Epipolar Geometry

EECS 442 – David Fouhey and Justin Johnson

Winter 2021, University of Michigan
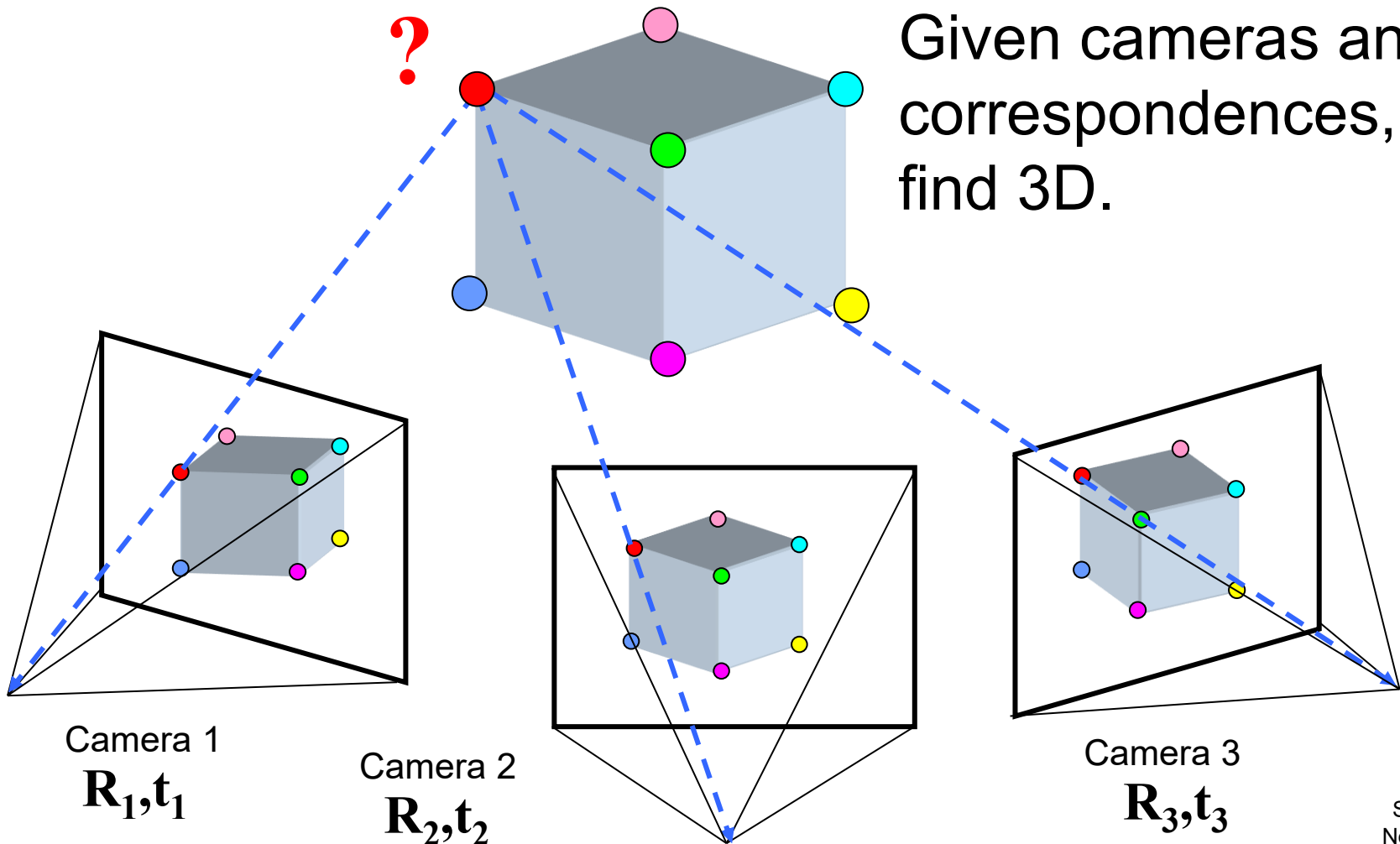
https://web.eecs.umich.edu/~justincj/teaching/eecs442/WI2021/

Multi-view geometry

Multi-view geometry problems

Recovering structure: Given cameras and correspondences, find 3D.

Camera 1
R_1, t_1

Camera 2
R_2, t_2

Camera 3
R_3, t_3

Multi-view geometry problems

Stereo/Epipolar Geomery:
Given 2 cameras and find where a point could be

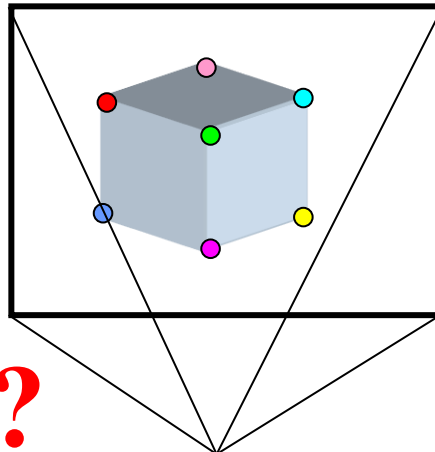Camera 1
R_1, t_1

Camera 2
R_2, t_2
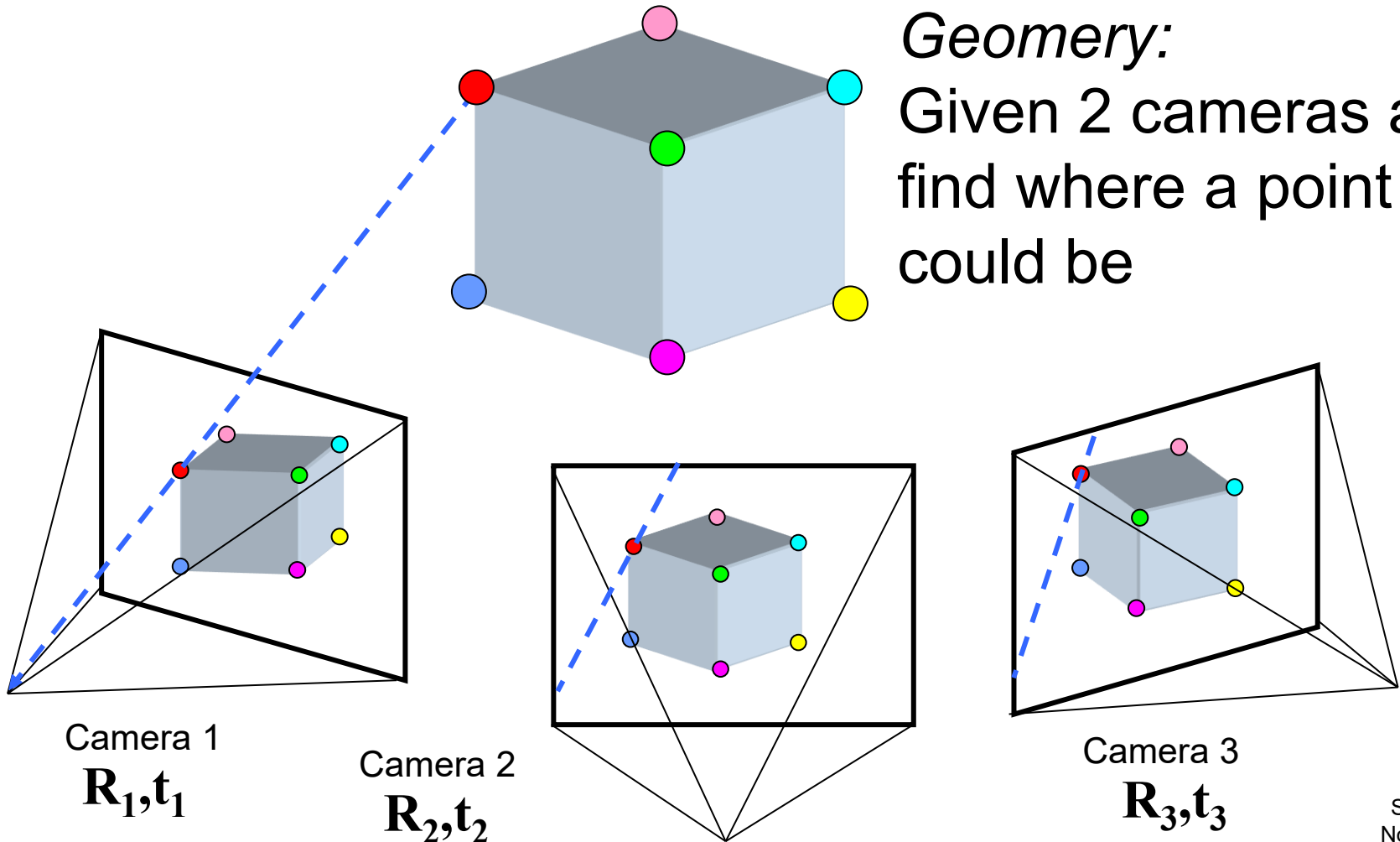
Camera 3
R_3, t_3

Multi-view geometry problems
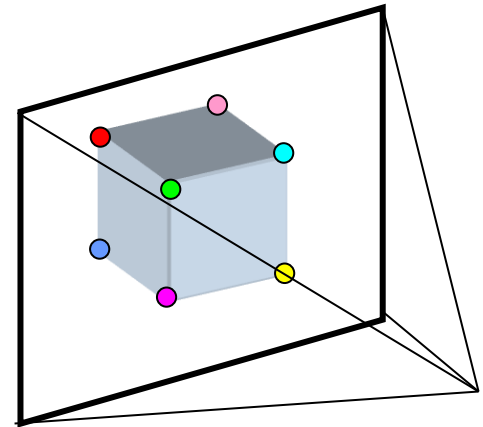
Motion:

Figure out R, t for a set of cameras given correspondences
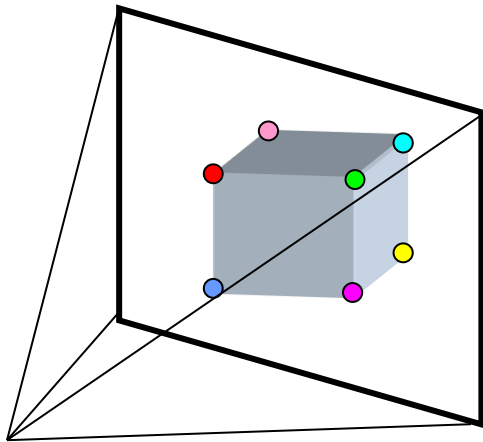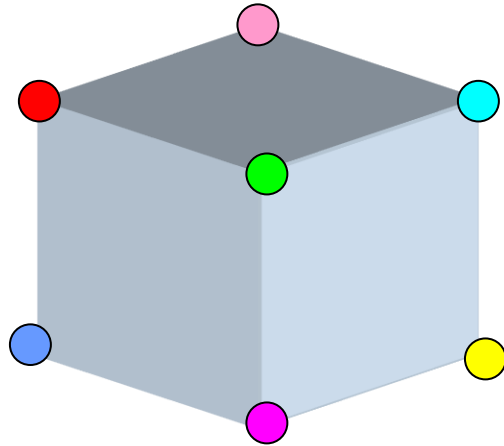
Camera 1
R_1, t_1 **?**

Camera 2
R_2, t_2 **?**

? Camera 3
R_3, t_3
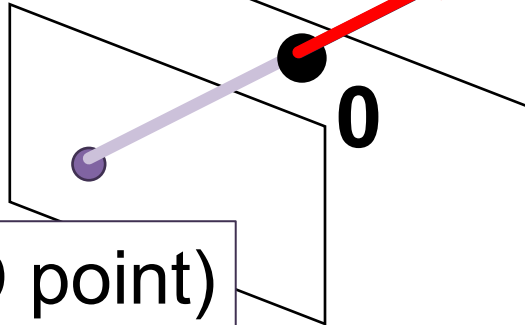
Two-view geometry

Camera Geometry Reminder

K⁻¹p (Ray)
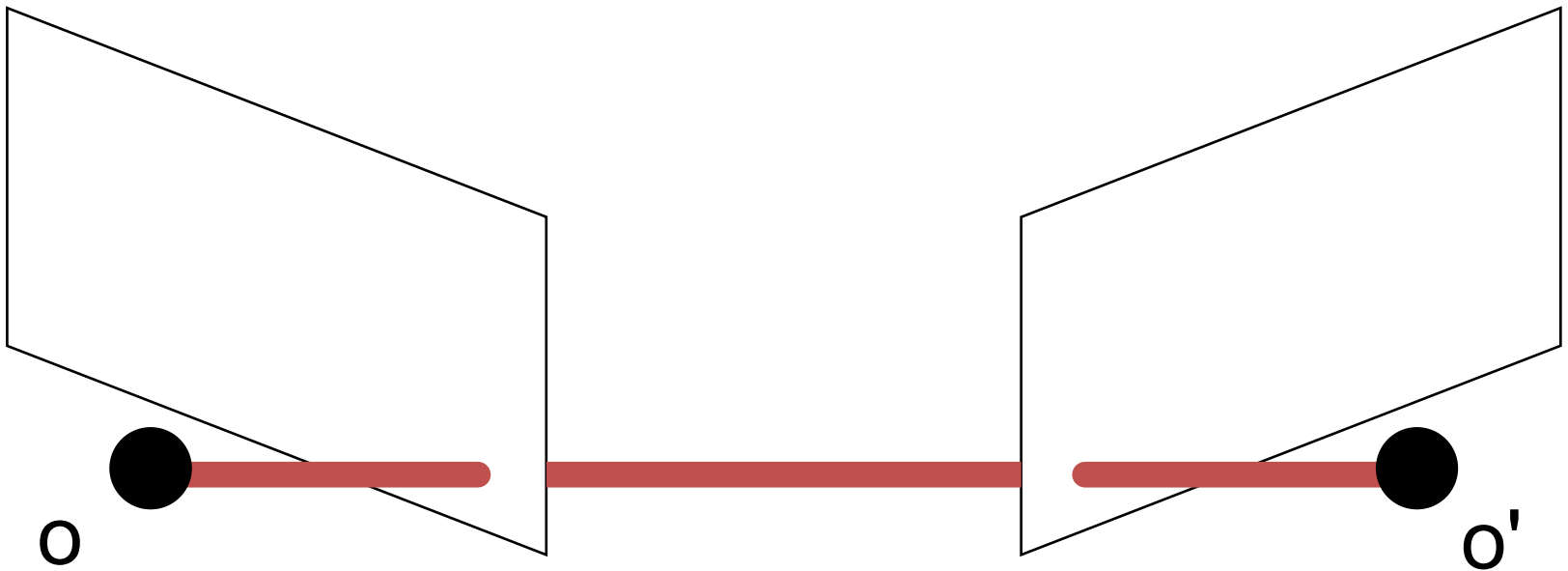3 h. coordinates

X (3D point)
4 h. coordinates

0

p (2D point)
3 h. coordinates
Pretending image
plane is in front

p (2D point)
3 h. coordinates
Actual location

Have camera with pinhole
at origin **0**

Epipolar Geometry

Suppose we have two cameras at origins o, o'
Baseline is the line connecting the origins

