EECS598: Prediction and Learning: It's Only a Game Fall 2013

Lecture 11: Perceptron and Universal Portfolio Selection

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Announcements

- Class on 10/16 in DOW 3150 9:00-10:30

1 Perceptron

Sequence of $(x^1, y^1), ..., (x^T, t^T) \in \mathbb{R}^d.\{-1, 1\}$ Assume $\exists$ unknown $w^* \in \mathbb{R}^d$ such that $\forall t$ we have the margin assumption:

$$\underline{w}^* \cdot \underline{x}^t\, y^t \geq 1 \tag{1.1}$$

$$\|w^*\| \leq \frac{1}{\gamma} \tag{1.2}$$

Perceptron Algorithm:

$$\underline{w}^1 = \overrightarrow{0} \in \mathbb{R}^d \tag{1.3}$$

for t=1,...,T

$$\begin{cases} if\, \underline{w}^t \cdot \underline{x}^t\, y^t > 0 \rightarrow \underline{w}^{t+1} = \underline{w}^t \\ o.w. \;\; \underline{w}^{t+1} = \underline{w}^t + y^t \underline{x}^t \end{cases}$$

Then: number of mistakes of perceptron $\leq \frac{1}{\gamma^2}$ assuming $\|x^t\|_2 \leq 1$

Proof: Use potential function:

$$\Phi_t = -\|\underline{w}^t - \underline{w}^*\|^2 \tag{1.4}$$

$$\begin{cases} If\ no\ mistake\ at\ t,\ \Phi_{t+1} = \Phi_t \\ Otherwise,\ \Phi_{t+1} - \Phi_t = \|\underline{w}^t - \underline{w}^*\|^2 - \|\underline{w}^{t+1} - \underline{w}^*\|^2 = 2y^t \underline{w}^* \cdot \underline{x}^t - 2y^t \underline{w}^t \cdot \underline{x}^t - \|y^t \underline{x}^t\|^2 \geq 1 \end{cases}$$

$$\sum_{t=1}^{T} \Phi_{t+1} - \Phi_t = \Phi_{T+1} - \Phi_1 \geq number\ of\ mistakes\ perceptron \tag{1.5}$$

$$\Phi_(T + 1 - \Phi_1 \leq -\Phi_1 = \|w^*\|^2 \tag{1.6}$$

Observations:

1) Perceptron $\leftrightarrow$ Gradient Descendant

$$lossFunc\ \ l(\underline{w}; (x, y)) := max(0, -y\underline{w} \cdot \underline{x}) \tag{1.7}$$

$$\nabla l(w) = \begin{cases} y, & \text{if } (w \cdot x)y \geq 0 \\ -y\underline{x}, & \text{otherwise} \end{cases} \tag{1.8}$$

2) After T rounds, $\underline{w}^{T+1}$ correctly classifies all (x^t, y^t)? NO!

3) Use perceptron to solve LPs (homework)

2 Universal Portfolio Selection

2.1 Online Learning Scenarios

1)Prediction with Experts 2)Online action / Game playing 3)Online Classification 4) Universal portfolio selection ↔ Online Convex Optimization

2.2 Betting: Horses

Given odds $r_1, ... r_m$, if I invest q dollars in horse i and he wins I earn qr_i. Expect: $\sum \frac{1}{r_i} \geq 1$. If $\sum \frac{1}{r_i} < 1$ there is arbitrage, then, invest $\frac{1}{r_i}$ in horse i. For any outcome i, $\frac{1}{r_i} r_i = 1$. Assume you now the true probability of winner $\underline{P} \in \Delta_n$.

$$\arg\max_{\underline{q} \in \Delta_n} E_{i \sim P}[q_i r_i] = \arg\max_{\underline{q}} \sum p_i r_i q_i \tag{2.1}$$

$$\rightarrow \text{Put all money on one horse: } i^* = \arg\max_i p_i r_i \tag{2.2}$$

$$\arg\max_{\underline{q} \in \Delta_n} E_{i\ P}[q_i r_i] = \arg\max_{\underline{q}} \sum p_i log(r_i q_i) \tag{2.3}$$

$$\arg\max_{\underline{q} \in \Delta_n} \sum p_i log(q_i) + f(r_i, p_i) \tag{2.4}$$

$$\arg\max_{\underline{q} \in \Delta_n} \sum p_i log(p_i) + \sum p_i log(\frac{p_i}{q_i}) \tag{2.5}$$

$$\arg\max_{\underline{q} \in \Delta_n} -H(p) - KL(\underline{p}||\underline{q}) \tag{2.6}$$

$$= P \tag{2.7}$$

2.3 Portfolios and Stocks

N stocks, prices fluctuate, $\underline{x}^t \in (0, \infty)^n$

$$x_i^t = \frac{Price_{t+1}(stock\ i)}{Price_t(stock\ i)} \tag{2.8}$$

Algorithm chooses portfolio $\underline{w}^t \in \delta_n$ on day t. W_i^t = fraction of wealth in stock i. Multi growth in weath on day t is $\underline{w}^t \cdot \underline{x}^t$ After T days, we define $Wealth_{T+1}(w) = c \prod_{t=1}^{T} (\underline{w} \cdot \underline{x}^t)$ CRT=Constant rebalanced portfolio: on each day, buy and sell stocks so that fraction of wealth of stock i is w_i.

