Non-uniqueness in Reverse Time of Hybrid System Trajectories

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Abstract. Under standard Lipschitz conditions, trajectories of systems described by ordinary differential equations are well defined in both forward and reverse time. (The flow map is invertible.) However for hybrid systems, uniqueness of trajectories in forward time does not guarantee flow-map invertibility, allowing non-uniqueness in reverse time. The paper establishes a necessary and sufficient condition that governs invertibility through events. It is shown that this condition is equivalent to requiring reverse-time trajectories to transversally encounter event triggering hypersurfaces. This analysis motivates a homotopy algorithm that traces a one-manifold of initial conditions that give rise to trajectories which all reach a common point at the same time.

1 Introduction

Uniqueness is a fundamental property of solutions of dynamical systems. Intuitively, uniqueness in forward time should imply reverse time uniqueness[1]. That is certainly the case for systems described by ordinary differential equations, as discussed in the background presentation in Section 2. However it is not necessarily true for hybrid systems.

Hybrid system solutions are composed of periods of smooth behaviour separated by discrete events [2]. Standard transversality conditions can be established to ensure transitions through events are well behaved. An overview is provided in Section 4. However those conditions are not sufficient to ensure reverse-time mappings through events are well defined. It is shown in the paper that another transversality-type condition must be satisfied to ensure uniqueness in reverse-time (or equivalently flow-map invertibility.)

Recent investigations have established conditions governing the well-posedness of solutions for various hybrid system formalisms. A complementarity modelling framework [3] underlies the characterization of solutions of linear relay systems [4, 5] and further extensions to piecewise-linear systems [6, 7]. A more general hybrid automata framework is considered in [8]. In all cases, well-posedness is

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1 In other words, invertibility of the flow map [1]. This is discussed further in Section 3.

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addressed in the context of forward solutions, i.e., whether there exists a unique (forward) solution for every initial state x_0. It is noted in [5], though without discussion, that forward-time well-posedness does not imply well-defined behaviour in reverse time. That reverse-time issue is addressed in this paper, in the context of flow-map invertibility. It is shown in Section 7 that non-invertibility gives rise to a manifold of initial conditions for trajectories which all reach a common point at the same time. Such concepts have not previously been explored.

Analysis of system dynamic behaviour is normally only concerned with trajectory evolution in forward time. In such cases, the issues raised here are inconsequential. However reverse-time trajectories form the basis for the adjoint system equations, which underlie algorithms for solving boundary value and dynamic embedded optimization problems [9, 10, 11]. Application of such algorithms to hybrid systems must therefore consider these uniqueness issues.

2 Background

Existence and uniqueness properties for systems of the form

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ are well known [12, 13]. In particular, if $f \in C^1$, i.e., is continuous in x and has continuous first partial derivatives with respect to x over $\mathbb{R}^n$, then (1) has a unique solution

$$x(t) = \phi(t, x_0) \equiv \phi_t(x_0), \tag{2}$$

with $\phi(0, x_0) = x_0$. Furthermore, the *flow map* $\phi(t, x_0)$ is differentiable with respect to x_0. The *sensitivity transition matrix* is defined as

$$\Phi(t, x_0) \triangleq \frac{\partial \phi(t, x_0)}{\partial x_0}. \tag{3}$$

It is obtained by differentiating (1) with respect to x_0 to give

$$\dot{\Phi}(t, x_0) = Df(t)\Phi(t, x_0), \qquad \Phi(0, x_0) = I \tag{4}$$

where

$$Df(t) \triangleq \left.\frac{\partial f(x)}{\partial x}\right|_{x=\phi(t,x_0)}.$$

The transition matrix $\Phi(t, x_0)$ is the solution of a set of linear time-varying differential equations (4), so has the property

$$\det \Phi(t, x_0) = \exp\left\{\int_0^t \mathrm{Trace}\{Df(\tau)\}d\tau\right\}, \tag{5}$$

which implies $\Phi(t, x_0)$ is nonsingular for all t.

