A LOWER BOUND ON PET TIMING ESTIMATION WITH PULSE PILEUP [1]

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ABSTRACT

Pulse pileup is an important limiting factor on coincidence detection and timing estimation performance in conventional and time of flight (TOF) PET. In this paper we derive a Cramer-Rao (CR) lower bound on timing estimator mean-square-error (MSE) when there are multiple overlapping single photon response (SPR) pulses within the photo-detector coincidence window. This bound is then used to study the loss in achievable timing resolution due to varying amounts of pulse overlap. The case of a bi-exponential BGO/Burle 8850 SPR is then considered. We show that in this case timing resolution need not degrade significantly even when pulse overlap is quite severe: up to approximately 20% overlap of the leading edges of the overlapping pulses can be tolerated with less than a 25% degradation in achievable timing resolution. This suggests that multiple event separation and improved coincidence resolving time can be attained well beyond the photon saturation limit of current scintillation detectors.

I. INTRODUCTION

Coincidence resolving time is an important factor affecting PET image reconstruction quality and parameter estimation performance. The ability to accurately resolve temporal positions of scintillator pulses allows for improved true coincidence detection, improved accidental coincidence rejection, and, for TOF-PET, improved time-of-flight estimates. Improvements in coincidence detection and rejection rates translate into a reduction in image artifacts and permit more accurate quantitative PET imaging [8, 7].

For low event rates, scintillation pulses are isolated in time and can be very accurately detected using simple thresholding such as first photo-electron timing (FPET), leading edge timing, or constant fraction timing methods [15, 12, 2, 14]. In particular for isolated mono-exponential pulses, no dark current, and direct photon counting measurements a bias corrected version of FPET achieves minimum mean-square timing error [1]. However, for high event rates scintillation pulses overlap in time, a phenomenon called *pulse pileup*, and these conventional methods perform poorly [3]. The high event rate regime occurs in several practical cases. One important case is in volumetric imaging where large area scintillation detectors are used and inter-slice septa are removed in order to improve detection efficiency. Several elegant methods for combatting pulse pileup have been implemented recently [11, 9, 10]. To gauge the performance of these and other methods it is useful to know the theoretical limits on timing resolution for the pulse pileup regime.

In this paper we compute a Cramer-Rao (CR) lower bound on timing estimation mean-square error (MSE) which accounts for pulse overlap and noisy photomultiplier output current measurements. CR bounds have been used previously to analyze isolated pulse performance for direct photon counting measurements [1, 4]. Since the CR bound is defined independently of the estimation algorithm it specifies limitations on estimator performance which are intrinsic to the estimation problem. On the other hand for many problems it is known that there exist estimators, for example the maximum likelihood (ML) estimator, whose MSE comes very close to the CR bound.

The pulse overlap CR bound is derived under the assumptions that the detector's mean single photon response (SPR) is known, photo-tube gain is large, and dark current is negligable. These assumptions are valid for the majority of PET photo-detectors currently implemented in practice. A general result of our study is the following. For two overlapping pulses timing resolution performance is naturally divided into several distinct regimes: 1) low overlap; 2) moderate overlap; and 3) high overlap. In the low overlap regime the CR bound for multiple pulses reduces to the CR bound for a single pulse: here timing resolution need not suffer from pulse pileup. In the moderate overlap regime, timing resolution gradually degrades as the amount of overlap increases. Finally, in the high overlap regime, a rapid and catastrophic degradation in resolution occurs with increasing overlap. The importance of this result is that the CR bound indicates just how much pulse overlap can be tolerated before achievable estimation performance degrades sharply. Since accurate rejection of accidentals depends on the ability to separate overlapped pulses, the CR bound is also relevant to detection of coincidences. The thresholds on pulse overlap separating these three regimes should therefore be useful for the design of high event rate detectors.

We apply the pulse overlap bound to evaluate the overlap sensitivity of timing resolution with BGO/Burle 8850 photo-detectors. We conclude that accurate pulse separation and timing resolution can be performed with up to approximately 20% overlap of the leading edges of the SPR pulses. This suggests that improved coincidence resolving time can be attained well beyond the pulse pileup saturation limit of current scintillation detectors.