Table 1:

Stock	t=1	2	3	...	n
MSFT	1/2	2	1/2	...	2
AAPL	2	1/2	2	...	1/2

Question: Is best CRP single stock? NO In the end, AAPL and MSFT so the wealth $(\frac{1}{2}, \frac{1}{2}) = (1.25)^T$

$$Wealth_{T+1}(\underline{w}^1, \underline{w}^2, ..., \underline{w}^T) = \prod_{t=1}^{T} (\underline{w} \cdot \underline{x}^t) \tag{2.9}$$

Want: Low regret to best CRP

$$max_{w^*} \sum_{t=1}^{T} log(\underline{w}^* \cdot \underline{x}^t) - \sum_{t=1}^{T} log(\underline{w}^t \cdot \underline{x}^t) \tag{2.10}$$

I want this to be SMALL.

Algorithm: Universal

- For every $\underline{w} \in \Delta_n$ invest CRD infenitesinal amount of money in $\underline{w}$.
- Rebalance money earned in CRD(w) within this portfolio.
- No sharing accross portfolios.

$$Wealth_{t+1}(Universal) = \int_{w \in \Delta_u} \frac{\prod_{s=1}^{t} (\underline{w} x^s)}{Vol(\Delta_n)} d\mu \tag{2.11}$$

$$Vol(\Delta_n) = \frac{\sqrt{n+1}}{n! \sqrt{2^n}} \tag{2.12}$$

$$W_{i,univ}^t = \frac{\int_{\underline{w} \in \Delta_n} w_i \prod_{s=1}^{t} (w \cdot x^S) d\mu}{\int Wealth_t(w) d\mu} \tag{2.13}$$

Analysis: Define $Ball_\epsilon(\underline{w})$:

$$Ball_\epsilon(\underline{w}) = \{w' \in \Delta_n : w' = (1-\epsilon)\underline{w} + \epsilon \underline{V} \; for \; any \; \underline{V} \in \Delta_n\} \tag{2.14}$$

Claim 1: $Vol(Ball_\epsilon(w)) = Vol(\Delta_n)\epsilon^{n-1}$ Claim 2:

$$\underline{w}' \in Ball_\epsilon(\underline{w}) \tag{2.15}$$

$$wealth_{T+1}(\underline{w}') = wealth_{T+1}(w(1-\epsilon) + \epsilon V) \geq (1-\epsilon)^T wealth(\underline{w}) \tag{2.16}$$

Observe:

$$wealth(universal) = \frac{1}{Vol(\Delta_n)} \int_{w \in \Delta_n} wealth_{T+1}(w) d\mu \tag{2.17}$$

$$\geq \frac{1}{Vol(\Delta_n)} \int_{w \in Ball_\epsilon(w^*)} wealth_{T+1}(w) d\mu = \frac{1}{Vol(\Delta_n)} \int_{\underline{w} \in Ball_\epsilon(w^*)} (1-\epsilon)^T wealth_{T+1}(\underline{w}^*) d\mu \tag{2.18}$$

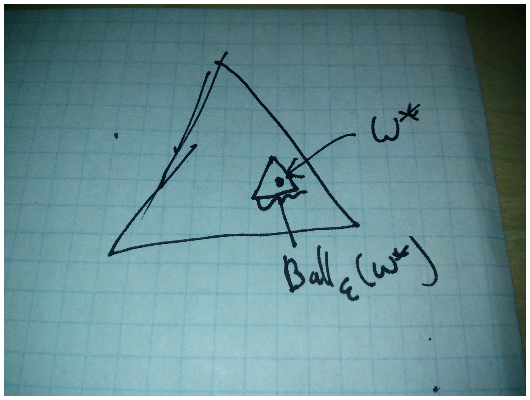

Figure 1: claim 2

$$= \frac{1}{Vol(\Delta_n)}(1-\epsilon)^T wealth_{T+1}(\underline{w}^*) \int_{\underline{w} \in Ball_\epsilon(\underline{w}^*)} d\mu \tag{2.19}$$

$$\int_{\underline{w} \in Ball_\epsilon(\underline{w}^*)} d\mu = Vol(Ball_\epsilon(\underline{w}^*)) \tag{2.20}$$

$$\frac{1}{Vol(\Delta_n)}(1-\epsilon)^T wealth_{T+1}(\underline{w}^*) \int_{w \in Ball_\epsilon(\underline{w}^*)} d\mu = (1-\epsilon)^T \epsilon^n wealth_{T+1}(\underline{w}*) \tag{2.21}$$

$$\text{For } \epsilon = 1/T, \ = (1-\frac{1}{T})^T \frac{1}{T}^N wealth_{T+1}(\underline{w}^*) \tag{2.22}$$

$$log(wealth(\underline{w})) \geq log(wealth(\underline{w}^*)) - NlogT + O(1) \tag{2.23}$$