DIRECT ASSESSMENT OF TRANSIENT SINGULARITY IN DIFFERENTIAL-ALGEBRAIC SYSTEMS

Chaman Singh

Dept. of Electrical & Computer Engineering
The University of Newcastle
Callaghan NSW 2308 Australia

Ian A. Hiskens

Dept. of Electrical & Computer Engineering
University of Illinois at Urbana-Champaign
Urbana IL 61801 USA

ABSTRACT

Characterization of the stability boundary of differential-algebraic (DAE) systems is more complicated than for systems described by ordinary differential equations (ODEs). In addition to unstable equilibria and periodic orbits, algebraic singularity plays an important role in defining the stability boundary. This paper presents a Lyapunov-based method for direct assessment of transient singularity, thus enabling estimation of the region of state-space where the system model remains valid.

1. INTODUCTION

The characterization of the stability boundary of systems that are modelled by ordinary differential equations (ODEs) is well documented [1]. For a large class of such systems the stability boundary is composed of the stable manifolds of unstable equilibrium points and unstable periodic orbits that lie on the stability boundary. Many systems cannot however be represented in an ODE form. A differential-algebraic (DAE) model is more appropriate. Trajectories of DAE systems are constrained to lie on a manifold described by the algebraic equations. When these constraints are present in the system, analysis of the stability boundary becomes more complicated. A detailed mathematical taxonomy theory for large DAE systems, such as power systems, is provided in [2, 3].

The state-space of DAE systems is divided into open components by surfaces where the algebraic equations are singular. The open components are often referred to as *causal regions*, and the singular surfaces form the *impasse surface* [4, 5]. Over causal components, system dynamic behaviour evolves according to a locally equivalent ODE system representation. However trajectories that reach the boundary of a causal component, i.e, that encounter the impasse surface, typically undergo loss of existence/uniqueness. The DAE model breaks down. Causal components usually contain regions of stability and instability. Of most interest are the regions of attraction of stable equilibria. It was shown in [3] that the boundary of the region of attraction of stable equilibria often includes segments of the impasse surface.

Algebraic singularity therefore plays a crucial role in assessing the stability/viability of DAE systems. System trajectories generally cannot continue after encountering the impasse surface. Hence, it is necessary to determine the region of state-space where the system is causal. Lyapunov (direct) methods have been widely used to estimate the stability boundary of ODE systems. However, they do not naturally take account of the situation where the boundary includes sections of the impasse surface. This paper presents an approach to direct assessment of algebraic singularity. It is shown that provided the system has less energy than a critical value, the trajectory will never encounter the impasse surface and causality will be ensured. The paper provides algorithms for determining that critical value of energy.

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2. MODEL AND STATE-SPACE STRUCTURE

Many systems are best modelled by a differential-algebraic representation of the form

$$\dot{x} = f(x,y), \qquad f : \Re^{n+m} \to \Re^n \tag{1}$$

$$0 = g(x,y), \qquad g : \Re^{n+m} \to \Re^m \tag{2}$$

where

$$x \in X \subset \Re^n, \quad y \in Y \subset \Re^m.$$

For model (1)-(2), the augmented state-space is $X \times Y$, where x are the dynamic states and y the algebraic (instantaneous) states respectively. In power systems for example, the dynamic state variables include generator flux linkages, rotor angles, control states and dynamic load states. Typical examples of algebraic variables include load bus voltages and phase angles.

Whilst the dynamics of state variables x are governed explicitly by (1), the dynamics of algebraic variables y are assumed to be so fast that the algebraic constraints (2) are always satisfied. In other words the states are constrained to the set

$$G = \{(x,y) \in X \times Y : g(x,y) = 0\}. \tag{3}$$

Every trajectory of the DAE model (1)-(2) must lie within G. However G often contains points where (2) does not have a unique local solution. These points are called singular or noncausal points [4, 5], and form the singular surface or impasse surface defined by

$$S = \{(x,y) \in G : \Delta(x,y) \equiv \det g_y(x,y) = 0\} \tag{4}$$

where $g_y = \partial g / \partial y$ is the Jacobian of algebraic equations with respect to algebraic variables. Trajectories that encounter S generally cannot continue. In power systems, the existence of the impasse surface is closely related to load models [5, 6].

Generically, the constraint manifold G is divided into open connected components C_i called causal regions that are separated by segments of the impasse surface S [2, 5]. The principal causal region C_0 is defined as the component where all eigenvalues of

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g_y are positive. This component will usually contain a stable equilibrium point (EP), though other components may also contain stable EPs.

