APPLICATIONS OF ERROR INTENSITY MEASURES TO BEARING ESTIMATION

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ABSTRACT

In this paper we will discuss a new class of performance approximations for maximum likelihood type bearing estimation systems. This class is based on the application of various point process models to a sequence of error prone points along the likelihood trajectory. The point process model is described by two quantities: the intensity function of the local maxima locations over the parameter space, and a selection rule for the global maximum from the set of local maxima. This class gives new estimators to the Mean-Square Error and specializes to the approximation techniques of [1] and [6].

I. INTRODUCTION

In this paper we present a new method for obtaining performance predictions for maximum likelihood type time delay estimators in passive bearing estimation. The general method includes the techniques [1], [3], [4], [5] and [6] as special cases. While the method does not in general give exact analytical predictions, it can provide simple and accurate approximations for mean-square error and probability of large error. As developed here, the approximations are based on the application of various models to the sequence of error prone local maxima of the sample correlation function. The locations, over correlation time, of these maxima are then related to the time delay estimate which occurs at the global maximum. Interpretation of the set of local maxima occurrence times as a stochastic point process gives the point process intensity function a primary role in our approximation. Accordingly, we call these approximations *error intensity measures.*

Our observation model is the following. Over an observation time interval $t \in \mathbf{T} = [0,T]$ we have:

$$X_1(t) = S(t) + N_1(t) \qquad (1)$$
$$X_2(t) = S(t-D) + N_2(t)$$

where D, $D \in [-D_M, D_M]$, is the time delay of interest and S, N_1, N_2 are mutually uncorrelated wide sense stationary random processes. We assume that S has bandwidth B and that N_1 and N_2 are broadband noises. The model (1) corresponds to the observation of a stationary source S at bearing b across two sensors in ambient incoherent noises.

The correlator-estimator of D, denoted $\hat{D}$, is simply the location over the a priori region $[-D_M, D_M]$ of the global maximum of the sample correlation function R_{12}:

$$R_{12}(\tau) = \frac{1}{T}\int_0^T X_1(t)X_2(t+\tau)d\tau \qquad (2)$$

For broadband flat observation spectra and large BT the correlator estimator is a maximum likelihood estimator for D.

To predict the accuracy of the correlator estimator two performance measures have been used, the Mean-Square-Error (MSE) and the Probability of Large Error (P_e):

$$MSE \triangleq \mathbf{E}[(\hat{D} - D)^2] \qquad (3)$$

$$P_e \triangleq Pr(|\hat{D} - D| > \delta) \qquad (4)$$

where δ is a parameter which characterizes the minimum magnitude of a large error. Several approaches to estimating MSE and P_e have been taken. In [1] a local, small error approximation to the MSE was derived. For broadband observation spectra and large observation time bandwidth product (BT) the small error MSE is equivalent to the Cramer-Rao Lower Bound (CRLB) on the variance of $\hat{D}$. The local approximation breaks down, however, for insufficiently large BT [2]. In [3] an approximation to P_e was proposed. The approximation is based on a calculation of the joint probability that $R_{12}(\tau)$ exceeds $R_{12}(D)$, which approximates $\max_{u \in [D-\delta, D+\delta]} R_{12}(u)$, within a set of predetermined "test points" $\tau = t_1, t_2, \ldots, t_n$. In [3] these test points were conveniently chosen to make $R_{12}(t_1), \ldots, R_{12}(t_n)$ approximately independent random variables for lowpass spectra and large BT. In [4] the technique of [3] was adapted to the approximation of MSE for lowpass spectra and in [5] it was generalized to broadband bandpass spectra. Finally in [6] a level crossing interpretation of the occurrence of large error led to a different approximation of MSE and P_e. In this approach a correspondance was made between the occurrence times $w_1, \ldots, w_N$ of level crossings of the random level $R_{12}(D)$ by $R_{12}(\tau)$ and the value of the estimate $\hat{D}$. Assumption of a uniform distribution of $\hat{D}$ over $w_1, \ldots, \tau_N$, and a Poisson model, gave expressions for MSE and P_e in terms of the level crossing intensity function.