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II. DERIVATION OF LOWER BOUND

Photo-Detector Model Assume a single γ-ray interaction at time τ_1 deposits energy η_1 into a scintillator crystal. Secondary photon emissions arrive at the first dynode of a photo-multiplier tube with intensity:

$$\lambda(t) \quad = \quad \eta_1 \lambda_S(t-\tau_1) + \lambda_d, \qquad (1)$$

where λ_S is a lumped time-varying signal-induced intensity, accounting for smoothing due to collection time dispersion of the scintillation photons and photo-tube transit time jitter, and λ_d is a constant dark current intensity due to ambient photon and photo-electron background levels. After photo-multiplication through several dynodes an induced electrical current $\mathrm{X} = \{\mathrm{X}(t)\}$ is produced at the anode of the photo-tube. Under a simple "standard pulse shape" model the phototube output current is the sum of a current pulse proportional to the intensity function of photo-electrons arriving at the anode plus a low power additive thermal instrumentation noise. This model has been determined to be accurate for Poisson statistics and high average photo-multiplication gain [2].

The above single-event model (1) can be applied to multiple-events when the induced anode pulses are isolated in time, i.e. the tails of pulses do not overlap so that the photo-tube output can be windowed over time to achieve pulse separation. On the other hand for overlapping pulses an intensity model λ different from (1) must be used. If effects such as scintillator crystal dispersion, dead-time, and photo-multiplier saturation can be neglected a natural extension of the single-event model is a superposition of multiple current pulses plus additive noise. With a time window $[0,T]$ over which n sucessive γ-ray/scintillator interactions occur the photo-detector output current will be modeled by:

$$\mathrm{X}(t) = \sum_{i=1}^{n} \mathbf{c}_i h(t-\tau_i) + \mathbf{W}_s(t) + \mathbf{W}_g(t), \quad t \in [0,T] \quad (2)$$

where $h(t)$ is a mean single photon response (SPR) pulse shape, the $\mathbf{c}_i$'s are random scale factors corresponding to the photo-detector output energy corresponding to each of the n events, $\mathbf{W}_s$ is a dark current induced shot noise, and $\mathbf{W}_g$ is a broadband Gaussian thermal noise of power level $N_o/2$. For the following analysis we will make the approximation that the $\mathbf{c}_i$'s are positive statistically independent random variables having truncated Gaussian densities with identical mean μ and variance σ^2, $\mu > \sigma > 0$. For simplicity we will also assume that the dark current term $\mathbf{W}_s$ is negligable. These assumptions along with the anode current model (2) can be mathematically justified by considering a Poisson/Gaussian superposition model [6, 5, 13] in the high intensity regime.

The timing resolution problem can now be stated as the following objective: form an estimate of one of the interaction times, τ_1 say, given the measurements X (2), the knowledge of the mean SPR pulse shape $h(t)$, and the statistics of the random quantities $\{\mathbf{c}_i\}$, $\mathbf{W}_g$. The objective in this paper is to study the intrinsic effect of pulse overlap on the achievable MSE of estimators of τ_1 through investigation of the CR bound.

For reference we summarize important definitions which will turn out to be important determining factors on performance:

- T: length of timing window over which X is observed;
- $h(t)$: mean photo-tube SPR pulse shape;
- T_d: SPR FWHM time width. $T_d \ll T$ is assumed;
- μ and σ^2: mean and variance of SPR;
- $N_o/2$: thermal noise spectral level;
- $\gamma = 2\mu^2/N_o$: SNR (mean signal to noise power);
- $\alpha = 2\sigma^2/N_o$: SVNR (signal variance to noise power).

The Cramer-Rao Bound

The CR bound on the MSE of an unbiased estimator $\hat{\tau}_1$ of τ_1 given that $\tau_2, \ldots, \tau_n$ are unknown has the general form:

$$\mathrm{MSE}_{\tau_1,\ldots,\tau_n}(\hat{\tau}_1) \geq B_{\tau_1,\ldots,\tau_n},$$
$$B_{\tau_1,\ldots,\tau_n} = \frac{1}{F_{11} - F_{12}^T F_{22}^{-1} F_{12}}, \qquad (3)$$

where F_{11}, F_{12} and F_{22} are 1×1, $(n-1)\times 1$ and $(n-1)\times(n-1)$ blocks of the ***Fisher Information Matrix***:

$$F \;=\; E\left[-\nabla^2 \ln f_{\tau_1,\ldots,\tau_n}\right] \qquad (4)$$
$$\;=\; \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix}.$$

Here $f_{\tau_1,\ldots,\tau_n} = f_{\tau_1,\ldots,\tau_n}(\mathrm{X})$ is the likelihood function (probability density of the waveform X) and $\nabla^2 f$ denotes the Hessian matrix of partial derivatives with respect to τ_i and τ_j.

