Solving Difficult SAT Instances In The Presence of Symmetry

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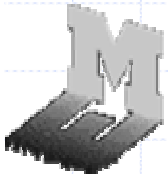
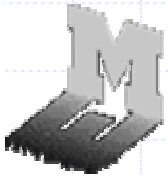


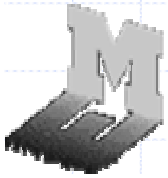
Highlights of Our Work

- No new SAT solvers are proposed
- We improve performance of existing complete SAT solvers *by preprocessing*
- Evaluate on carefully chosen SAT benchmarks
  - ignore easy benchmarks
  - only worry about benchmarks with symmetries (but the symmetries may not be given!)
  - show applicability to chip layout (200x speed-ups) and derive new hard SAT benchmarks
  - show asymptotic improvements

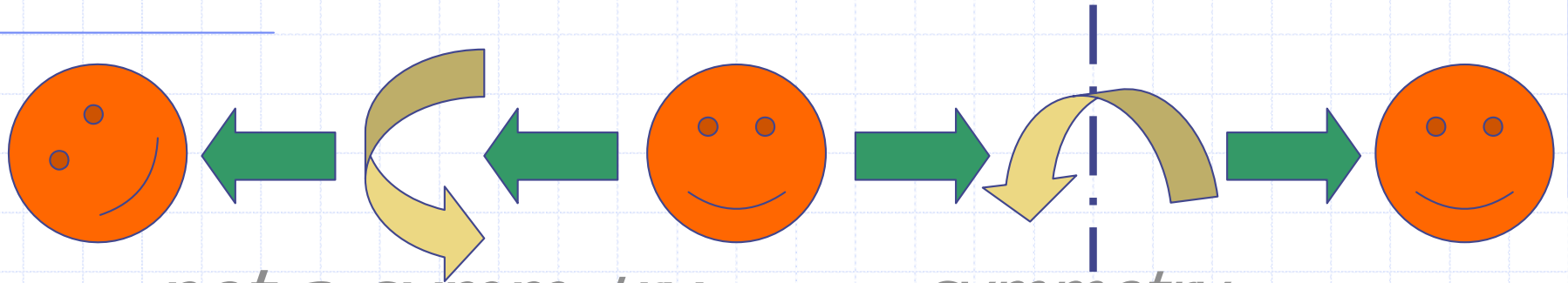


Outline

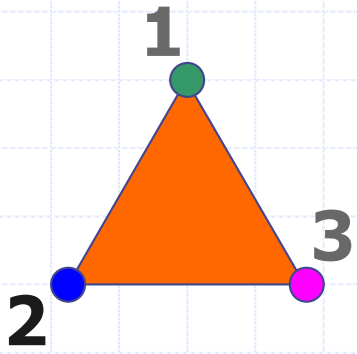
- Symmetries and permutations
  - Compact representations of symmetries
  - Computational group theory
- Symmetries of CNF instances
  - Detection via Graph Automorphism
  - Syntactic versus semantic symmetries
  - Using symmetries to speed up search
- Opportunistic symmetry detection
- Empirical results



Symmetries and Permutations



Symmetries of the triangle:



$1\to2,\ 2\to3,\ 3\to1$	(123)
$1\to3,\ 3\to2,\ 2\to1$	(132)
$1\to2,\ 2\to1,\ 3\to3$	(12)
$1\to1,\ 2\to3,\ 3\to2$	(23)
$1\to3,\ 3\to1,\ 2\to2$	(13)
$1\to1,\ 2\to2,\ 3\to3$	"do nothing"