II. A GENERAL REPRESENTATION FOR ERROR

Define the error penalty $g(\hat{D}-D)$ where $g(x)$ is a non-decreasing function of $|x|$ with $g(0)=0$. The MSE and P_e correspond to $g(x)=x^2$ and $g(x)=I[|x|>\delta]$ respectively. Let N be the number of error prone local maxima of R_{12}, that is the number of local maxima which exceed the random level $R_{12}(D)$ over the a priori interval $[-D_M,D_M]$. Let $m_1,...,m_N$ be the times of occurrence of these maxima (see Fig. 1). Note that in general N and m_i are random variables and that $N \geq 1$ since $\hat{D}$ is never exactly equal to D. The contraction property of iterated expectations relates the mean penalty to the sequence of error prone local maxima:

$$\mathbf{E}[g(\hat{D}-D)] = \mathbf{E}[\mathbf{E}[g(\hat{D}-D) \mid m_1, \ldots, m_N]] \quad . \quad (5)$$
$$= \mathbf{E}[\sum_{k=1}^{N} g(m_k-D)P_k^N]$$

Where P_k^N is the conditional probability that $\hat{D}=m_k$ given the random variables N, $m_1,...,m_N$:

$$P_k^N \triangleq \begin{cases} Pr(\hat{D}=m_k \mid m_1,...,m_N) \ , & N=1,2,... \\ I(\hat{D}=D) \ , & N=0 \end{cases} \quad (6)$$

The representation (5) is the fundamental relation between the estimator error and the point process $m_1, \cdots, m_N$ on which our class of approximations is based. Essentially, the representation hinges on the fact that $\hat{D}$ must take on one of the values: D, $m_1,...,m_N$. Thus the continuous time problem of time delay estimation over $[-D_M,D_M]$ has been imbedded into a discrete time estimation problem over the true delay time and the error prone local maxima occurrence times.

It is possible to develop approximations based on (5) in two different directions. On the one hand, one can specify a simple set of fixed "test points", $\{m_1, \ldots, m_N\}$ in (5), and concentrate on developing models for the distribution of $\hat{D}$ over the $\{m_k\}$, $P_1^N, \ldots, P_N^N$. This is equivalent to the approach of [3], [4] and [5]. On the other hand, one can focus on developing models for the distribution of the random sequence $\{m_1, \ldots, m_N\}$, while fixing simple distributions $\{P_1^n, \ldots, P_k^n\}_{n=1}^{\infty}$. This is the approach taken in [6]. In Section III we will further develop the latter approach in the context of error intensity measures.

III. ERROR INTENSITY MEASURES

Fix a sequence of distributions $\{P_1^n, \ldots, P_k^n\}_{n=1}^{\infty}$. Then $\mathbf{E}[g(\hat{D}-D)]$ is solely a function of the $\{m_k\}$. A convenient and simple model for the sequence of m_k is a Markov model such as an independent increments model. This can also be theoretically justified for large BT and large SNR [6]. An important special case is given by specification of the $\{m_k\}$ as the occurrence times of an inhomogeneous Poisson point process. In this case the mean penalty is characterized by the error intensity function λ over $[-D_M,D_M]$:

$$\lambda(\tau) \triangleq \lim_{h \to 0} \frac{1}{h} \ Pr(m_j \in [\tau,\tau+h) \ , \ some \ j) \quad (7)$$

$\lambda(\tau)$ can be interpreted as the pre-disposition of the estimator to giving an error $\hat{D}=\tau$. More generally λ is the mean number of error prone local maxima per unit time in the sense that the expected number of maxima over $[-D_M,t)$, denoted $\Lambda(t)$, is given by: $\Lambda(t) = \int_{-D_M}^{t} \lambda(\tau)d\tau$, $t \in [-D_M,D_M]$. In particular $\Lambda(D_M) = \mathbf{E}[N]$ the expected total number of error prone local maxima over $[-D_M,D_M]$.

We now specialize (5) to the cases of P_e and MSE.

$\mathbf{P_e}$: Let $g(x)=I[|x|>\delta]$. Under the model

- $\{m_i : m_i \notin [D-\delta,D+\delta]\}_{i=1}^{N}$ *is an inhomogeneous Poisson process:*

$$P_e = 1 - e^{-\Lambda_\delta(D_M)} \quad (8)$$

where

$$\Lambda_\delta(t) = \int_{-D_M}^{t} \lambda(\tau) I[|\tau-D|>\delta] d\tau \quad (9)$$

is the integrated intensity associated with the *large* error prone local maxima locations $\{m_i : m_i \notin [D-\delta,D+\delta]\}_{i=1}^{N}$ [7]. It is to be noted that (8) is valid independent of any assumptions on the P_k^N of (6). This is simply because the mere existence of an m_i outside of the small error region $|\tau-D| \leq \delta$ is sufficient for a large error.