On the other hand, the CR bound on the MSE of $\hat{\tau}_1$ for a single isolated γ-ray event is simply:

$$\mathrm{MSE}_{\tau_1}(\hat{\tau}_1) \geq B_{\tau_1} \qquad (5)$$
$$B_{\tau_1} = \frac{1}{F_{11}},$$

where $F_{11} = E[-\ln f_{\tau_1}'']$ and f_{τ_1}'' is the second derivative of the probability density $f_{\tau_1} = f_{\tau_1}(\mathrm{X})$ with respect to τ_1. Similar CR bounds to (3) and (5) can be developed for biased estimators of τ_1.

Given the model (2), the assumptions invoked in the previous section, and $\mu \gg \sigma$, the likelihood functions $f_{\tau_1,\ldots,\tau_n}$ and f_{τ_1} can be derived using standard methods [16]. Neglecting unimportant constants, the log likelihood function $\ln f_{\tau_1,\ldots,\tau_p}$ has the representation for any p, $1 \leq p \leq n$:

$$\ln f_{\tau_1,\ldots,\tau_p}(\mathrm{X}) \;= -\tfrac{1}{2}\ln|I+\alpha R| \qquad (6)$$
$$+\frac{\mu^2}{2\sigma^2}[(\alpha\underline{\mathrm{Y}}+\underline{1})^T[I+\alpha R]^{-1}(\alpha\underline{\mathrm{Y}}+\underline{1})-p],$$

where similarly to [6]: $\underline{Y} = [Y(\tau_1), \ldots, Y(\tau_p)]^T$ is a vector of time samples of the matched filter output:

$$Y(t) \stackrel{\text{def}}{=} \frac{1}{\mu} \int_0^T X(u)h(t-u)du, \qquad (7)$$

R is a $p \times p$ matrix of SPR correlation lags $R_{ij} = r(\tau_i - \tau_j)$

$$r(\tau) \stackrel{\text{def}}{=} \int_0^T h(t-\tau)h(t)dt,$$

and $\underline{1}$ is a $p \times 1$ vector of ones.

Isolated Events ($n = 1$)

For isolated events the Fisher information F_{11} in (5) is simply computed using (6) with $p = 1$, and the isolated pulse CR bound (5) takes the analytic form:

$$B_{\tau_1} = \left[\gamma + \alpha \frac{\alpha}{1+\alpha}\right]^{-1} \frac{1}{-r''(0)}, \qquad (8)$$

where $-r''(0)$ is the magnitude of the curvature of $r(\tau)$ at $\tau = 0$.

The form of B_{τ_1} indicates several factors determining timing performance in the isolated pulse regime. First observe that the CR bound is inversely proportional to the sharpness of the photo-peak as measured by the magnitude of $r''(0)$. Second note that the bound is also inversely proportional to a nonlinear combination of the conventional SNR γ and the SVNR α. If the SPR statistical mean μ dominates the SPR statistical variation σ then this nonlinear combination is approximated by γ. At the other extreme, if σ dominates μ then this nonlinear combination is approximated by α

Two Events ($n = 2$)

For two events F_{11}, F_{12} and F_{22} in (3) are scalars which can be computed using (6) with $p = 2$. These quantities can then be plugged into (3) to obtain an exact but complicated expression for the CR bound B_{τ_1,τ_2}. While we will use this exact expression to compute the numerical results presented in the next section, considerable simplification occurs when μ dominates σ and when $\Delta = \tau_1 - \tau_2$ is small. In this case we can show:

$$B_{\tau_1,\tau_2} \approx \frac{1}{-\gamma r''(0)} \left[1 - \frac{r''(\Delta)}{r''(0)}\right]^{-1} \qquad (9)$$

Note that the isolated pulse CR bound (8) appears as the first factor $[-\gamma r''(0)]^{-1}$ on the right side of (9) ($\gamma \gg \alpha$). Since $r''(\Delta)/r''(0) > 0$ for small Δ the second factor in (9) amplifies the isolated pulse CR bound. Since $r''(\Delta) \approx 0$ for $\Delta \gg T_d$, this amplification factor decreases to unity as the distance between the two pulses increases significantly beyond the FWHM T_d of the SPR. Hence, as expected, the CR bound (9) predicts that for sufficiently well separated pulses isolated pulse timing performance is achievable. On the other hand, as the pulses begin to overlap $\Delta \leq T_d$ and isolated pulse timing performance becomes impossible. Unfortunately since (9) is only a small Δ approximation quantitative assesment of the maximum amount of tolerable pulse overlap for a given level of timing resolution cannot be evaluated from this expression.