Epipolar Geometry
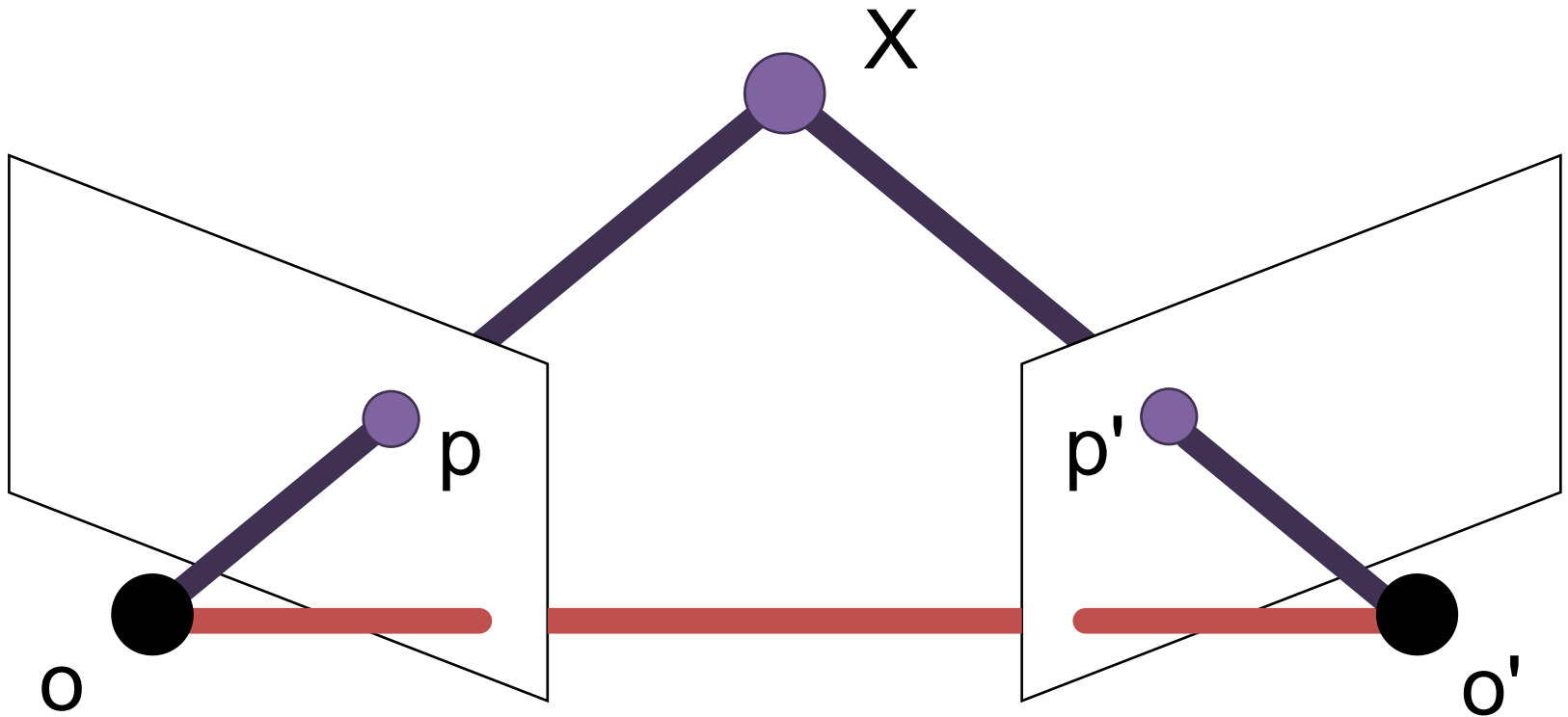
Now add a **point X**, which projects to p and p'

Epipolar Geometry
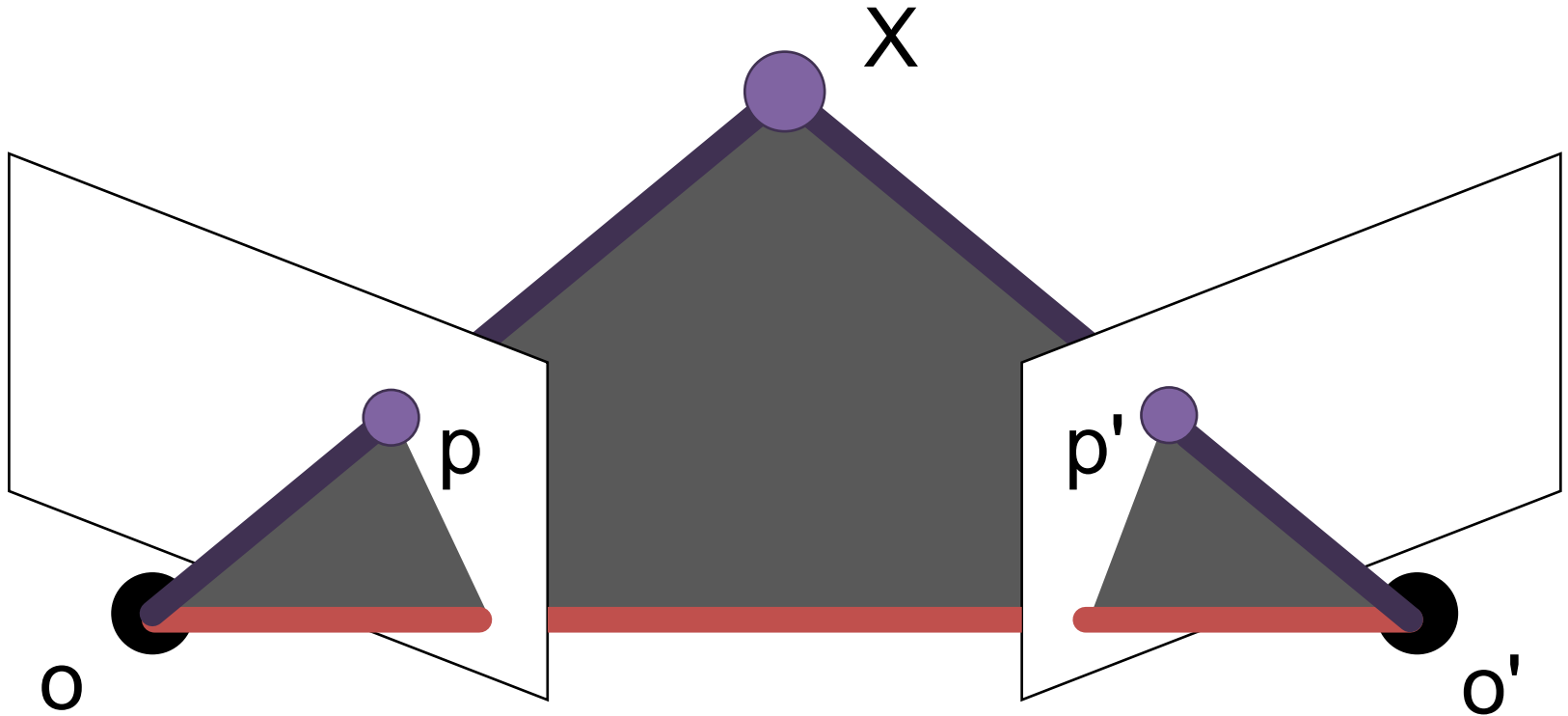
The plane formed by X, o, and o' is called the
epipolar plane
There is a family of planes per o, o'

Epipolar Geometry
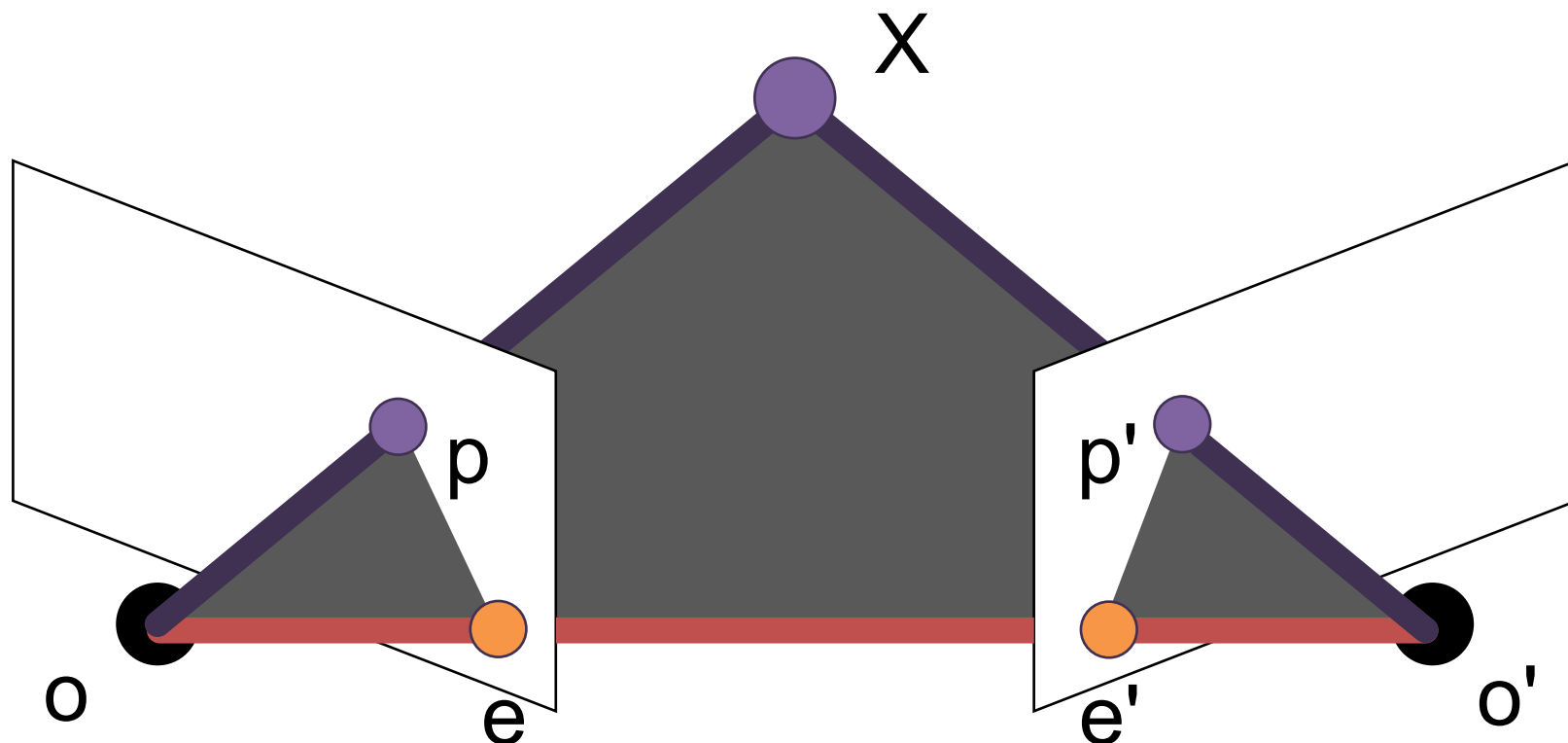
- Epipoles e, e' are where the baseline intersects the image planes
- Projection of other camera in the image plane