MSE: Let $g(x)=x^2$. Then under the model

- $\{m_i : m_i \notin [D-\delta,D+\delta]\}_{i=1}^{N}$ *is an inhomogeneous Poisson process,*
- $P_k^N = P(\hat{D}=m_k \mid m_1, \ldots, m_N) = P(\hat{D}=m_k \mid N)$:

$$MSE = \sigma_{loc}^2 e^{-\Lambda_\delta(D_M)} + \int_{-D_M}^{D_M} (\tau-D)^2 \lambda_\delta(\tau) h(\tau) d\tau \quad (10)$$

where

$$h(\tau) \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} P_{k+1}^{n+1} \binom{n}{k} \Lambda_\delta^k(\tau)[1-\Lambda_\delta(\tau)]^{n-k} \quad (11)$$

$$\lambda_\delta(\tau) \triangleq \frac{d\Lambda_\delta(\tau)}{d\tau} \quad (12)$$

$$\sigma_{loc}^2 \triangleq \mathbf{E}[(\hat{D}-D)^2 \mid |\hat{D}-D| \leq \delta] \quad (13)$$

The expression (10) is the general form of an approximation to the MSE. It involves two terms, a small error contribution σ_{loc}^2 plus a large error contribution. We will be mainly interested in the latter large error term. The P_k^n in (11) are the distributions of D over the possible large error values $\{m_i : m_i \notin [D-\delta,D+\delta]\}_{i=1}^{n}$ for $n=1,2,\ldots$. Consider the following examples:

Maximal Model Poisson (MMP) MSE: Under the distribution $P_k^n \triangleq I[k = \underset{j=1,\ldots,n}{argmax} |m_j - D|]$:

$$MSE = \sigma_{loc}^2 e^{-\Lambda_\delta(D_M)} + \int_{-D_M}^{D_M} (\tau - D)^2 \lambda_\delta(\tau) e^{\int_{-|\tau-D|}^{|\tau-D|} \lambda_\delta(u)du} d\tau \; e^{-\Lambda_\delta(D_M)} \tag{14}$$

The maximal model is conservative in that it maximizes the MSE over all choices of P_k^n for $n = 1,2,\ldots$. A less pessimistic model is the following:

Uniform Model Poisson (UMP) MSE: Under the distribution $P_k^n = \frac{1}{n}$, $n = 1,2,\ldots$

$$MSE = \sigma_{loc}^2 e^{-\Lambda_\delta(D_M)} + \int_{-D_M}^{D_M} (\tau - D)^2 \frac{\lambda_\delta(\tau)}{\Lambda_\delta(D_M)} d\tau \, (1 - e^{-\Lambda_\delta(D_M)}) \tag{15}$$

The expression (15) involves a natural measure of error dispersion: the radius of gyration of the error intensity function about the point D, $\int(\tau - D)^2 \hat{\lambda}_\delta(\tau) d\tau$, where $\hat{\lambda}_\delta$ is a PDF normalized version of λ_δ. Likewise (14) involves the radius of gyration of $\lambda_\delta(\tau)\exp[\int_{-|\tau-D|}^{|\tau-D|} \lambda_\delta(u)du$, which is a measure of the tail behavior of λ appropriate for the characterization of the largest magnitude error. Finally, P_e (8) involves only the integral of the unnormalized error intensity function.

IV. APPLICATIONS

The approximations (8), (14) and (15) are functions of the error intensity λ associated with a set of possible locations for $\hat{D}$: $\{m_1, \ldots, m_N\}$. In Section III these were the large error prone local maxima locations of R_{12}. In general, however, the $\{m_k\}$ can be defined as an arbitrary sequence of test points for the value of the error $\hat{D}$. This provides an additional degree of freedom in designing an approximation to the mean penalty using error intensity measures. Consider the following examples:

