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The Decision Problem for Linear Temporal Logic

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Introduction The aim of *temporal logic* is the analysis of arguments about events and processes in time. To achieve this aim, truth-functional logic is enriched by certain *tense operators,* among them:

Pp	p was the case (at least once)
Fp	p will be the case (at least once)
$S(p, q)$	there has been an occasion when p was the case, ever since which q has been the case
$U(p, q)$	there is going to be an occasion when p will be the case, up until which q is going to be the case.

Which sentences involving tense operators express valid principles of reasoning?

That turns out to depend on what is assumed about the structure of time. Consider, for instance, this example:

$$(Fp \wedge F(p \wedge \sim Pq) \wedge \sim(Pq \vee q)) \rightarrow Fq$$

"If p is sometime going to be the case,
but not until q has previously been the case,
and q hasn't yet been the case,
then q is sometime going to be the case."

If it is assumed (as it will be throughout this paper) that the earlier/later relation *linearly orders* the instants of time, then the above example counts as valid; but not, in general, otherwise. There are even examples (see the survey

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[3]) whose validity depends on what additional assumptions (density? discreteness? etc.) are made about the temporal order: Each important class of linear orders gives rise to its own temporal theory.

The application of temporal logic to reasoning about the execution of programs has been suggested by several computer science theorists; see for instance [5], where it is shown that the tense operator U above is just what is needed to express such properties of programs as *responsiveness* and *fair scheduling.* (A related development is so-called *dynamic logic,* surveyed in [13].) In this connection, two models of time are especially important: likening the flow of time to the *integers* in their natural order is appropriate when reasoning about basic features of the sequential functioning of a *digital* machine, and likening time to the *real numbers* is appropriate to the continuous workings of an *analog* device (and to certain special features of the digital case, e.g., *asynchronous parallel processing*).

Our concern in this paper will be with the *decidability* of the temporal theories of various classes of linear orders. It turns out that decidability for such theories can often be immediately derived from results in so-called *monadic* logic. For instance, we get the decidability of the temporal theory of *arbitrary* linear orders (and of every *elementary* class of linear orders) from a result of Gurevich [6]. For the *integral* order, we get decidability from a result of Büchi [2]. (The methods of [6] are model-theoretic, those of [2] are automata-theoretic.) The results of [6] and [2] are unified and extended by a powerful result of Rabin [14], which gives us decidability for some new cases, e.g., *well*-orders. (The original proof in [14] is automata-theoretic; a model-theoretic alternative proof is given in [16].) A case not covered by [14] is that of *complete* orders, which (along with elementary classes of complete orders) is covered by Gurevich [7] (where the methods are again model-theoretic, following [12]).

The main result of this paper is the decidability of the temporal theory of the *real* order. In Section 1 a detailed proof is given, involving an indirect reduction to the main result of [14]. In Section 2 an alternative proof, by the method of [7], is outlined. In Section 1 it is also shown that whenever a sentence of temporal logic fails to be valid for the real order, then there is a counterexample to validity which is fairly simple topologically speaking (viz., Borel). In Section 2 it is also shown that the temporal theory of the real order differs from that of arbitrary *continuous* (unbounded, dense, complete) orders.[1]

1 Decidability and Borel counterexamples

1.1 Syntax The set L of (well-formed) sentences of temporal logic is defined inductively:

The atoms $p_0, p_1, p_2, \ldots$ are sentences
If ϕ, ψ are sentences, then so are $\sim\phi$, $(\phi \wedge \psi)$, $S(\phi, \psi)$, $U(\phi, \psi)$.

In terms of negation ($\sim$) and conjunction ($\wedge$) we can in the usual way define disjunction ($\vee$), conditional ($\rightarrow$), biconditional ($\longleftrightarrow$), constant truth ($\uparrow$). In terms of 'since' (S) and 'until' (U) we can define 'was' (P), 'will' (F), 'some-times' ($\Diamond$), and 'always' ($\Box$); thus:

$P\phi$ abbreviates $S(\phi, \uparrow)$
$F\phi$ abbreviates $U(\phi, \uparrow)$
$\Diamond\phi$ abbreviates $Pp \vee p \vee Fp$
$\Box\phi$ abbreviates $\sim\Diamond\sim\phi$.

