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Alternative Strategies for Designing Stabilizing Model Predictive Controllers

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***Abstract*— In this article, we propose two stabilizing discrete-time model predictive control (MPC) strategies, which are alternatives to other classical (e.g. terminal cost/constraint-based) approaches. Both proposed strategies take advantage of a known stabilizing controller and its associated Lyapunov function. The first strategy allows optimization of an arbitrary cost function at each stage, but guarantees stability by enforcing a decrease in the known Lyapunov function at the first step of each MPC state. The second strategy uses an averaged/summed Lyapunov function as the objective function. A combined strategy that enforces a decrease in a summed Lyapunov function while optimizing an arbitrary cost is also considered. The proposed strategies are applied to an example drawn from the class of linear systems subject to actuator saturation constraints.**

I. INTRODUCTION

Model predictive control (MPC) methods have been widely used in solving on-line optimal control problems with constraints, cf. [1]–[4]. One motivation for using MPC is that many real-world processes requiring real-time control have constrained, highly nonlinear dynamics that are subject to disturbances, and hence classical optimal control methodologies (which usually require a pre-calculated controller) are difficult to apply directly. MPC often can relieve the complexity inherent in designing high-performance controllers for these systems: because a finite-horizon problem is solved, MPC can exploit efficient algorithms such as linear programming tools to solve the underlying optimization problem. Specifically, given the current state at each time-step, MPC aims to numerically find inputs that optimize a finite-horizon cost function subject to the constraints. Since the process is repeated at each successive sample time-step, MPC generates a form of feedback control law for the system, and does so in real time (or on-line).

In many domains where MPC is applied, asymptotic or long-term dynamics of the closed-loop system are of importance. Guaranteeing stability via MPC while still taking advantage of its performance-optimization and computational benefits is critical for both applications and control theoretical developments. Because MPC feedback controllers are not expressed explicitly as functions of the system state, asymptotic stability analysis of the closed-loop system becomes challenging. The stability of MPC has received considerable attention over the last several decades, cf. [5]–[9], many of which are based on using a *terminal-constraint-set*. For example, a dual-mode controller proposed in [11] guarantees stability by employing MPC outside a terminal region and a linear feedback controller inside it, which is a relaxation of earlier approaches based on terminal equality constraints. The survey article [10] establishes a rather general set of sufficient conditions for terminal-constraint-based MPC stability. Specifically, the terminal state constraint guarantees the existence of a feasible control sequence for successive MPC steps. Further, the terminal cost requirement allows the total cost to be employed as a Lyapunov function. Some other classical stabilizing MPC strategies are worth mentioning: [12] presents a quasi-infinite horizon approach with an additional terminal penalty constraint; [13] introduces a stable MPC scheme by adding an inequality contraction constraint to the state vector; in [14], [15], stability is guaranteed by requiring a global control Lyapunov function (CLF) to be decreasing along the state trajectory; among many others.

These MPC stability analyses/methods are a starting point for evaluating feedback-control performance of MPC. Specifically, the analyses show that, when the cost function and MPC receding horizon are specified properly, closed-loop stability can be achieved while the optimality properties of MPC are retained. However, the conventional methods introduce several concerns with respect to the tradeoff between stability and performance/computation:

1) Selection of the MPC receding horizon often depends on the ball of possible initial conditions (domain of attraction). For stability, the terminal-constraint-based MPC schemes are required to bring the state trajectory into a terminal set within the MPC time horizon at each stage. For different initial conditions, the time horizon required for a controller to achieve this requirement may vary; even for the same initial condition, different time horizons may lead to stability or instability (see [14]). To ensure MPC stability, either this MPC time horizon has to be changed based on the initial-condition set, or the local controller and cost function have to be redesigned. Hence, there is an inherent tradeoff between stability and performance or computational burden.

2) Different forms of cost functions also have an impact on stability. The terminal-constraint-based MPC stability conditions require that the terminal cost decreases sufficiently quickly (usually to compensate possible growth in the stage cost) to guarantee stability [10]. Similarly, the quasi-infinite horizon MPC method uses a quadratic terminal cost to upper bound an infinite horizon cost within a terminal region to achieve close-loop stability [12]. The fact that the cost function has to be redesigned, or is restricted by the specifics of the problem, sometimes may limit the computational performance. For example, the presence of a special terminal penalty in the cost function may lead to slower settling. This

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also means that the optimization horizon can become long, leading to high computational complexity.

These concerns, to some extent, motivate the study described in this article. We explore alternative approaches to MPC stability, that may allow greater freedom in selecting the time horizon (e.g. possible duration to bring the state trajectory into the terminal constraint set) and cost function (e.g. terminal penalty), and hence improve tradeoffs between stability, settling performance (total cost), and computational effort. Keeping in mind that previous classical MPC stability results require certain system knowledge (e.g. a local stable controller and its Lyapunov function [10]), the development here is also built on the assumption of a pre-known globally stabilizing controller and its associated Lyapunov function for the nonlinear system. Based on this assumption, two different stability design schemes are proposed, both of which take advantage of the Lyapunov function for the controlled system, but in a way that differs from the classical MPC stability results.