The Epipole

Photo by Frank Dellaert

Epipolar Geometry
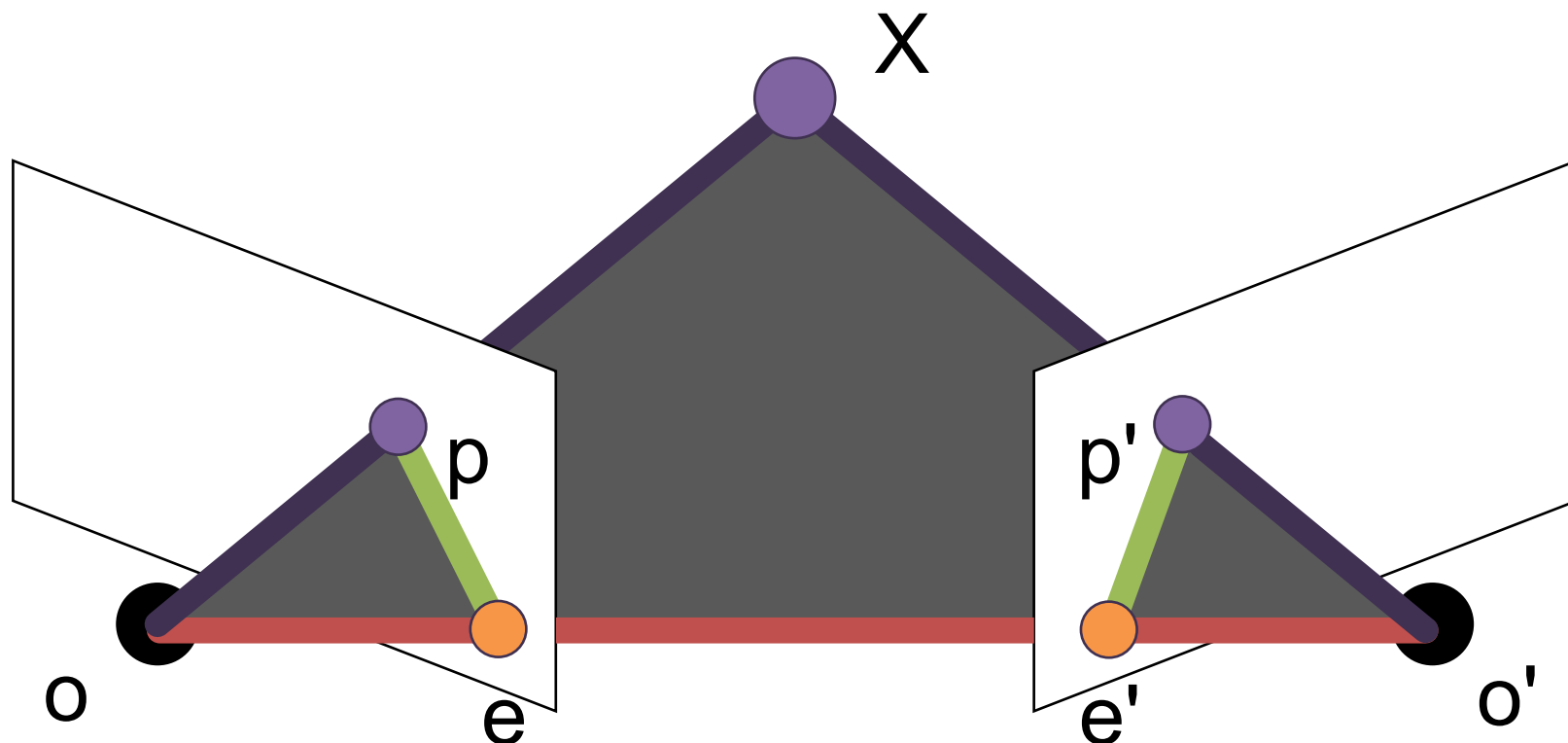
- Epipolar lines go between the epipoles and the projections of the points.
- Intersection of epipolar plane with image plane

Example: Converting Cameras

Epipoles finite, maybe in image; epipolar lines converge

Example: Converging Cameras
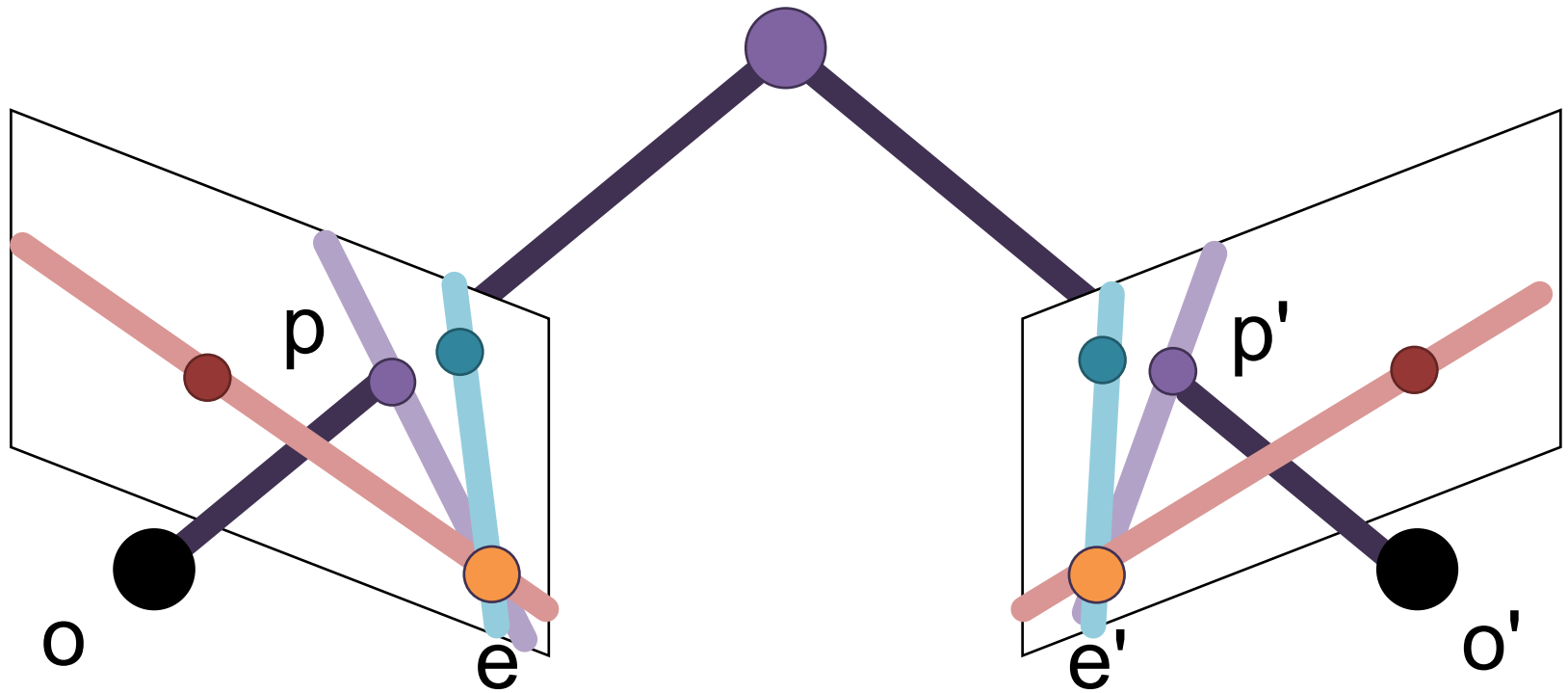
Epipolar lines come in pairs: given a point p, we can construct the epipolar line for p'.

Example 1: Converging Cameras

Image Credit: Hartley & Zisserman

Example: Parallel to Image Plane
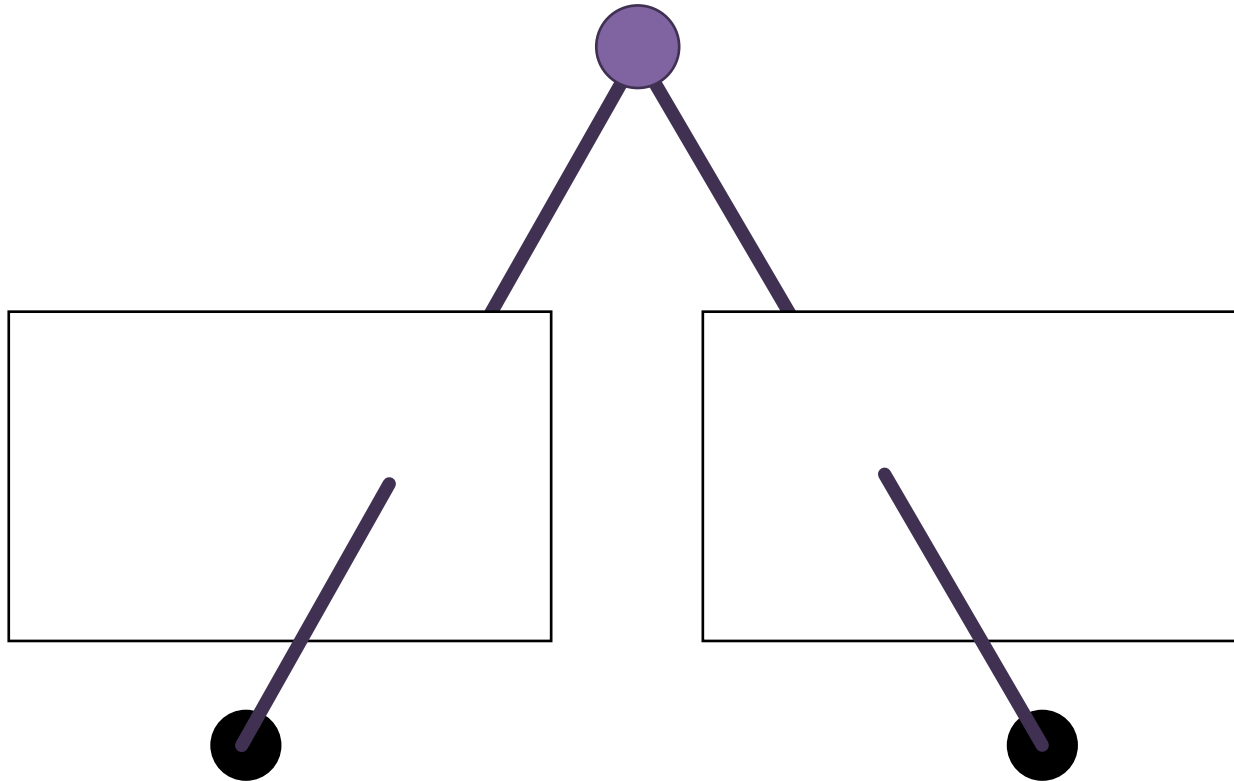
Suppose the cameras are both facing outwards.
Where are the epipoles (proj. of other camera)?

Example: Parallel to Image Plane
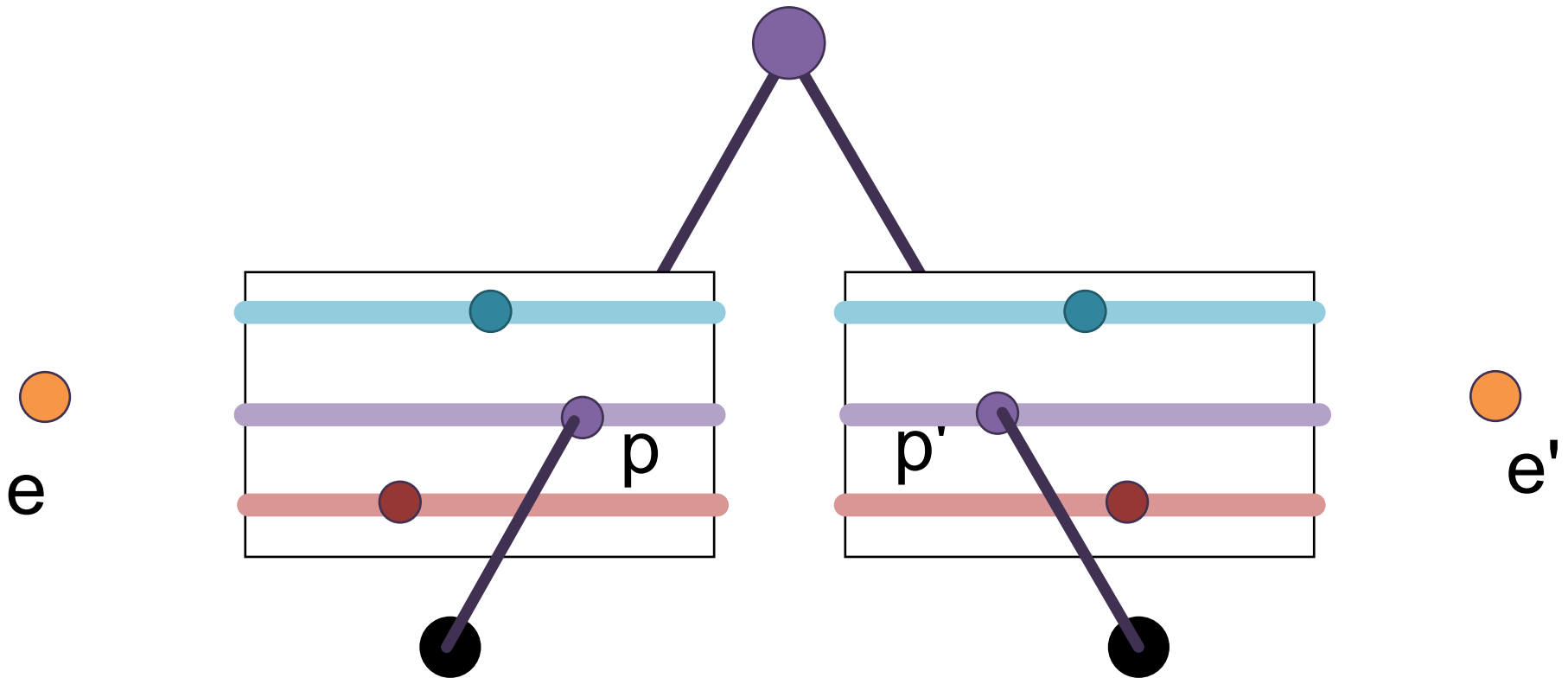
Epipoles *infinitely* far away, epipolar lines parallel

Example: Parallel to Image Plane

Image Credit: Hartley & Zisserman
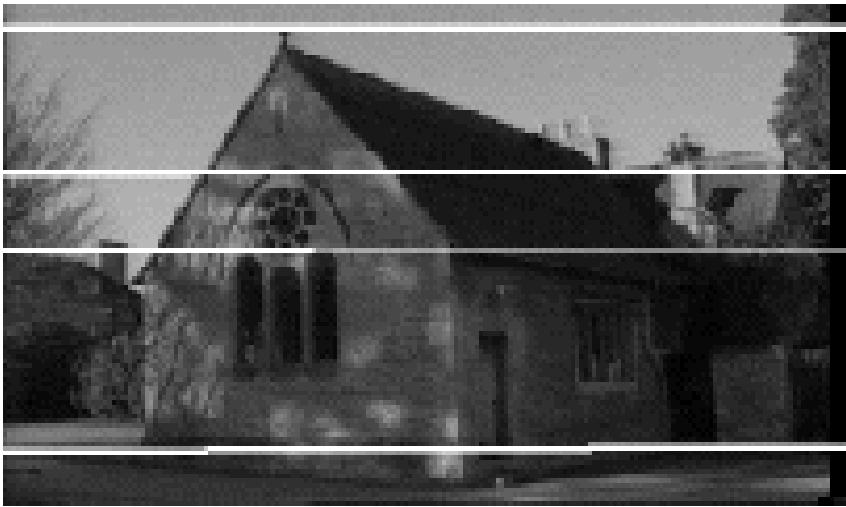
Example: Forward Motion

Example: Forward Motion

Example: Forward Motion

Epipole is focus of expansion / principal point of the camera.