In DAE systems, the region of attraction of each stable EP is a subset of the associated causal region. To describe the boundary of the region of attraction, it is helpful to establish the sets [2, 3]

$$\Psi = \{(x,y) \in S : h(x,y) \equiv \text{adj}(g_y(x,y))g_x(x,y)f(x,y) = 0\}$$
$$\Xi = \{(x,y) \in S \setminus \Psi : \Delta_y(x,y)h(x,y) = 0\}$$

where adj(g_y) refers to the adjoint (adjugate) of g_y. These sets define *pseudo-equilibria* and *semi-singular* points respectively. The equilibrium nature of points in Ψ follows from a time-scale transformation of the original DAE model (1),(2), i.e,

$$\dot{x} = f(x,y)\Delta(x,y) \tag{5}$$
$$\dot{y} = -\text{adj}(g_y(x,y))g_x(x,y)f(x,y). \tag{6}$$

The boundary of the region of attraction of a stable EP, denoted ∂A, is composed of:

- Stable manifolds of unstable equilibria and periodic orbits, $e \in \partial A$,
- Stable manifolds of particular pseudo-equilibria called transverse pseudo-saddles, $\psi \in \Psi \cap \partial A$,
- Stable manifolds of particular semi-singular points called semi-saddles, $\xi \in \Xi \cap \partial A$, and
- Singular boundary pieces, $S \cap \partial A$.

Complete details of this categorization can be found in [2, 3].

3. DIRECT ASSESSMENT OF SINGULARITY

3.1. Background

As indicated in Section 2, the stability boundary of stable equilibria can include sections of the impasse surface. Stability assessment should therefore take account of algebraic singularity. A Lyapunov (energy) function approach is adopted in this paper.

Energy functions form the basis for many direct stability assessment algorithms [7]. Many of these techniques use the energy of the controlling unstable equilibrium point (UEP) to estimate the stability boundary. These methods have been developed for ODE systems though, and so do not take algebraic singularity into account. However it has been shown in [3] that for DAE systems, UEPs may not even exist within a particular causal region C_i. In such a case, the stability boundary is solely determined by segments of the impasse surface. A technique which accounts for the impasse surface in therefore important.

It was shown in [3] that the minimum energy on the stability boundary occurs at a UEP, a pseudo-saddle or a semi-saddle. The energy of such a point can be used as the critical energy to estimate the stability boundary. These concepts are consistent with the closest UEP method of ODE systems, which typically gives conservative estimates of stability.

Another interesting energy function approach was proposed in [8], in the context of power system applications. It shares some similarities with the potential energy boundary surface (PEBS) method [7], in that the initial estimate of the critical energy is related to the energy at the point where the sustained fault trajectory intersects the impasse surface. This initial guess is refined via an optimization algorithm that approximates pseudo-saddle conditions. The final estimate does not truly reflect the energy over the impasse surface. Therefore the initial and final estimates of critical energy provide lower and upper bounds on the critical clearing time. An approach which considers the actual energy over the impasse surface is desirable. Such a method is proposed in the sequel.

3.2. Problem Formulation

The method presented here is motivated by ideas proposed in [9]. Suppose a Lyapunov (energy) function $\mathcal{V}$ exists for the system (1)-(2). An estimate of the critical energy (that ensures the post-fault trajectory never encounters the impasse surface) is calculated by minimizing the energy over the impasse surface. This minimization can be formulated as

$$\min_{(x,y)} \mathcal{V}(x,y) \tag{7}$$
$$\text{s.t. } g(x,y) = 0 \tag{8}$$
$$\det g_y(x,y) = 0. \tag{9}$$

Equation (9) can also be expressed as

$$g_y v = 0 \tag{10}$$
$$v_1^2 + v_2^2 + \cdots + v_n^2 = 1 \tag{11}$$

where v is the right eigenvector corresponding to a zero eigenvalue of the algebraic Jacobian g_y. Condition (11) ensures a nonzero eigenvector in (10).