We stress that our assumption of a globally-stabilizing controller is strong. We realize that such an assumption may not always be practical since it can be difficult to establish a stabilizing controller for a nonlinear system subject to constraints. On the other hand, the strong assumption on the system's stability helps to free the limitations and constraints on the selection of the time horizon and cost function. Also, several important classes of nonlinear systems, for example linear systems with actuator saturation, do admit global controllers and associated Lyapunov functions. A key feature of our formulation is the freedom to optimize any cost function with a relatively small look-ahead horizon. We finally stress that the effort here is not intended to provide a general and complete analysis for MPC stability; in fact, the goal of this article is to provide different conceptual perspectives for achieving MPC stability which may provide interesting design freedoms, and to further relate the proposed schemes back to some of the existing MPC stability results.

The article is organized as follows. Section II briefly reviews the MPC optimization problem. Section III introduces the two proposed stable MPC design strategies, and provides conceptual comparisons between these two methods and existing designs. Several illustrative examples are included, and a third approach that combines the two proposed strategies is also briefly discussed. Finally, Section IV concludes the article by summarizing the contributions.

II. Problem Formulation

Let us introduce a particular control problem that we will address using MPC. Formally, we consider the following nonlinear discrete-time system subject to input constraints:

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state vector at time step k, $\mathbf{u}_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the input/actuator vector at time step k, and $\mathbb{U}$ is compact. We assume that f satisfies the equilibrium condition $f(\mathbf{0},\mathbf{0}) = \mathbf{0}$. The MPC cost function at each stage k is defined as:

$$J_k = \sum_{i=k+1}^{k+N} \mathbf{x}_i^T Q_i \mathbf{x}_i + \sum_{i=k}^{k+N-1} \mathbf{u}_i^T R_i \mathbf{u}_i, \tag{2}$$

where $N \geq 1$ is the MPC time horizon, and Q_i, R_i are positive semi-definite weighting matrices. For convenience, let the control sequence $U_k \stackrel{\triangle}{=} \{\mathbf{u}_k, \cdots, \mathbf{u}_{k+N-1}\}$. At each stage k, the MPC optimization goal is:

$$\min_{U_k} \quad J_k, \text{ subject to } \mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \text{ and } \mathbf{u}_k \in \mathbb{U}, \tag{3}$$

where the compact set $\mathbb{U}$ is a set of admissible values for the input. To actually obtain an MPC solution, we need to solve the stage optimization problem (3) by using some optimization techniques (e.g., linear, quadratic, or nonlinear programming tools). The resulting control sequence $U_k^* = \{\mathbf{u}^(k)_k, \cdots, \mathbf{u}^(k)_{k+N-1}\}$ achieves optimality while satisfying the system dynamics and constraints. MPC implements only the first input $\mathbf{u}^(k)_k$ of the sequence to the system.

As mentioned above, our development is based on the assumption that a stabilizing state-feedback controller of the original nonlinear system (1) subject to constraints has already been established, and a Lyapunov function for the closed-loop system is known. Our proposed MPC algorithms take advantage of this known control capability, while also exploiting the advantages of MPC (optimal control over a finite horizon, on-line design in the face of disturbances and noise). Formally, let $F(\mathbf{x}_k) = \mathbf{u}_k$ be a stabilizing feedback controller of the original system (1), with $F(0) = 0$. The Lyapunov function $V(\cdot)$ of the closed-loop system corresponding to $F(\cdot)$ satisfies $V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) < 0$ for $\mathbf{x}_k \neq 0$, and $V(0) = 0$. For clarity, we will subsequently use the notation $V_k \equiv V(\mathbf{x}_k)$.

III. Two Alternative Strategies for Stabilizing MPC Design

In this section, we introduce two different stable MPC design approaches: one is based on forcing a first-step-Lyapunov constraint on the MPC problem (which is an extension of an idea presented in [16]), and the second is based on using a special summed Lyapunov function as the cost function. We also discuss some comparisons between these two methods, and present an MPC design version that combines both alternative strategies.

A. Multi-Step MPC with a First-Step Lyapunov Constraint

Controller design for nonlinear systems is usually either based on using a linearized model to approximate the original nonlinear dynamics, or restricted to limited classes of nonlinear plants for which systematic designs are possible. For the latter case, a representative example is the class of linear systems subject to input/actuator saturation. Designing feedback controllers that can stabilize such saturated linear systems has received much attention, e.g. [18]. Classical stabilizing feedback designs include low-gain control, scheduled low-gain control, and low-and-high-gain control.