Expanding $\phi(t, x_0)$ in a Taylor series, and neglecting higher order terms, results in

$$\phi(t, \bar{x}_0) - \phi(t, x_0) \approx \Phi(t, x_0)(\bar{x}_0 - x_0) \tag{6}$$

$$\Rightarrow \delta x(t) \approx \Phi(t, x_0)\delta x_0. \tag{7}$$

In other words a change δx_0 in initial conditions[2] induces a change $\delta x(t)$ in the trajectory at time t, with that change described (approximately) by $\Phi(t, x_0)$. Because $\Phi(t, x_0)$ is nonsingular for all t, it may be concluded that given any $\delta x(t)$, it is always possible to find the corresponding δx_0, i.e.,

$$\delta x_0 = \Phi(t, x_0)^{-1}\delta x(t). \tag{8}$$

3 Reverse Time Trajectories

For systems of the form (1), the map $\phi_t \in C^1$. Furthermore, according to (5), its derivative $D\phi_t(x) = \Phi(t, x)$ is always invertible. Therefore, by the inverse function theorem [1], ϕ_t is a one-parameter family of diffeomorphisms. It follows that ϕ_t has a C^1 inverse ϕ_{-t}, such that $\phi_{-t}(\phi_t(x)) = \phi_0(x) = x$. This inverse ϕ_{-t} is referred to as the *reverse time* trajectory.

4 Hybrid Systems

Hybrid systems have the form

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P} \tag{9}$$

where $f_p : \mathbb{R}^n \to \mathbb{R}^n$, and $\mathcal{P}$ is some finite index set. Transitions between the various subsystems $f_i \to f_j$ occur when the state x evolves to a point that satisfies an event triggering condition,

$$s_{ij}(x) = 0 \tag{10}$$

where $s_{ij} : \mathbb{R}^n \to \mathbb{R}$. We shall assume $s_{ij} \in C^1$. A more elaborate differential-algebraic model, that incorporates switching and impulse effects, is described in [14].

Assume all f_p satisfy the differentiability condition of f in (1), and x is continuous at events, i.e., impulses do not occur. Furthermore, assume that event triggers are encountered transversally[3],

$$\nabla s_{ij}^T \dot{x} = \nabla s_{ij}^T f_i \neq 0 \tag{11}$$

[2] Parameter sensitivity can be incorporated through initial conditions by introducing trivial equations

$$\dot{\lambda} = 0, \quad \lambda(0) = \lambda_0.$$

[3] Tangential encounters are associated with grazing phenomena [15].

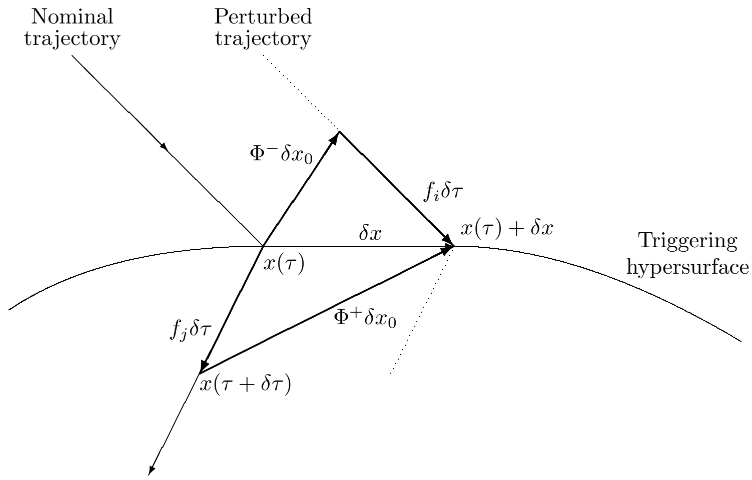

Fig. 1. Jump conditions

and that event switching is well defined, in the sense that accumulation effects do not occur. Under those conditions, (9) has a unique solution that can be expressed in the same form as (2).

Away from events, the sensitivity transition matrix $\Phi(t, x_0)$ is defined according to (4). It is shown in [14] that at an event $i \rightarrow j$, occurring at time τ, sensitivities Φ generically jump[4] according to

$$\Phi(\tau^{+}, x_0) = \Phi(\tau^{-}, x_0) + (f_j - f_i)\frac{\nabla s_{ij}^{T}\Phi(\tau^{-}, x_0)}{\nabla s_{ij}^{T} f_i} \tag{12}$$

$$= \left(I + (f_j - f_i)\frac{\nabla s_{ij}^{T}}{\nabla s_{ij}^{T} f_i}\right)\Phi(\tau^{-}, x_0) \tag{13}$$

$$= \Phi(\delta, x(\tau^{-}))\Phi(\tau^{-}, x_0) \tag{14}$$

where δ in (14) signifies the time increment $\tau^{+} - \tau^{-}$. Notice that the transversality condition (11) ensures that the denominator of (12) is non-zero.