Using Lagrangian multiplier theory, the constrained minimization problem (7)-(8) together with (10)-(11) can be transformed into an unconstrained minimization of the Lagrangian function $\mathcal{L} : \Re^{n+4m+1} \to \Re$,

$$\mathcal{L}(x,y,\lambda_1,\lambda_2,\mu,v) = \mathcal{V}(x,y) + \lambda_1^t g + \lambda_2^t g_y v + \mu(v_1^2 + v_2^2 + \cdots + v_n^2 - 1) \tag{12}$$

where λ_1 and λ_2 are m-dimensional (column) vectors of Lagrangian multipliers associated with constraints (8) and (10) respectively, and μ is the the Lagrangian multiplier associated with constraint (11). The minimum of $\mathcal{L}$ is given by the first order optimality conditions [10],

$$\nabla_{x,y,\lambda_1,\lambda_2,\mu,v}\mathcal{L}(x^*,y^*,\lambda_1^*,\lambda_2^*,\mu^*,v^*) = 0 \tag{13}$$

where $(x^*,y^*,\lambda_1^*,\lambda_2^*,\mu^*,v^*)$ is a local minimum of $\mathcal{L}$. These conditions represent $n+4m+1$ equations in the same number of unknowns, viz., the states x, y, the Lagrangian multipliers λ_1, λ_2 and μ, and the elements of the eigenvector v. Equation (13) can be written explicitly as

$$\nabla_x\mathcal{L} = \nabla_x\mathcal{V} + g_x^t\lambda_1 + (g_y v)_x^t\lambda_2 = 0 \tag{14}$$
$$\nabla_y\mathcal{L} = \nabla_y\mathcal{V} + g_y^t\lambda_1 + (g_y v)_y^t\lambda_2 = 0 \tag{15}$$
$$\nabla_{\lambda_1}\mathcal{L} = g(x,y) = 0 \tag{16}$$
$$\nabla_{\lambda_2}\mathcal{L} = g_y(x,y)v = 0 \tag{17}$$
$$\nabla_\mu\mathcal{L} = v_1^2 + v_2^2 + \cdots + v_n^2 - 1 = 0 \tag{18}$$
$$\nabla_v\mathcal{L} = g_y^t\lambda_2 + 2\mu v = 0 \tag{19}$$

where $g_x \equiv \partial g/\partial x$, and other partial derivatives follow the same convention.

The system (14)-(19) constitutes a set of nonlinear equations that can be solved for the optimum solution $(x^*,y^*,\lambda_1^*,\lambda_2^*,\mu^*,v^*)$ using an iterative nonlinear solver such as the Newton method.

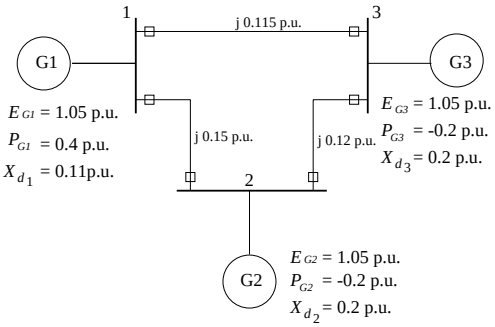

Figure 1: Three-machine three-bus system.

The critical energy $\mathcal{V}(x^*, y^*)$ can be used to estimate the stability boundary. If a system at point (x_0, y_0) has less energy than this critical value, i.e., $\mathcal{V}(x_0, y_0) < \mathcal{V}(x^*, y^*) \equiv \mathcal{V}_{cr}$, then the trajectory starting from (x_0, y_0) will never encounter the impasse surface.

3.3. Example

A power system example is used to illustrate the minimization proposed in Section 3.2. For power systems, the energy function is composed of kinetic energy $\mathcal{V}_{ke}$ and potential energy $\mathcal{V}_{pe}$ terms

$$\mathcal{V} = \mathcal{V}_{ke}(\omega) + \mathcal{V}_{pe}(\bar{x}, y)$$

where $\bar{x}$ includes all dynamic state variables except generator speeds ω. The kinetic energy $\mathcal{V}_{ke}(\omega)$ is quadratic in ω, and ω does not appear in the algebraic equations g. Therefore by (14), $\omega^* = 0$ at the optimal point, and hence $\mathcal{V}_{ke}(\omega^*) = 0$. It follows that only the potential energy $\mathcal{V}_{pe}$ needs to be minimized in (7)-(9).