Recently, an alternative design for stable MPC algorithms has been proposed, though only for the class of linear systems with actuator saturation and a single-step receding horizon [16]. The proposed design strategy is summarized as follows: 1) first confirm that there exists a stable low-and-high-gain controller whose corresponding Lyapunov function is decreasing along state trajectories; 2) next, pose a set of extra sufficient constraints on the input, which guarantee a decrease in the same Lyapunov candidate (corresponding to the stable controller) during a single time-step; 3) solve the original MPC optimization problem with these extra constraints. Since the existing low-and-high-gain controller is a feasible solution to the MPC problem and the imposed extra conditions guarantee a decrease of the Lyapunov function, it is thus shown that MPC is stabilizing. It is worth noting that at each MPC stage, the stabilization conditions are obtained through solving an algebraic Riccati equation. This pre-calculation process may increase the complexity of the MPC algorithm even though the constraints are linear.

Here, we argue that the MPC design technique given in [16] can be extended to allow multi-step optimization horizons, and also that the design philosophy can be adapted for control of a broader class of nonlinear plants (not only linear systems with saturation). We refer to the new stabilizing MPC design as *multi-step MPC with first-step Lyapunov constraint*. Specifically, since only the first input from the optimal sequence is implemented, the main philosophy of this design strategy is still to impose extra constraints on the first input to guarantee a decrease of a Lyapunov function, while optimizing a multi-step cost. This new MPC design scheme is formalized in the following theorem:

Theorem 1: Consider a nonlinear system (1) subject to the input constraint $\mathbf{u}_k \in \mathbb{U}$, and say that the cost function J_k for MPC is as defined in (2). Assume that there exists an admissible stabilizing feedback control law $\mathbf{u}_k = F(\mathbf{x}_k) \in \mathbb{U}$, such that the closed-loop system has a Lyapunov function V_k. Choose any positive function $p(\mathbf{x}_k)$, such that $\mathbb{U} \cap \mathbb{U}^p \neq \emptyset$ where $\mathbb{U}^p$ is a set of inputs $\mathbf{u}_k$ that satisfies $V_{k+1} - V_k = V(f(\mathbf{x}_k, \mathbf{u}_k)) - V(\mathbf{x}_k) \leq -p(\mathbf{x}_k)$. Then, the MPC solution to (3) is asymptotically stable if J_k is optimized subject to an extra constraint $\mathbf{u}_k \in \mathbb{U} \cap \mathbb{U}^p$.

Proof: First, we show that the MPC optimization problem (3) has a feasible solution. Noticing that the system (1) is time invariant, there exists a positive function $p^*(\mathbf{x}_k)$ such that $V_{k+1} - V_k \leq -p^*(\mathbf{x}_k)$ when $F(\mathbf{x}_k) \in \mathbb{U}$ is applied. Then, for any chosen $p(\mathbf{x}_k)$ such that $0 < p(\mathbf{x}_k) \leq p^*(\mathbf{x}_k)$, the feedback controller $F(\mathbf{x}_k)$ must satisfy $V_{k+1} - V_k \leq -p^*(\mathbf{x}_k) \leq -p(\mathbf{x}_k)$. Therefore, $F(\mathbf{x}_k) \in \mathbb{U}^p$. But $F(\mathbf{x}_k) \in \mathbb{U}$ by definition, so $\mathbb{U} \cap \mathbb{U}^p \neq \emptyset$. We thus have shown that there exists at least one feasible solution (i.e. $F(\mathbf{x}_k)$) to the MPC optimization problem (3) subject to $\mathbf{u}_k \in \mathbb{U} \cap \mathbb{U}^p$.

Next, we argue that such MPC is stable. Note that MPC first finds a set of admissible inputs subject to $\mathbf{u}_k \in \mathbb{U} \cap \mathbb{U}^p$, and then selects a sequence of inputs that optimizes J_k. The optimum sequence U_k^* therefore automatically falls within set $\mathbb{U}^p$, which guarantees $V_{k+1} - V_k \leq -p(\mathbf{x}_k)$. The decrease reflected in V_k indicates that the MPC design is stable. ■

In the above theorem, a positive function $p(\mathbf{x}_k)$ needs to be pre-selected, and the stability conditions are then obtained by solving the inequality equation $V_{k+1} - V_k \leq -p(\mathbf{x}_k)$. In general, $V_{k+1} - V_k < 0$ for $\mathbf{x}_k \neq 0$ is sufficient to guarantee stability. However, $V_{k+1} - V_k < 0$ may yield a constraint set that is not closed, and hence the solution to the optimization problem may not be in the set (i.e., optimization achieves infimum rather than minimum) and may not achieve $V_{k+1} - V_k < 0$. Therefore, by forcing $V_{k+1} - V_k \leq -p(\mathbf{x}_k)$, the possibility for an infimum can be avoided, which also sometimes makes the optimization easier to implement.