$\{m_k\}$ **= level crossing locations:** The incidence of a level crossing of $\max_{u \in [D-\delta, D+\delta]} R_{12}(u)$ by the sample correlation R_{12} implies that an error of magnitude greater than δ has occurred (see Fig. 2). Therefore the intensity of the level crossing times $w_1, \ldots, w_N$ is an appropriate error intensity measure. The Gaussian approximation to R_{12} and an approximation of $\max_{u \in [D-\delta, D+\delta]} R_{12}(u)$ by $R_{12}(D)$ gives the well known formula for the intensity of the zero up-crossings by the Gaussian random process $R_{12}(\tau) - R_{12}(D)$:

$$\lambda^{(1)}(\tau) = (1-\rho^2)^{1/2} \frac{\dot{\sigma}_\tau}{\sigma_\tau} \phi(\frac{\mu_\tau}{\sigma_\tau})[\phi(\xi_\tau) - \xi_\tau \Phi(\xi_\tau)] \tag{16}$$

where $\xi_\tau \triangleq (\sigma_\tau \dot{\mu}_\tau - \rho_\tau \dot{\sigma}_\tau \mu_\tau) / [(1-\rho_\tau^2)^{1/2} \sigma_\tau \dot{\sigma}_\tau]$ and $\mu, \dot{\mu}, \sigma^2, \dot{\sigma}^2$ are the means and variances of $R_{12}(\tau) - R_{12}(D)$ and its derivative, respectively, and ρ is the correlation between them. In (16) Φ and ϕ are the Gaussian CDF and PDF respectively. For more detailed treatment see [6].

$\{m_k\}$ **= local maxima locations:** The exact form for the intensity function of the error prone local maxima is more complicated than (16) On the other hand a simple approximation [7] yields:

$$\lambda^{(2)}(\tau) = \frac{\ddot{\sigma}_\tau}{\sigma_\tau} \phi(\frac{\mu_\tau}{\sigma_\tau}) \Phi(\frac{\sigma_\tau \dot{\mu}_\tau - \rho_\tau \dot{\sigma}_\tau \mu_\tau}{(1-\rho_\tau^2)^{1/2} \sigma_\tau \dot{\sigma}_\tau}) \tag{17}$$

For flat lowpass observation spectra of bandwidth B radians and large BT $\lambda^{(1)}(\tau)$ and $\lambda^{(2)}(\tau)$ are similar and virtually constant over $|\tau - D| >> \delta$. It can be shown that $\lambda^{(2)}(\tau) \approx \alpha B \Phi(-\gamma\sqrt{BT})$, $|\tau| > \delta$, where $\gamma^2 \triangleq [2(1+SNR^{-1})^2+1]^{-1}$. A SNR threshold SNR_t occurs when $\gamma\sqrt{BT} \approx 1$ or equivalently $SNR_t \approx \sqrt{2/BT}$ ($BT >> 1$). Below SNR_t the error intensity increases abruptly and uniformly over the a priori region.

In Fig. 3 the intensity $\lambda^{(1)}$ is plotted over time and SNR for a flat bandpass observation spectrum. Note the time varying nature of the error intensity over time as the SNR decreases. In Figs. 4 and 5 we plot the MSE given by the approximations (14) and (15), using $\lambda = \lambda^{(1)}$ for lowpass and bandpass flat observation spectra. Also plotted for comparison is the Ziv-Zakai Lower Bound (ZZLB) [2] and, where applicable, the approximation of [3]. Note the similar behavior of the approximations and the ZZLB in that they all indicate the presence of multiple thresholds of SNR SNR_{t1}, SNR_{t2} and SNR_{t3}.

V. CONCLUSION

We have developed a general framework for the approximation to the performance of maximum likelihood type estimators in the context of time delay and bearing estimation which includes several previously developed methods. In this paper we focused on a particular category of approximations within this framework: the error intensity measures. These are functions of a point process intensity associated with a sequence of error prone points along the correlator trajectory. The flexibility and generality of the technique should appeal to investigators who are modeling the error performance of general maximum likelihood type estimators.

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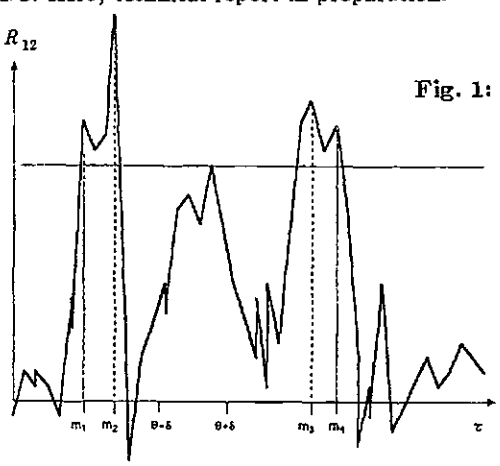

Fig. 1: Local maxima $\{m_k\}$ indicate error outside of small error region $[\theta-\delta,\theta+\delta]$ ($\theta \triangleq D$ in figure).