1.2 ***Semantics*** Let $\mathfrak{T} = (T, <_T)$ be a linear order. A *valuation* in $\mathfrak{T}$ is a function assigning each atom p_i a subset of T. Intuitively, think of T as the set of instants of time, $<_T$ as the earlier/later relation, and $V(p_i)$ as the set of times when p_i is true. V can be extended to a function (by abuse of language still called V) defined on *all* sentences thus:

$$\begin{aligned}
V(\sim\phi) &= T - V(\phi)\\
V(\phi \wedge \psi) &= V(\phi) \cap V(\psi)\\
V(S(\phi, \psi)) &= \{t \in T\colon \exists u(u <_T t \wedge u \in V(\phi) \wedge \forall v(u <_T v <_T t \to v \in V(\psi)))\}\\
V(U(\phi, \psi)) &= \{t \in T\colon \exists u(t <_T u \wedge u \in V(\phi) \wedge \forall v(t <_T v <_T u \to v \in V(\psi)))\}.
\end{aligned}$$

Our definitions then tell us:

$$\begin{aligned}
V(P\phi) &= \{t \in T\colon \exists u(u <_T t \wedge u \in V(\phi))\}\\
V(F\phi) &= \{t \in T\colon \exists u(t <_T u \wedge u \in V(\phi))\}.
\end{aligned}$$

We read "$t \in V(\phi)$" as "ϕ is true at t in $\mathfrak{T}$ under V". We say that ϕ is *satisfiable* (respectively, valid) in $\mathfrak{T}$ *under* V if it is true at some (respectively, every) $t \in T$. We say that ϕ is *satisfiable* (respectively, *valid*) in $\mathfrak{T}$ if it is so under some (respectively, every) valuation V in $\mathfrak{T}$. We say that ϕ is *satisfiable* (respectively, *valid*) in a class $\mathfrak{K}$ of linear orders if it is so in some (respectively, every) $\mathfrak{T} \in \mathfrak{K}$. The set of sentences of L valid in $\mathfrak{K}$ is called the *temporal theory* of $\mathfrak{K}$. **Z**, **Q**, **R** will denote the sets of integers, rationals, and reals with their natural order; superscripts ($^+$,$^-$) will denote restriction to positive or negative elements. In this notation our promised result is:

1.3 Theorem *The temporal theory of* **R** *is decidable.*

We will offer two proofs, one in this section and one in the next, for this, our main result. The method of the first proof will be familiar to specialists from other applications. We show that: (1) a formula satisfiable in **R** is satisfiable in **Q** under a "nice" valuation; (2) a formula satisfiable in **Q** under a "nice" valuation is satisfiable in **R**; (3) the set of formulas satisfiable in **Q** under "nice" valuations is decidable. (1) is easy; (3) is easy given the main result of [14]; and (2) is proved by showing that a "nice" valuation in **Q** satisfying a formula ϕ can be extended to a valuation in **R** satisfying ϕ. (Our rather complicated definition of "niceness" is just what is needed to make this extension lemma hold.) Unfortunately, the proof of this extension lemma, which is quite easy for P, F-temporal logic, becomes more difficult for S, U-temporal logic.

1.4 Definitions L_n is the set of $\phi \in L$ containing no atoms p_i with $i \geqslant n$. For $\phi \in L$ or $\Phi \subseteq L$, $N(\phi)$ or $N(\Phi)$ is the least n with $\phi \in L_{n+1}$ or $\Phi \subseteq L_{n+1}$. For $\Phi \subseteq L$ with $N(\Phi) = n$ and with conjunction $\bigwedge\Phi$, $\Phi^{\#}$ will denote the sentence $\Box(p_n \wedge \bigwedge\Phi)$. A sentence of the *first species* is one having one of the six forms:

$$p_i \quad \sim p_i \quad S(p_i, p_j) \quad \sim S(p_i, p_j) \quad U(p_i, p_j) \quad \sim U(p_i, p_j).$$

A sentence of the *second species* is one having one of the four forms:

$$p_i \longleftrightarrow \sim p_j \quad p_i \longleftrightarrow (p_j \wedge p_k) \quad p_i \longleftrightarrow S(p_j, p_k) \quad p_i \longleftrightarrow U(p_j, p_k).$$

A *special* sentence is one of form $\Phi^{\#}$ where Φ is a finite set of sentences of the second species.