Remark 1: Compared to the MPC method in [16], the above stable design does not impose a limit on the time horizon of the MPC. By taking advantage of the existence of a stabilizing controller and the fact that only the first input will be used, a constraint can be imposed that guarantees a decrease of the same Lyapunov function without impacting the multi-step optimization. The remaining inputs $\{\mathbf{u}_{k+1}, \cdots, \mathbf{u}_{k+N-1}\}$ are designed to achieve the multi-step optimization, without respect to the Lyapunov-function-based constraint on $\mathbf{u}_k$. We also note that for the terminal penalty on the state $\mathbf{x}_{k+N}$, [16] chooses the weight matrix Q_{k+N} in the cost function (2) as the unique positive definite solution of an algebraic Riccati equation. In our formulation, such a limitation can also be released, since the decrease in the Lyapunov function has already been considered at the first step and a particular terminal state penalty is not necessary in this situation.

Remark 2: The above first-step Lyapunov-constraint-based MPC design can be viewed in terms of the classical terminal-constraint-based MPC method as summarized in [10]. The classical method requires knowledge of a stabilizing controller and its associated Lyapunov function for only the terminal constraint set. It also requires that the state trajectory actually should be steered into this set within the MPC time horizon. Our formulation, however, extends the terminal constraint set to the full state set $\mathbb{R}^n$, without any limitation on the receding horizon. In other words, for any initial condition at each MPC stage, it automatically lies within this "terminal constraint set" and no control action is needed to bring the trajectory to this set. Our formulation, of course, requires existence of a global Lyapunov condition; however, given this strong condition, it does afford considerable freedom in the choice of the cost function and look-ahead horizon, and their selection becomes independent of the initial-condition set in contrast with the classical approach.

Remark 3: The proposed scheme also has strong similarities with the CLF-based method presented in [14], in which a continuous-time nonlinear system is considered. In [14], a global stabilizing controller is also required (through finding a CLF). Similarly to our first-step constraint, stability is guaranteed in [14] by forcing the derivative of the CLF to be negative along the trajectory.

To better illustrate this design approach, we consider the example presented in [16], and use the new approach to design a multi-step MPC for a linear system with actuator saturation.

Example 1: Linear Systems with Actuator Saturation

As mentioned above, a stable MPC method with $N = 1$ has been successfully implemented for the class of linear systems subject to input saturation [16]. Specifically, [16] uses the fact that a low-high-gain controller can be designed for such plants, and so a Lyapunov function can be found. Based on this recognition, the authors impose a set of extra linear constraints on the input (in addition to the saturation constraints) that guarantees a decrease in a Lyapunov function, and hence ensures stability. To better illustrate the generality provided by our method, we consider an extended version of the example from [16], where the MPC time horizon is multi-step. Specifically, we consider the same system plant as in [16]:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\begin{bmatrix}\sigma(u_{1,k})\\ \sigma(u_{2,k})\end{bmatrix}, \tag{4}$$

where

$$A = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}, \qquad B = \begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix},$$

and $\sigma(\mathbf{u}_{i,k})$ is the saturated input of the system (i.e., $\sigma(\mathbf{u}_{i,k}) = sign(\mathbf{u}_{i,k})min\{1, |\mathbf{u}_{i,k}|\}$) for $i = 1,2$.

We consider an N-step MPC, with the following objective function:

$$J_k = \sum_{i=k+1}^{k+N} \mathbf{x}_i^T Q_i \mathbf{x}_i + \sum_{i=k}^{k+N-1} \mathbf{u}_i^T R_i \mathbf{u}_i. \tag{5}$$

For simplicity, let $Q_i = I$ and $R_i = I$ (chosen as identity matrices with appropriate sizes). Then, the cost function becomes

$$J_k = \sum_{i=k+1}^{k+N} \mathbf{x}_i^T \mathbf{x}_i + \sum_{i=k}^{k+N-1} \mathbf{u}_i^T \mathbf{u}_i. \tag{6}$$

We note that, in the above cost function (6), the terminal state penalty matrix Q_{k+N} can be any arbitrary positive matrix. It is a key difference from the result in [16], where the terminal penalty matrix Q_{k+N} is selected as the unique solution of an algebraic Riccati equation.

The MPC problem is to minimize the N-step cost function J_k (6) at each stage, subject to the system dynamics (4) and the input saturation constraint. In addition, we also pose a set of linear constraints on the first step to guarantee a decrease of the low-high-gain controller's Lyapunov function. These are the same as in [16], so we do not repeat the derivation and simply provide the result:

$$sign(D_{i,k}\mathbf{x}_k)(\mathbf{u}_{i,k} - F_{i,k}\mathbf{x}_k) \geq 0,$$

$$sign(D_{i,k}\mathbf{x}_k)[2D_{i,k}\mathbf{x}_k - C_{i,k}(\mathbf{u}_k - F_k\mathbf{x}_k)] \geq 0,$$

for $i = 1,2$. The following notation applies:
1) $D_{1,k}$, $D_{2,k}$ are the first and second rows of matrix D_k, where $D_k = RF_k$;
2) $C_{1,k}, C_{2,k}$ are the first and second rows of matrix C_k, where $C_k = B^T P_k B$;
3) $F_k = -(R + B^T P_k B)^{-1} B^T P_k A$;
4) P_k is the solution to the algebraic Riccati equation $P_k = A^T P_k A + \varepsilon(\mathbf{x}_k)I - A^T P_k B(I + B^T P_k B)^{-1} B^T P_k A$, where $\varepsilon(\mathbf{x}_k) = \max\{r \in (0,1] \mid (\mathbf{x}_k^T P_k \mathbf{x}_k)trace(P_k) \leq \frac{1}{2trace(BB^T)}\}$.