Permutations can have multiple (disjoint) cycles

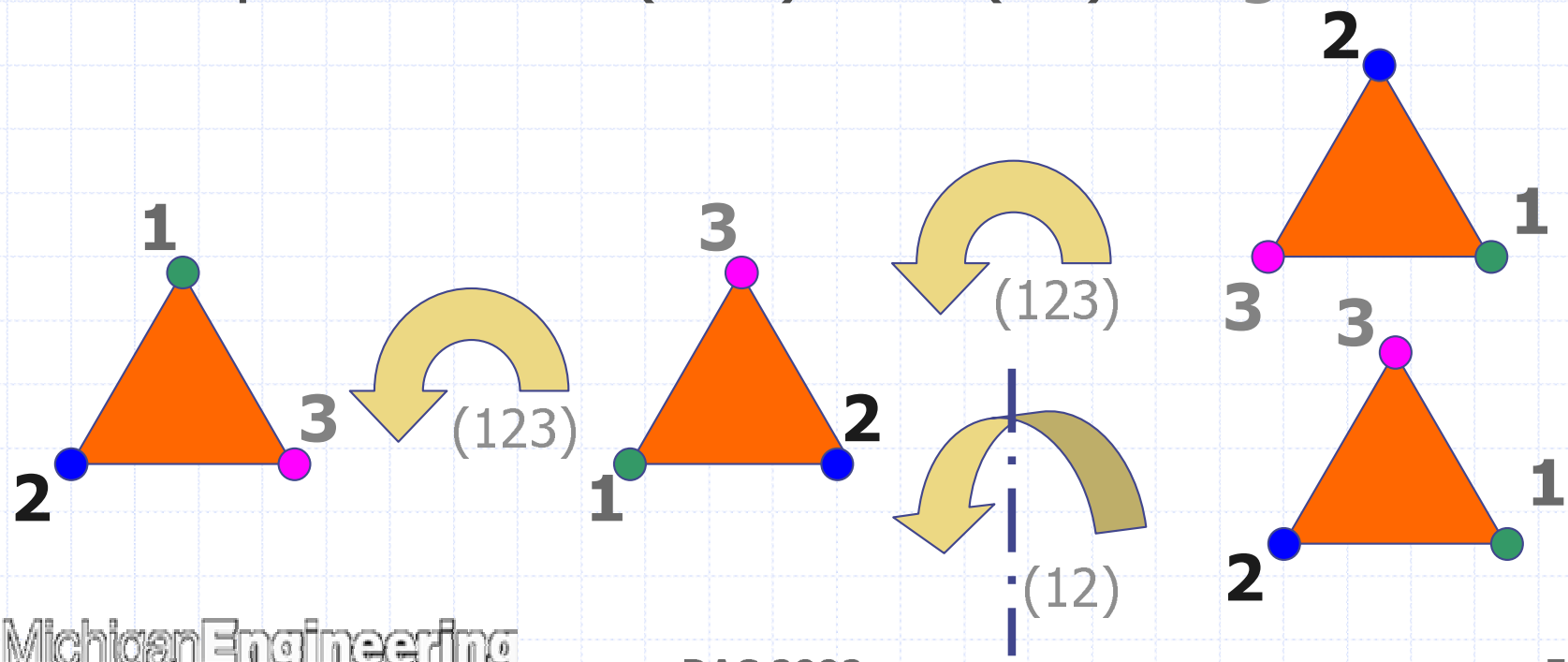
Symmetries and Permutations (2)

apply (123) and then again (123): get (132)

apply (123) and then (12) : get (23)

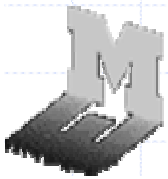
all non-trivial symmetries

are products of (123) and (12) - "generators"



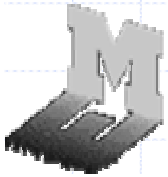
Symmetries and Permutations (3)

- Idea: represent symmetries of an object by permutations that preserve the object
- Composition of symmetries is modeled by composition of permutations
  - Composition is associative
  - Every symmetry has an inverse
  - The do-nothing symmetry is the identity
- This enables applications of group theory



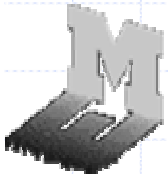
Compact Representations

- Represent the group of all symmetries
  - Do not list individual symmetries
  - List generating permutations (generators)
- Elementary group theory proves:
  - If redundant generators are avoided,
  - A group with N elements can be represented by at most $log_2(N)$ generators
- Guaranteed exponential compression



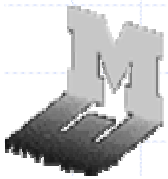
Compact Representations (2)

- Sometimes can do better than $log_2(N)$
- E.g., consider the group S_k of all $k!$ permutations of $1..k$
  - Can be generated by (12) and (123..k)
  - Or by (12), (23), (34),…, (k-1 k)
- To use this guaranteed compression, we need algorithms in terms of permutation generators



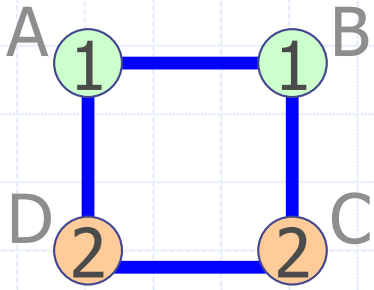
Computational Group Theory

- Algorithms for group manipulation in terms of generators are well known
  - Published by Sims, Knuth, Babai and others
  - Especially efficient for permutation groups
- High-quality implementations available
  - The GAP package – free, open-source (GAP="Groups, Algebra, Programming")
  - The MAGMA package – commercial

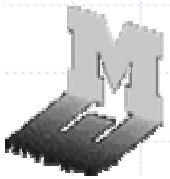


Finding Symmetries of Graphs

- Symmetry (automorphism) of a graph
  - Permutation of vertices that maps edges to edges

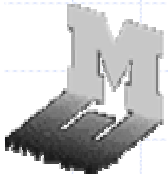


- Additional constraints
  - Vertex colors (labels): integers
  - Every vertex must map into a vertex of same color
- Computational Graph Automorphism
  - Find generators of a graph's group of symmetries
  - GraphAuto $\in$ NP, and is believed to $\notin$ P and $\notin$ NPC
  - Linear average-case runtime (but that's irrelevant!)
  - Algorithms implemented in GAP(GRAPE(NAUTY))