1.5 Reduction Our definitions tell us:

$V(\Diamond\phi) = T$ or ϕ according as ϕ is satisfiable in $\mathfrak{T}$ under V or not
$V(\Box\phi) = T$ or ϕ according as ϕ is valid in $\mathfrak{T}$ under V or not.

Hence ϕ is valid iff $\sim\phi$ is not satisfiable iff $\Diamond\sim\phi$ is not satisfiable iff $\Box\Diamond\sim\phi$ is not satisfiable; and to prove Theorem 1.3 it suffices to provide a decision procedure for satisfiability in **R** of sentences beginning with $\Box$.

If $\Box\phi$ is such, enumerate the subformulas of ϕ as $\phi_0, \phi_1, \ldots, \phi_m$ where $\phi_m = \phi$ itself and where for $i \leqslant N(\phi)$ $\phi_i = p_i$. Let Φ consist of just the following biconditionals:

$p_i \longleftrightarrow \sim p_j$ for i, j with $\phi_i = \sim\phi_j$
$p_i \longleftrightarrow (p_j \wedge p_k)$ for i, j, k with $\phi_i = (\phi_j \wedge \phi_k)$
$p_i \longleftrightarrow S(p_j, p_k)$ for i, j, k with $\phi_i = S(\phi_j, \phi_k)$
$p_i \longleftrightarrow U(p_j, p_k)$ for i, j, k with $\phi_i = U(\phi_j, \phi_k)$.

Unpacking our definitions, a little thought shows that satisfiability for $\Box\phi$ is equivalent to satisfiability for $\Phi^{\#}$; thus to prove Theorem 1.3 it suffices to provide a decision procedure for satisfiability in **R** of special sentences. The remainder of this section will be devoted to that task. We need some technical notions.

1.6 Definitions Recall that an *open* subset of **R** is one that can be written as a countable union of open intervals $]u, v[= \{t: u < t < v\}$. Intervals *maximal* with respect to the property of being contained in a given open set U are called its *components*. The *interior* of a set A is the largest open set contained in it. A set is *closed* if its complement is open, and is F_σ if it can be written as a countable union of closed sets. For basic information about such topological notions we refer the reader to [11].

It is easy to define a primitive recursive function f such that $f(n)$ properly bounds the number of subsets of L_{n+1} consisting of formulas of the first and second species. We call a subset of R *n-rudimentary* if it is a union of no more than $f(n)$ sets each of which is a difference of two F_σ sets. A valuation V in **R** is *n-rudimentary* if $V(p_i)$ is always an n-rudimentary set. A sentence ϕ is *rudimentarily satisfiable* if it is satisfiable in **R** under an $N(\phi)$-rudimentary valuation.

Recall that a linear order $\mathfrak{T} = (T, <_T)$ is (*Dedekind*) *complete* if the following *inf* and *sup* axioms hold in it:

$$\forall X \subseteq T(X \neq \phi \wedge \exists y \sim\exists x(x \in X \wedge x <_T y) \rightarrow$$
$$\exists y(\sim\exists x(x \in X \wedge x <_T y) \wedge \forall z(y <_T z \rightarrow \exists x(x \in X \wedge x <_T z))))$$

$$\forall X \subseteq T(X \neq \phi \wedge \exists y \sim\exists x(x \in X \wedge y <_T x) \rightarrow$$
$$\exists y(\sim\exists x(x \in X \wedge y <_T x) \wedge \forall z(z <_T y \rightarrow \exists x(x \in X \wedge z <_T x)))).$$

Though $\mathbf{Q}$, unlike $\mathbf{R}$, is incomplete, a valuation V in $\mathbf{Q}$ will be called *ϕ-complete* if these axioms hold for $X \subseteq \mathbf{Q}$ of either of the forms:

$$\{t \in \mathbf{Q}: t < c \wedge \forall u \in \mathbf{Q}(t < u < c \rightarrow u \in V(\psi))\}$$
$$\{t \in \mathbf{Q}: c < t \wedge \forall u \in \mathbf{Q}(c < u < t \rightarrow u \in V(\psi))\}$$

where c may be any element of $\mathbf{Q}$, ψ any element of F = first-species sentences that are subformulas of ϕ or negations of such.

Let $\psi \in F$ have form p_i or $\sim p_i$. Note that if V is a ϕ-complete valuation in $\mathbf{Q}$, then the following conditions are equivalent for any irrational $\gamma \in \mathbf{R} - \mathbf{Q}$.