Equation (12) can be rewritten

$$\Phi^{+} = \Phi^{-} - (f_j - f_i)\frac{\partial \tau}{\partial x_0} \tag{15}$$

where

$$\frac{\partial \tau}{\partial x_0} = -\frac{\nabla s_{ij}^{T}\Phi^{-}}{\nabla s_{ij}^{T} f_i}$$

[4] No jump occurs if $f_i = f_j$ or $\nabla s_{ij}^{T}\Phi(\tau^{-}, x_0) = 0$.

gives the sensitivity of event triggering time to initial conditions. For a perturbation δx_0, (15) gives

$$\delta x = \Phi^- \delta x_0 + f_i \delta\tau = \Phi^+ \delta x_0 + f_j \delta\tau,$$

which is illustrated in Figure 1.

5 Uniqueness in Forward and Reverse Time

Generalizing (14) to a sequence of events occurring at times $0 < \tau_1 < \tau_2 < \cdots < \tau_\ell$ results in the sensitivity transition matrix at $t > \tau_\ell$ having composition

$$\begin{aligned}\Phi(t, x_0) = \Phi(t - \tau_\ell^+, x(\tau_\ell^+)) \times \Phi(\delta, x(\tau_\ell^-)) \times \Phi(\tau_\ell^- - \tau_{\ell-1}^+, x(\tau_{\ell-1}^+)) \times \ldots \\ \times \Phi(\tau_1^-, x_0)\end{aligned}$$

where $\Phi(\tau_\ell^- - \tau_{\ell-1}^+, x(\tau_{\ell-1}^+))$ corresponds to transitions along smooth sections of the flow, and $\Phi(\delta, x(\tau_\ell^-))$ describes the transition through an event at time τ_ℓ. Property (5) ensures that matrices $\Phi(\tau_\ell^- - \tau_{\ell-1}^+, x(\tau_{\ell-1}^+))$ are always nonsingular. However the following theorem establishes conditions governing the singularity of transition matrices $\Phi(\delta, x(\tau_\ell^-))$.

Theorem 1. *The sensitivity transition matrix $\Phi(\delta, x(\tau^-))$ is singular if and only if $\nabla s_{ij}^T f_j = 0$.*

Proof: The proof makes use of the fact that $\det(I + ab^T) = 1 + b^T a$, which is a special case of $\det(I + AB) = \det(I + BA)$ [16]. Then

$$\begin{aligned}\det \Phi(\delta, x(\tau^-)) &= \det \left(I + (f_j - f_i) \frac{\nabla s_{ij}^T}{\nabla s_{ij}^T f_i} \right) \\ &= 1 + \frac{\nabla s_{ij}^T}{\nabla s_{ij}^T f_i} (f_j - f_i) \\ &= \frac{\nabla s_{ij}^T f_j}{\nabla s_{ij}^T f_i},\end{aligned}$$

which is zero if and only if $\nabla s_{ij}^T f_j = 0$.

□

Therefore, for a hybrid system, $\Phi(t, x_0)$ will be singular if the conditions of Theorem 1 occur at *any* event. But if $\Phi(t, x_0)$ is singular, the hybrid system flow ϕ_t is not a diffeomorphism, and the reverse time trajectory ϕ_{-t} is not well defined.

Recalling (7), a perturbation δx_0 in initial conditions will always result in a well defined perturbation $\delta x(t)$ in the trajectory at time t. However if $\Phi(t, x_0)$ is not invertible, the reverse mapping (8) is not valid. A general perturbation $\delta x(t)$ cannot be mapped backwards to a corresponding unique δx_0. More specif-

