Uncertainty, information and time-frequency distributions *

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ABSTRACT

The well-known uncertaintly principle is often invoked in signal processing. It is also often considered to have the same implications in signal analysis as does the uncertainty principle in quantum mechanics. The uncertainty principle is often incorrectly interpreted to mean that one cannot locate the time-frequency coordinates of a signal with arbitrarily good precision, since, in quantum mechanics, one can not determine the position and momentum of a particle with arbitrarily good precision. Rényi information of the third order is used to provide an information measure on time-frequency distributions. The results suggest that even though this new measure tracks time-bandwidth results for two Gabor logons separated in time and/or frequency, the information measure is more general and provides a quantitative assessment of the number of resolvable components in a time frequency representation. As such the information measure may be useful as a tool in the design and evaluation of time-frequency distributions.

1. TIME-FREQUENCY UNCERTAINTY

Time frequency uncertainty is defined in various ways with slightly different results. Gabor's [10] definition will be used for the purposes of this paper. It is:

$$\Delta t \Delta \omega \geq \frac{1}{2} \tag{1.1}$$

The proper interpretation[3] of the Δt $\Delta\omega$ is that they are the standard deviations (respectively)of the time and frequency marginals of the time-frequency distribution. This assumes that the time and frequency margins are 'correct', in that they reflect the instantaneous power and energy spectrum of the signal. If a signal is stretched in time its Fourier transform representation is compressed inversely in frequency. Thus the Δt $\Delta\omega$ product remains constant. It is correct that one cannot independently reduce the time spread and frequency spread simultaneously. These quantities are tightly linked via the scaling property of the Fourier transform. That is,

$$F[s(at)] = \frac{1}{a} S(\frac{\omega}{a}) \tag{1.2}$$

1.1 Representation, localization and resolution

It is certainly true that the concept of time-bandwidth product (TBP) is useful in characterizing the concentration of energy in time and frequency. It may seem that the most concentrated signal offers the best localization in time-frequency. This is not true. Skolnik[14] points out that there are many signals which have large

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TBPs, yet offer excellent localization properties. The problem is that ambiguities arise.

One may use a pair of chirps in time-frequency. One chirp may rise linearly in frequency (up-chirp) while the other falls linearly in frequency (down-chirp). Each chirp will produce a thin "knife edge" in time frequency. The time frequency distribution will be in an "X" form due to the crossing of the up-chirp and down-chirp. The crossing point defines both the frequency and time location of the "center" of the compound signal and it defines these locations quite precisely. This definition may be made ever more precise by increasing the time and frequency extent of the chirps. Thus we have the supposedly paradoxical result that increasing the TBP improves time and frequency localization. There is a price to pay, however. With multiple versions of these "X"s there may be overlap of the chirps from the different versions. The problem is to distinguish the "true" crossings which yield the time-frequency localizations of each "X" from the "false" crossings due to overlap of the "X"s. Ambiguities have been introduced.

Resolution of signals is another important aspect of signal processing. It is here that TBP and uncertainty in the signal processing context makes sense. The concept of an elementary signal is needed. There are several possibilities, prolate spheriodal,[15,16] Hermit functions [6] and special compact bases[11] are among them. Gabor logons will be used in this paper due to their minimum uncertainty properties. If there are two elementary signals how much separation between these two elementary signals must one achieve in order to be able to conclude that there are two signals present rather than one? One would like an objective count of the number of resolvable elementary signals present.

An information theoretic approach has some appeal in this context. Such an approach would produce a result expressed in "bits". One elementary signal would yield zero bits of information (2^0), two well separated elementary signals would yield one bit of information (2^1), four well separated elementary signals would yield two bits of information (2^2), and so on. The strategy for this paper is to apply information measures to time-frequency distributions. Unfortunately, the well known Shannon Information[13] can not be applied to some time-frequency distributions due to their negative energy values[3]. There is a generalized form of information due to Rényi which admits negative values in the distribution. This allows a treatment of many time-frequency distributions in terms of information measures even though Rényi formally requires that the probabilities be non-negative in his formulation. As will be seen later in this paper violation of that rule does not prevent "reasonable" answers from being obtained.

1.2 Time-frequency distributions

There are many types of time-frequency distributions. For purposes of this paper the distributions utilized will be members of a general class of distributions commonly called Cohen's class[3]. Cohen's class is defined as[3]

$$\begin{aligned} C_s(t,\omega;\phi) &= \frac{1}{2\pi}\iiint e^{j(-\theta t-\tau\omega+\theta u)} \\ &\qquad \phi(\theta,\tau)s(u+\frac{\tau}{2}) \\ &\qquad s^*(u-\frac{\tau}{2})du\,d\tau\,d\theta, \end{aligned} \quad (1.3)$$

and has been frequently used as a unified framework in time-frequency signal analysis. The *kernel* $\phi(\theta,\tau)$ determines the distribution characteristics. Different kernels produce different distributions[3], such as the WD, spectrogram Exponential (ED)[2] and the RID.[9,17,18,19] The kernel has a unity value for the WD. The RID (Reduced Interference Distribution) enjoys almost all of the nice properties of the WD, but has the advantage of greatly reducing cross-term interference that often plagues the WD when it is used to study multicomponent signals.