(1) $\exists a \in \mathbf{Q}(a < \gamma \wedge \forall u \in \mathbf{Q}(a < u < \gamma \rightarrow u \in V(\psi)))$
(2) $\exists b \in \mathbf{Q}(\gamma < b \wedge \forall u \in \mathbf{Q}(\gamma < u < b \rightarrow u \in V(\psi)))$.

If they hold we say *V imposes ψ* on γ.

Let $\psi \in F$ have form $S(p_i, p_j)$. We say *V imposes* ψ or *imposes* $\sim\psi$ on γ according as the following condition holds or not:

(3) $\exists a \in \mathbf{Q}(a < \gamma \wedge a \in V(p_i) \wedge \forall u \in \mathbf{Q}(a < u < \gamma \rightarrow u \in V(p_j)))$.

For $\psi \in F$ of form $U(p_i, p_j)$ or $\sim U(p_i, p_j)$ the definition of imposition is similar. Let $I(\phi, V, \gamma)$ denote the set of all $\psi \in F$ imposed on γ by V.

V is called *ϕ-good* if it is ϕ-complete and for each $\gamma \in \mathbf{R} - \mathbf{Q}$ the set $\Phi \cup I(\phi, V, \gamma)$ is truth-functionally consistent. ϕ is *well-satisfiable* if it is satisfiable in $\mathbf{Q}$ under a ϕ-good valuation.

We may suppose we have fixed in advance a method of associating to any finite truth-functionally consistent set K of sentences of the first and second species a *maximal* truth-functionally consistent extension K^+. Let K^* denote the set of atoms that both appear in K and belong to K^+. The association can be so arranged that K^* will be, in an appropriate sense, a primitive recursive function of K. If V is a ϕ-good valuation in $\mathbf{Q}$, we define its *canonical extension* to a valuation W in $\mathbf{R}$ by setting:

$$W(p_i) = V(p_i) \cup \{\gamma \in \mathbf{R} - \mathbf{Q}: p_i \in (\Phi \cup I(\phi, V, \gamma))^*\} \ .$$

To clarify these definitions we note the following: Suppose that ϕ is satisfied in $\mathbf{Q}$ under the ϕ-complete valuation V. Suppose that Φ implies $p \longleftrightarrow S(q,r)$ and that V imposes p on γ. Then (1) above will hold for $\psi = S(q,r)$, but in general this does *not* imply that V will impose ψ on γ according to definition (3) above; that only follows if we assume in addition that V is ϕ-good. Suppose that Φ implies $s \longleftrightarrow (t \vee u)$ and that V imposes s on γ. Then even if we assume that V is ϕ-good this in general does *not* imply that V will impose either t or u on γ. However, our definition of the canonical extension W guarantees that we will have either $\gamma \in W(t)$ or $\gamma \in W(u)$.

1.7 Lemma *The canonical extension of a ϕ-good valuation V in $\mathbf{Q}$ is an $N(\phi)$-rudimentary valuation W in $\mathbf{R}$.*

Proof: With notation as above, conditions (1)-(3) each have the form "$\exists x \in \mathbf{Q} \forall y \in \mathbf{Q} \ldots$" and hence by techniques used repeatedly in [11] define F_σ sets. Hence for each pertinent K, the set $\{\gamma \in \mathbf{R} - \mathbf{Q}: I(\phi, V, \gamma) = K\}$ is (the restriction to the irrationals of) a finite intersection of sets each of which is an F_σ or the complement of one. This makes it a difference of two F_σ sets. Now

$W(p_i)$ has the form:

$$V(p_i) \cup \bigcup_{\text{certain } K} \{\gamma \in \mathbf{R} - \mathbf{Q}: I(\phi, V, \gamma) = K\}$$

(the union being over those K with $p_i \in K^*$). $V(p_i)$, like any countable set, is an F_σ. The number of pertinent K is properly bounded by $f(n)$, where $n = N(\phi)$. Hence $W(p_i)$ is n-rudimentary, as required.