Consider the small system shown in Figure 1. Machines were represented by the classical machine model. Terminal buses were retained, introducing algebraic (power balance) equations and algebraic variables, viz., bus voltage magnitudes and angles. Active power loads were zero. All reactive power loads were modelled as voltage dependent, i.e., $Q(V) = Q_0 V^\eta$, with Q_0 taking the values 1.5, 1.0 and 1.0 per unit at buses 1, 2 and 3 respectively. Algebraic singularity is more likely to occur with lower values of voltage indices η_i. Therefore indices of $\eta_i = 0.5$ were chosen to induce singularity. These values resulted in the impasse surface shown in Figure 2 as a dark line[1]. The critical energy given by minimizing $\mathcal{V}_{pe}$ over the impasse surface, i.e., the solution of (14)-(19), was $\mathcal{V}_{cr}^{min} = 2.65$. This minimum energy point is marked by `MinEP` in Figure 2.

A three-phase fault was applied at bus 3. The critical clearing time predicted using $\mathcal{V}_{cr}^{min}$ was $t_{cr}^{min} = 0.324$s. From simulation, the critical value of energy was found to be $\mathcal{V}_{cr}^{sim} = 2.76$. The corresponding critical clearing time was $t_{cr}^{sim} = 0.332$s. Figure 2 shows the case where clearing was delayed just slightly beyond that critical value, to $t_{cl} = 0.333$s. The trajectory encountered the impasse surface at point A and could not continue. The critically cleared case is shown in Figure 3. In this case the trajectory approached point A but swung around before touching the impasse surface, allowing the trajectory to continue.

Note that $\mathcal{V}_{cr}^{min} < \mathcal{V}_{cr}^{sim}$, i.e., the energy of the system can be greater than the critical value predicted using the minimization approach, yet the system is stable. This is a consequence of the conservativeness inherent in Lyapunov theory, which provides sufficient but not necessary conditions. However the predicted fault clearing time of $t_{cr}^{min} = 0.324$s compares well with the actual value of $t_{cr}^{sim} = 0.332$s.

[1]The impasse surface was obtained using a continuation method [11]. An initial point on the impasse surface was identified by monitoring the eigenvalues of g_y along a fault-on trajectory.

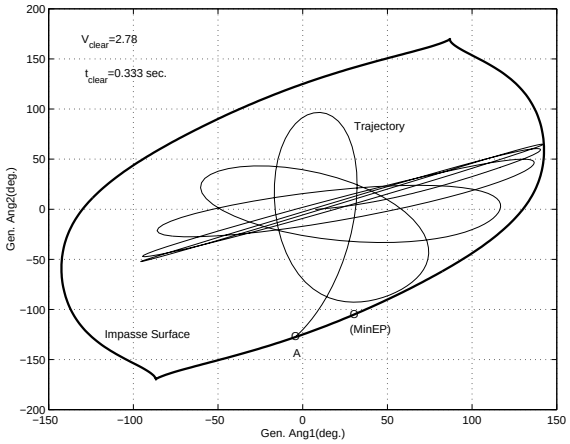

Figure 2: Trajectory encounters the impasse surface at A.

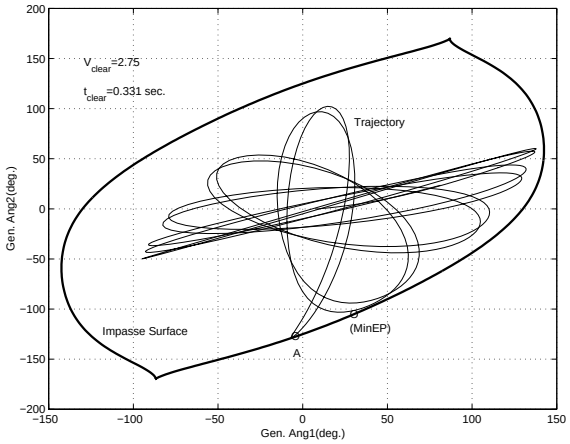

Figure 3: Critically cleared trajectory.

In this example trajectories have been constrained to the principal causal region C_0, i.e., the component of the constraint manifold G where all eigenvalues of g_y are positive. The concepts illustrated here extend naturally though to allow stability assessment on any causal component C_i that contains a stable equilibrium point.

4. MINIMIZATION PRACTICAL CONSIDERATIONS

The set of equations (14)-(19) that follows from the optimality conditions can be written in the form

$$F(z) = 0$$

where $z = [x^t\ y^t\ \lambda_1^t\ \lambda_2^t\ \mu\ v^t]^t$. A Newton algorithm solves these equations using the iterative process

$$z^{k+1} = z^k - \alpha^k\ J^{-1}(z^k) F(z^k) \tag{20}$$

to move from the k^{th} to the $(k+1)^{th}$ iteration. In (20), α is the stepsize (acceleration factor) and J is the Jacobian matrix of (14)-(19) calculated at the k^{th} iteration. It can be shown that J is also the Hessian matrix of the Lagrangian function $\mathcal{L}$ [10].