Epipolar lines go out from principal point
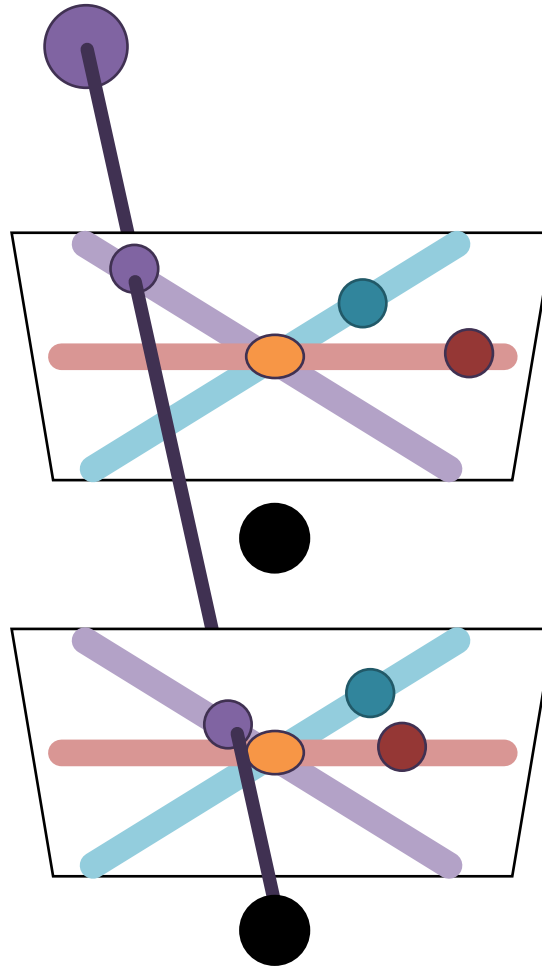
Motion perpendicular to image plane

http://vimeo.com/48425421

HW6 Preview

Image 2 points and epipolar lines from image 1

Image 2 points and epipolar lines from image 1

Image 2 points and epipolar lines from image 1
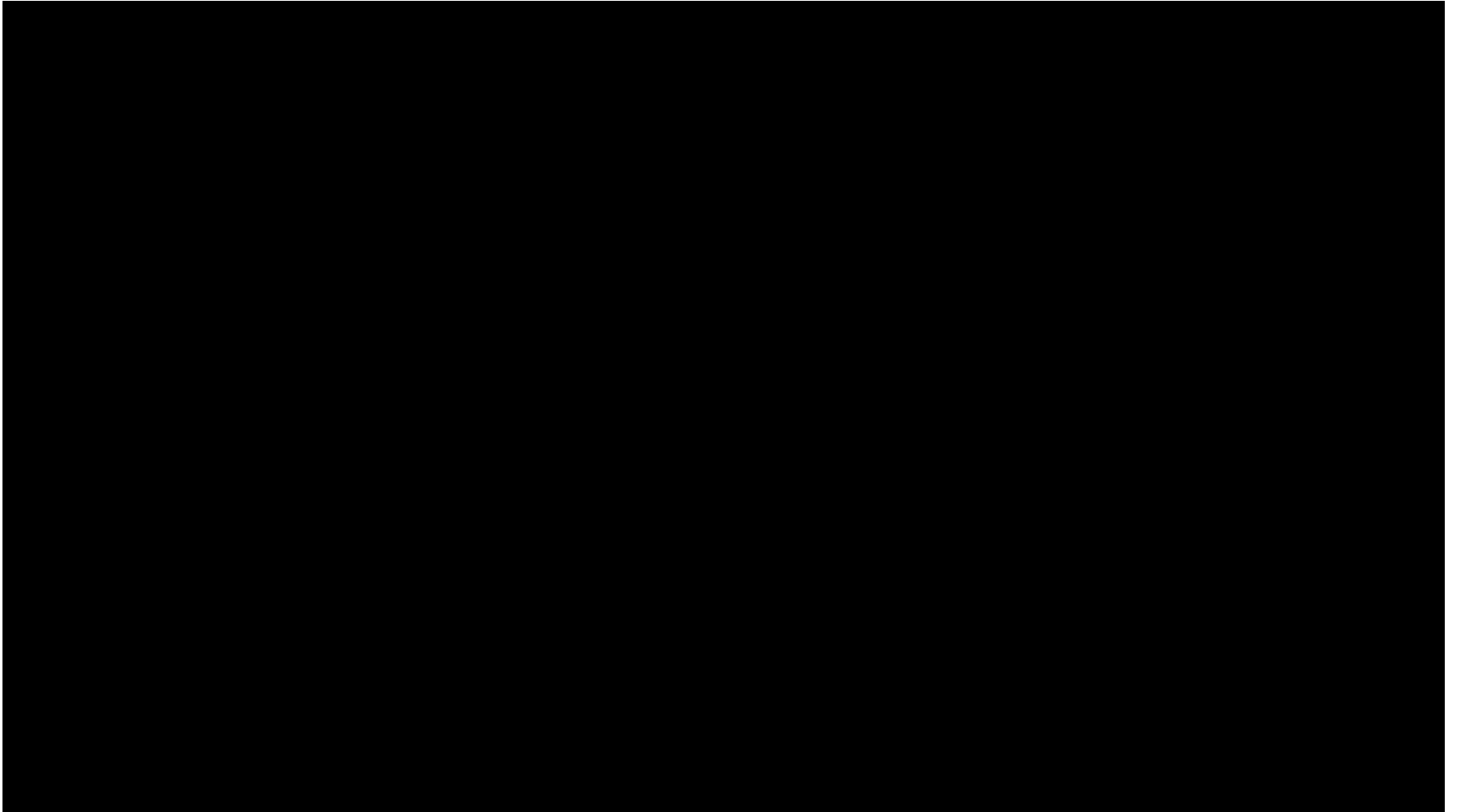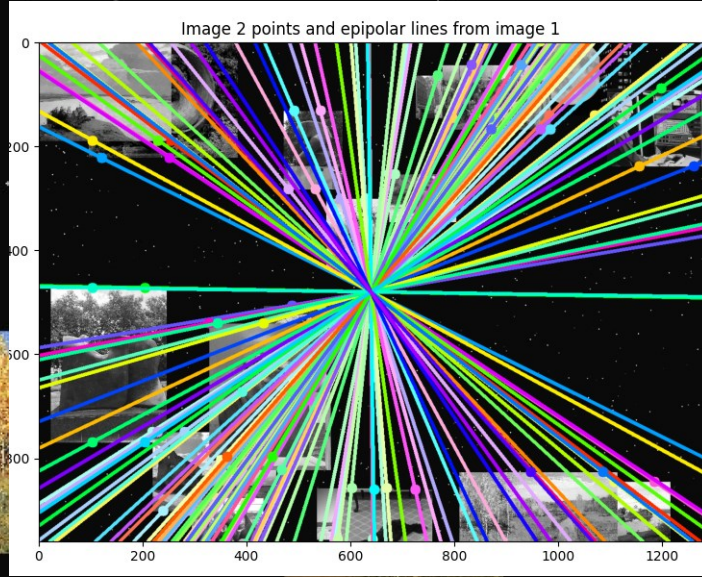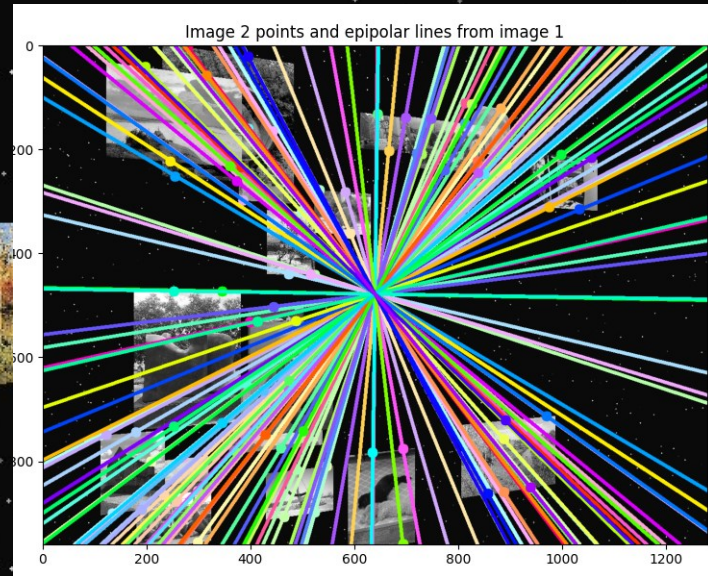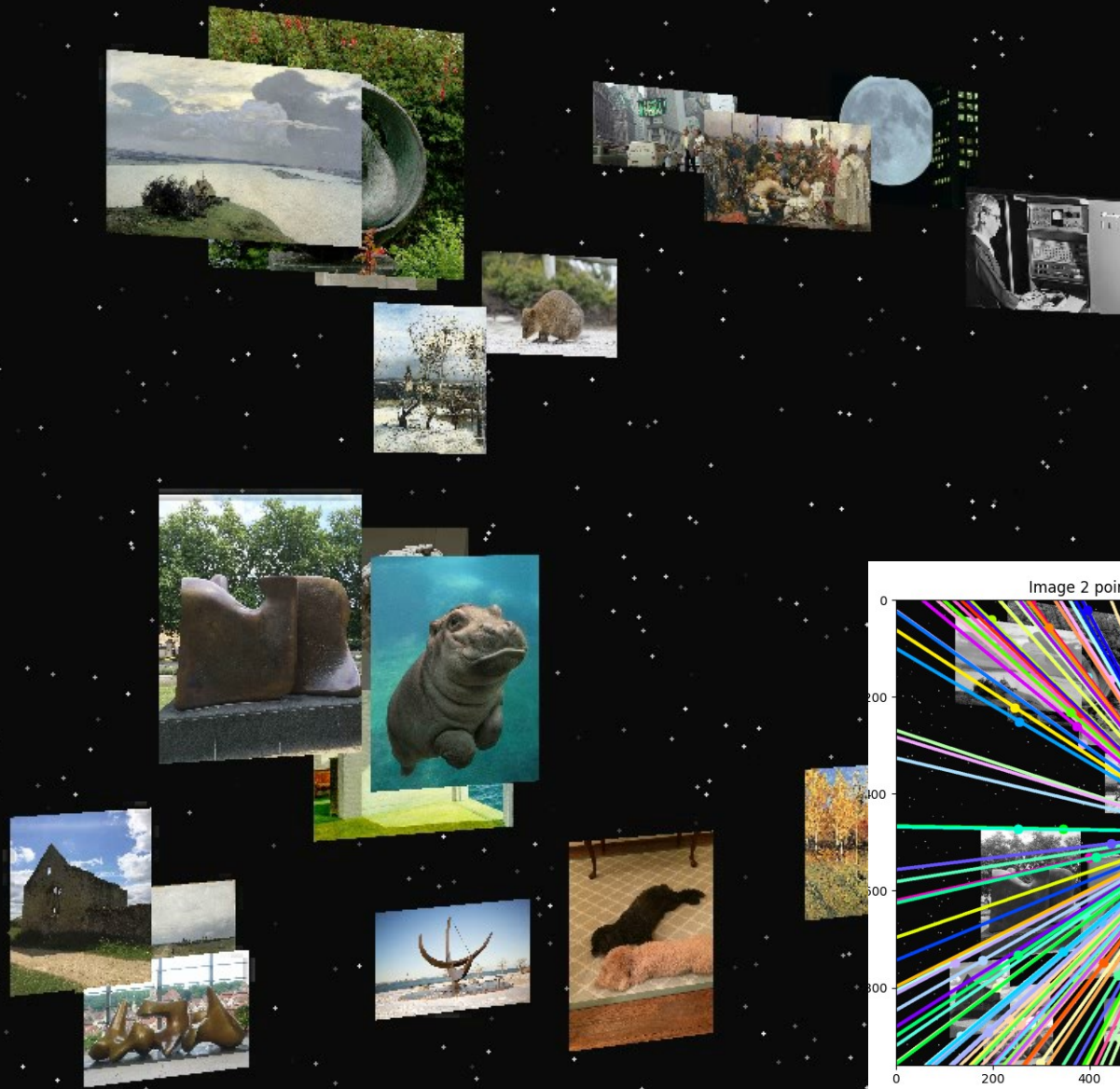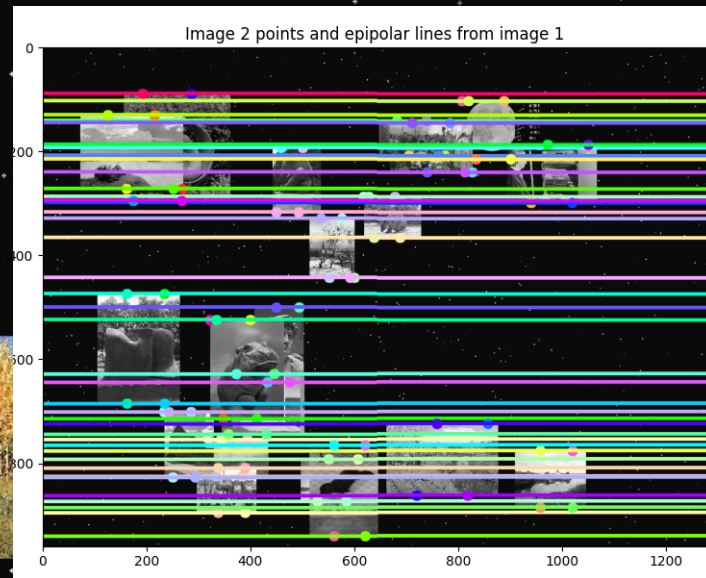
So?
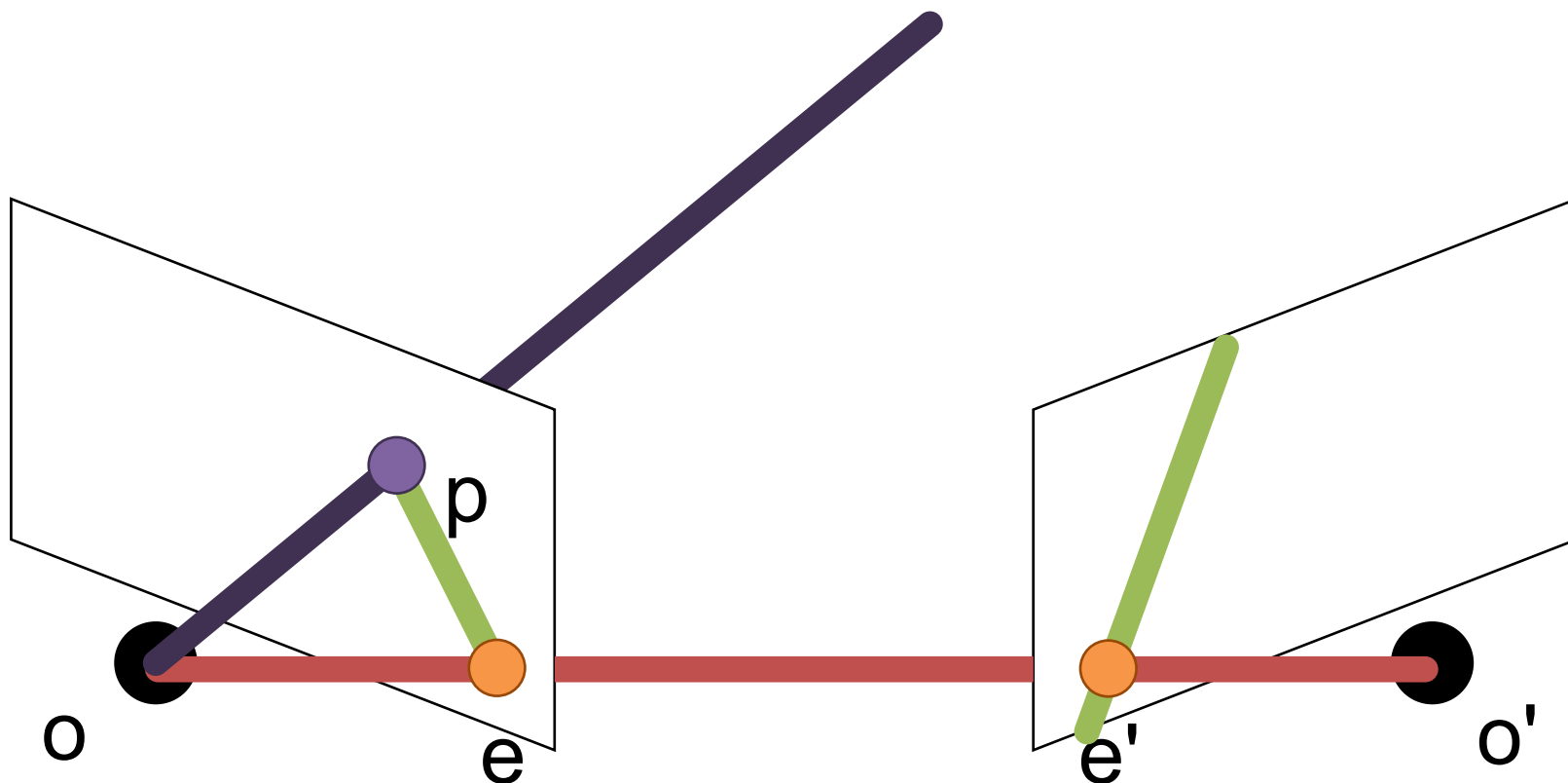
Epipolar Geometry