1.3 The Gabor logon

The Gabor logon[10] is a popular candidate for representation of the elementary signal element. It is possible to expand signals in terms of such elementary signals and such an approach has been useful in the detection of transient signals.[7] Using a Gabor logon defined as:

$$s(t) = \frac{1}{\sqrt[4]{\pi\sigma^2}}\, e^{-t^2/2\sigma^2} \tag{1.4}$$

The instantaneous power (the time marginal) is:

$$p(t) = s^2(t) = \frac{1}{\sqrt{\pi\sigma^2}}\, e^{-t^2/\sigma^2} \tag{1.5}$$

Power is normalized to 1:

$$\int_{-\infty}^{\infty} p(t)\, dt = \frac{1}{\sqrt{\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-t^2/\sigma^2}\, dt = 1 \tag{1.6}$$

The Fourier transform of $s(t)$ is:

$$\begin{aligned} S(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, dt \\ &= (\sigma^2/\pi)^{1/4}\, e^{-\omega^2\sigma^2/2} \end{aligned} \tag{1.7}$$

Energy density in the frequency domain (the frequency marginal) is:

$$P(\omega) = S^2(\omega) = \sqrt{\frac{\sigma^2}{\pi}}\, e^{-\omega^2\sigma^2} \tag{1.8}$$

Total energy in the frequency domain is also normalized:

$$\int_{-\infty}^{\infty} P(\omega)\, d\omega = \sqrt{\frac{\sigma^2}{\pi}} \int_{-\infty}^{\infty} e^{-\omega^2\sigma^2}\, d\omega = 1 \tag{1.9}$$

Equation 1.1 takes the equality value in this case as can readily be seen by determining the product of the standard deviations of the time and frequency marginals. The time and frequency marginals are shown in Figure 1a,b.

1.4 Wigner distribution for one Gabor logon

The Wigner distribution (WD) for the Gabor Logon with a σ^2 of one is, using the standard definition[3]:

$$\begin{aligned} W(t,\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} s(t+\tau/2) \\ &\qquad s^*(t-\tau/2)\, e^{-j\omega t}\, d\tau \\ &= \frac{1}{\pi}\, e^{-\omega^2} e^{-t^2} \end{aligned} \tag{1.10}$$

The Wigner distribution is also normalized; energy obtained by integrating over all time and frequency is 1:

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t,\omega)\, d\omega\, dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\omega^2}\, d\omega \int_{-\infty}^{\infty} e^{-t^2}\, dt \\ &= 1 \end{aligned} \tag{1.11}$$

This would be true for other values of σ^2 as well. The WD of the Gabor logon is shown in Figure 1c. Note that the distributions must always be normalized for the purposes of this study. We wish to treat the distributions using tools usually applied to probability density functions, hence they must be properly normalized.

2. INFORMATION MEASURES

Information is most often related to random events. Distributions considered in the information context are usually distributions of random variables. In this paper we wish to appropriate information concepts and apply them to distributions which may be derived from deterministic signals. This is done in order to benefit from certain very desirable properties of information theoretic measures. There are six properties of information that are desirable. For the purposes of this paper the properties of additivity, continuity and symmetry are particularly useful. A brief presentation of Shannon and Rényi's information formulations and their relationship sets the stage for the application of these ideas.

Shannon Information Definition

The definition of Shannon information for continuous functions is:

$$I_x = -\int_{-\infty}^{\infty} f(x)\log_2 f(x)\,dx \qquad (2.1)$$

The definition of Shannon information for discrete functions is:

$$I = -\sum_n p_i \log_2 p_i \qquad (2.2)$$

Rényi Information Definition

The general defintion for α^{th} order Rényi information of a continuous function is:

$$R_x^\alpha = \frac{1}{1-\alpha}\log_2 \int_{-\infty}^{\infty} f^\alpha(x)\,dx \qquad (2.3)$$

The discrete equivalent is:

$$R^\alpha = \frac{1}{1-\alpha}\log_2 \sum_n p_i^\alpha \qquad (2.4)$$

First order Rényi Information reduces to Shannon Information for $\alpha = 1$. Third order Rényi Information is defined as:

$$R_x^3 = -\frac{1}{2}\log_2 \int_{-\infty}^{\infty} f^3(x)\,dx \qquad (2.5)$$

3. SHANNON INFORMATION OF THE DISTRIBUTIONS

Now we wish to compare the information in the marginals against the information in the time-frequency distribution. Using Shannon's information definition, information for the time marginal is:

$$\begin{aligned} I_t &= -\int_{-\infty}^{\infty} p(t)\log_2 p(t)dt \\ &= -\frac{1}{\sqrt{\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/\sigma^2} \\ &\quad \log_2\left[\frac{1}{\sqrt{\pi\sigma^2}}\mathrm{e}^{-t^2/\sigma^2}\right]dt \\ &= \frac{\log_2\pi}{2} + \frac{\log_2 \mathrm{e}}{2} + \log_2\sigma \qquad (3.1) \end{aligned}$$