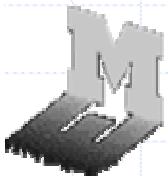
Symmetries of CNF Formulae

- Permutations of variables that map clauses to clauses
  - E.g., symmetries of (a+b+c)(d+e+f) include (ab), (abc) as well as (ad)(be)(cf)
  - Considering single swaps only is not enough
- Ditto for variable negations (a→a') and compositions with permutations
  - E.g., symmetries of (a+b+c)(d+e'+f') include (de') as well as (ad)(be')(cf')



Reduction to Graph Automorphism

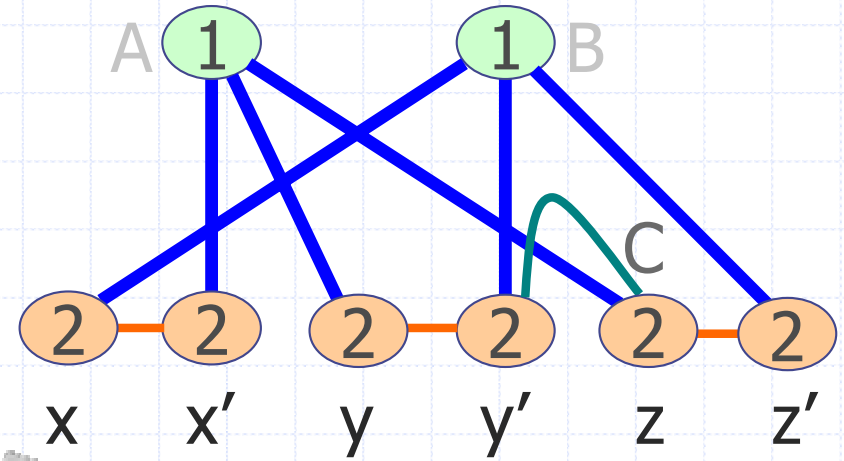
- CNF formula $\rightarrow$ colored graph
  - Linear time and space
- Find graph's [colored] symmetries
  - Worst-case exponential time
- Interpret graph symmetries found as symmetries of the CNF formula
  - Permutational symmetries
  - Phase-shift symmetries



Reduction to Graph Automorphism

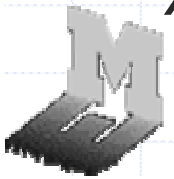
- Vertices of two colors: clauses and vars
  - One vertex per clause, two per variable
- Edges of three types: (i) incidence, (ii) consistency, and (iii) 2-literal clauses

Clauses: A (x' + y + z) , B (x + y' + z'), C (y' + z)



Symmetry:
(x x')(y z')(y' z)

1	2	3	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



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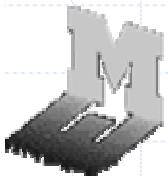
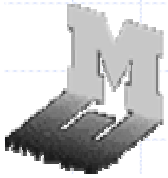
Syntactic and Semantic Symmetries

- CNF formula versus Boolean function
- Syntactic symmetries
  - symmetries of representation
- Semantic symmetries of the object
  - E.g., permutations and negations of variables that preserve the value of the function for all inputs
- Any syntactic symmetry is also semantic
  - but not vice versa, example: (a)(a')(a+b)



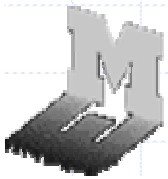
Speeding up SAT Search

- Search space may have symmetries
  - May have regions that map 1:1
  - This makes search redundant
  - (a+d)(b+d)(a+b)(…)…(…) ab′⇔a′b
- Ideas for speed-ups
  - Consider equivalence classes under symmetry
  - Pick a representative for each class
  - Search only one representative per class
- This restricted search is ⇔ to original



Symmetry-breaking Predicates

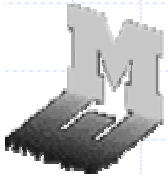
- To restrict search
  - Add clauses to the original CNF formula ("symmetry-breaking" clauses)
  - They will pick representatives of classes and restrict search
- Our main task is to find those clauses
  - Use only permutations induced by generators
  - Permutation $\rightarrow$ group of clauses (a "symmetry-breaking" predicate)



Construction of S.-b. Predicates

- Earlier work:
  - By Crawford, Ginsberg, Roy and Luks (92,96)
  - Not based on cycle notation for permutations
- Our construction is more efficient
  - Every cycle considered separately
  - In practice almost all cycles are 2- or 3-cycles
    - Two types of 2-cycles: (aa') and (ab)
    - Symm.-breaking predicates: (a) and (a'+b) resp.
  - For multiple cycles
    - Procedure to chain symmetry-breaking predicates

CGRL



Details: Individial Cycles (1)

- Use an ordering of all variables (arbitrary)
  - To prevent transitivity violations: (a+b')(b+c')(c+a)

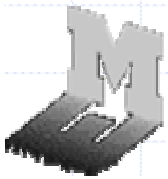
    (the construction by CGRL uses an ordering as well)
- Symmetry-breaking predicate for cycle (ab):
  - (a⇒b) aka (a≤b), if a precedes b in the ordering
  - Think of partial variable assignments to b and a
    - Must choose one from 01 and 10

      → 00

      → 01

      ~~10~~ (a'+b)

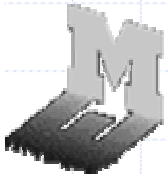
      → 11



Details: Individial Cycles (2)

000	~~100~~	(a'+b)(b'+c)
001	~~101~~	
~~010~~	~~110~~	
011	111	

- S.-b. predicate for cycle (abc) is (a≤b≤c)
  - For 3-var partial assignments, can cycle all 0s to front
- For longer cycles, still can improve upon CGRL
- Does ordering affect overall performance?