- Suppose we don't know X and just have p
- Can construct the epipolar line in the other image

Epipolar Geometry
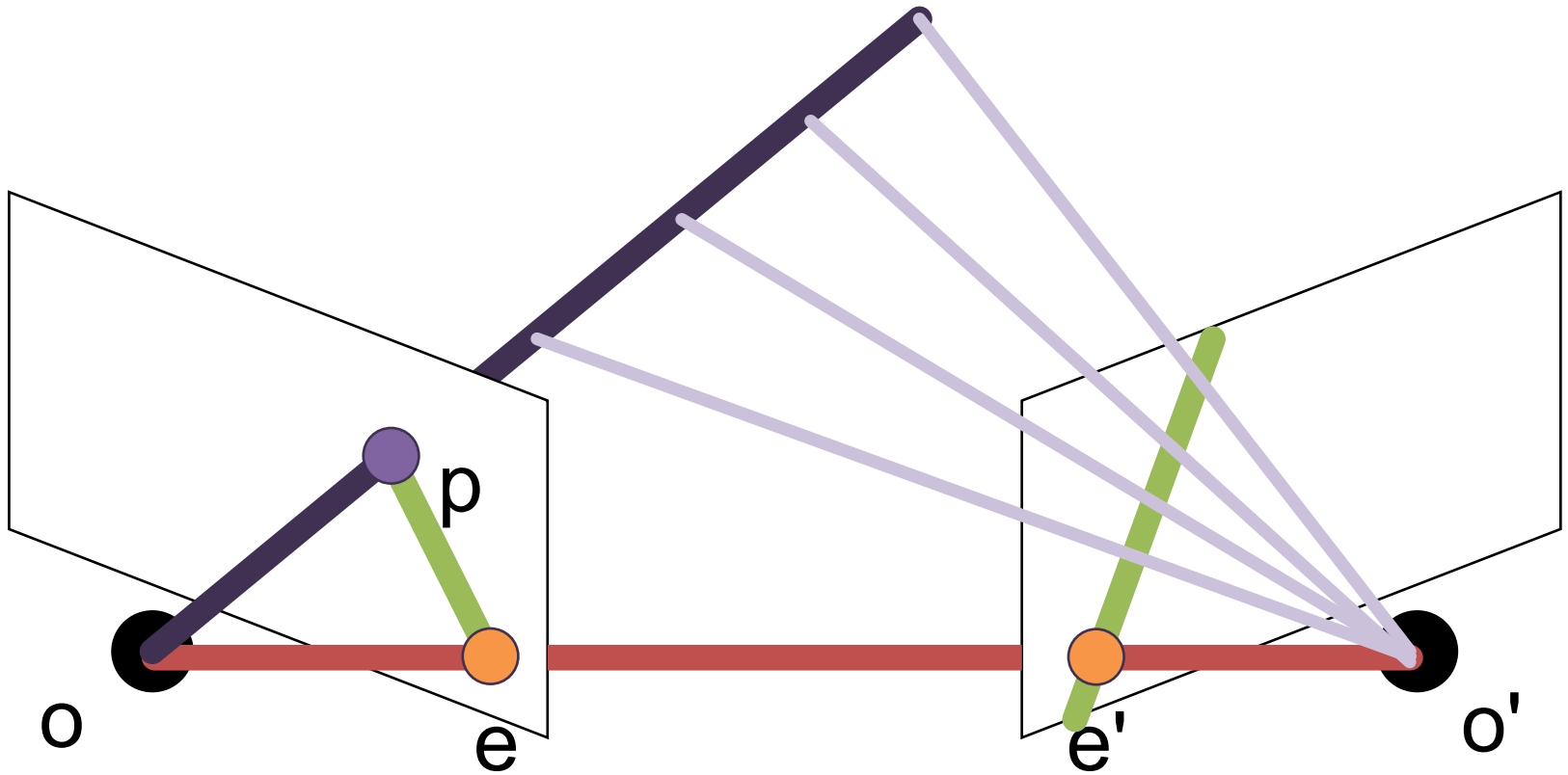
- Suppose we don't know X and just have p
- Corresponding p' is on corresponding epipolar line
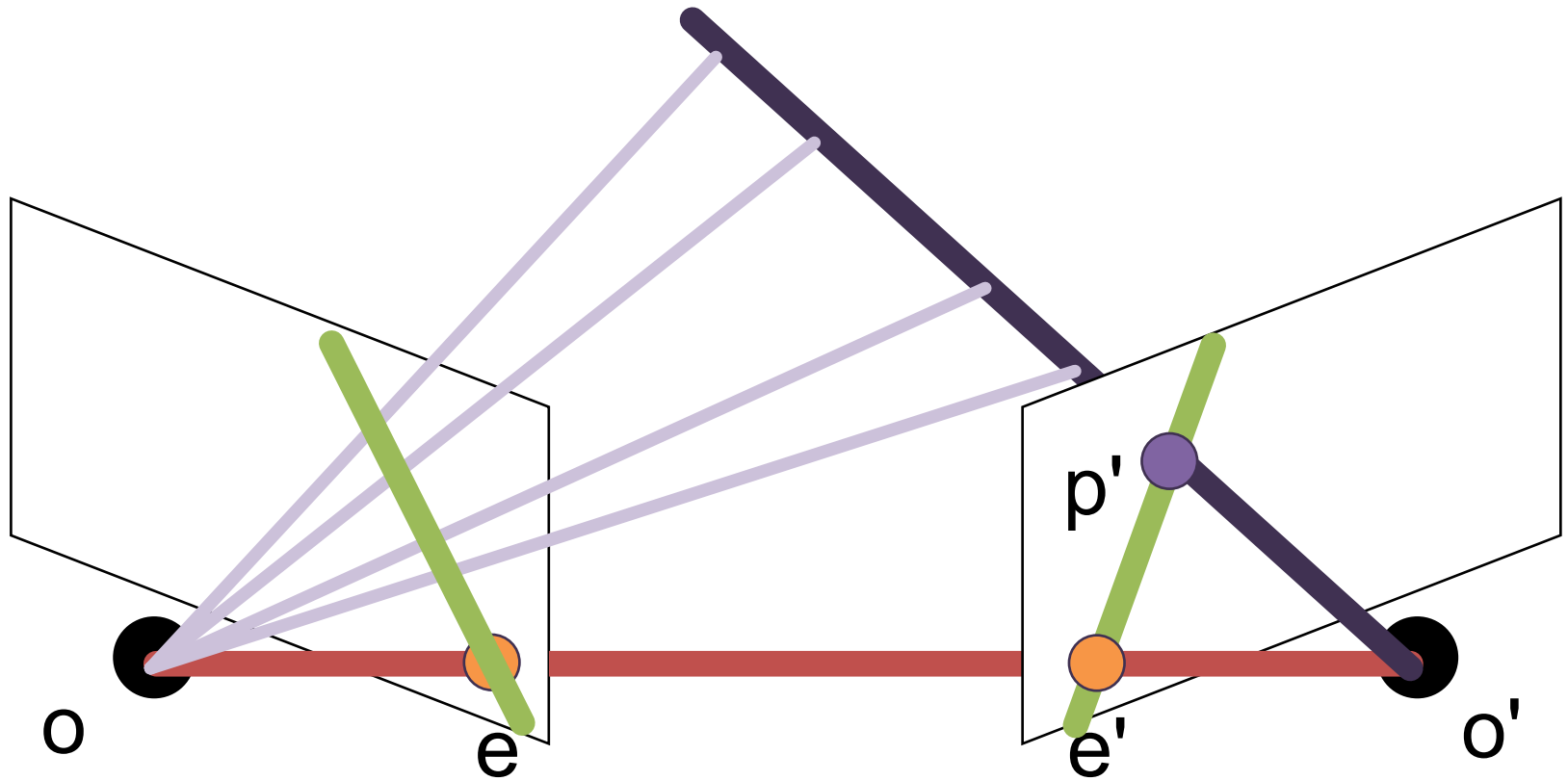
Epipolar Geometry

- Suppose we don't know X and just have p'
- Corresponding p is on corresponding epipolar line

Epipolar Geometry

- If I want to do stereo, I want to find a corresponding pixel for each pixel in the image:

- Naïve search:
  - For each pixel, search every other pixel

- With epipolar geometry:
  - For each pixel, search along each line (1D search)

Epipolar constraint example
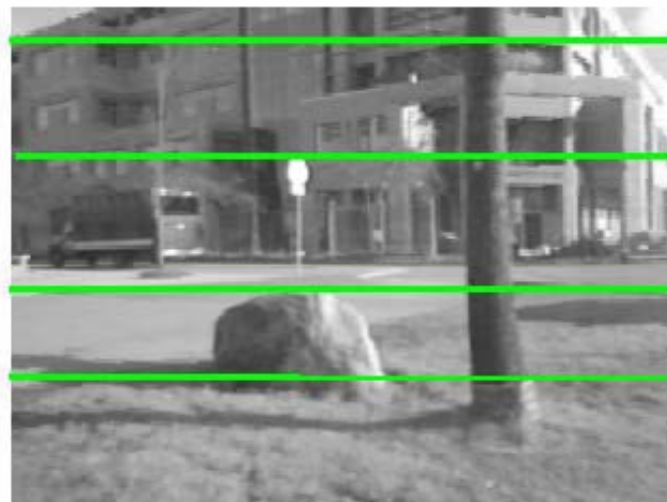
Epipolar Constraint: One Note

- This year, I've updated my slides to match the more common convention. If you read earlier slides from me, they may not match

- I'm pretty sure I got them all right but I apologize for any p vs p' typos!