1.8 Lemma *Let ϕ be a special formula. If ϕ is satisfiable in* $\mathbf{R}$ *under some valuation W, then ϕ is satisfiable in* $\mathbf{Q}$ *under some ϕ-good valuation V.*

Proof: Assume the hypotheses of the lemma. Let F be as in Definition 1.6. For each $\psi \in F$ and each open interval with rational bounds $]a, b[$ in $\mathbf{R}$ containing a point of $W(\psi)$, choose one such point, and let the set of points so chosen be E_0. Let E_1 be the set of bounds α, β of components $]\alpha, \beta[$ of the interiors of the sets $W(\psi)$ for $\psi \in F$. $E_0 \cup E_1$ being countable and dense in $\mathbf{R}$ (between any two points of $\mathbf{R}$ lies a point of this set), there is an order-isormorphism of $\mathbf{R}$ carrying this set onto $\mathbf{Q}$. So we may as well assume $E_0 \cup E_1 = \mathbf{Q}$ to begin with. This assumption more or less immediately implies the following for open intervals $]\alpha, \beta[$ in $\mathbf{R}$ and $\psi \in F$:

1.9 Claim If $]\alpha, \beta[\cap W(\psi) \neq \phi$, then $]\alpha, \beta[\cap W(\psi) \cap \mathbf{Q} \neq \phi$. And if $]\alpha, \beta[\cap \mathbf{Q} \subseteq W(\psi)$, then $]\alpha, \beta[\subseteq W(\psi)$.

1.10 Claim If $]\alpha, \gamma[\subseteq W(\psi)$, then there exist $a, c \in \mathbf{Q}$ with $]\alpha, \gamma[\subseteq]a, c[\subseteq W(\psi)$.

Let now V be the restriction of W to $\mathbf{Q}$. We must prove: (a) that V is ϕ-complete, (b) that V is ϕ-good, (c) that ϕ is satisfied in $\mathbf{Q}$ under V. Now (a) is more or less immediate from Claim 1.10. As for (b) and (c), it will suffice to establish:

1.11 Claim If $\psi \in F$ and $\gamma \in \mathbf{R} - \mathbf{Q}$ and V imposes ψ on γ, then $\gamma \in W(\psi)$.

1.12 Claim If $\psi \in F$ and $c \in \mathbf{Q}$ and $c \in W(\psi)$, then $c \in V(\psi)$.

As they are similar we treat only the former. Even it is more or less immediate from Claim 1.9, except for the case of ψ of form $\sim S(p_j, p_k)$ (or, similarly, $\sim U(p_j, p_k)$). In this case we attack the contrapositive, assuming that $\gamma \in W(S(p_j, p_k))$, that is:

(4) $\exists \alpha(\alpha < \gamma \wedge \alpha \in W(p_j) \wedge \forall \beta(\alpha < \beta < \gamma \to \beta \in W(p_k)))$

in order to prove that V imposes $S(p_j, p_k)$ on γ, that is:

(5) $\exists a \in \mathbf{Q}(a < \gamma \wedge a \in V(p_j) \wedge \forall b \in \mathbf{Q}(a < b < \gamma \to b \in V(p_k)))$.

Fixing α as in (4) and assuming it to be irrational, Claim 1.10 gives us a rational $a_0 < \alpha$ with $]a_0, \gamma[\subseteq W(p_k)$. Fixing $a_1 \in \mathbf{Q}$ with $\alpha < a_1 < \gamma$, since $\alpha \in]a_Q, a_1[\cap W(p_j)$, Claim 1.9 gives us a rational a belonging to the same set. This will do for (5).

1.13 Lemma *Let ϕ be a special formula. If ϕ is satisfiable in* $\mathbf{Q}$ *under a ϕ-good valuation V, then ϕ is satisfiable in* $\mathbf{R}$ *under the canonical extension W of V.*

Proof: Assume the hypotheses of the lemma. Let $\phi = \Phi^{\#}$, $N(\phi) = n$. What we must show is that if $\gamma \in \mathbf{R}$ and θ is a conjunct of $p_n \wedge \bigwedge\Phi$, then $\gamma \in W(\theta)$. We treat the case where γ is irrational, the rational case being similar. Let $J = (\Phi \cup I(\phi, V, \gamma))^+$. The facts needed from the construction are summed up by:

(6) $\theta \in J$
(7) $\psi \in J$ whenever V imposes ψ on γ
(8) $p_i \in J$ iff $\gamma \in W(p_i)$
(9) $\sim\sigma \in J$ iff $\sigma \notin J$
(10) $(\sigma \wedge \tau) \in J$ iff $\sigma \in J$ and $\tau \in J$.