Details: Multiple Cycles(1)

- Solution space reduction
  - By **2x** when (a) is added to break cycle (aa')
  - Still by **2x** if permutation has cycles (aa') and (bb')
  - By **4/3x** when (a'+b) is added to break cycle (ab)
  - **What if** a permutation has cycles (ab) and (cd) ?
  - By **2x** when (a≤b≤c) is added to break (abc)
- Suppose you have cycles (aa') and (uvt)
  - Adding both predicates cuts solution space by **4x**
- Rule of thumb: after breaking a 2-cycle, symmetry-break the square of the permutation

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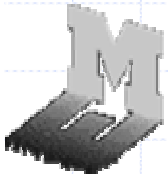
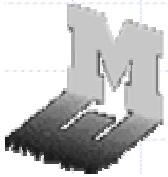


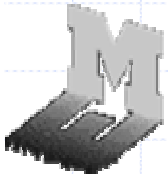
Details: Multiple Cycles(2)

- Rule of thumb: after breaking a 3-cycle, symmetry-break the cube of the permutation
- What if we have both (xy) and (uv) ?
  - Squaring will kill the second cycle, so don't square!
  - Look at partial assignments for x,y: 00, 01, 10 and 11
  - For 10 or 01, (x'+y) is all we can do
  - For 00 or 11, can add (u'+v)
  - Adding (x≤y) and (x=y)⇒(u≤v)
    cuts the solution space by **8/5x** (better than **4/3x**)
- For 3-cycles, add (x=y=z)⇒(u≤v≤w) or the like
- For multiple cycles ((x=y=z)&(a=b))⇒(u≤v), etc



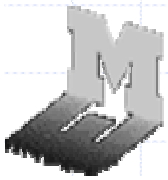
Discussion

- We detect syntactic symmetries only
  - If more semantic symmetries available, can use them in the same way
- Symmetry-detection can take long time
  - Sometimes longer than solving SAT
- In some cases the only symmetry is trivial
  - Symm. detection is often fast in these cases
- Symmetry-breaking using generators only is not exhaustive (remark by CGRL)
  - But makes symmetry-breaking practical (our result)
  - Pathological cases are uncommon:why?(future work)