III. NUMERICAL EVALUATIONS

Here we will consider the case of timing estimation for a pair of overlapping bi-exponential pulses ($n = 2$) at time delays τ_1 and τ_2. In Fig. 1, three representative pulse shapes are plotted for a falling edge exponential time constant of $\beta_2 = 100ns$ and rising edge exponential time constants of $\beta_1 = 20ns$, $10ns$, and $3.3ns$, respectively.

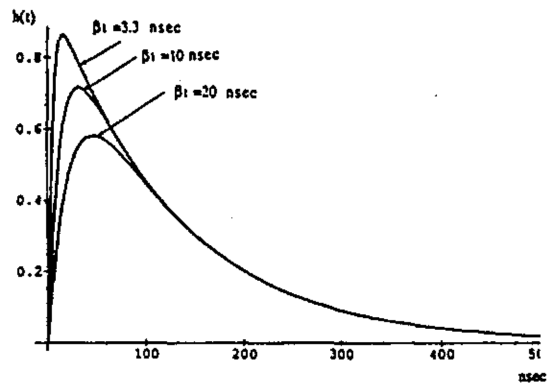

Figure 1: Bi-exponential mean pulse shapes. $\beta_2 = 100ns$.

In Fig. 2 the CR bound on timing resolution ($\sqrt{MSE}$) for τ_1 is plotted as a function of the time difference $\Delta = |\tau_1 - \tau_2|$ between the two pulses for the cases of $\beta_1 = 20ns$ and $\beta_1 = 10ns$. Also plotted for comparison are the isolated pulse CR bounds (horizontal lines labelled "no overlap") for each case. The plots correspond to typical parameters for a moderately high gain BGO scintillator and a Burle RCA 8850 PMT. Notice from Fig. 2 that there are multiple distinct regions of MSE timing performance as a function of the pulse separation Δ: 1) the timing MSE is identical to the isolated pulse CR bound for Δ greater than approximately $2\beta_2 = 200ns$; 2) a gradual degradation in MSE occurs as Δ decreases below this first $200ns$ threshold until a second threshold at $\Delta \approx 2\beta_1$ is attained; 3) there is little additional degradation in MSE for Δ between the second threshold $\Delta \approx 2\beta_1$ and a lower third threshold value at $\Delta \approx \frac{2}{3}\beta_1$; 4) finally as Δ decreases beyond $\frac{2}{3}\beta_1$ the leading edges of the two pulses interfere with each other and an abrupt increase in MSE occurs.

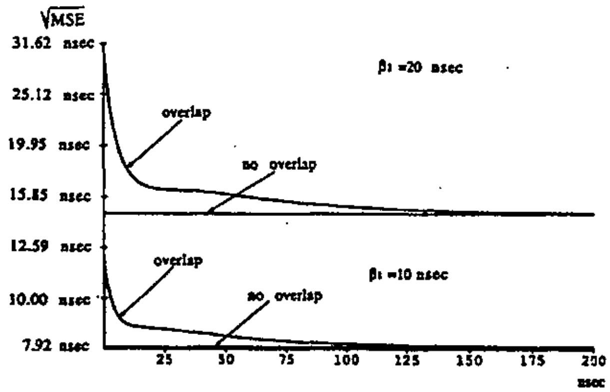

Figure 2: Plots of the CR bounds as a function of $\Delta = |\tau_1 - \tau_2|$. $\beta_2 = 100ns$, $\gamma = 10dB$, $\mu = 1.0$, $\sigma^2 = 0.2$.

The significance of the result reported in Fig. 2 is that it suggests timing resolution similar to that which is achievable for single isolated pulses is possible even for quite severe pulse overlap: up to about 20% overlap of the leading edges of the photo-detector responses can be tolerated with less than a factor of 25% degradation in achievable timing resolution.

IV. FUTURE WORK

The CR bounds developed here are useful for determining pulse pileup thresholds beyond which achievable timing resolution degrades rapidly relative to the isolated pulse regime. In the future the CR bound can be used as a benchmark against which practical pulse pileup implementations are compared. Also, the achievability of the CR bound can be investigated by deriving and simulating the maximum likelihood estimator obtained by maximizing the log-likelihood function (6).

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