Information for the frequency marginal is:

$$\begin{aligned} I_f &= -\int_{-\infty}^{\infty} P(\omega)\log_2 P(\omega)d\omega \\ &= -\sqrt{\frac{\sigma^2}{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\omega^2\sigma^2} \\ &\quad \log_2\left[\sqrt{\frac{\sigma^2}{\pi}}\mathrm{e}^{-\omega^2\sigma^2}\right]d\omega \\ &= \frac{\log_2\pi}{2} + \frac{\log_2 \mathrm{e}}{2} - \log_2\sigma \qquad (3.2) \end{aligned}$$

Information in the Wigner distribution is:

$$\begin{aligned} I_{tf} &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t,\omega)\log_2 W(t,\omega)dtd\omega \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\omega^2\sigma^2}\mathrm{e}^{-t^2/\sigma^2} \\ &\quad \log_2\left(\frac{1}{\pi}\mathrm{e}^{-\omega^2\sigma^2}\mathrm{e}^{-t^2/\sigma^2}\right)dtd\omega \\ &= \log_2\pi + \log_2 \mathrm{e} \qquad (3.3) \end{aligned}$$

Thus the differential information is zero:

$$I_{dif} = I_{tf} - (I_t + I_f) = 0.0 \qquad (3.4)$$

The Wigner distribution provides no additional information about the signal above and beyond that provided by the marginals.

4. RÉNYI INFORMATION FOR THE DISTRIBUTIONS

A similar result is found using third order Rényi information. For the time marginal the information is:

$$R_t^3 = -\frac{1}{2}\log_2 \int_{-\infty}^{\infty} p^3(t)dt$$
$$= -\frac{1}{2}\log_2 \int_{-\infty}^{\infty} (\pi\sigma^2)^{-\frac{3}{2}} e^{-3t^2/\sigma^2} dt$$
$$= \frac{\log_2 \pi}{2} + \frac{\log_2 3}{4} + \log_2 \sigma \quad (4.1)$$

For the frequency marginal the information is:

$$R_f^3 = -\frac{1}{2}\log_2 \int_{-\infty}^{\infty} P^3(\omega)d\omega$$
$$= -\frac{1}{2}\log_2 \int_{-\infty}^{\infty} (\sigma^2/\pi)^{\frac{3}{2}} e^{-3\omega^2\sigma^2} d\omega$$
$$= \frac{\log_2 \pi}{2} + \frac{\log_2 3}{4} - \log_2 \sigma \quad (4.2)$$

The Rényi information for the Wigner distribution is:

$$R_{tf}^3 = -\frac{1}{2}\log_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi^3} e^{-3\omega^2\sigma^2} e^{-3t^2/\sigma^2} dt d\omega$$
$$= \log_2 \pi + \frac{1}{2}\log_2 3 \quad (4.3)$$

The differential information for the Rényi information is zero.

$$R_{dif}^3 = R_{tf}^3 - \left[R_t^3 + R_f^3\right] = 0 \quad (4.4)$$

Again, the Wigner distribution provides no addition information about the signal above and beyond that provided by the marginals.

4.1 Wigner distribution for two Gabor logons

A signal containing two Gabor logons separated in time by t_0 is defined as:

$$s(t) = \frac{1}{\sqrt[4]{4\pi\sigma^2}\sqrt{1 + e^{-t_0^2/4\sigma^2}}}$$
$$= \left[e^{-(t+t_0/2)^2/2\sigma^2} + e^{-(t-t_0/2)^2/2\sigma^2}\right] \quad (4.5)$$

The instantaneous power is:

$$p(t) = s^2(t)$$
$$= \frac{1}{\sqrt{4\pi\sigma^2}(1 + e^{-t_0^2/4\sigma^2})} \left[e^{-(t+t_0/2)^2/\sigma^2} + e^{-(t-t_0/2)^2/2\sigma^2} + 2e^{-(t^2+t_0^2/4)/\sigma^2}\right] \quad (4.6)$$

The spectrum of $s(t)$ is:

$$S(\omega) = \frac{2\sqrt[4]{\sigma^2}}{\sqrt[4]{4\pi}\sqrt{1 + e^{-t_0^2/4\sigma^2}}} e^{-\omega^2\sigma^2/2} \cos(\omega t_0/2) \quad (4.7)$$

Power in the frequency domain is:

$$P(\omega) = S^2(\omega) = \frac{2\sigma}{\sqrt{\pi}(1 + e^{-t_0^2/4\sigma^2})} e^{-\omega^2\sigma^2} \cos^2(\omega t_0/2) \quad (4.8)$$

The Wigner distribution for this signal is:

$$W(t,\omega) = \frac{1}{2\pi(1 + e^{-t_0^2/4\sigma^2})} \cdot \left[e^{-(t+t_0/2)^2/\sigma^2} e^{-\omega^2\sigma^2} + e^{-(t-t_0/2)^2/\sigma^2} e^{-\omega^2\sigma^2} + 2e^{-t^2/\sigma^2} e^{-\omega^2\sigma^2} \cos(\omega t_0)\right] \quad (4.9)$$

4.2 Rényi information for two Gabor logons

$$R_t^3 = -\frac{1}{2}\log_2 \int_{-\infty}^{\infty} p^3(t)dt$$

$$R_f^3 = -\frac{1}{2}\log_2 \int_{-\infty}^{\infty} P^3(\omega)d\omega$$

$$R_{tf}^3 = -\frac{1}{2}\log_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot$$
$$W^3(t,\omega)dtd\omega \qquad (4.10)$$

The formulation becomes quite complex for two Gabor logons under Rényi Information. Results for the information measures were obtained using numerical techniques in practice.