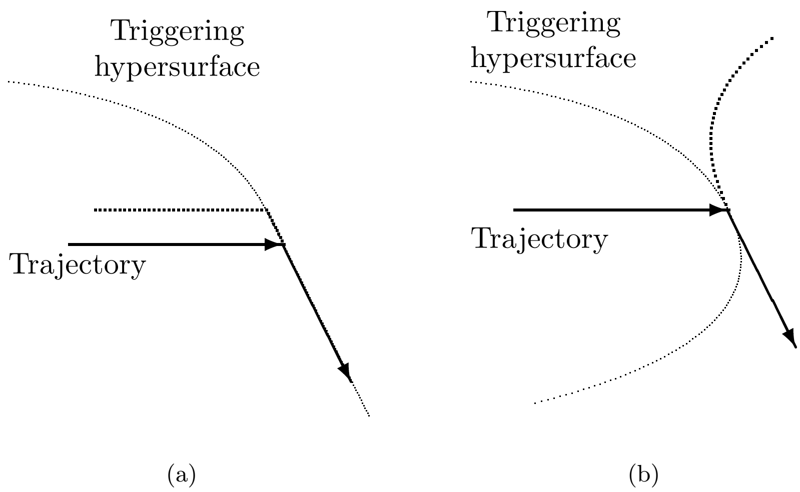

Fig. 2. Conditions inducing singularity

ically though, if $\delta x(t)$ lies in the range space of $\Phi(t, x_0)$, then it can be mapped backwards to a continuum of δx_0[5].

It may be concluded that for hybrid systems, uniqueness of trajectories in forward time does not guarantee uniqueness in reverse time.

Note that there is a subtle but important difference between the transversality condition (11) and the singularity condition of Theorem 1, even though they have a similar form. Transversality (11) ensures the trajectory has a well defined (forward) encounter with the triggering hypersurface. Theorem 1 establishes conditions that relate to the trajectory's departure from the event. Furthermore, it should be emphasised that the triggering condition $s_{ij}(x) = 0$ is only active for subsystem i, before the event. After the event, in subsystem j, it is no longer relevant.

Figure 2 illustrates ways in which the singularity condition of Theorem 1, $\nabla s_{ij}^T f_j = 0$, may arise. In Figure 2(a), the post-event trajectory remains on the triggering hypersurface for a non-zero time interval. This situation is relatively common in practice, for example the action of anti-wind-up limits [17]. In Figure 2(b), the post-event trajectory leaves the triggering hypersurface tangentially. The examples of Section 8 consider both situations further.

These two cases motivate an interesting corollary of Theorem 1.

Corollary 2. *The sensitivity transition matrix $\Phi(\delta, x(\tau^-))$ is nonsingular if and only if the reverse-time trajectory $\phi_{-t}(x(\tau^+))$ is transversal to the triggering hypersurface s_{ij} that induces the event at time τ.*

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[5] Let u and v be the left and right eigenvectors of $\Phi(t, x_0)$ corresponding to a zero eigenvalue. Then if $u^T \delta x(t) = 0$, δx_0 will lie in the one-dimensional subspace defined by $\delta x_0 = w + \alpha v$ where $\delta x(t) = \Phi(t, x_0)w$ and α is a scalar.

In other words, to ensure uniqueness in reverse time, the reverse-time trajectory must "encounter" triggering hypersurfaces transversally. (Though keep in mind these hypersurfaces are really only defined for the forward trajectory.) The situations presented in Figure 2 illustrate reverse-time non-uniqueness when this transversality condition is not satisfied. In both illustrations, the post-event segment of the trajectory could have originated from the dotted trajectory, rather than the actual pre-event (solid) trajectory.

Note that the two cases depicted in Figure 2 are structurally quite different. In Figure 2(a), reverse-time non-uniqueness persists under perturbations in the initial conditions, whereas for Figure 2(b), perturbations destroy that property. However this latter case has an interesting sliding interpretation when the triggering hypersurface is common to both the pre- and post-trigger subsystems. Referring to Figure 2(b), consider trajectories that emanate from either subsystem and encounter the triggering hypersurface just above the switching point of the nominal trajectory (the switching that induces reverse-time non-uniqueness). Those trajectories will slide along the hypersurface until they reach that pivotal switching point. From there they will depart the hypersurface and follow the post-switching trajectory shown in the figure. The pivotal switching point separates the sliding region from that associated with well-behaved switching.

Keep in mind that this sliding interpretation is only appropriate when the triggering hypersurface is common to both subsystems. Corollary 2 is more generally applicable.

6 Impulses at Events

The hybrid system model established in Section 4 and used through Section 5 assumed continuity of x at events. However results can be generalized to allow impulses at events. Assume the impulse mapping at event $i \rightarrow j$ has the form

$$x^+ = h_{ij}(x^-)$$

where $h_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism, and x^+, x^- refer to the values of the state just after, and just prior to, the event respectively[6].