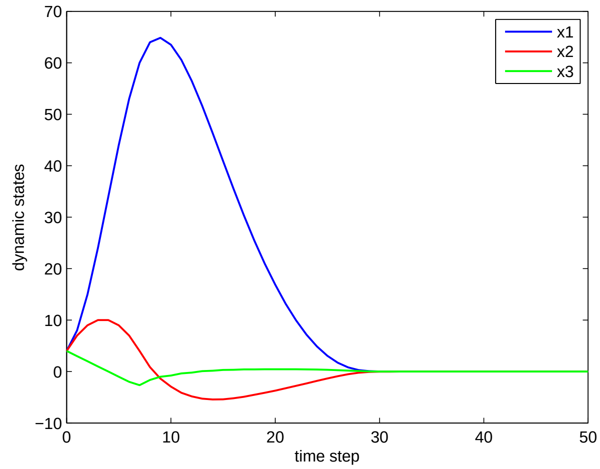

Fig. 1. The state dynamics under low-high-gain feedback control for Example 1.

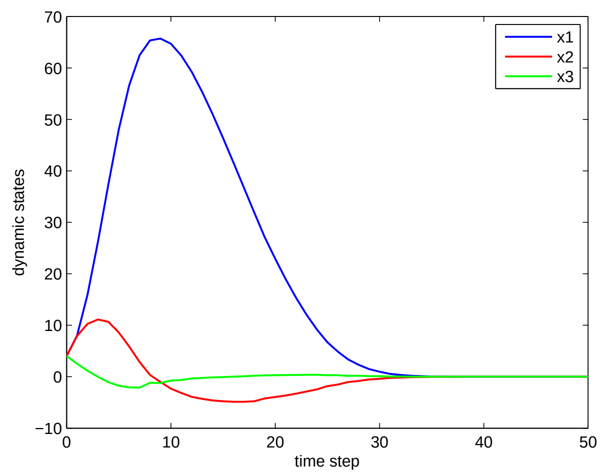

Fig. 2. The state dynamics under MPC with $N = 1$ for Example 1.

These linear constraints guarantee that the Lyapunov function associated with a stable feedback controller (in this case, a classical low-and-high-gain controller) is decreasing. We simulate the system dynamics under both the one-step and two-step MPC, with the same initial condition $\begin{bmatrix}4 & 4 & 4\end{bmatrix}^T$ and the specified cost function. For comparison, we also simulate the system trajectory under the original low-and-high-gain control. The state trajectories for all three cases are shown in Figures 1, 2, and 3. We observe that the two-step MPC has a significant performance improvement in terms of the settling time, relative to both the low-and-high-gain design and the one-step MPC design.

Remark: We note that classical low-high-gain feedback control, the one-step MPC given in [16], and our proposed multi-step MPC all guarantee closed-loop stabilization. Compared to [16], however, our approach asserts that there is

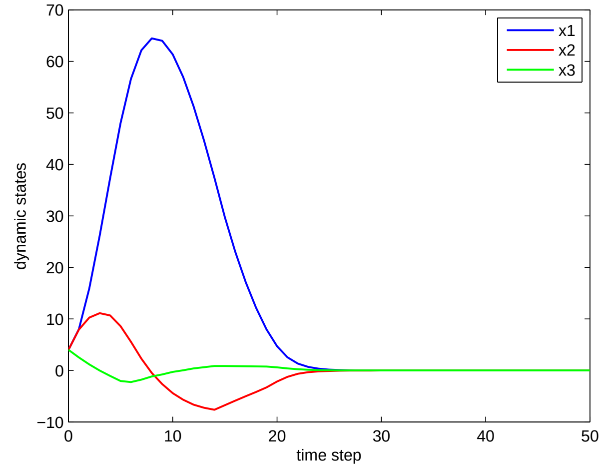

Fig. 3. The state dynamics under MPC with $N = 2$ for Example 1.

freedom in selecting the cost function and MPC horizon, thus allowing optimization of a desirable cost and a trade-off between performance and computational effort.

B. MPC Design Using A Summed Lyapunov-Function Cost

Here, we explore further the use of known Lyapunov functions to guarantee MPC stability. Rather than imposing an initial-step or final-step constraint/cost, we consider defining the cost function based on a summed or averaged Lyapunov function over an optimization horizon of interest. Defining the cost in this way permits the design of controllers that cause the Lyapunov function (and hence state norm) to rapidly and globally decrease over the long term, while also allowing temporary increases in the Lyapunov energy in seeking for an optimal solution. This result can be formally stated as the following theorem.