5. INFORMATION RESULTS

An information measure based on Rényi Information of the third order was applied to Wigner Distribution (WD) and Reduced Interference Distribution (RID) time-frequency distributions of two Gabor logons and four Gabor logons separated by various amounts in time and frequency.

5.1 Two Gabor logons

Two Gabor logons with unity variance as previously described were separated in time according to units of time standard deviation and in frequency according to units of frequency standard deviation. Various combinations of time and frequency separation were used. Figure 2 compares the results of separating two Gabor logons in time in terms of the Rényi third order information measure and the time-bandwidth product of the two Gabor logons. One can see that the both the information and the TBP remain near zero until about two units of separation. At that point both measures begin to increase. However, the information measure levels off at one bit above the zero separation value. The information gain is thus one bit. This is quite appropriate for the two logons. The TBP continues to increase with increasing separation, giving no indication of the number of signal entities present. Between time separation values of zero and three, TBP and information follow a very similar curve if scale and offset are taken into account. This implies that both measures proving something useful concerning the resolvability of the two logons. However, TBP does not provide any indication of the time separation at which the two logons are essentially resolved. The information measure shows that this occurs at a separation of about six units. Equivalent results are obtained if the separation is changed to frequency separation of the logons. If combinations of time and frequency separation are used and the Wigner distribution is derived for the resulting signal, it is found that the Rényi information measure performs well in assessing the situation. If time or frequency separation are applied singularly the results are very similar to the marginal results. If time and frequency separations are applied together there is an interesting overestimation of information for a combination of two units of separation for both time and frequency. In this case the information gain was considered. The information obtained for a single logon was subtracted from the result. Thus the information for zero separation of two logons is zero.The maximum value observed was 1.35 bits. However, with greater separation the information value approachs one bit as it should. These results are shown in Figure 3a. Figure 3b shows the WD of the two logons at the separations of maximum information overestimation.

The same analysis was carried out using RID instead of the WD to determine the time-frequency distribution. Similar results were obtained. However, the RID does not produce nearly as much overestimation of the information as does the WD. In fact, the maximum overestimation was 1.08 bits. These results are shown in Figure 4. Note that the information has a residual value above zero for zero separation in this case. That is because the gain was obtained using the WD result for one logon. This was done to show that the RID has slightly more spread for a logon than does the WD. If the information gain had been determined using the RID result for one logon the overestimation of information under the RID would be less.

Figures 3b and 4b may be compared to gain some insight into the cause of information overestimation. The larger WD interference terms may be the reason for this.

5.2 Four Gabor logons

A more ambitious application of these ideas was then carried out. Four Gabor logons were placed at the corners of a rectangle in time-frequency. One side of the rectangle was time separation and the orthogonal side was frequency separation. When the length of both sides was set to zero the four logons completely overlapped forming one logon. This should yield zero information gain. By increasing the length of the sides of the rectangle it was possible to explore the effects of different types of separations. When the time separation was zero, but the frequency separation was increased the effect was that of two logons separating in frequency. Likewise when frequency separation was zero and time separation was increased the effect was that of two logons separating in time. Finally, when both time and frequency separation was applied the four logons separated as at the corners of a rectangle expanding in size. The results for the WD and the RID are shown in Figure 5. The results are as one might expect. Separation along either the time or frequency dimension produces a rise to one bit as was the case in the two logon experiment. Separation in both time and frequency produced a rise to one bit continuing with a rise to two bits with a small plateau for some combinations of time and frequency separation. Again, the WD produced an overestimation of information in some cases and the RID produced a smaller overestimation of information. Again also, since the RID information gain was based on the WD result for one logon, there was a small amount of non-zero information at zero separation in time and frequency.

6. INFORMATION INVARIANT TIME-FREQUENCY DISTRIBUTIONS

Here we introduce a new requirement for time-frequency distributions. Information invariant time-frequency distributions are defined to be time-frequency distributions which possess an invariant information measure under time and frequency shift and time and frequency scaling. This is a useful property because it allows the information measure to be employed usefully in several ways. For example,

- The information could be minimized under selection of kernel parameters, providing the highest resolution or optimum result.
- The information measure provides an objective assessment of the number of equal energy resolvable elementary signals present.
- The information measure may provide a means of judging the effectiveness of several different distributions.

Possession of a product kernel ($\psi(\theta,\tau) = \psi(\theta\tau)$) by a distribution is sufficient for it to be information invariant.The WD and the RID enjoy this property among others. However, many interesting time-frequency distributions do not. The spectrogram, for example does not ordinarily preserve the volume in time-frequency under time scaling of the signal. This is because of interactions with the window. It would be possible to design a special window which adjusts, perhaps, to account for this, but it is probably not useful.