It is shown in [14] that with the inclusion of impulse effects, the sensitivity transition matrix jump conditions (12)-(14) become

$$\Phi(\delta, x(\tau^-)) = Dh + (f_j - Dhf_i)\frac{\nabla s_{ij}^T}{\nabla s_{ij}^T f_i} \tag{16}$$

where $Dh \triangleq \frac{\partial h_{ij}}{\partial x}$. In this case, Theorem 1 takes a slightly modified form.

Theorem 3. *For nonsingular Dh, the sensitivity transition matrix $\Phi(\delta, x(\tau^-))$ is singular if and only if $\nabla s_{ij}^T Dh^{-1} f_j = 0$.*

[6] An implicit impulse mapping $\check{h}_{ij}(x^+, x^-) = 0$ is also acceptable, though not used here.

Proof: The proof is similar to that of Theorem 1. With Dh nonsingular,

$$\begin{aligned}\det \Phi(\delta, x(\tau^-)) &= \det(Dh)\det\left(I + (Dh^{-1}f_j - f_i)\frac{\nabla s_{ij}^T}{\nabla s_{ij}^T f_i}\right)\\ &= \det(Dh)\left(1 + \frac{\nabla s_{ij}^T}{\nabla s_{ij}^T f_i}(Dh^{-1}f_j - f_i)\right)\\ &= \det(Dh)\left(\frac{\nabla s_{ij}^T Dh^{-1} f_j}{\nabla s_{ij}^T f_i}\right).\end{aligned}$$

Given that Dh is nonsingular, singularity of $\Phi(\delta, x(\tau^-))$ corresponds to $\nabla s_{ij}^T Dh^{-1} f_j = 0$.

□

The condition established in Theorem 3 has a very similar interpretation to that of Theorem 1. Now though, the post-event vector field f_j is translated via Dh^{-1} back to a pre-event coordinate system, where transversality is again required for nonsingularity.

7 Homotopy Algorithm

To first order, deviations in a trajectory at time t are given by (7). If $\Phi(t, x_0)$ is singular then a deviation δx_0 that coincides with the null-space[7] of $\Phi(t, x_0)$ results in $\delta x(t) = 0$. As mentioned in Section 5, under such conditions $\phi(t, x_0)$ maps a continuum of x_0 to a single point $x(t)$. In fact, if $\Phi(t_*, x_0)$ has rank deficiency k, then $x(t_*) = \phi(t_*, x_0)$ defines a k-manifold.

If $\Phi(t_*, x_0)$ has a single zero eigenvalue, then

$$\Sigma = \{x_0 : \phi(t_*, x_0) - x(t_*) = 0\} \tag{17}$$

describes a 1-manifold, or curve. Homotopy methods can be used to generate successive points along such curves. An Euler homotopy provides a robust predictor-corrector algorithm [18].

Assume a point $\bar{x}_0$ on Σ is known. (This is a straightforward initial value problem.) The first step of the homotopy algorithm is the (first order) prediction of the next point on the curve. This is achieved by finding the vector that is tangent to Σ at $\bar{x}_0$. This tangent vector is nothing more than the (normalized) right eigenvector v of $\Phi(t_*, \bar{x}_0)$ corresponding to the zero eigenvalue. The prediction of the next point is obtained by moving along v a predefined distance τ,

$$x_{0,pred} = \bar{x}_0 + \tau v,$$

where

$$\Phi(t_*, \bar{x}_0)v = 0 \tag{18}$$

$$\|v\| = 1. \tag{19}$$

[7] The null-space is spanned by the right eigenvectors corresponding to zero eigenvalues.

Having found the prediction point, we now need to correct to a point x_0 on the curve. The Euler method does this by solving for the point of intersection of the curve and a hyperplane that passes through $x_{0,pred}$ and that is orthogonal to v. Points x_0 on this hyperplane are given by,

$$(x_0 - \bar{x}_0)^T v = \tau. \tag{20}$$

The point of intersection of the curve and the hyperplane is therefore given by

$$\phi(t_*, x_0) - x(t_*) = 0 \tag{21}$$

$$(x_0 - \bar{x}_0)^T v = \tau. \tag{22}$$

Note though that (21)-(22) describe $n + 1$ equations in n unknowns. However the rank deficiency of (21) suggests that one of those equations is redundant, and so can be discarded. It remains to determine which equation to discard.