Theorem 2: Consider the nonlinear system (1) subject to the input constraint $\mathbf{u}_k \in \mathbb{U}$. Let $\mathbf{u}_k = F(\mathbf{x}_k) \in \mathbb{U}$ be a stabilizing feedback controller of the system (1), and let $V(\mathbf{x}_k) \equiv V_k$ be the Lyapunov function for the associated closed-loop system. Let the MPC objective function at time k be $J_k = \sum_{i=k+1}^{k+N} V_i$, for $N \geq 1$. If the MPC optimization goal is to minimize J_k subject to $\mathbf{u}_k \in \mathbb{U}$, then the MPC solution stabilizes the system (1).

Proof: To begin, let $\{\mathbf{u}_k^{(k)}, \cdots, \mathbf{u}_{k+N-1}^{(k)}\}$ be the optimal input sequence that minimizes the objective function $J_k = \sum_{i=k+1}^{k+N} V_i$ at MPC stage k, and let $J_k^* = \sum_{i=k+1}^{k+N} V_i^{(k)}$ be the minimum objective-function value at stage k, where $V_i^{(k)}$ is the i-th portion of this minimum cost. Now consider a non-optimal input sequence $\{\mathbf{u}_k^{(k)}, \cdots, \mathbf{u}_{\widetilde{k}}^{(k)}, F(\mathbf{x}_{\widetilde{k}+1}), \cdots, F(\mathbf{x}_{k+N-1})\}$ for $k \leq \widetilde{k} < k+N-1$, where the first $\widetilde{k}-k+1$ inputs are from the optimal sequence, and the last $(k+N-1)-\widetilde{k}$ inputs are provided by the stable feedback controller. For such an input sequence, the cost functions for the first $\widetilde{k}-k+1$ time-steps remain the same as those of J_k^*, i.e. $V_i = V_i^{(k)}$ for $i = k+1, \cdots, \widetilde{k}+1$. For the remaining $(k+N-1)-\widetilde{k}$ cost functions, assuming $\mathbf{x}(\widetilde{k}+1) \neq 0$, we have $V_{i-1} > V_i$ for $i = \widetilde{k}+2, \cdots, k+N$, since the controller F is stable and V_i is its corresponding Lyapunov function. The decrease in V_i yields,

$$\sum_{i=\widetilde{k}+2}^{k+N} V_i < (k+N-1-\widetilde{k})V_{\widetilde{k}+1} = (k+N-1-\widetilde{k})V_{\widetilde{k}+1}^{(k)}.$$

Because of optimality, the following inequality also holds: $\sum_{i=\widetilde{k}+2}^{k+N} V_i^{(k)} \leq \sum_{i=\widetilde{k}+2}^{k+N} V_i$. Combining these two inequalities, we obtain

$$\sum_{i=\widetilde{k}+2}^{k+N} V_i^{(k)} < (k+N-1-\widetilde{k})V_{\widetilde{k}+1}^{(k)}, \tag{7}$$

for any $k \leq \widetilde{k} < k+N-1$. With some effort, the set of inequalities in (7) can be used to obtain a simple relationship: $V_i^{(k)} > V_{k+N}^{(k)}$ for all $i = k+1, \cdots, k+N-1$. That is, the last term in the optimal J_k^* is the smallest of all its terms.

Next, MPC implements the first input $\mathbf{u}_k^{(k)}$ from the optimal sequence, and then moves to the next time-step $k+1$. At the $k+1$ stage, one possible control sequence is $\{\mathbf{u}_{k+1}^{(k)}, \cdots, \mathbf{u}_{k+N-1}^{(k)}, F(\mathbf{x}_{k+N})\}$, where the first $N-1$ inputs are the same as the last $N-1$ inputs from the previous optimal sequence, and the last input corresponds to the stable controller. The cost function now becomes

$$J_{k+1} = V_{k+2}^{(k)} + \cdots V_{k+N}^{(k)} + V_{k+N+1} = J_k^* - V_{k+1}^{(k)} + V_{k+N+1}.$$

Again, at time-step $k+N$, the controller F causes the Lyapunov function to decrease: $V_{k+N+1} < V_{k+N}^{(k)}$. Since $V_{k+N}^{(k)} < V_i^{(k)}$ for $i = k+1, \cdots, k+N-1$, we specifically obtain $V_{k+N+1} < V_{k+N}^{(k)} < V_{k+1}^{(k)}$. Thus, $J_{k+1} < J_k^*$, which indicates $J_{k+1}^* < J_k^*$. Also, it is straightforward to show that $J_k^* = 0$ for $\mathbf{x}_k = 0$.