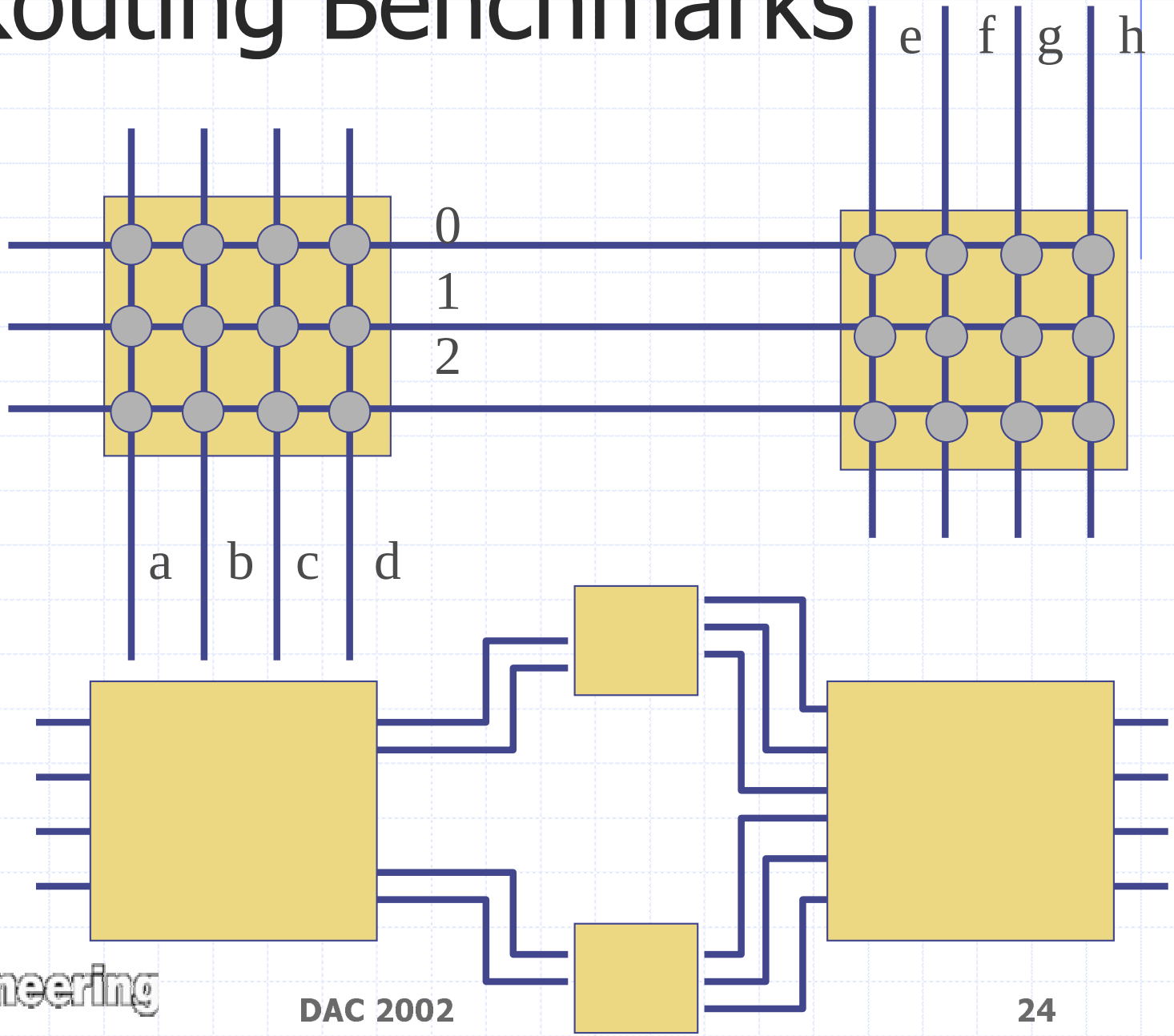


Evaluation and Benchmarks

- Most of DIMACS benchmarks are easy for existing solvers
- We focus on difficult CNF instances
  - Pigeon-hole-n (PHP-n), Urquhart, etc.
- Observe that PHP-n can appear in apps
- EDA layout apps (routing) $\rightarrow$ symmetry
- We generate satisfiable and unsatisfiable CNF instances related to PHP-n

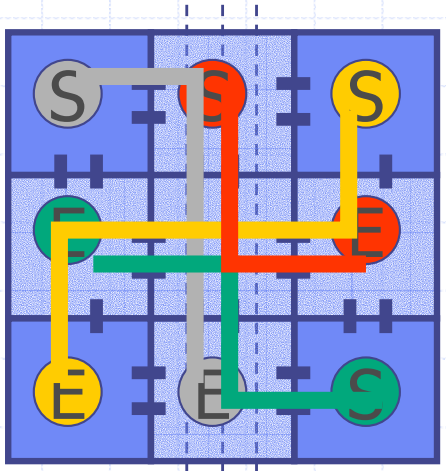


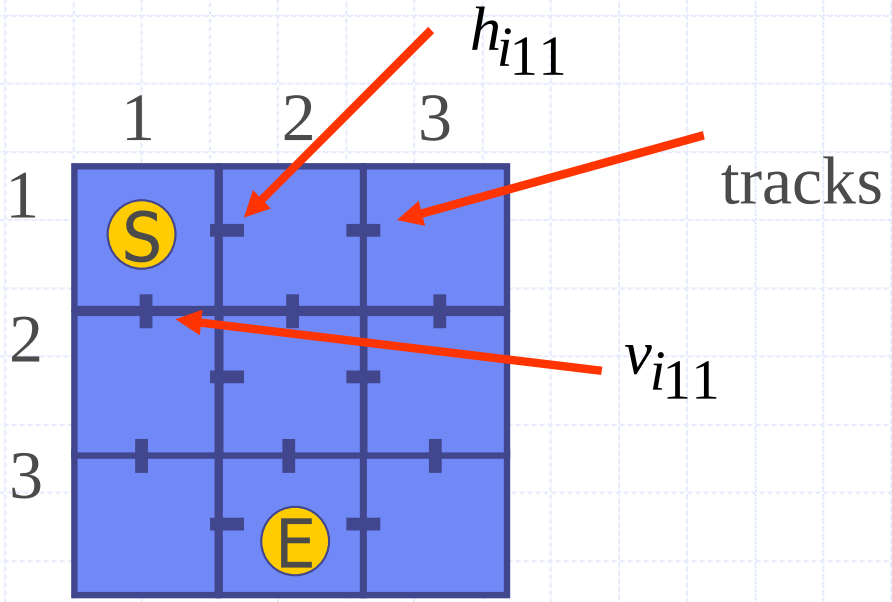
FPGA Routing Benchmarks



Global Routing Benchmarks

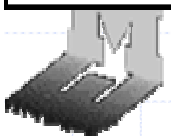
- Construct difficult grid-routing instances by "randomized flooding"
- Then convert to CNF