Use of the information measure to optimize kernels may not produce a unique choice. There may be several choices which produce local minima in the information.

It is interesting to note that the analyzing wavelet [5] of the wavelet transform is information invariant in time-frequency under the definitions of this paper. This may be useful in certain applications of wavelet transforms.

Despite some of the problems the information concept may be useful if applied with care. One possible application is in instantaneous frequency applications.

6.1 Instantaneous frequency

Instantaneous frequency is an important aspect of time-frequency studies [3]. Boashash has provided many useful insights into matters of instantaneous frequency [1]. Instantaneous frequency may be defined to be the mean of frequency under the time-frequency distribution conditioned on time. The spread of instantaneous frequency is based on the variance of the instantaneous frequency about that value. Cohen has provided many interesting insights on instantaneous frequency, its spread and instantaneous bandwidth [4]. In the same sense as Cohen has proposed one may formulate an uncertainty measure for instantaneous frequency. It is:

$$R^3(t) = -\frac{1}{2}log_2 \left(\int_{-\infty}^{\infty} \left(\frac{C_x(t,\omega,\phi)}{s^2(t)} \right)^3 \cdot d\omega \right) \quad (6.1)$$

This is an alternative to Cohen's local bandwidth idea. It indicates the number of resolvable entities in frequency for each time. Ideally it should be zero, indicating one component which is very narrow in frequency. Then,

$$R^3_{ave} = \int_{-\infty}^{\infty} R^3(t)s^2(t)dt \quad (6.2)$$

The selection of kernels for instantaneous frequency representation [8] may be aided by this form of the information measure.

7. CONCLUSIONS

The new information measure derived in this paper may be a useful tool in time-frequency analysis. It has properties similar to the time-bandwidth product in that it indicates the required separation to resolve two elementary signals. In addition it indicates the separation required to completely resolve multiple elementary signals. It may be useful in assessing candidates for the elementary signal role as well. The terms mono-component and multi-component are used in time-frequency analysis. Perhaps a mono-component signal should only be considered to be the one with the minimum uncertainty under an information measure such as described here. The new measure may be a useful tool in designing kernels and comparing distributions. Since complicated signals may be derived from assemblages of elementary signals this new measure should also provide useful information concerning the resolution of combinations of these complicated signals. Rényi information of the third order was utilized in this study. It may be profitable to consider other orders as well. Second order Rényi information would count the cross terms as well as the auto terms. This may serve as an index of cross term interference. Few objective measures exist to answer these questions. It is worthwhile to explore a variety of possibilities.

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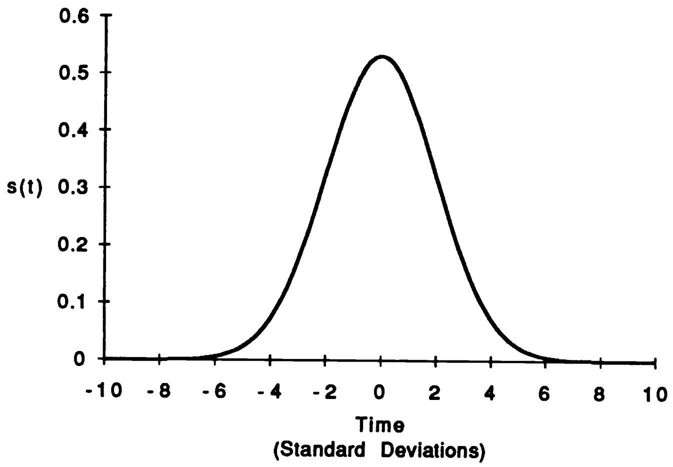