Finally, let us argue that J_k^* is a Lyapunov function for the MPC-controlled system. Note that J_k^* is a nonnegative function and that $J_k^* > 0$ if the state is nonzero. Also, we have shown that $J_{k+1}^* < J_k^*$. This is sufficient to ensure that J_k^* will asymptotically decrease to the origin. Hence, the MPC-controlled system is asymptotically stable. ■

Example 2: Revisiting the Linear System with Actuator Saturation

We consider the same linear plant with saturation considered in Example 1. The MPC time horizon is chosen as $N = 2$. As discussed in Example 1, the Lyapunov function associated with the classical low-and-high-gain controller is $V_k = \mathbf{x}_k^T P_k \mathbf{x}_k$, where P_k is the same solution to the algebraic Riccati equation as described in Example 1. Then, the corresponding cost function at time-step k is a summed Lyapunov function over two time-steps:

$$J_k = \sum_{i=k+1}^{k+2} x_i^T P_i x_i. \tag{8}$$

The MPC problem is to minimize J_k subject to the input saturation constraint only. Figure 4 shows the simulation of the state trajectories when MPC is applied.

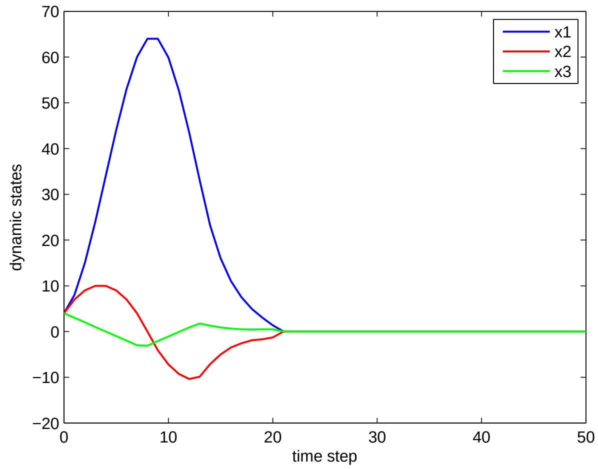

Fig. 4. The state dynamics for the Lyapunov-cost-based MPC with $N = 2$.

We observe in Figure 4 that the summed-Lyapunov-function cost-based method achieves an even faster convergence rate than the first-step-Lyapunov-constraint approach. This is because the cost function (8) considered here is a quadratic form of only the states, and the input magnitude does not impact the cost. Not surprisingly, a design is obtained that quickly drives the state to the origin, using relatively large inputs. We note that the summed Lyapunov cost also permits a short-term increase in the cost even while guaranteeing stability, which may provide the algorithm with further freedom to find a low-cost solution.

C. Comparisons

As mentioned above, for many of the classical stability-achieving MPC methods, stability is guaranteed based on pre-knowledge of a terminal state constraint. In other words, if the state can be steered into a "safety" set with known locally-stabilizing controller during the MPC time horizon, it will eventually be trapped in this set and converge to the origin. In practice, such a set, and hence the terminal constraint, can be difficult to select properly. Furthermore, the terminal constraint limits the choice of cost functions that can be optimized, making selection of an appropriate cost function complicated. The two alternative stabilizing MPC strategies that we propose also require some pre-knowledge (i.e. a stabilizing controller and associated global controller are known). Such pre-knowledge essentially provides MPC with at least one control sequence that achieves stability, and yields a design that guarantees eventual decrease of a Lyapunov function. The benefit of our approach, when such global stabilization is possible, is the performance improvement afforded by the freedom to choose the cost function and time horizon at will (i.e., independently of the stabilizing controller).

By utilizing the Lyapunov function (in two different ways), MPC can thus asymptotically bring the state trajectory to the origin and the cost function is optimized at the same time. In particular, for some nonlinear system classes (e.g., linear systems with input saturation), stabilizing controllers and associated Lyapunov functions are well known. By taking advantage of the existing knowledge on stabilization of such systems, we can also implement these two alternative MPC methods. Meanwhile, if the cost function is properly chosen, the two MPC methods can achieve faster convergence than other stable controllers. For example, in Examples 1 and 2, both MPC methods (Figures 3 and 4) show a faster convergence rate than the classical low-and-high-gain controller (Figure 1).

It is worth stressing that the mechanisms by which the two methods achieve stability are different. The first-step-Lyapunov-constraint-based MPC design forces a decrease in the Lyapunov function during the first time-step at each MPC stage. Since only the first time-step's control is applied during the MPC stage, system stability is thus guaranteed, regardless of the optimization solution. One significant advantage of this approach is that the cost function can be selected at will: as long as the Lyapunov function is decreasing with time, the cost function J_k and the MPC time horizon N do not impact stability. However, they may significantly alter dynamic characteristics such as settling time and other performance metrics. For the second MPC design, we used a summed Lyapunov function over N time-steps as the cost function. In this case, the optimization problem is to minimize the cost function at each time-step. The decreasing property of the Lyapunov function ensures that the cost function is also eventually decreasing, though short-term increases in the cost are allowed.

D. A Combined Version of The Two Design Approaches

Thus far, we have introduced two alternative schemes for building asymptotically-stable MPC for nonlinear plants subject to constraints, with both schemes avoiding the use of terminal constraints and costs. Both are applicable to plants for which stabilizing controllers are already known, but MPC solutions are sought to enable more desirable characteristics such as performance optimization and disturbance rejection.