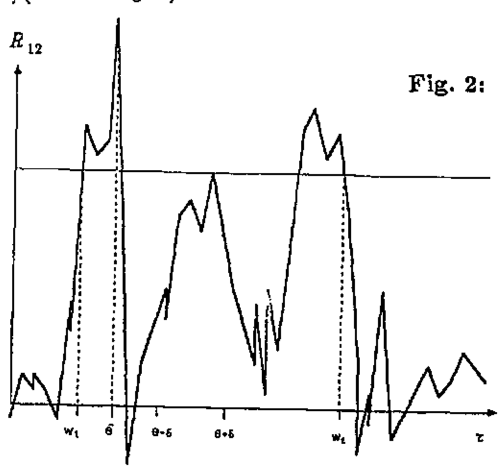

Fig. 2: Level crossings $\{w_k\}$ indicate error outside of $[\theta-D,\theta+D]$ ($\theta \triangleq D$).

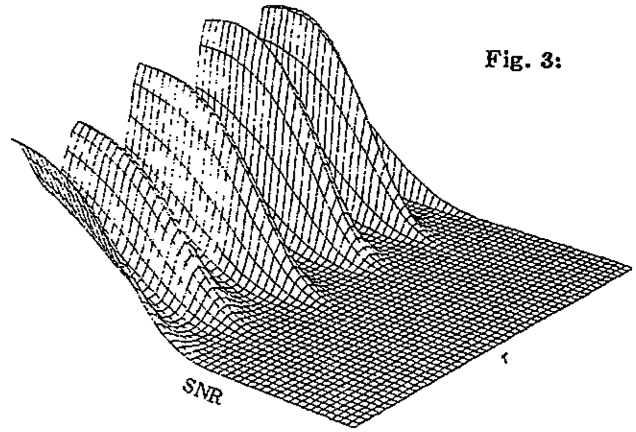

Fig. 3: Intensity surface $\lambda^{(1)}(\tau)$ over SNR and $\tau < D-\delta$ for bandpass spectrum. SNR increases from SE to NW, time decreases from NE to SW.

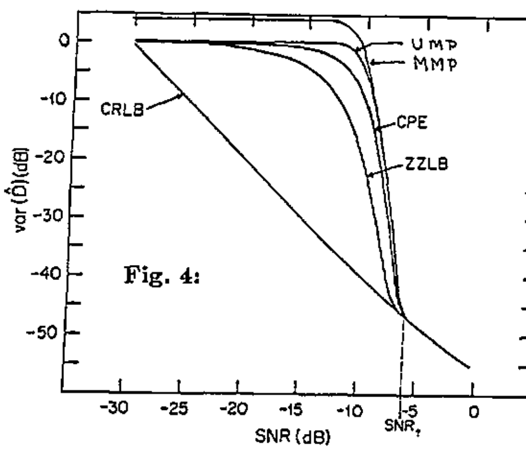

Fig. 4: MSE using UMP, MMP, CPE [3], CRLB and ZZLB for $\lambda^{(1)}$ and lowpass signal spectrum. $BT=1600, BD_M=25, D=0$.

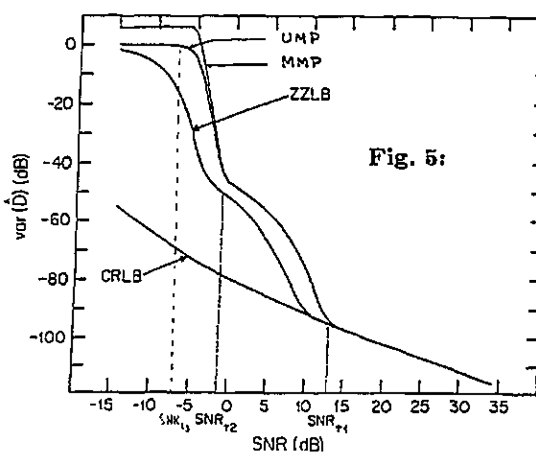

Fig. 5: MSE using UMP, MMP, CRLB and ZZLB for $\lambda^{(1)}$ and bandpass signal spectrum. $w_o/B=10, BD_M=25, BT=200, D=0$.