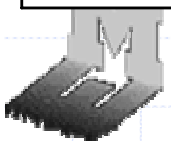
Empirical Results - Chaff

Instance	S/U	#V	#CL	Plain Chaff	Time-out	Symmetries				Speedup	
						Finding	Number	#generators	Search	Total	Search
				sec	%	sec	of	cycles	Time		only
hole7	U	56	204	0.37	0%	0.1	2.03E+08	all 13	0.01	3.32	36.50
hole8	U	72	297	1.27	0%	0.07	1.46E+10	all 15	0.01	15.22	94.15
hole9	U	90	415	3.79	0%	0.1	1.32E+12	all 17	0.02	32.00	204.97
hole10	U	110	561	22.44	0%	0.15	1.45E+14	all 19	0.02	130.07	997.18
hole11	U	132	738	212.73	0%	0.13	1.91E+16	all 21	0.03	1329.54	7090.88
hole12	U	156	949	1000	100%	0.24	2.98E+18	all 23	0.04	3597.12	26315.79
Urq3_5	U	46	470	232.44	10%	0.48	5.37E+08	all 29	0.00	484.16	2.32E+06
Urq4_5	U	74	694	250.01	25%	1.35	8.80E+12	all 43	0.00	185.18	2.50E+06
Urq5_5	U	121	1210	1000	100%	13.15	4.72E+21	all 72	0.00	76.05	1.00E+07
Urq6_5	U	180	1756	1000	100%	62.93	6.49E+32	all 109	0.00	15.89	1.00E+07
Urq7_5	U	240	2194	1000	100%	176.62	1.12E+43	all 143	0.00	5.66	1.00E+07



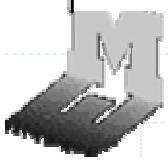
Empirical Results - Chaff

Instance	S/U	#V	#CL	Plain Chaff sec	Time-out %	Symmetries: Finding sec	Number of	#generators	cycles	Search Time	Speedup: Total	Search only
grout3.3-01	S	864	7592	**19.01**	0%	**4.79**	8.71E+09	10	26	**0.67**	**3.48**	**28.37**
grout3.3-03	S	960	9156	**44.35**	0%	**8.94**	6.97E+10	10	29	**0.40**	**4.75**	**110.89**
grout3.3-04	S	912	8356	**19.36**	0%	**6.81**	2.61E+10	10	27	**0.36**	**2.70**	**53.79**
grout3.3-08	S	912	8356	**21.30**	0%	**7.14**	3.48E+10	10	28	**0.67**	**2.73**	**31.80**
grout3.3-10	S	1056	10862	**28.18**	0%	**10.65**	3.48E+10	10	28	**0.85**	**2.45**	**33.15**
chnl10x11	U	220	1122	**22.17**	0%	**0.45**	4.20E+28	all	39	**0.11**	**39.91**	**210.13**
chnl10x12	U	240	1344	**81.88**	0%	**0.61**	6.04E+30	all	41	**0.12**	**111.63**	**663.00**
chnl10x15	U	300	2130	**657.61**	25%	**1.28**	4.50E+37	all	47	**0.17**	**454.78**	**3961.49**
chnl11x12	U	264	1476	**207.37**	0%	**0.75**	7.31E+32	all	43	**0.15**	**231.31**	**1415.51**
chnl11x13	U	286	1742	**788.32**	20%	**1.08**	1.24E+35	all	45	**0.16**	**633.45**	**4792.24**
chnl11x20	U	440	4220	**1000**	100%	**4.4**	1.89E+52	all	59	**0.31**	**212.49**	**3267.97**



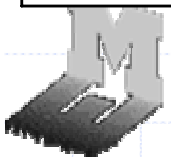
Empirical Results - Chaff

Instance	S/U	#V	#CL	Plain Chaff sec	Time-out %	Symmetries: Finding sec	Number of	#generators	cycles	Search Time	Speedup: Total	Search only
fpga10_8	S	120	448	**7.56**	0%	**0.63**	6.00E+71	all	62	**0.05**	**11.15**	**157.56**
fpga10_9	S	135	549	**3.80**	0%	**0.88**	6.33E+77	all	68	**0.03**	**4.16**	**113.39**
fpga12_11	S	198	968	**694.00**	50%	**3.76**	7.18E+77	all	95	**0.06**	**181.63**	**11377.05**
fpga12_12	S	216	1128	**80.20**	0%	**5.31**	7.44E+77	all	104	**0.13**	**14.74**	**616.92**
fpga12_8	S	144	560	**246.70**	10%	**1.23**	8.41E+77	all	72	**0.08**	**188.39**	**3103.14**
fpga12_9	S	162	684	**885.00**	80%	**1.7**	2.25E+77	all	79	**0.05**	**504.56**	**16388.89**
fpga13_9	S	176	759	**550.00**	85%	**2.57**	2.56E+77	all	84	**0.06**	**208.81**	**8593.75**
fpga13_10	S	195	905	**1000**	100%	**4.04**	5.76E+77	all	93	**0.08**	**242.60**	**12195.12**
fpga13_12	S	234	1242	**1000**	100%	**6.9**	8.85E+77	all	110	**0.08**	**143.23**	**12195.12**