(Here (9) and (10) hold for *all* formulas by maximal truth-functional consistency.) Using (6)–(10) one can more or less immediately show $\gamma \in W(\theta)$ except in the case where θ is of form $p_i \longleftrightarrow S(p_j, p_k)$ (or, similarly, $p_i \longleftrightarrow U(p_j, p_k)$). Even in this case, (6)–(10) reduce the proof that $\gamma \in W(\theta)$ to the two claims below:

1.14 Claim If V imposes $S(p_j, p_k)$ on γ, then $\gamma \in W(S(p_j, p_k))$.

This says that (5) above implies (4) above. And indeed the a of (5) can serve as the α of (4). To see this, just note that if $a < \beta < \gamma$, then by (5) (and definition (1)) V imposes p_k on β, whence, by (7) and (8), $\beta \in W(p_k)$.

1.15 Claim If V imposes $\sim S(p_j, p_k)$ on γ, then $\gamma \in W(\sim S(p_j, p_k))$.

The antecedent and consequent respectively say:

(11) $\forall a \in \mathbf{Q}(a < \gamma \wedge a \in V(p_j) \rightarrow \exists b \in \mathbf{Q}(a < b < \gamma \wedge b \notin V(p_k)))$
(12) $\forall \alpha(\alpha < \gamma \wedge \alpha \in W(p_j) \rightarrow \exists \beta(\alpha < \beta < \gamma \wedge \beta \notin W(p_k)))$.

In case V does *not* impose $\sim p_j$ on γ, then given α for which the antecedent of (12) holds, definition (1) provides an $a \in \mathbf{Q}$ with $\alpha < a < \gamma$ and $a \in V(p_j)$. Then (11) provides a b which can serve as the β of (12).

In case V *does* impose $\sim p_j$ on γ, then taking a as in definition (1) and applying (11) to obtain a b, we have:

$$b \in \mathbf{Q} \wedge b < \gamma \wedge b \notin V(p_k) \wedge \forall u \in \mathbf{Q}(b < u < \gamma \rightarrow u \notin V(p_j)).$$

This in effect says that V imposes $S(\sim p_k, \sim p_j)$ on γ and, arguing as in the proof of Claim 1.14, we see that $\gamma \in W(S(\sim p_k, \sim p_j)$ and $\gamma \notin W(S(p_j, p_k)))$.

Lemmas 1.7, 1.8, and 1.13 above show that for special sentences satisfiability in $\mathbf{R}$, well-satisfiability, and rudimentary satisfiability coincide. We will see in the next section that decidability for well-satisfiability and equally for rudimentary satisfiability follows directly from the main theorem of [14], thus completing the proof of Theorem 1.3. Let us make explicit a bonus implicit in this proof: A *Borel* set is one obtainable from open sets by iterated application of complementation and countable union, e.g., rudimentary sets are Borel. *Borel-satisfiability* is satisfiability in $\mathbf{R}$ under a valuation V for which $V(p_i)$ is always Borel. We have:

1.16 Corollary *Any sentence of temporal logic satisfiable in* $\mathbf{R}$ *is Borel-satisfiable.*

2 Continuous orders and decidability again

2.1 Universal monadic theories *First-order* logic, the logic of the textbooks, allows quantification only over individual elements of the universe of discourse. Quantification over subsets of the universe as well is allowed by *monadic* logic, a fragment of second-order logic. We follow the terminology and notation (e.g., $\models$ for modeling, $\binom{y}{x}$ for substitution) of [1], which is recommended for developing intuition about the scope and limits of first-order logic. To illustrate, let $L^{1\text{st}}$ and L^{mono} be respectively the sets of sentences of first-order and of monadic logic involving a single nonlogical symbol, the binary predicate $<$. Then the usual axioms for linear order (irreflexivity, transitivity, trichotomy) belong to $L^{1\text{st}}$:

$$\forall x \sim(x < x)$$
$$\forall x \forall y \forall z(x < y \wedge y < z \rightarrow x < z)$$
$$\forall x \forall y(x < y \vee x = y \vee y < x),$$

as do the axioms for density, discreteness, etc. (which we leave to the reader). By contrast, the axiom for *well-ordering* only belongs to L^{mono}:

$$\forall X(\exists x(x \in X) \rightarrow \exists x(x \in X \wedge \sim\exists y(y \in X \wedge y < x))) .$$

So, too, for the *inf* and *sup* axioms for completeness given in the preceding section.