Epipolar Constraint: Calibrated Case
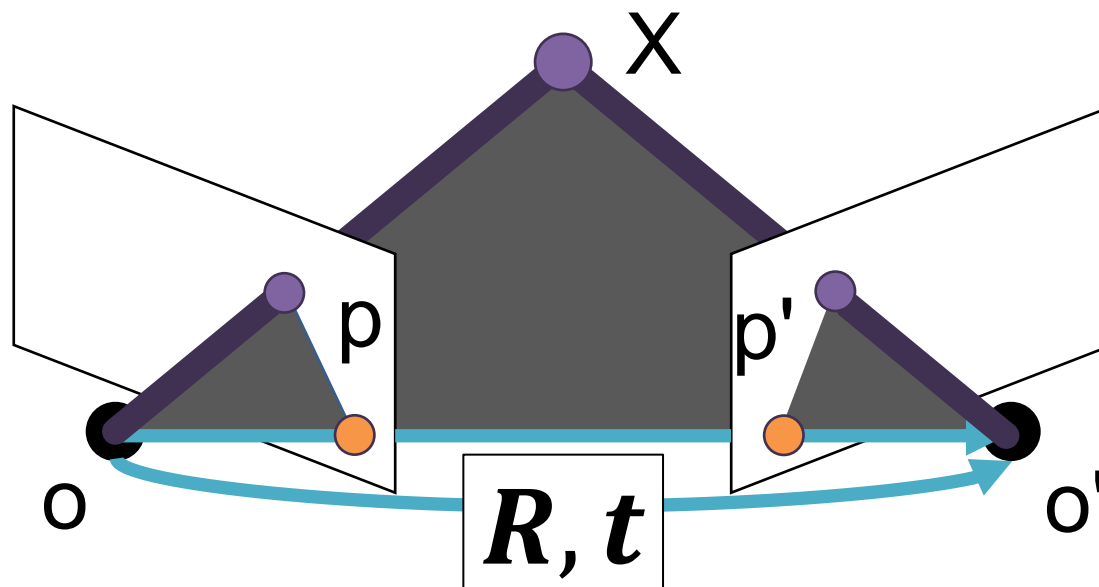
- If we know intrinsic and extrinsic parameters, set coordinate system to first camera
- Projection matrices: $M_1 = K[I, 0]$ and $M_2 = K'[R, t]$
- **What are:**

$$M_1 X \qquad M_2 X \qquad K^{-1} p \qquad K'^{-1} p'$$

Epipolar Constraint: Calibrated Case

$$\widehat{\boldsymbol{p}} = \boldsymbol{K}^{-1}\boldsymbol{p}$$

$$\widehat{\boldsymbol{p}}' = \boldsymbol{K}'^{-1}\boldsymbol{p}'$$

X

p

p'

o

o'

$$\boldsymbol{R}, \boldsymbol{t}$$
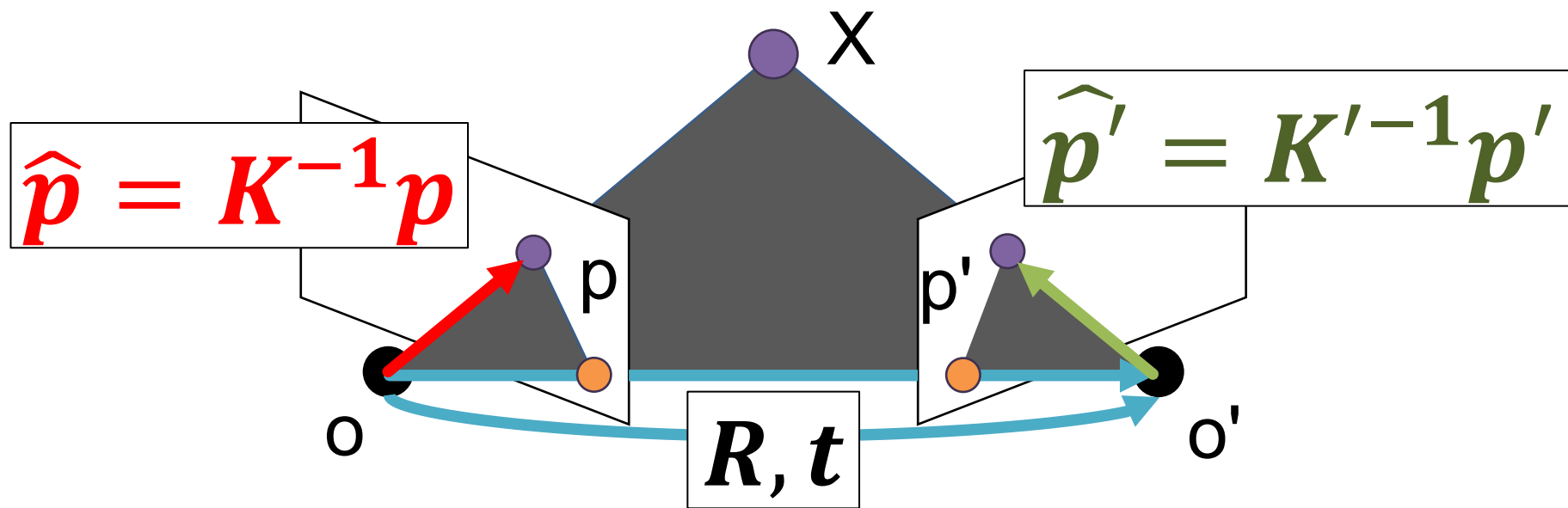
- Given calibration, $\widehat{\boldsymbol{p}} = \boldsymbol{K}^{-1}\boldsymbol{p}$ and $\widehat{\boldsymbol{p}}' = \boldsymbol{K}'^{-1}\boldsymbol{p}'$ are "normalized coordinates"
- Note that $\widehat{\boldsymbol{p}}'$ is actually translated and rotated to o'

Epipolar Constraint: Calibrated Case

$$\widehat{p} = K^{-1}p$$

$$\widehat{p}' = K'^{-1}p'$$

X

p

p'

o

o'

$$R, t$$
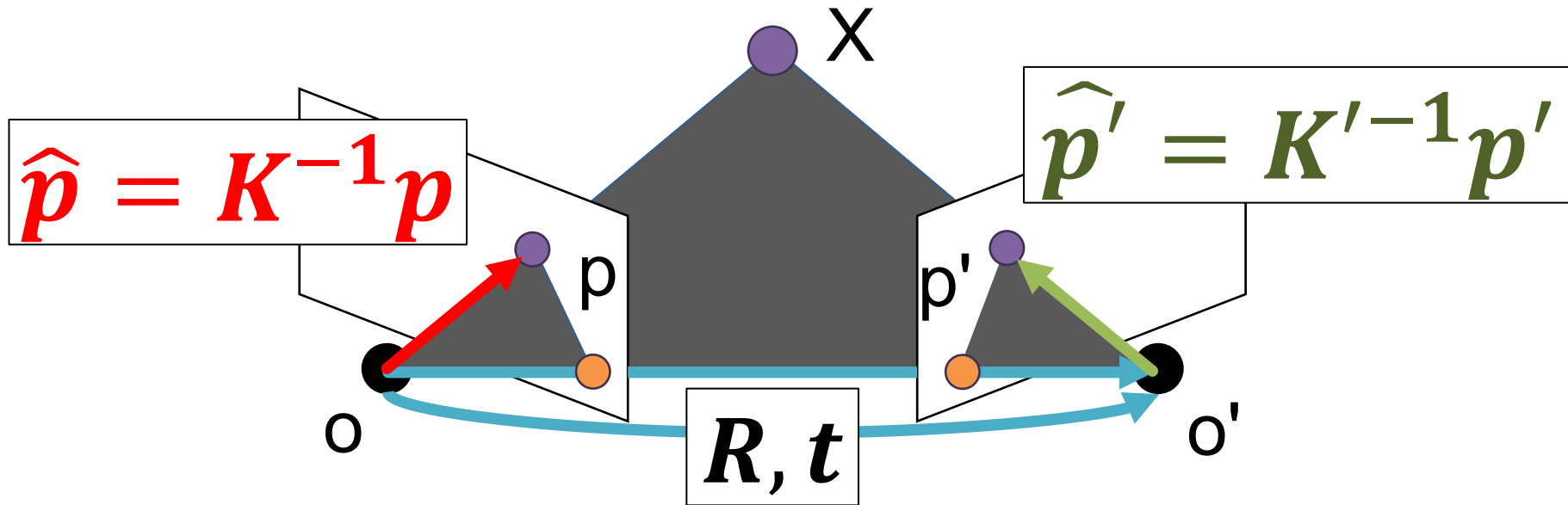
- The following are all co-planar: $\mathbf{R}\widehat{p}$, t, $\widehat{p}'$ (can ignore translation for co-planarity here)
- One way to check co-planarity (triple product):
$$\widehat{p}'^{T}(t \times \mathbf{R}\widehat{p}) = 0$$

Epipolar Constraint: Calibrated Case

$$\hat{p} = K^{-1}p$$

$$\hat{p}' = K'^{-1}p'$$
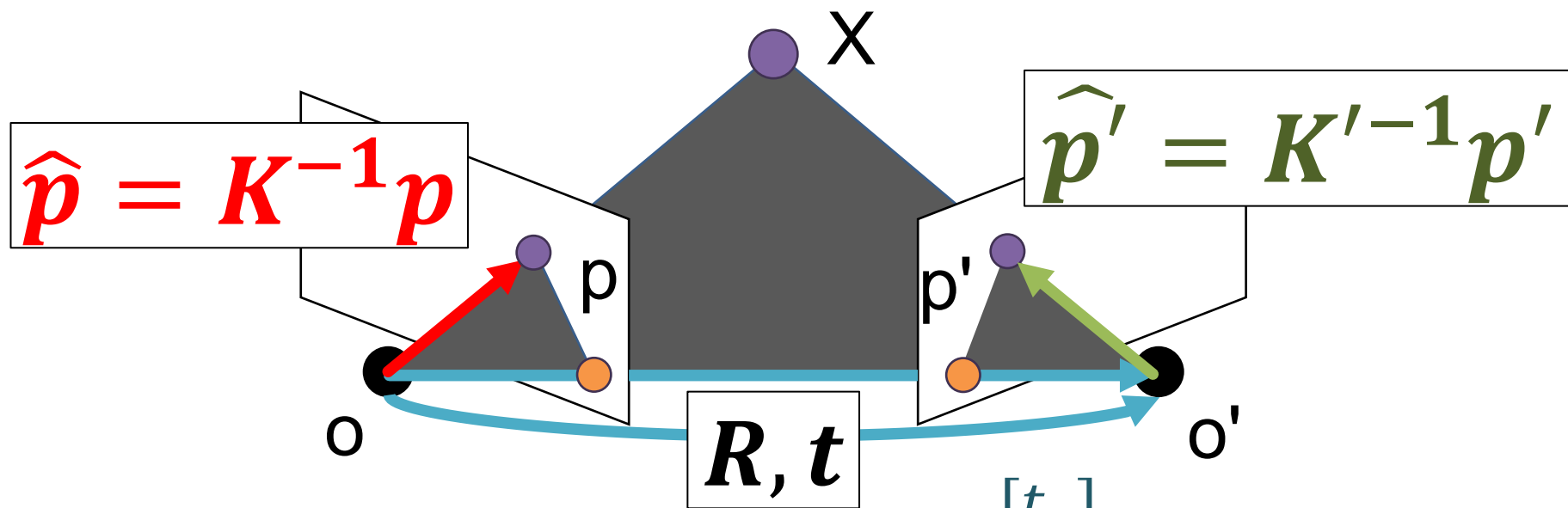
$$\hat{p}'^{T}(t \times R\hat{p}) = 0 \quad \Longrightarrow \quad \hat{p}'^{T} \overset{[t_x]}{\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}} R\hat{p} = 0$$

Want something like $\mathbf{x^T A y}$=0. **What's A?**

Epipolar Constraint: Calibrated Case

$$\hat{p} = K^{-1}p$$

$$\hat{p}' = K'^{-1}p'$$

X

p

p'

o

o'

$$R, t$$
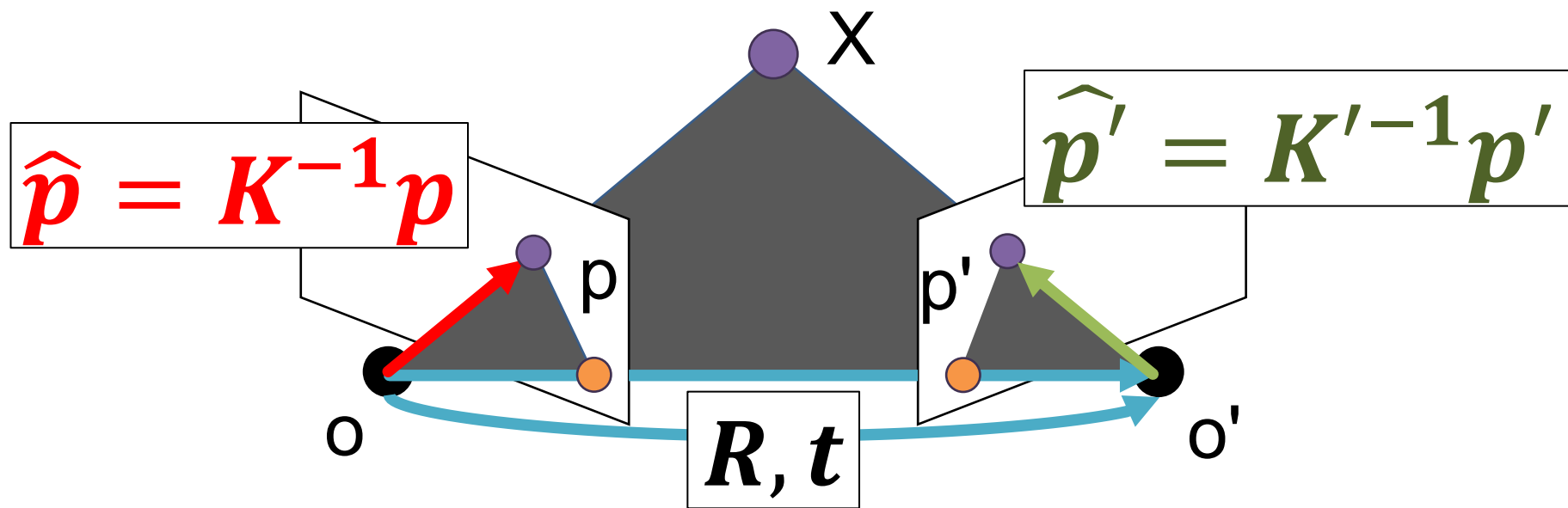
Essential matrix (Longuet-Higgins, 1981): $E = [t_x]R$