Newton-Raphson solution of (21)-(22) proceeds via the iteration formula

$$\begin{bmatrix} \Phi(t_*, x_0) \\ v^T \end{bmatrix} \Delta x_0 = \begin{bmatrix} \phi(t_*, x_0) - x(t_*) \\ (x_0 - \bar{x}_0)^T v - \tau \end{bmatrix} \tag{23}$$

where $\Phi(t_*, x_0)$ is singular, with a single zero eigenvalue. Solution of (23) requires that $u^T(\phi(t_*, x_0) - x(t_*)) = 0$, where u is the left eigenvector of $\Phi(t_*, x_0)$ corresponding to the zero eigenvalue. In other words, u describes the linear dependence between the first n equations of (23). This implies that the best equation to discard from (21) is that corresponding to the element of u with the largest absolute value.

The next point on the curve is therefore given by Newton-Raphson solution of

$$F(x_0) \equiv \begin{bmatrix} \underline{\phi}(t_*, x_0) - \underline{x}(t_*) \\ (x_0 - \bar{x}_0)^T v - \tau \end{bmatrix} = 0 \tag{24}$$

which utilizes the (nonsingular) Jacobian

$$DF(x_0) = \begin{bmatrix} \underline{\Phi}(t_*, x_0) \\ v^T \end{bmatrix}, \tag{25}$$

where underlining in (24) and (25) indicates that the appropriate equation has been discarded.

8 Examples

8.1 Example 1

As indicated in Section 5, anti-wind-up limits provide a common situation where reverse-time transversality is not possible. This can be illustrated using a simple example that consists of a linear continuous-time system

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} x$$

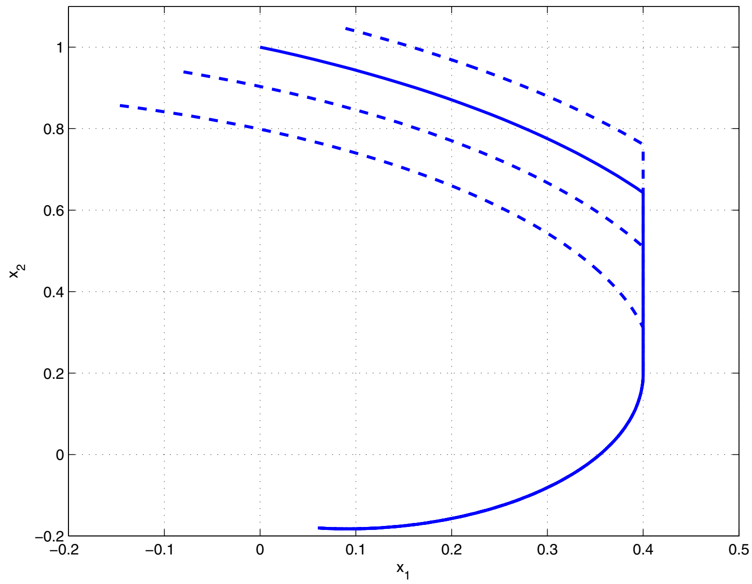

Fig. 3. Example 1 response

together with an anti-wind-up limit restricting $x_1 \leq 0.4$. In terms of the hybrid system representation of Section 4, this system may be modelled as

$$\dot{x} = f_1(x) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} x \qquad \text{(subsystem 1)}$$

$$\dot{x} = f_2(x) = \begin{bmatrix} 0 & 0 \\ -2 & -1 \end{bmatrix} x \qquad \text{(subsystem 2)}$$

with transitions from subsystem 1 to 2 triggered when

$$s_{12}(x) = x_1 - 0.4 = 0$$

and from subsystem 2 to 1 when

$$s_{21}(x) = [-1\ 2]x = 0.$$

This latter condition ensures $\dot{x}_1 < 0$ after switching, so behaviour is directed away from the limit surface. The response of this system for initial conditions $x_0 = [0\ 1]^T$ is shown as a solid line in Figure 3.