A monadic sentence is *universal* if it consists of zero, one, or more universally quantified set-variables followed by a formula without further bound set-variables. For example, the well-ordering and completeness axioms are universal. The first-order (respectively, universal monadic) (respectively, (full) monadic) *theory* of a class $\mathfrak{K}$ of linear orders is the set of all first-order (respectively, universal monadic) (respectively, monadic) sentences that hold in all $\mathfrak{T} \in \mathfrak{K}$. To connect this subject with temporal logic we have:

2.2 Reduction To each sentence ϕ of temporal logic we will associate a formula ϕ^+ of monadic logic. If the atoms of ϕ are $p_0, p_1, \ldots, p_n$, then ϕ^+ will have the free individual variable x and free set-variables $X_0, X_1, \ldots, X_n$, and no bound set-variables:

$$\begin{aligned}
p_i^+ &= x \in X_i \\
(\sim\phi)^+ &= \sim(\phi^+) \\
(\phi \wedge \psi)^+ &= (\phi^+ \wedge \psi^+) \\
(S(\phi, \psi))^+ &= \exists y\left(y < x \wedge \phi^+\binom{y}{x} \wedge \forall z\left(y < z < x \rightarrow \psi^+\binom{z}{x}\right)\right) \\
(U(\phi, \psi))^+ &= \exists y\left(x < y \wedge \phi^+\binom{y}{x} \wedge \forall z\left(x < z < y \rightarrow \psi^+\binom{z}{x}\right)\right).
\end{aligned}$$

Here the variables y, z substituted for x are the alphabetically first ones that have not already been used. To each sentence ϕ of temporal logic we will associate a universal monadic sentence ϕ^*, namely the universal closure $\forall X_0 \forall X_1 \ldots \forall X_n \forall x \phi^+$ of ϕ^+. Comparison of the semantics for temporal logic (Subsection 1.2 above) with that for first- and higher-order logic (as in [1]) discloses:

2.3 Proposition *Let $\phi(p_0, \ldots, p_n)$ be a sentence of temporal logic,*

$\phi^+(x, X_0 \ldots, X_n)$ and ϕ^* its transforms. Let $\mathfrak{T} = (T, <_T)$ be a linear order, V a valuation in $\mathfrak{T}$, $a \in T$, $A_i = V(p_i) \subseteq T$. Then:

(a) $a \in V(\phi)$ *iff* $\mathfrak{T} \models \phi^+[a, A_0, \ldots, A_n]$
(b) ϕ *is valid in* $\mathfrak{T}$ *iff* $\mathfrak{T} \models \phi^*$.

Since the transforms are effective we have:

2.4 Corollary *For any class $\mathfrak{K}$ of linear orders, the decision problem for the temporal theory of $\mathfrak{K}$ is reducible to the decision problem for the universal monadic theory of $\mathfrak{K}$.*

There exists a partial converse to Proposition 2.3 and Corollary 2.4, but unlike them it is no trivial consequence of the definitions. Its complicated original proof [9] has recently been simplified [4].

2.5 Theorem (J. W. A. Kamp) *Let $\sigma(x, X_0, \ldots, X_n)$ be a formula of monadic logic with the free variables shown and no bound set-variables. Then there exists a sentence $\phi(p_0, \ldots, p_n)$ of temporal logic such that σ is equivalent to ϕ^+ over all complete linear orders. Hence for any class $\mathfrak{K}$ of complete linear orders, the decision problem for the universal monadic theory of $\mathfrak{K}$ is reducible to the decision problem for the temporal theory of $\mathfrak{K}$.*

The main theorem of [14] has as direct corollaries many previously known and many new decidability results; see the survey [15]. For our purposes, what is most important is:

2.6 Theorem (M. O. Rabin) *The following are decidable:*

(a) *The monadic theory of* $\mathbf{Q}$
(b) *The monadic theory of* $\mathbf{R}$ *with set variables restricted to range only over F_σ sets.*

Some remarks are in order:

(i) The full monadic theory of $\mathbf{R}$ is known to be undecidable. Indeed, allowing just one quantification over arbitrary sets of reals produces undecidability even if all other set-quantifiers are restricted to range only over sets of rationals. A new proof of this negative result will be found in [8], superseding the old proof of [16], which used the Continuum Hypothesis.