Figure 1.a. Gabor logon time marginal

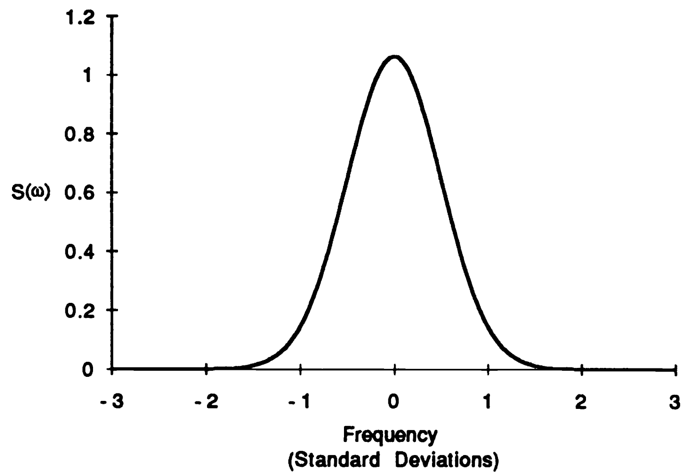

Figure 1.b. Gabor logon frequency marginal

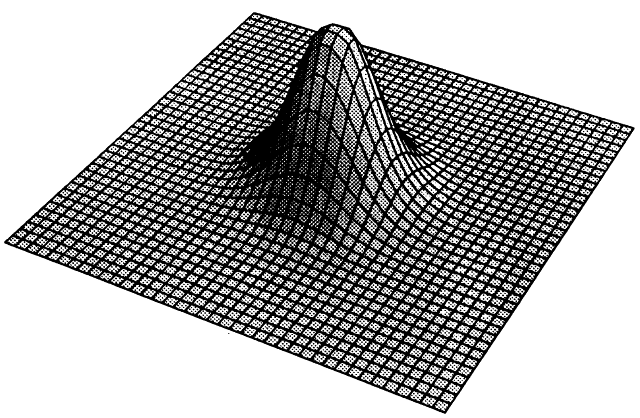

Figure 1.c. Gabor logon WD

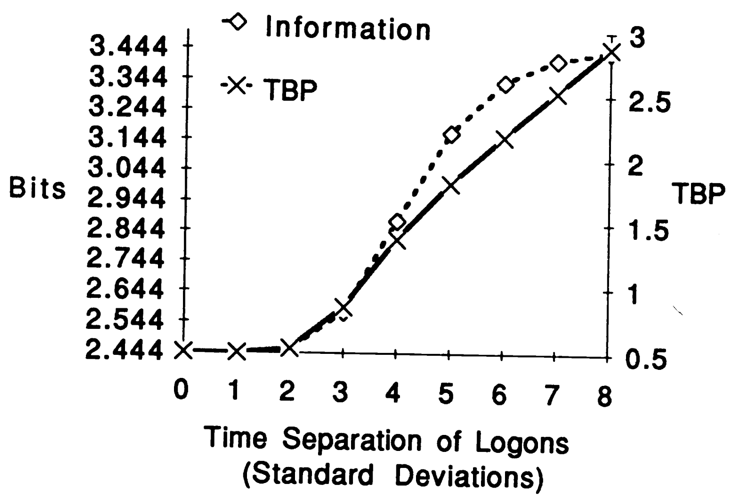

Figure 2. Information and time-bandwidth product (TBP) as a function of logon separation in time standard deviations for two logons.

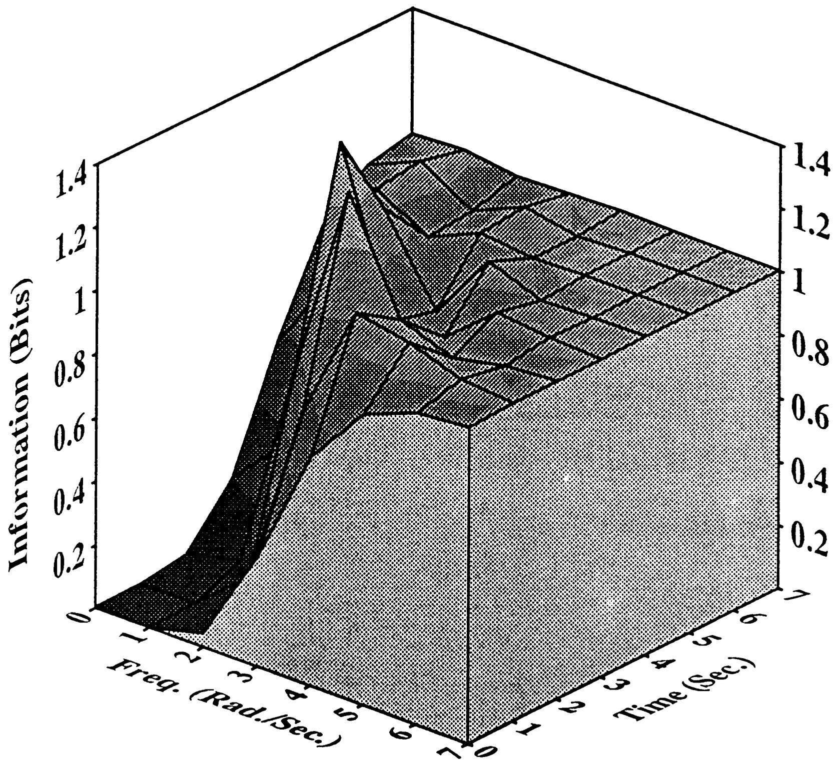

Figure 3.a. Information in the WD as a function of time separation and frequency separation in standard deviations for two logons.