If you have a normalized point $\hat{p}$, its correspondence $\hat{p}'$ **must** satisfy $\hat{p}'^T E \hat{p} = 0$

Essential Essential Matrix Facts
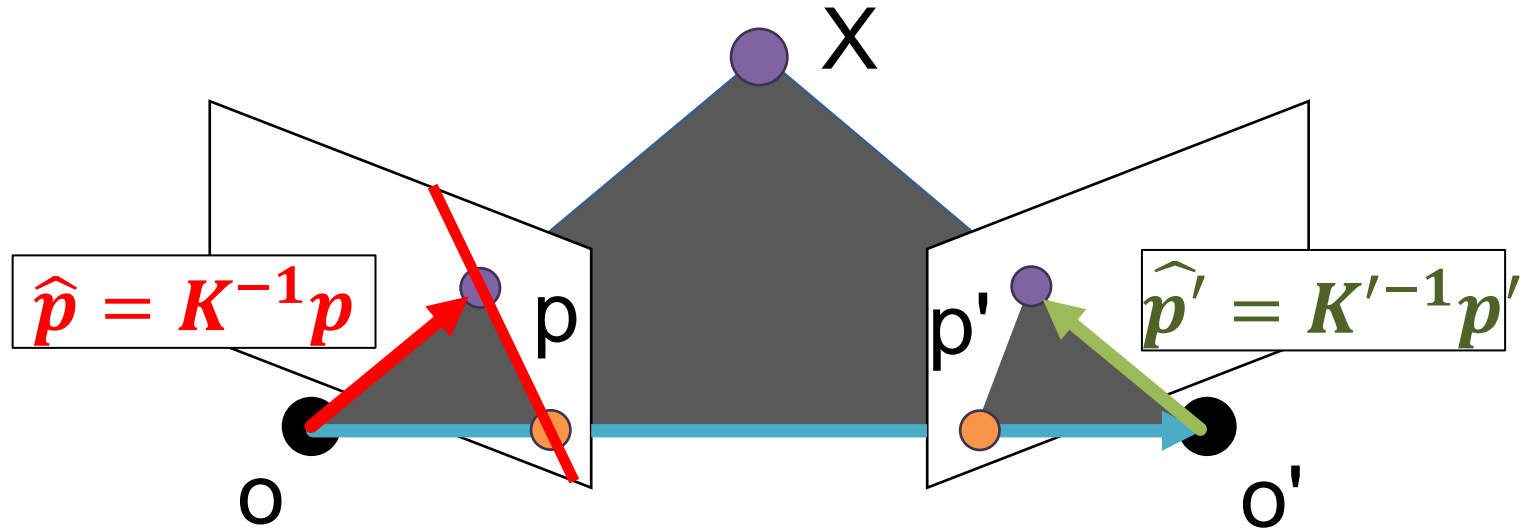
- Suppose we know **E** and $\widehat{p}'^{T} E \widehat{p} = 0$. What is the set $\{x : \widehat{p}'^{T} E x = 0\}$?

- $\widehat{p}'^{T} E$ gives equation of the epipolar line (in ax+by+c=0 form) in image for o.

- What's $E^{T} \widehat{p}'$?

Essential Essential Matrix Facts

$\widehat{p} = K^{-1}p$

$\widehat{p}' = K'^{-1}p'$
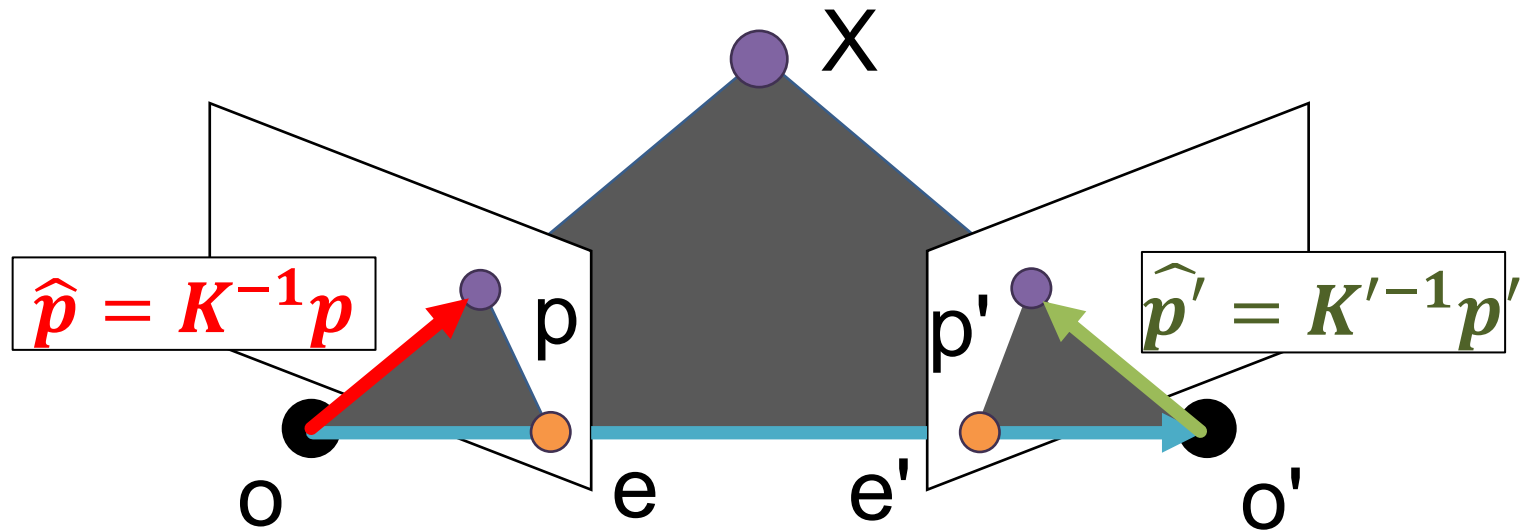
- $E\widehat{e} = 0$ and $E^T\widehat{e}' = 0$ (epipoles are the nullspace of E – note all epipolar lines pass through epipoles)
- **Degrees of freedom (Recall $E = [t_x]R$)?**
- 5 – 3 (R)+ 3 (t) – 1 due to scale ambiguity
- E is singular (rank 2); it has two non-zero and *identical* singular values

Essential Essential Matrix Facts
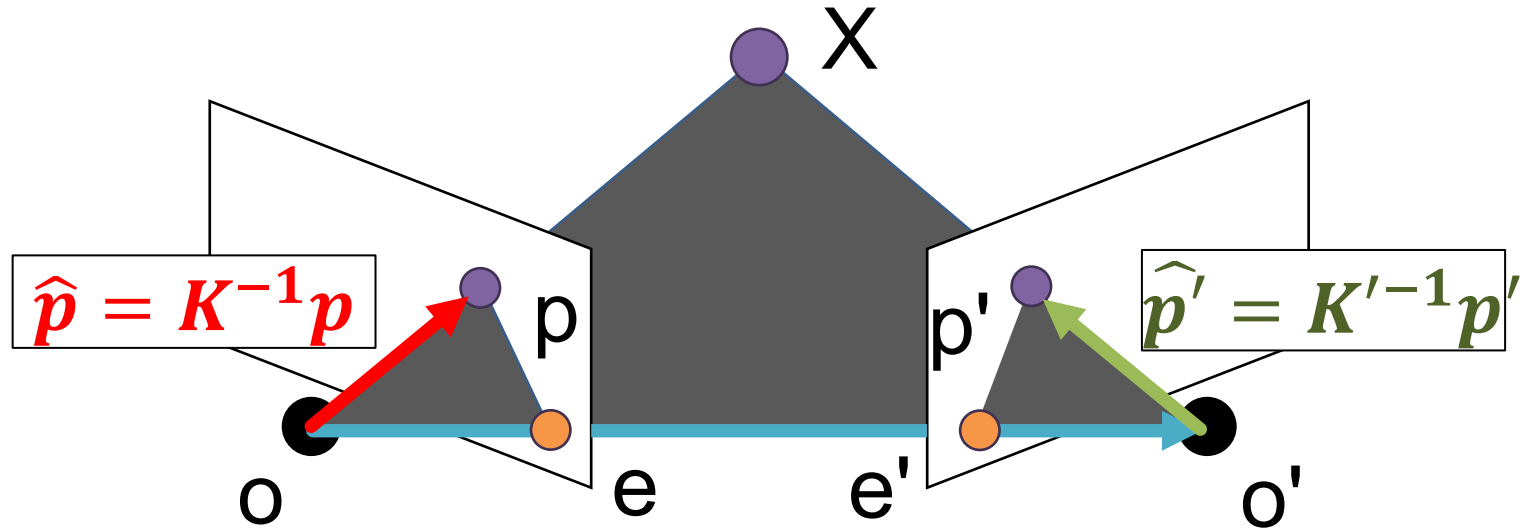
$$\widehat{\boldsymbol{p}} = \boldsymbol{K}^{-1}\boldsymbol{p}$$

$$\widehat{\boldsymbol{p}'} = \boldsymbol{K}'^{-1}\boldsymbol{p}'$$

- One nice thing: if I estimate E from two images (more on this later), it's unique up to easy symmetries

What if we don't know K?
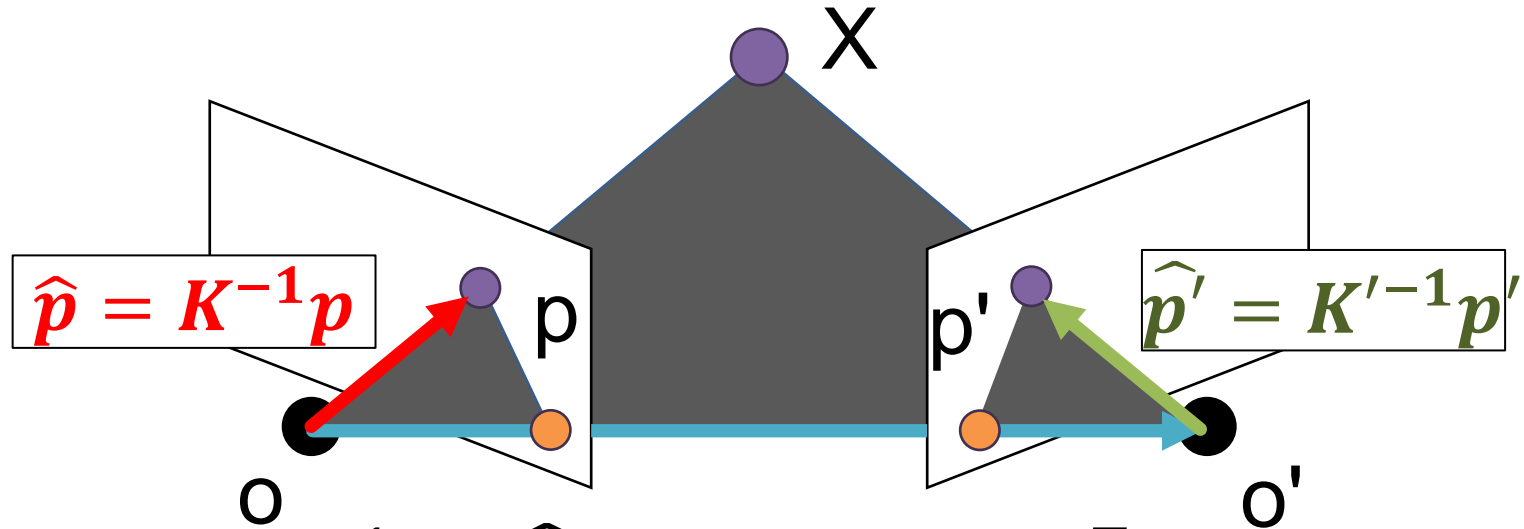
Have: $\hat{p} = K^{-1}p$, $\hat{p}' = K'^{-1}p'$, $\hat{p}'^T E \hat{p} = 0$

$$\left(K'^{-1}p'\right)^T E\left(K^{-1}p\right) = \mathbf{0} \implies p'^T K'^{-T} E K^{-1} p = 0$$

Set: $\underbrace{F = K'^{-T} E K^{-1}}$ Then: $p'^T F p = 0$

Fundamental Matrix (Faugeras and Luong, 1992)