The first approach, which is based on constraining a candidate Lyapunov function to decrease during the first MPC stage, is compelling in that it permits optimization of a flexible multi-step cost function while guaranteeing stability. However, the constraint imposed on the optimization is quite strong, in the sense that the candidate function (which serves as a Lyapunov function for the known controller) must decrease at each time-step. As such, the resulting design may be rather similar in performance to the original controller. Meanwhile, the second approach – which uses the sum of a candidate function (which is a Lyapunov function for a known controller) over the MPC horizon as the MPC cost function – permits flexibility in the state evolution (i.e., the function value need not decrease monotonically), but there is no flexibility in choosing the cost function. Here, we introduce a methodology which constrains the multi-step sum of a candidate function (which again is a Lyapunov function for a known controller) to decrease, while an arbitrary cost function is optimized. This approach replicates the benefits of both previous approaches (flexibility in choosing the cost function, and looser constraints on the state trajectory). It is formalized in the following main result:

Theorem 3: Consider a nonlinear system (1) subject to the input constraint $\mathbf{u}_k \in \mathbb{U}$, and consider applying an N-step MPC with objective function J_k as defined in (2). Let $\mathbf{u}_k = F(\mathbf{x}_k) \in \mathbb{U}$ be a stabilizing feedback controller of the system (1), and let $V(\mathbf{x}_k) \equiv V_k$ be a Lyapunov function for the associated closed-loop system. Choose the same positive function $p(\mathbf{x}_k)$ as in Theorem 1. For each stage k, given the current state $\mathbf{x}_k$, let $\mathbb{V}_k \triangleq \sum_{i=k+1}^{k+N} V_i$ be a summed Lyapunov function over the next N steps when the input sequence $U_k \equiv \{\mathbf{u}_k, \cdots, \mathbf{u}_{k+N-1}\}$ is applied. Define the MPC solution at stage k to be the optimal input sequence U_k^* that minimizes the cost function, subject to $\mathbb{V}_k - \mathbb{V}_{k-1} \leq -p(\mathbf{x}_k)$ (for all $k \geq 1$), where $\mathbb{V}_{k-1}$ is obtained from the previous stage $k-1$ when the optimal MPC schedule U_{k-1}^* is applied. The closed-loop system is stable when this MPC algorithm is applied.

Proof: First of all, we verify that the MPC optimization at each stage has a feasible solution. At each stage $k \geq 1$, consider the control sequence $U_k = \{F(\mathbf{x}_k), \cdots, F(\mathbf{x}_{k+N-1})\}$, where each $F(\cdot)$ is the stabilizing controller. Then, we have $\mathbb{V}_k = \sum_{i=k+1}^{k+N} V_i^F$, where $V_k^F, \cdots, V_{k+N}^F$ are the values of the Lyapunov function associated with $F(\cdot)$. Because V_k is the Lyapunov function of the closed-loop dynamics when the controller $F(\cdot)$ is applied, we have $V_{k+1}^F - V_k^F \leq -p(\mathbf{x}_k)$ for $k \geq 1$. Therefore, $\mathbb{V}_k - \mathbb{V}_{k-1} = V_{k+N}^F - V_k^F \leq -p(\mathbf{x}_k)$ for $k \geq 1$. Hence, the control sequence $\{F(\mathbf{x}_k), \cdots, F(\mathbf{x}_{k+N-1})\}$ is always a feasible solution to the MPC problem.

Now, we argue that $\mathbb{V}_k$ itself is a Lyapunov function for the closed-loop system when MPC is applied. Noting that $\mathbb{V}_k$ is a summation of a sequence of Lyapunov functions, $\mathbb{V}_k > 0$ if $\mathbf{x}_k \neq 0$ and $\mathbb{V}_k = 0$ only if $\mathbf{x}_k = 0$. Also, $\mathbb{V}_k$ is decreasing along trajectories since the MPC solution forces $\mathbb{V}_k - \mathbb{V}_{k-1} \leq -p(\mathbf{x}_k)$. Therefore, MPC achieves asymptotic stability. ■

IV. Conclusions

We have introduced two alternative stabilizing MPC design strategies for nonlinear systems with constraints, and a third approach that combines these two strategies. The two MPC approaches use the fact that the original system can be stabilized, and that in fact we know a stabilizing controller and its associated Lyapunov function. By taking advantage of this knowledge, MPC can be designed to optimize a desired cost function while still achieving stabilization. The simulation results show some interesting features: 1) the performance of the various MPC approaches (in terms of the convergence rate) varies with the MPC time horizon and cost function; 2) if the cost function is chosen properly, the proposed MPC strategies can provide faster convergence to the origin than other stable controllers. In practice, if the MPC time horizon is large, the optimization at each stage becomes more complex since more free input values need to be calculated. For real applications where the convergence rate is important, the length of the MPC time horizon becomes crucial.

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