Alternatives to the Generalized Cross Correlater for Time Delay Estimation

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ABSTRACT

An alternative method for estimating the time delay between two noisy waveforms containing a common signal is presented. The estimate is obtained by means of an approximation to the center of symmetry of a certain correlation function. For narrowband signals preliminary results indicate that the procedure is less sensitive to peak ambiguity which is inherent in the classical optimal estimator.

I. Introduction

In the passive time delay estimation problem, the Generalized Cross Correlator (GCC) estimate of time delay is given by the location in time of the global peak of the GCC output trajectory. Under a Gaussian assumption, the maximum likelihood estimate (MLE) of the delay has been shown to be implementable as a GCC for known observation spectra [1]. The MLE asymptotically achieves minimum variance over all unbiased estimators. However, for finite observation time, the MLE can only be said to achieve minimum local variance, that is, when the variation in the estimate is such that the estimate is highly unlikely to fall outside of the immediate vicinity of the true delay at the global maximum. This assumption is especially tenuous when the spectra of the observations contain a narrowband component [2].

Here an alternate estimation scheme is presented which, by design, takes into account non-local error and appears to be less sensitive to narrowband components. The idea is to substitute a center of symmetry estimate (CSE) of the cross-correlation function in place of the global peak estimate used in the conventional GCC. We introduce two variants on this idea which are implementable as the Local CSE (LCSE), which acts on the GCC waveform, and the Modulus CSE (MCSE), which acts on the absolute magnitude of the GCC waveform. After displaying an approximate expression for the variance of the LCSE, for a general filtered sample cross-correlation function, we optimize the filter to give minimum local variance and compare the result to the MLE. Results of a simulation of the CSE estimators are presented and compared to two popular GCC's.

II. Problem Statement

Two Gaussian wide sense stationary processes $X_1(t)$ and $X_2(t)$ are observed over a time interval $[0, \mathrm{T}]$, where $X_1(t)$ and $X_2(t)$ are assumed to be of the form

$$X_1(t) = S(t) + N_1(t) \tag{2.1}$$

$$X_2(t) = S(t - D_o) + N_2(t)$$

Here $S(t)$ is a random signal with an autocorrelation function $R_{ss}(\tau)$ which falls off to zero for τ greater than T_c, the correlation time of the signal. The noises $N_1(t)$ and $N_2(t)$ are uncorrelated and are taken as broadband with respect to $S(t)$. We assume that the sensor observation time, T, is much greater than T_c.

The conventional GCC estimate of the time delay D_o uses a suitably filtered sample cross-correlation function to yield the estimate $\hat{D}_{GCC}$. Specifically the GCC is implemented as follows. First the cross-spectrum between the observations $X_1(t)$ and $X_2(t)$, $G_{12}(\omega)$, is estimated using one of a number of various approaches [3]. This estimate, called the sample cross-spectrum, $\hat{G}_{12}(w)$, is then weighted in frequency by the real function $W(\omega)$ and transformed into the time domain to form the GCC trajectory, $R^g(\tau)$. The absolute maximum of $R^g(\tau)$ is then taken as the GCC estimate of time delay. Symbolically

$$R^g(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}_{12}(w) W(w) e^{jw\tau} dw \qquad \tau \in [-T, T]$$
$$\hat{D}_{GCC} = \underset{\tau \in [-T,T]}{\mathrm{argmax}} \left[R^g(\tau)\right] \tag{2.2}$$

Here T is some fraction of the sensor observation time T and will henceforth be referred to simply as the "observation time". We associate a correlation time $T_c^{\prime}$ with R^g as some time beyond which the magnitude of the expected value of R^g is close to zero.

In the case where the underlying observation spectra are known, $W(\omega)$ can be chosen to minimize the local variance of the time delay estimate when $\hat{G}_{12}(w)$ is obtained by averaging periodogram type estimates of the cross-spectrum according to the Bartlett procedure [3]. The resulting frequency weight is known as the Hannan-Thomson processor (HT) [8]

$$W_{HT}(w) = \frac{1}{G_{ss}(w)} \frac{|\gamma_{12}(w)|^2}{1 - |\gamma_{12}(w)|^2} \tag{2.3}$$

$$\gamma_{12}(w) = \frac{G_{12}(w)}{\sqrt{G_{11}(w)\, G_{22}(w)}}$$

$\gamma_{12}(w)$ is called the coherence between $X_1(t)$ and $X_2(t)$, and $G_{11}(w)$ and $G_{22}(w)$ are their respective autospectra. Under assumptions of sufficiently large observation time the HT processor is the maximum likelihood estimator for the unknown delay D_o [1].