Fundamental Matrix Fundamentals
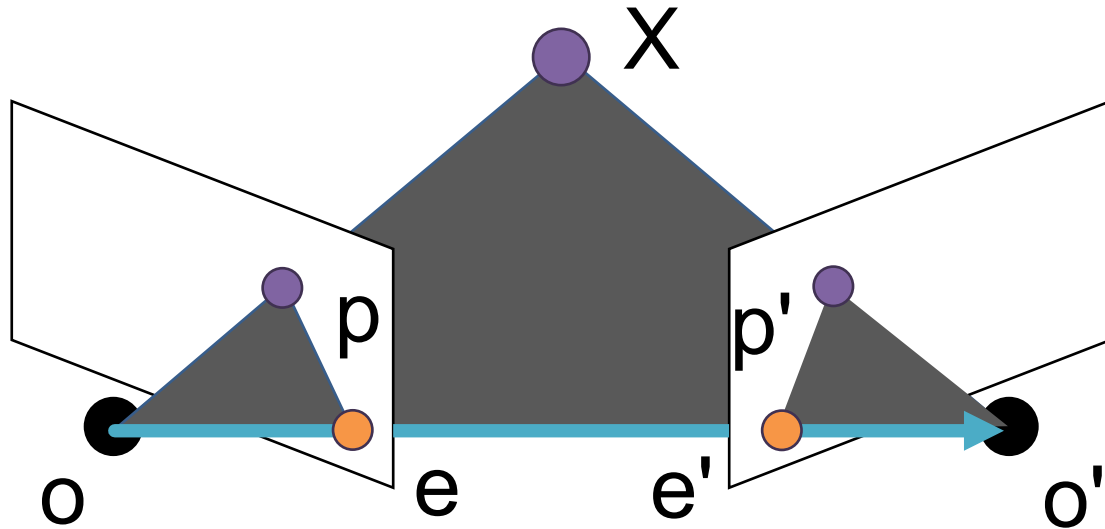
- $\boldsymbol{Fp}, \boldsymbol{F^T p'}$ are epipolar lines for p', p
- $\boldsymbol{Fe} = 0, \boldsymbol{F^T e'} = 0$
- F is singular (rank 2)
- F has seven degrees of freedom
- F definitely not unique

Estimating the fundamental matrix

Estimating the fundamental matrix

- F has 7 degrees of freedom so it's in principle possible to fit F with seven correspondences, but it's a slightly more complex and typically not taught in regular vision classes

Estimating the fundamental matrix

Given correspondences $\boldsymbol{p} = [u, v, 1]$ and $\boldsymbol{p}' = [u', v', 1]$ (e.g., via SIFT) we know: $\boldsymbol{p}'^T \boldsymbol{F} \boldsymbol{p} = 0$

$$[u', v', 1] \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

$$\begin{aligned} &[u'u, u'v, u', v'u, v'v, v', u, v, 1] \cdot \\ &[f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}] \end{aligned} = 0$$

How do we solve for f?
How many correspondences do we need?
Leads to the **eight point algorithm**

Eight Point Algorithm

Each point gives an equation:
$$[u'u, u'v, u', v'u, v'v, v', u, v, 1] \cdot$$
$$[f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}] \quad = 0$$

Stack equations to yield **U**:

$$\boldsymbol{U} = \begin{bmatrix} \vdots \\ u_i'u_i & u_i'v_i & u_i' & v_i'u_i & v_i'v_i & v_i' & u_i & v_i & 1 \\ \vdots \end{bmatrix}$$

Usual eigenvalue stuff to find **f** (**F** unrolled):

$$\arg \min_{\|f\|=1} \|\boldsymbol{Uf}\|_2^2 \quad \longrightarrow \quad$$ Eigenvector of $\boldsymbol{U^T U}$ with smallest eigenvalue

Eight Point Algorithm – Difficulty 1

If we estimate F, we get some 3x3 matrix F.
We know F needs to be singular/rank 2. How do we force F to be singular?

$$U\Sigma V^T = F_{init}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \rightarrow \Sigma' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
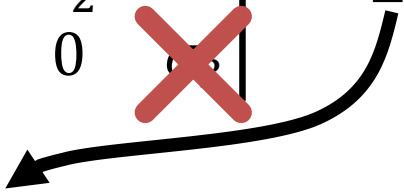
Open it up with SVD, mess with singular values, put it back together.

$$F = U\Sigma'V^T$$

See Eckart–Young–Mirsky theorem if you're interested

Eight Point Algorithm – Difficulty 1

Estimated F
(Wrong)

Estimated+SVD'd F
(Correct)

Eight Point Algorithm – Difficulty 2

$$[u'u, u'v, u', v'u, v'v, v', u, v, 1] \cdot$$
$$[f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}] = 0$$

Recall: u,u' are in pixels. Suppose image is 1Kx1K

How big might uu' be? How big might u' be?

Each row looks like:

$$U = \begin{bmatrix} & & & & \vdots & & & & \\ 10^6 & 10^6 & 10^3 & 10^6 & 10^6 & 10^3 & 10^3 & 10^3 & 1 \\ & & & & \vdots & & & & \end{bmatrix}$$

Then: $U^T U_{1,1}$ is ~10^{12}, $U^T U_{2,9}$ is ~10^3

Eight Point Algorithm – Difficulty 2

Numbers of varying magnitude → instability

Remember: a floating point number (float/double) isn't a "real" number: for sign, coefficient, exponent integers

$(-1)^{sign}$ * coefficient * $2^{exponent}$

Exercise to see how this screws up: add up Gaussian noise (mean=100, std=10), divide by number you added up
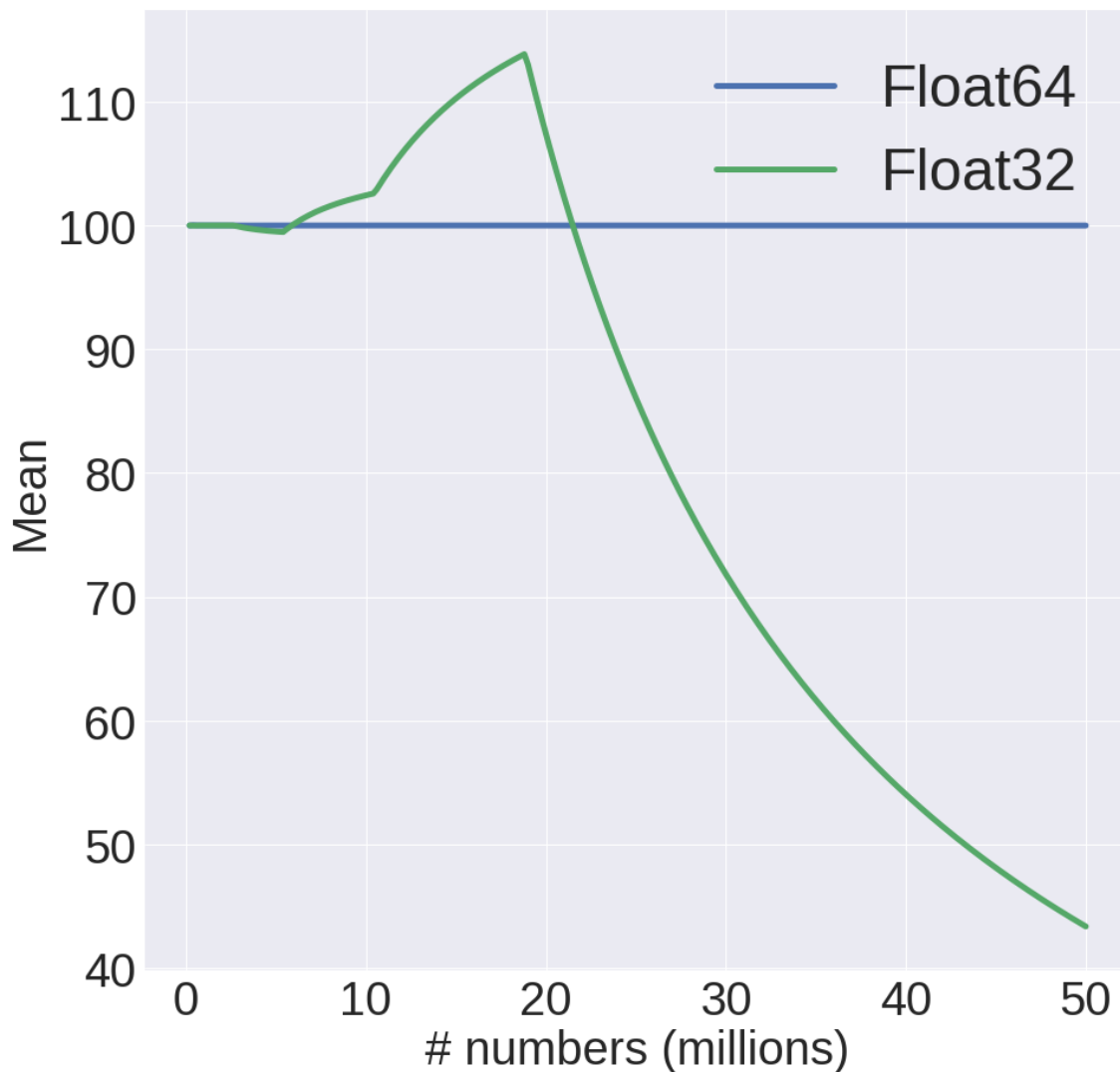
Remember Numerical Instability?

```
Code:

x += N(100,10)
i += 1
mean = x/I
```

Only change is the # of bits in accumulator x

Note: 50M is 50 1Kx1K images

Solution: Normalized 8-point

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels

- Use the eight-point algorithm to compute **F** from the normalized points

- Enforce the rank-2 constraint (for example, take SVD of **F** and throw out the smallest singular value)

- Transform fundamental matrix back to original units: if **T** and **T'** are the normalizing transformations in the two images, than the fundamental matrix in original coordinates is **T'^{T} F T**
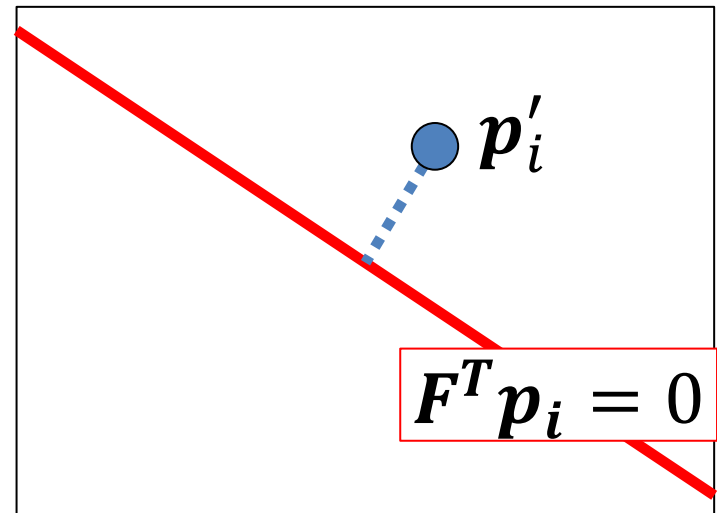
R. Hartley
In defense of the eight-point algorithm TPAMI 1997

Last Trick

Minimizing via U^{T}U minimizes sum of squared *algebraic* distances between points $\boldsymbol{p}_i$ and epipolar lines $\boldsymbol{Fp'}_i$ (or points $\boldsymbol{p'}_i$ and epipolar lines $\boldsymbol{F}^T\boldsymbol{p}_i$):

$$\sum_i \left(\boldsymbol{p'}_i^T \boldsymbol{F} \boldsymbol{p}_i\right)^2$$

May want to minimize *geometric* distance:

$$\sum_i \begin{array}{l} d(\boldsymbol{p'}_i, \boldsymbol{F}\boldsymbol{p}_i)^2 + \\ d(\boldsymbol{p}_i, \boldsymbol{F}^T\boldsymbol{p'}_i)^2 \end{array}$$

$$\boldsymbol{F}^T\boldsymbol{p}_i = 0$$

Comparison
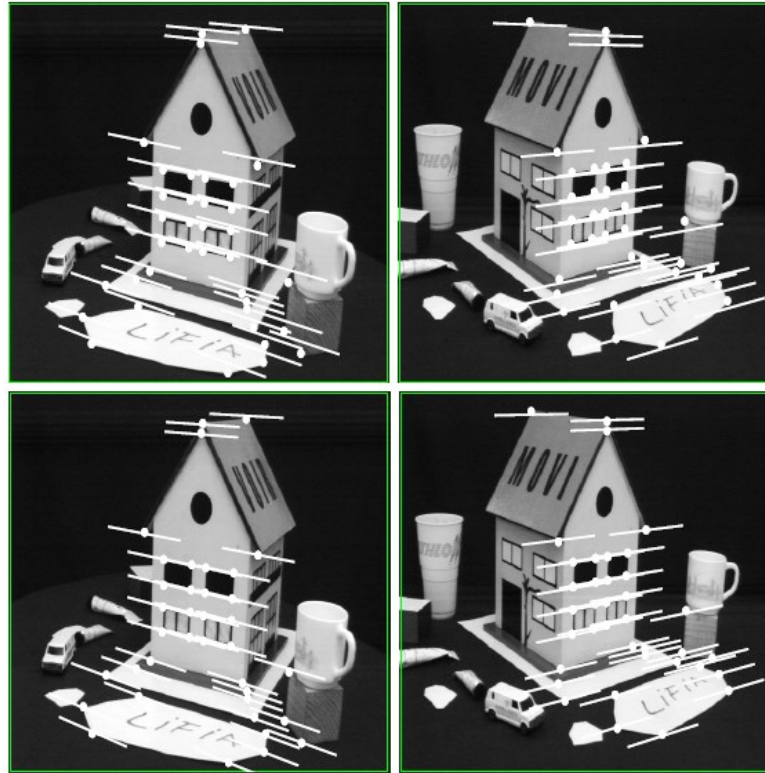
	8-point	Normalized 8-point	Nonlinear least squares
Av. Dist. 1	2.33 pixels	0.92 pixel	0.86 pixel
Av. Dist. 2	2.18 pixels	0.85 pixel	0.80 pixel

The Fundamental Matrix Song

From Epipolar Geometry to Calibration

- Estimating the fundamental matrix is known as "weak calibration"
- If we know the calibration matrices of the two cameras, we can estimate the essential matrix: $E = K'^T F K$
- The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters
- Alternatively, if the calibration matrices are known, the five-point algorithm can be used to estimate relative camera pose