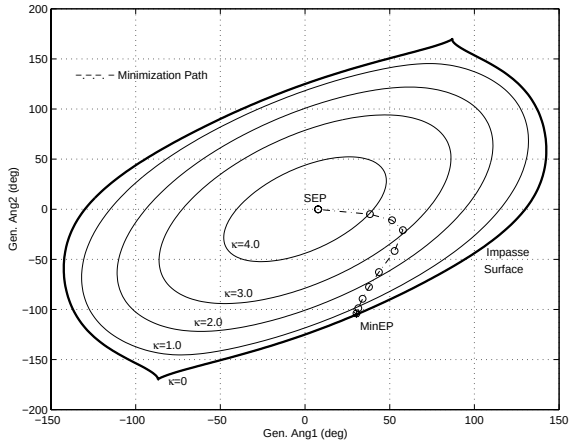

Figure 4: Illustration of the continuation algorithm.

A disadvantage of the the Newton algorithm is that a good initial guess is required. The difficulty of finding the initial values is compounded by the presence of Lagrangian multipliers and elements of the eigenvector v. Unfortunately there is no straightforward way of assigning good initial values for ensuring fast convergence.

To deal with this problem, a continuation algorithm developed in [12] has been adapted. This algorithm computes a sequence of points along a continuation path. At each successive point, reliable initial guesses are computed from the solution at the previous point. The algorithm solves a parameterized problem that is initialized at the stable equilibrium point (SEP). The minimization problem (14)-(19) can be parameterized by an eigenvalue κ of g_y that turns zero on the impasse surface.

Let v_ℓ be the eigenvector corresponding to the (parameterized) eigenvalue[2] κ_ℓ, i.e., $g_y v_\ell = \kappa_\ell v_\ell$. The desired minimization can be achieved by solving the parameterized version of (14)-(19)

$$\nabla_x \mathcal{V} + g_x^t \lambda_1 + (g_y v_\ell)_x^t \lambda_2 = 0 \quad (21)$$

$$\nabla_y \mathcal{V} + g_y^t \lambda_1 + (g_y v_\ell)_y^t \lambda_2 = 0 \quad (22)$$

$$g(x, y) = 0 \quad (23)$$

$$g_y(x, y) v_\ell - \kappa_\ell v_\ell = 0 \quad (24)$$

$$v_{\ell 1}^2 + v_{\ell 2}^2 + \cdots + v_{\ell n}^2 = 1 \quad (25)$$

$$g_y^t \lambda_2 + 2\mu v_\ell - \kappa_\ell \lambda_2 = 0 \quad (26)$$

for a sequence of values of the parameter (eigenvalue) κ_ℓ stepping from the SEP value κ_ℓ^{sep} to 0. Note that the derivatives of the energy function $\mathcal{V}_x$ and $\mathcal{V}_y$ vanish at the SEP. So when $\kappa_\ell = \kappa_\ell^{sep}$, all equations in (21)-(26) are trivially satisfied for $\lambda_1 = 0$, $\lambda_2 = 0$ and $\mu = 0$. Therefore the SEP serves as a good starting point for the continuation process. To move from the SEP, the parameter κ_ℓ is reduced towards zero. Equations (21)-(26) are solved for each successive value of κ_ℓ using the previous solution as the initial guess. The desired minimum, i.e., the solution of (7)-(8) together with (10)-(11), is obtained when $\kappa_\ell = 0$. This process is illustrated in Figure 4.

5. CONCLUSIONS

Trajectories of differential-algebraic systems are constrained to lie on a manifold described by the algebraic equations. This complicates the stability assessment of such systems, as the stability boundary includes segments of algebraic singularity. This paper proposes a Lyapunov (energy) method for direct assessment of transient singularity.

Transient singularity will be avoided if the system has less energy than a critical value given by minimizing energy over the relevant segment of the impasse surface. This minimization is complicated by the initialization of Lagrangian multipliers. A continuation algorithm is presented. The critical energy enables an estimate of the stable causal region surrounding a stable equilibrium point. A power system example has been used for illustration.

[2]The eigenvalue chosen is that which reduces towards zero along the fault-on trajectory.

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