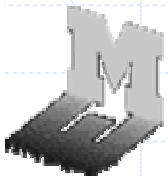
Empirical Results - Chaff

Instance	S/U	#V	#CL	Plain Chaff sec	Time-out %	Symmetries Finding sec	Number of	#generators		Search Time	Speedup Tot	Speedup Search only
2dlx_ca_mc	U	3250	24640	**6.54**	0%	**38.36**	9.36E+77	10	66	**6.30**	**0.15**	**1.04**
2pipe	U	892	6695	**2.08**	0%	**10.74**	2.26E+45	10	38	**1.56**	**0.17**	**1.33**
2pipe_1_ooo	U	834	7026	**2.55**	0%	**9.37**	8.00E+00	10	3	**1.80**	**0.23**	**1.41**
2pipe_2_ooo	U	925	8213	**3.43**	0%	**11.14**	3.20E+01	10	5	**2.82**	**0.25**	**1.22**
3pipe	U	2468	27533	**36.44**	0%	**463.57**	7.29E+77	10	85	**19.65**	**0.08**	**1.85**
2dlx_ca_mc	U	3250	24640	**6.54**	0%	**3.17**	2.34E+77	10	64	**5.42**	**0.76**	**1.21**
2pipe	U	892	6695	**2.08**	0%	**10.47**	2.26E+45	10	38	**1.30**	**0.18**	**1.60**
2pipe_1_ooo	U	834	7026	**2.55**	0%	**9.02**	8.00E+00	10	3	**1.80**	**0.24**	**1.41**
2pipe_2_ooo	U	925	8213	**3.43**	0%	**11.09**	3.20E+01	10	5	**2.80**	**0.25**	**1.23**
3pipe	U	2468	27533	**36.44**	0%	**3.63**	1.42E+77	10	78	**36.20**	**0.91**	**1.01**
4pipe	U	5237	80213	**337.61**	0%	**9.32**	1.03E+78	10	142	**334.00**	**0.98**	**1.01**
5pipe	U	9471	195452	**325.92**	0%	**29.42**	3.64E+78	10	227	**290.50**	**1.02**	**1.12**