At the instant prior to the limit being encountered (event triggering), the sensitivity transition matrix $\Phi(\tau^-, x_0)$ had eigenvalues $0.64 \pm j0.40$. However for this event,

$$\Phi(\delta, x(\tau^-)) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

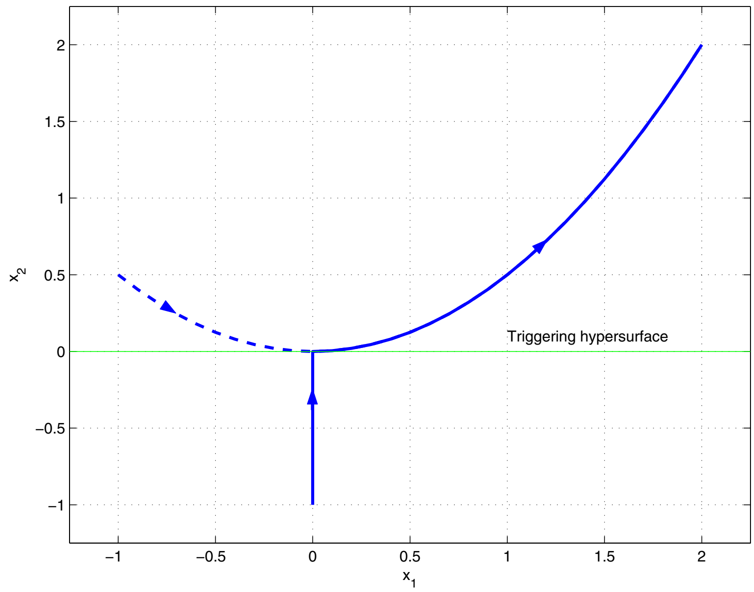

Fig. 4. Example 2 response

which clearly has a zero eigenvalue, with corresponding left eigenvector $\nabla s_{ij} = [1\ 0]^T$.

The homotopy algorithm was used to locate other initial points that reached the same final point at the same time. These are shown in Figure 3 as dashed lines. Once the limit is encountered, the trajectories are indistinguishable. All are well defined in forward time, but there is no unique reverse-time trajectory.

8.2 Example 2

Behaviour of the form shown in Figure 2(b) can be illustrated using the simple hybrid system,

$$\dot{x} = f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(subsystem 1)}$$

$$\dot{x} = f_2(x) = \begin{bmatrix} 1 \\ x_1 \end{bmatrix} \qquad \text{(subsystem 2)}$$

with transitions from subsystem 1 to 2 triggered when

$$s_{12}(x) = x_2 = 0.$$

The solid line in Figure 4 shows the trajectory given by initial conditions $x_0 = [0\ -1]^T$. The singularity condition of Theorem 1 occurs at the switching point