The SCOT processor is an ad hoc scheme introduced to desensitize the GCC procedure to the bandwidth properties of the signal [4]. The SCOT uses a GCC weighting

ACKNOWLEDGEMENT This research was supported in part by the Office of Naval Research under contract N00014-81-K-0146

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function

$$W_{SCOT}(w) = \frac{1}{\sqrt{G_{11}(w)G_{22}(w)}} \quad (2.4)$$

which essentially prewhitens the spectrum of the sample cross-correlation associated with the sample cross-spectrum through the inverse Fourier transform.

The local variance is a measure which is only sensitive to "small errors". That is, it characterizes estimator performance for high signal-to-noise ratio in the immediate region of the true delay D_o. However, it significantly underestimates the actual variance when the global peak is likely to be far removed from D_o, as can occur for even moderately high signal-to-noise ratio for narrowband signals using the HT [2]. Thus the optimality of the HT can only be asserted in a small error sense. The maximum value of the trajectory of the HT is in general highly unstable in the sense that small variations in signal-to-noise ratio can translate into large and abrupt changes in the location of the absolute maximum (e.g. waveform in Fig. 3.1). This discontinuous behavior of the estimate is characteristic of GCC's in general due to the peak detection based estimation procedure. It is this undesirable property of peak detection oriented time delay estimation schemes that motivates the present work.

III. Center of Symmetry Estimates

The GCC type processors can be interpreted as estimators which utilize the property that, asymptotically, the cross-correlation function assumes its absolute maximum at the true time delay D_o. Here we exploit a different asymptotic property of the cross-correlation to motivate another approach: the delay D_o occurs at the center of symmetry. As an approximation to the center of symmetry of R^g we define the local center of symmetry for a real parameter L as any point $\tau \in [-T+L, T-L]$ where the regions of length L to either side of τ correspond to equal area under R^g. Specifically this estimate, the LCSE, is formed by finding the zero, $\tau = \hat{D}$, of the function

$$I_L(\tau) = \frac{1}{L^2}\left\{\int_{\tau-L}^{\tau} R^g(\sigma)d\sigma - \int_{\tau}^{\tau+L} R^g(\sigma)d\sigma\right\} \quad (3.1)$$

L is the width of the sliding window over which backward and forward integrations are performed (see Fig. 3.1). By convention we will take the median of the zeros of the expression (3.1) as the estimate of D_o if there are multiple solutions.

In principle the window parameter L can be chosen to optimize the theoretical and/or practical performance of the estimate. Indeed in the limit as L becomes small, $L = \Delta$ say, the LCSE can be looked upon as a generalization of a peak discriminator by noting that we have from Eqn. (3.1) (assuming R^g is sufficiently smooth)

$$I_L(\tau) = \frac{R^g(\tau+\Delta) - R^g(\tau)}{\Delta} = \frac{dR^g(\tau)}{d\tau} \quad (3.2)$$

Hence, for small L, $I_L(\tau) = 0$ whenever there is a local peak at τ, obviously not a good attribute for our purposes. On the other hand if the maximum admissible deviation of D_o from zero, D_m, is such that $D_m + T_c' < T$ then , asymptotically, one can integrate over the majority of the positive extent of R^g by setting $L = T_c'$. This choice should yield a statistic, I_L, which has the least chance of getting hung up on a local maximum of R^g.

Another approach of interest is to perform the center of symmetry procedure on the modulus (absolute value) of R^g. This estimate, denoted the MCSE, is obtained by finding the location of the zero crossing, $\tau = \hat{D}$, of

$$I_M(\tau) = \frac{1}{L^2}\left\{\int_{\tau-L}^{\tau} \mid R^g(\sigma) \mid d\sigma - \int_{\tau}^{\tau+L} \mid R^g(\sigma) \mid d\sigma\right\} \quad (3.3)$$

Note that for both of the above CSE methods, unlike the GCC estimates, a small random change in the detailed structure of R^g will not cause instability in $\hat{D}$, since only large changes in the area under $R^g(\tau)$, $\tau > D_o$ or $R^g(\tau)$, $\tau < D_o$ can significantly change the location of a zero of $I_L(\tau)$ or $I_M(\tau)$. This can be attributed to the fact that the LCSE (MCSE) is essentially linear in the waveform R^g ($|R^g|$) while the GCC estimate is nonlinear. Thus it may be expected that gross error instability in the CSE will be less of a problem than with the GCC.