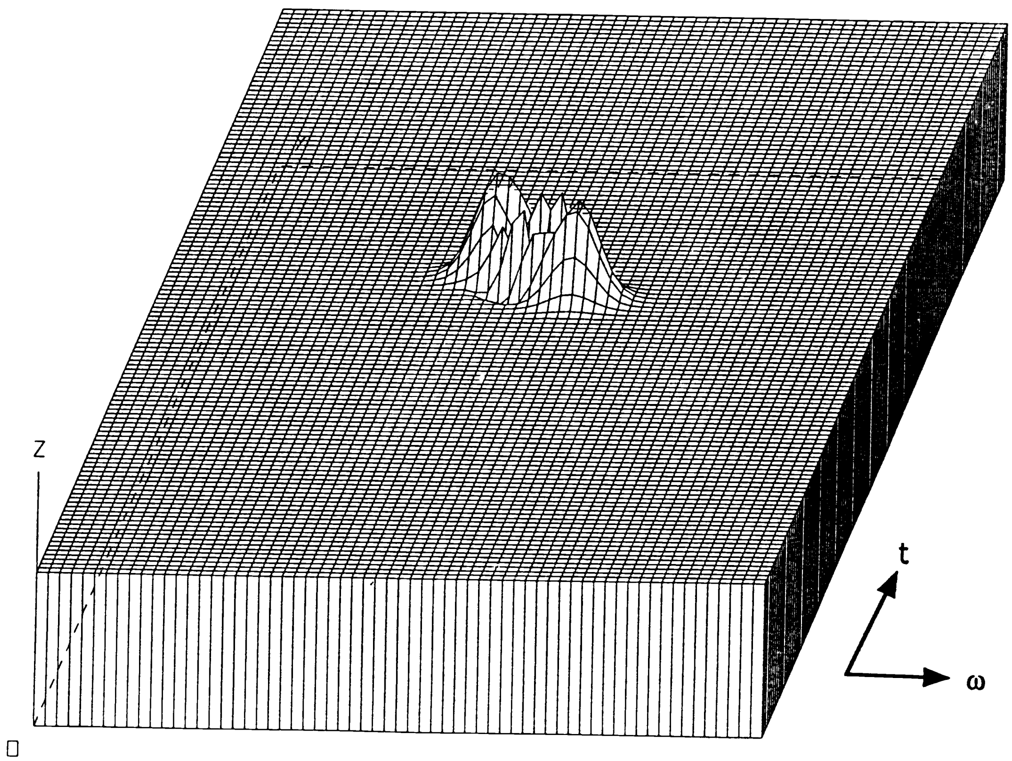

Figure 3.b. WD result for the most over-estimated amount of information (1.35 bits).

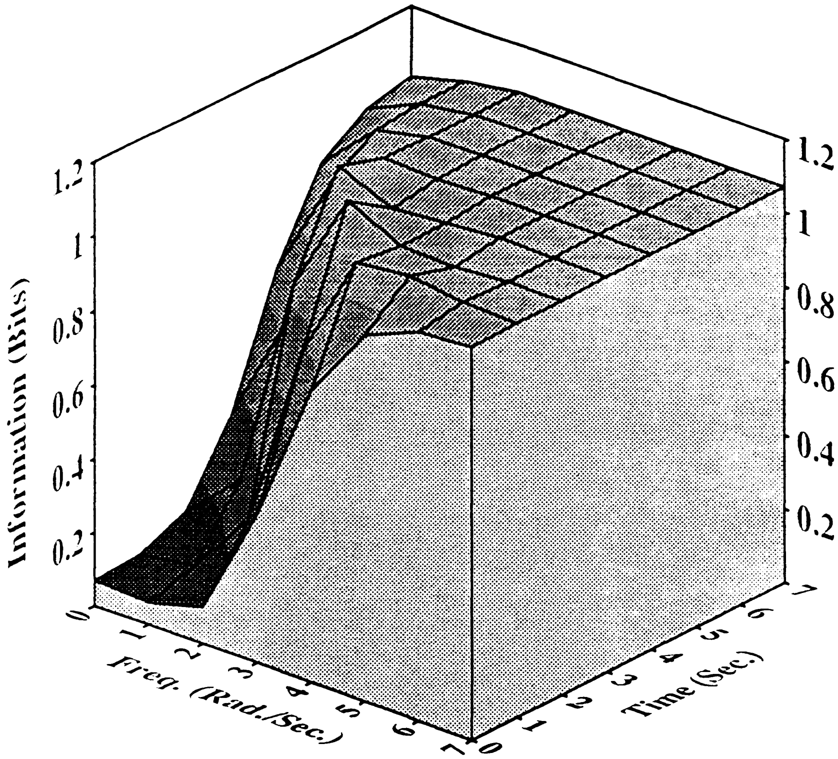

Figure 4.a. Information in the RID as a function of time separation and frequency separation in standard deviations for two logons.