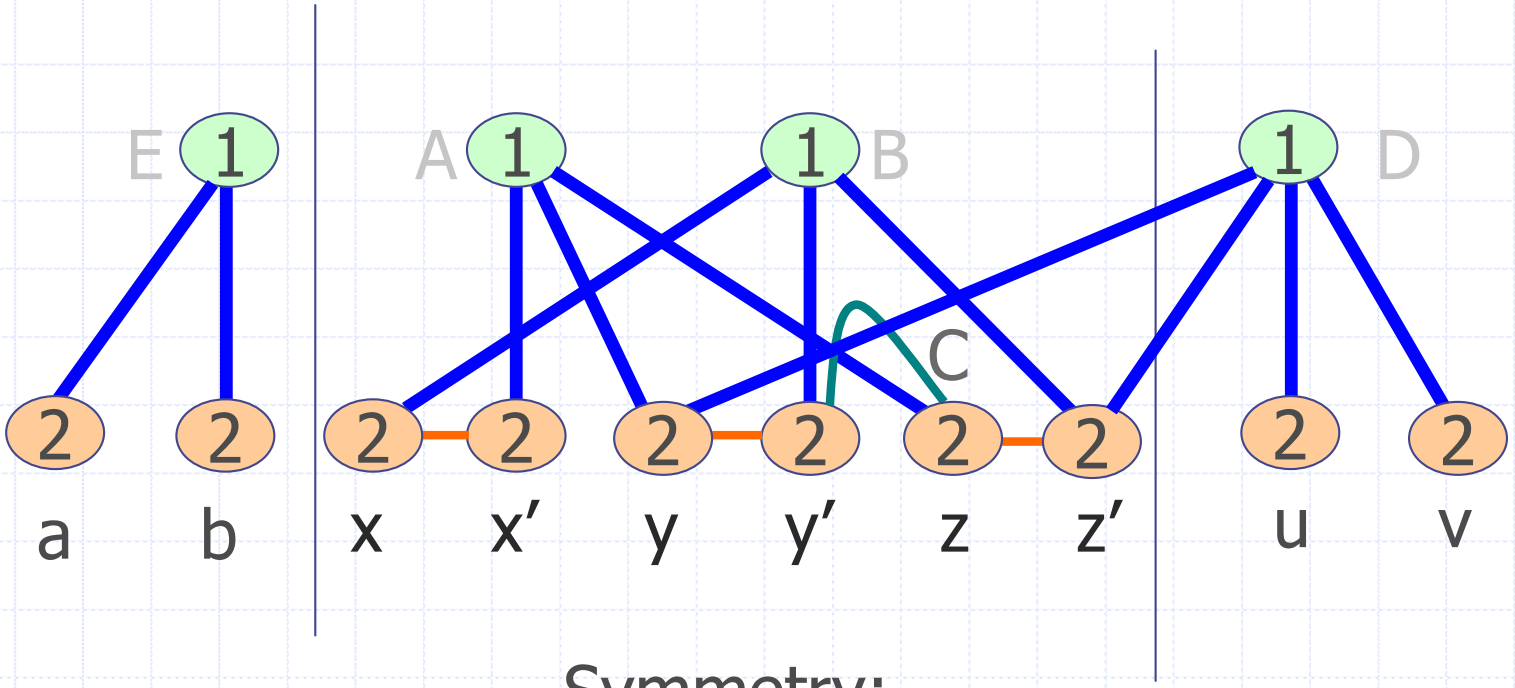


Domain-specific Symmetry-Breaking Predicates

- We looked at symmetry generators for global routing benchmarks
- Those symmetries were permutations of routing tracks
- Symmetry-breaking clauses can be added when converting to CNF
  - Serious speed-up for Chaff in all cases
- No symmetries left after that



Fast Symmetry Detection

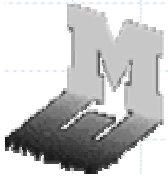


Symmetry:

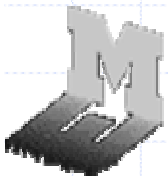
$(x\ x')(y\ z')(y'\ z)$

Conclusions

- Pre-processing speeds up SAT solvers on difficult instances with symmetries
  - Strong empirical results on new and old BMs
- Improved constructions
  - Reduction to graph automorphism
  - Symmetry-breaking predicates
    - Cycle-based construction
    - Using generators only
- Many important questions not answered
- Significant on-going work



Thank you



Instance	S/U	#V	#CL	Plain Chaff sec	Time-out %	Symmetries				Speedup	
						Finding sec	Number of	#generators cycles		Search Time	Total	Search only
grout3.3-01	S	864	7592	**19.01**	0%	**4.79**	8.71E+09	10	26	**0.67**	**3.48**	**28.37**
grout3.3-03	S	960	9156	**44.35**	0%	**8.94**	6.97E+10	10	29	**0.40**	**4.75**	**110.89**
grout3.3-04	S	912	8356	**19.36**	0%	**6.81**	2.61E+10	10	27	**0.36**	**2.70**	**53.79**
grout3.3-08	S	912	8356	**21.30**	0%	**7.14**	3.48E+10	10	28	**0.67**	**2.73**	**31.80**
grout3.3-10	S	1056	10862	**28.18**	0%	**10.65**	3.48E+10	10	28	**0.85**	**2.45**	**33.15**
chnl10x11	U	220	1122	**22.17**	0%	**0.45**	4.20E+28	all	39	**0.11**	**39.91**	**210.13**
chnl10x12	U	240	1344	**81.88**	0%	**0.61**	6.04E+30	all	41	**0.12**	**111.63**	**663.00**
chnl10x15	U	300	2130	**657.61**	25%	**1.28**	4.50E+37	all	47	**0.17**	**454.78**	**3961.49**
chnl11x12	U	264	1476	**207.37**	0%	**0.75**	7.31E+32	all	43	**0.15**	**231.31**	**1415.51**
chnl11x13	U	286	1742	**788.32**	20%	**1.08**	1.24E+35	all	45	**0.16**	**633.45**	**4792.24**
chnl11x20	U	440	4220	**1000**	100%	**4.4**	1.89E+52	all	59	**0.31**	**212.49**	**3267.97**
fpga10_8	S	120	448	**7.56**	0%	**0.63**	6.00E+71	all	62	**0.05**	**11.15**	**157.56**
fpga10_9	S	135	549	**3.80**	0%	**0.88**	6.33E+77	all	68	**0.03**	**4.16**	**113.39**
fpga12_11	S	198	968	**694.00**	50%	**3.76**	7.18E+77	all	95	**0.06**	**181.63**	**11377.05**
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fpga12_8	S	144	560	**246.70**	10%	**1.23**	8.41E+77	all	72	**0.08**	**188.39**	**3103.14**
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2dlx_ca_mc	U	3250	24640	**6.54**	0%	**38.36**	9.36E+77	10	66	**6.30**	**0.15**	**1.04**
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2pipe	U	892	6695	**2.08**	0%	**10.47**	2.26E+45	10	38	**1.30**	**0.18**	**1.60**
2pipe_1_ooo	U	834	7026	**2.55**	0%	**9.02**	8.00E+00	10	3	**1.80**	**0.24**	**1.41**
2pipe_2_ooo	U	925	8213	**3.43**	0%	**11.09**	3.20E+01	10	5	**2.80**	**0.25**	**1.23**
3pipe	U	2468	27533	**36.44**	0%	**3.63**	1.42E+77	10	78	**36.20**	**0.91**	**1.01**
4pipe	U	5237	80213	**337.61**	0%	**9.32**	1.03E+78	10	142	**334.00**	**0.98**	**1.01**
5pipe	U	9471	195452	**325.92**	0%	**29.42**	3.64E+78	10	227	**290.50**	**1.02**	**1.12**