We have derived approximate expressions for the variance of the LCSE and MCSE using identical assumptions to those in [5]. The details are contained in [9]. The expression for the local variance of the LCSE is reproduced below

$$var_L \hat{D} = \frac{L}{2k} \frac{G_{11}(0)G_{22}(0)\,(1-|\gamma_{12}(0)|^2)\,|W(0)|^2}{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{12}(w)|\,W(w)\,dw\right]^2} \quad (3.4)$$

where k is proportional to T.

For comparison we display the expression derived in [5] for the local GCC estimator variance

$$var_L \hat{D}_{GCC} = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} w^2 G_{11}(w)G_{22}(w)(1-|\gamma_{12}(w)|^2)\,|W(w)|^2 dw}{k\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} w^2 |G_{12}(w)|\,W(w)dw\right]^2} \quad (3.5)$$

The quantity $G_{11}(w)G_{22}(w)(1-|\gamma_{12}(w)|^2)\,|W(w)|^2$ can be interpreted as the spectral density of the component of R^g which corrupts the global peak at D_o. This we refer to as correlator noise. Comparing the numerators of (3.4) and (3.5), it is evident that the variance of the LCSE depends on the correlator noise variance only through its D.C. component, while for the GCC it varies as the second moment of the noise spectrum. This reflects the time averaging criterion which forms the crux of the center of symmetry estimate. In other words, only the average value of $R^g(\tau)$ to the left and right of $\tau = \hat{D}$ are utilized by $I_L(\tau)$ in its search for a zero. The above comments imply that the (local) variance of the LCSE will be insensitive to changes in the detailed, i.e. high frequency, structure of the noise spectra as long as the average, or D.C., power remains the same. Thus a certain robustness to the underlying noise in the observations is indicated.

The form of Eqn. (3.4) is amenable to the following optimization problem. Assume we impose an energy constraint on the GCC weight $W(w)$. Then the left hand side of Eqn. (3.4) is minimized by applying the Schwarz inequality giving the solution

$$W(w) = q\,|G_{12}(w)| = q\;G_{ss}(w) \quad (3.6)$$

where q in Eqn. (3.6) is chosen to satisfy

$$\int_{-\infty}^{\infty} | W(w) |^2 dw = \sigma_W^2$$

and σ_W^2 is the energy associated with the filter $W(w)$. This filter can be interpreted as a variant of the "matched" filter in the sense that $W(w)$ is equal to the average value of the sample-cross-correlation function, i.e. the signal component. (Note this does not define a matched filter in the conventional sense since the signal is non-deterministic and no prewhitening is employed). Although optimality cannot be claimed one may expect that the choice of weighting in Eqn. (3.6) would also be a fortuitous one in the MCSE procedure. In the next section we discuss the experimental performance of this implementation, hereafter referred to as the MF-MCSE.

IV. Simulation Results

A comprehensive simulation study is currently in progress and we only give a preliminary sample of the results here for the MF-MCSE. We synthesized two sensor sequences which contain a common narrowband component in uncorrelated white Gaussian noises to exercise and compare the various processors discussed in sections II and III. The spectrum and auto-correlation function of the signal sequence are displayed in Figs. 4.1 and 4.2 which correspond to a suppressed carrier A.M. waveform $S(k)$. The relative delay D_0 of the signal component in the sensor sequences X_1 and X_2 was set to 25 bins. 5120 samples of these two records were divided into 5 distinct groups and the averaged cross-spectral estimate was constructed. This estimate was then weighted with the MF, SCOT and HT functions, Eqns. (3.6), (2.4) and (2.3). Finally peak detection was performed over the 1024 samples of the GCC trajectory $\{R^g(k)\}_{k=-511}^{512}$ for the various processors, and Eqn. (3.3) was implemented in discrete time.