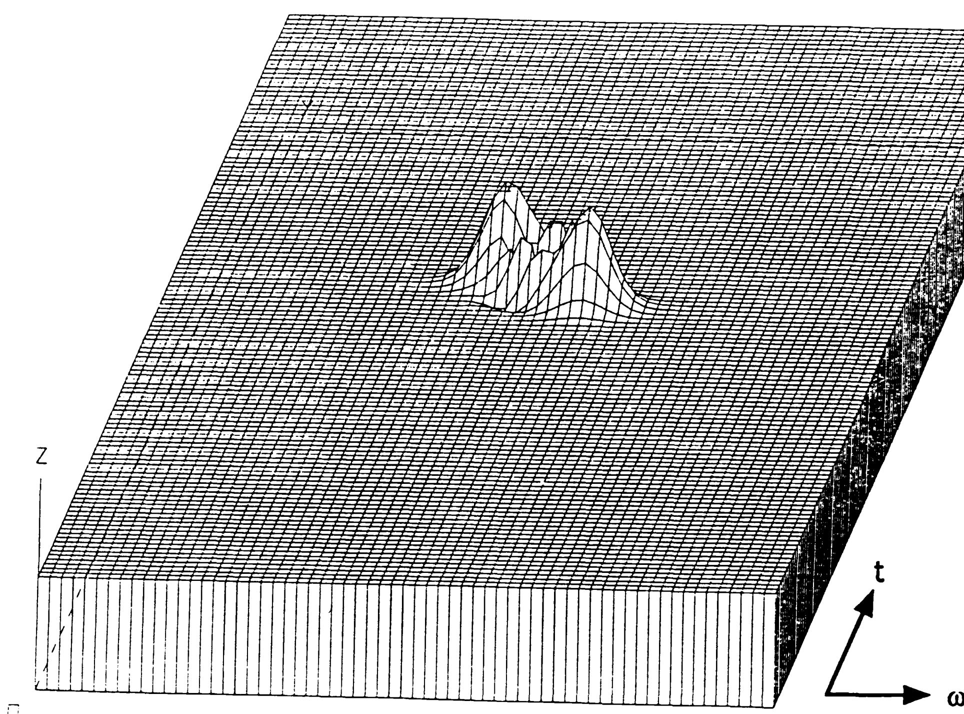

Figure 4.b. RID result for the time-frequency separation which produced the maximum overestimation in information for the WD (1.08 bits for RID).

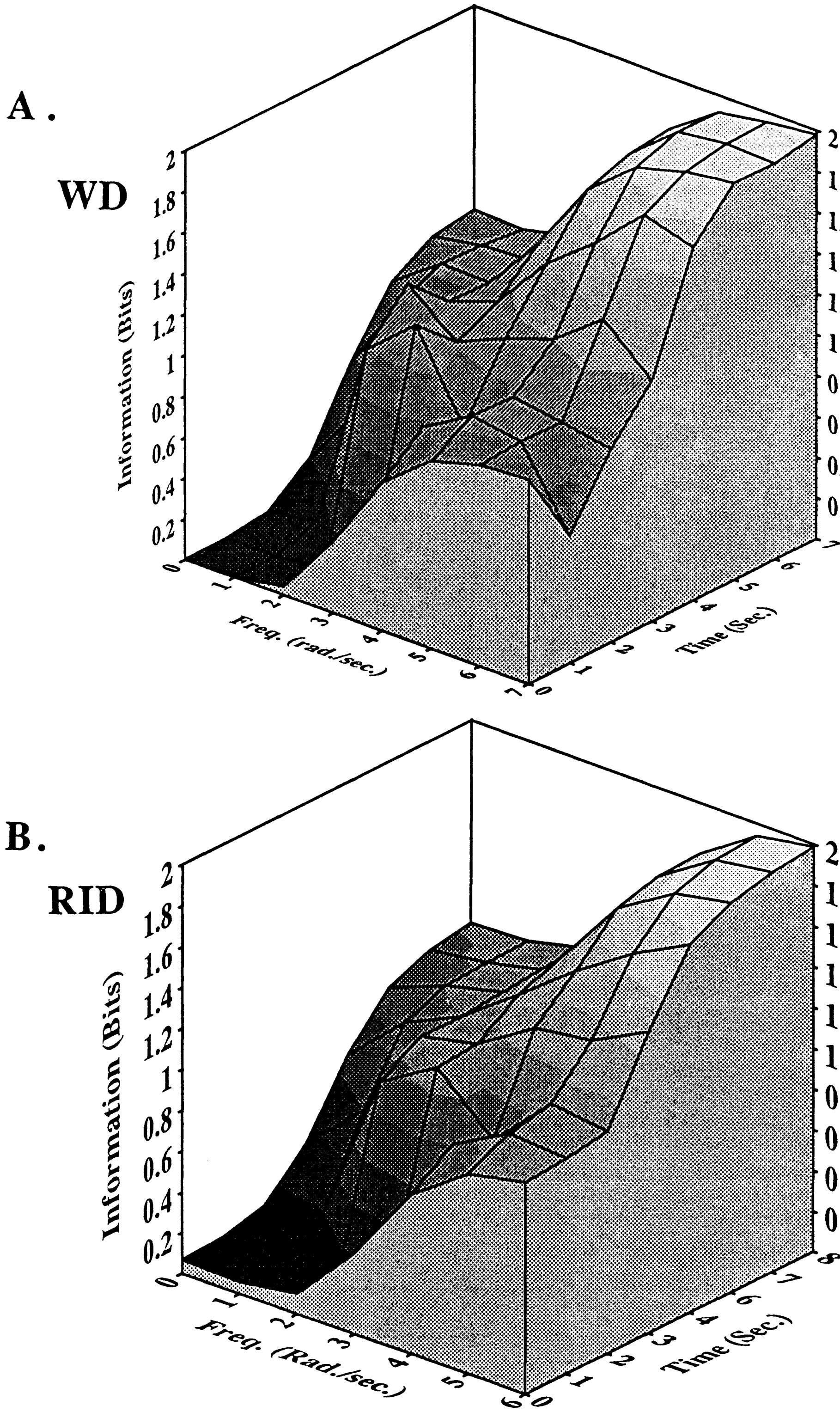

Figure 5. Information for four logons placed at the corners of a $\Delta t \times \Delta\omega$ rectangle, where Δt and $\Delta\omega$ are in units of standard deviation. **a.** WD result. **b.** RID result.