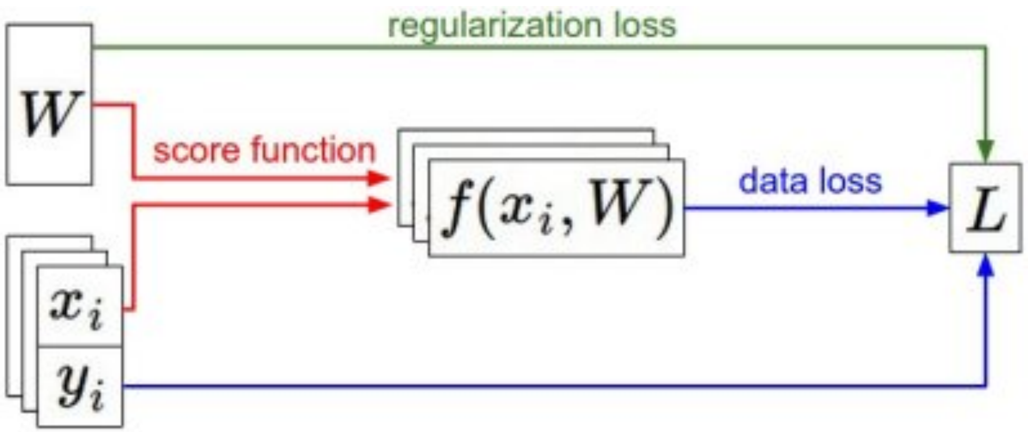
Linear classifier

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$ Softmax

SVM

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$ Full loss



Next time:
Optimization