The full length window was used for the MCSE when a priori the true delay lies within 100 bins of the 0-th bin corresponding to the middle 200 indices of the sequence $\{R^g(k)\}$. That is we summed over 411 bins to the left and to the right of each point in the interval $[-100,100]$.

For signal-to-noise ratios (SNR) between -20dB and 20dB the HT and SCOT gave exact estimates of D_0 while the MF-MCSE was observed to be in error by 1 bin on the average. Figs. 4.3, and 4.4 show the GCC trajectories for the HT and the SCOT respectively for a SNR of -23dB (The MCSE has a trajectory virtually identical to that of the MCSE for this SNR). Note that the HT and SCOT fail to resolve a global peak anywhere near the true delay of 25 bins, and that within 100 bins of 0 the highest peaks occur at 0 and -48 respectively. Fig. 4.5 shows the statistic I_N, Eqn (3.5), for the MF-MCSE, as a function of the parameter $k = -100, \ldots, 0, \ldots 100$. The zero crossings of I_N are clustered close to the 25-th bin at values 18, 20, 21, 22 and 23 giving an estimate of 21 bins, a significant improvement over the HT and SCOT processors.

V. Conclusion

In this study we have presented several forms of the center of symmetry estimator, which are simple modifications of the Generalized Cross Correlation method for time delay estimation. Based on a variance approximation an optimization of the LCSE weight yielded a "matched" filter function. Even though we could not show the optimality of the "matched" filter for the MCSE the simulation results indicate that the magnitude CSE may have better performance than the GCC for narrowband signal spectra at low signal-to-noise ratio. We believe that this results from the insensitivity of the MCSE to peak ambiguity relative to the GCC.

In general the HT is a more accurate estimator for small errors than any other unbiased estimator[8]. This is borne out in the simulations by the high SNR performance of the "matched" filter implementation of the MCSE. One possible estimation strategem would be to use the optimal GCC and CSE together as a possible improvement over either processor alone. This would be a variant of the so called "gated mode" implementation of the GCC [6]. That is one would censure the GCC output outside of some region determined by the rough estimate obtained using the CSE, estimating the delay as the location of the highest peak within the gated region. Further study of this and other uses of the CSE remains to be undertaken.

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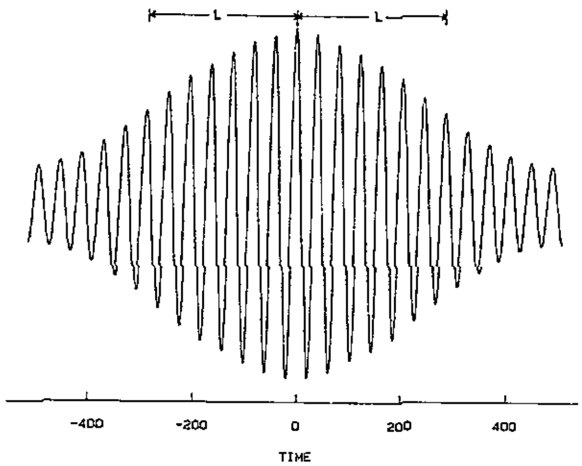

Figure 3.1
Mean value of $R^g(\tau)$ for a narrowband signal spectrum. CSE compares the integrals a distance L to the left and to the right of each point in the admissible region of delay.

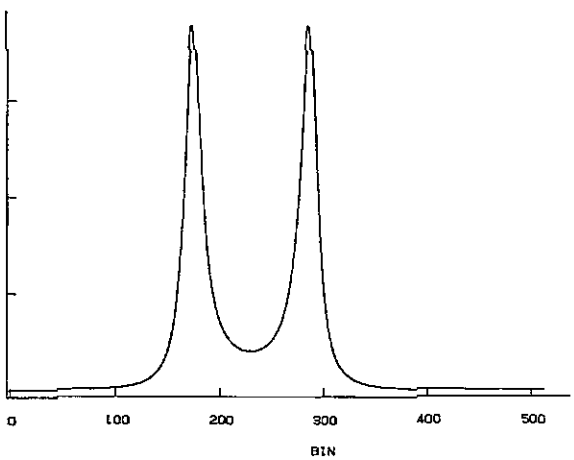

Figure 4.1
Spectrum of the simulated signal corresponds to an A.M. waveform with carrier-to-signal ratio of -6dB.