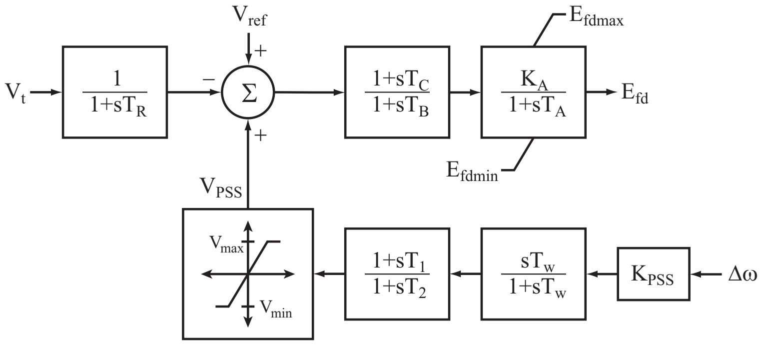

Fig. 5. Excitation system (AVR/PSS) representation

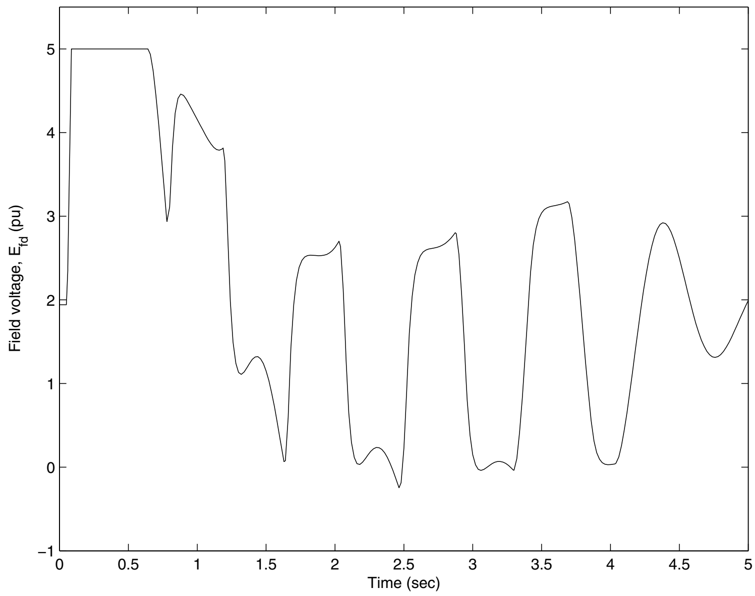

Fig. 6. Generator field voltage response

$x = [0\ 0]^T$, implying the trajectory is not unique in reverse time. This non-uniqueness is confirmed by the dashed trajectory which starts at $x_0 = [-1\ 0.5]^T$, but coincides with the nominal trajectory from the point $x = [0\ 0]^T$ onwards.

If transitions from subsystem 2 to 1 were triggered when

$$s_{21}(x) = s_{12}(x)$$

then all trajectories emanating from below the dashed line would slide along the x_1-axis until reaching the point $x = [0\ 0]^T$. From there they would all follow the nominal trajectory shown.

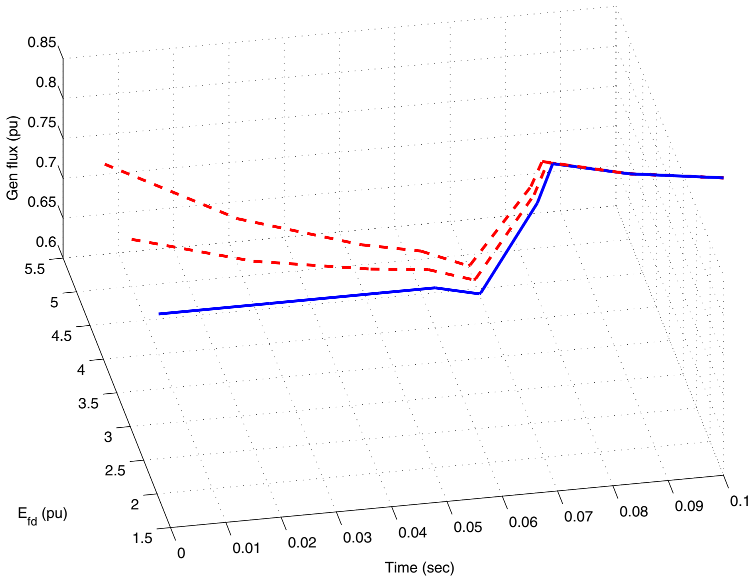

Fig. 7. Trajectories given by homotopy algorithm

8.3 Example 3

A more elaborate power system case has also been considered. In this case, generators were represented by a sixth order nonlinear model [19], and equipped with the excitation system shown in Figure 5. This system includes clipping limits on the stabilizer output V_{PSS}, and anti-windup limits on the field voltage E_{fd}. The response of the generator field voltage E_{fd} to a fault is shown in Figure 6. Note that this trajectory is quite non-smooth, as is typical for power systems.

At 0.086 sec, the anti-windup limit was encountered. As anticipated, the sensitivity transition matrix $\Phi(\delta, x(0.086^-))$ was singular, with a single zero eigenvalue at that event. The homotopy algorithm was again used to locate initial points that converged to the same final point at the same final time. Results are shown in Figure 7. The solid line corresponds to the original case. The dashed trajectories originate from points given by the homotopy. Note that all curves converge at 0.086 sec.

9 Conclusions

Hybrid system solutions are composed of periods of smooth behaviour separated by discrete events. Standard transversality conditions can be established to ensure transitions through events are well behaved. However certain applications, such as the adjoint equations in dynamic embedded optimization, require the evaluation of behavior in reverse time. Standard transversality conditions are not sufficient in that case. It is shown in the paper that another *reverse-time* transversality-type condition must be satisfied to ensure a unique reverse-time mapping through events.

When reverse-time trajectories are not well-posed, two situations may arise. In the first case, a continuum of initial conditions can be found for trajectories that all reach the same point in state-space at the same time. It has been shown that when this continuum is a 1-manifold, a predictor-corrector homotopy method can be used to trace that curve. Alternatively, trajectories exhibiting reverse-time non-uniqueness may be isolated. In that case, under the special condition that the event triggering hypersurface is common to both the pre- and post-event subsystems, the ill-posed trajectory separates (reverse-time non-unique) sliding behaviour from well-defined switching.

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