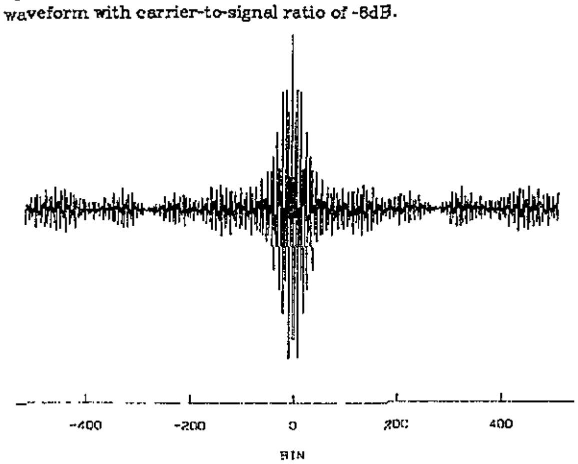

Figure 4.2
Auto-correlation function of simulated signal.

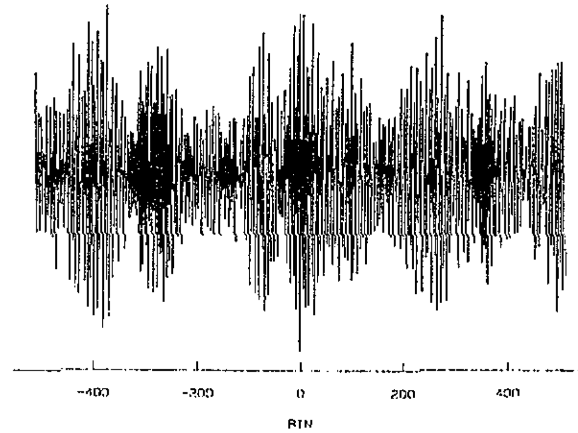

Figure 4.3
HT trajectory for SNR of -23dB. Max over $[-100, 100]$ at 0.

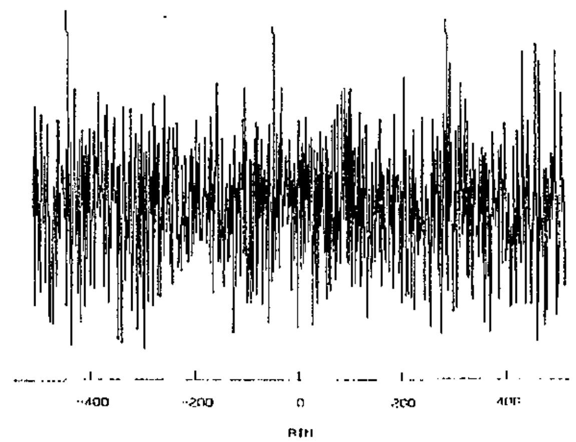

Figure 4.4
SCOT trajectory for SNR of -23dB. Max over $[-100, 100]$ at -48.

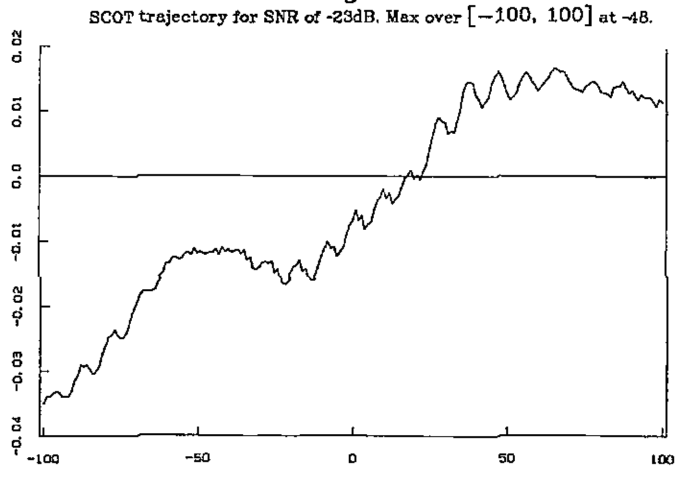

Figure 4.5
CSE statistic $I_M(k)$. Median of zero locations at 21.