(ii) By the transform method (Proposition 2.3, Corollary 2.4), parts (a) and (b) of Theorem 2.6 yield, respectively, the decidability of the set of well-satisfiable and rudimentarily satisfiable sentences of temporal logic, as required for Theorem 1.3.

(iii) Theorem 2.6(a) has as immediate corollaries decidability for the monadic theories of $\mathbf{Q}^+$, $\mathbf{Q}^-$, $\mathbf{Z}$, etc. Less immediate corollaries are obtainable by combining Theorem 2.6(a) with *Cantor's Theorem* (to the effect that any countable linear order is isomorphic to a suborder of $\mathbf{Q}$) and the *Löwenheim-Skolem Theorem* (which implies that when a universal monadic sentence fails in a linear order, then it fails in some countable suborder). Some corollaries are summed up in Corollary 2.7 below. As noted there (and in [15]), many such corollaries were previously known (e.g., from [2] or [6]). A subclass $\mathfrak{K}'$ of a class $\mathfrak{K}$ of linear orders is *elementary* relative to $\mathfrak{K}$ (or when $\mathfrak{K}$ = all linear orders, simply *elementary*) if $\mathfrak{K}'$ is the class of all $\mathfrak{T} \in \mathfrak{K}$ in which some given first-order sentence holds. For example, the class of dense orders, discrete orders, and of orders with (without) minimum (maximum) elements are elementary.

(iv) Shelah has recently announced the analogue of Theorem 2.6(b) with Borel in place of F_σ.

2.7 Corollary *The universal monadic theories of the following classes of linear orders are decidable:*

(a) *(Gurevich) arbitrary linear orders*
(b) *(Gurevich) any elementary class of linear orders*
(c) *(Rabin) well-orders*
(d) *(Büchi) the integral order* $\mathbf{Z}$.

2.8 Definitions A linear order will be called *unbounded* if it has neither a minimum nor a maximum; *continuous* if it is unbounded, dense, and complete; and *separable* if it has a countable dense subset. Any separable continuous order is isomorphic to $\mathbf{R}$. A typical inseparable continuous order is $\mathfrak{S} = (S, <_S)$, the horizontal unit strip in the plane with the lexicographic order:

$$S = \{(\alpha, \beta) \in \mathbf{R}^2 \colon 0 \leqslant \beta \leqslant 1\}$$
$$(\alpha, \beta) <_S (\alpha', \beta') \text{ iff } (\alpha < \alpha' \vee (\alpha = \alpha' \wedge \beta < \beta')) \ .$$

The following does *not* follow directly from results in [14]:

2.9 Theorem *The universal monadic theories of the following classes of complete linear orders are decidable:*

(a) *arbitrary complete linear orders*
(b) *continuous orders*
(c) *separable complete linear orders*
(d) *the real order* $\mathbf{R}$.

Note that (a) and (c) imply the decidability of any relatively elementary class of continuous orders. This gives (b) and (d). Alternatively, (d) can be derived in a very roundabout fashion by combining Theorems 1.3 and 2.5. We defer the proof of (a) and (c) until we have shown that the universal monadic theories in (b) and (d) differ from each other.

2.10 Counterexample Let $\mathfrak{T} = (T, <_T)$ be a linear order. *Degenerate* will mean: having only one element. A *convex* set $U \subseteq T$ in $\mathfrak{T}$ is one for which $a <_T b <_T c$ and $a, c \in U$ always imply $b \in U$. An *antichain* in $\mathfrak{T}$ is a collection of pairwise disjoint nondegenerate convex sets. A *congruence* on $\mathfrak{T}$ is an equivalence relation $\equiv$ whose equivalence classes are convex. The quotient $\mathfrak{T}/\equiv$ is then the linear order $(T', <')$ obtained as follows, where $\bar{a}$ is the equivalence class of a:

$$T' = \{\bar{a} \colon a \in T\}$$
$$\bar{a} <' \bar{b} \text{ iff } a \not\equiv b \wedge a <_T b \ .$$

Let $\mathfrak{T}(x, y, X)$ be the formula:

$$(x = y) \vee (x < y \wedge \forall z(x \leqslant z \leqslant y \rightarrow (z \in X \longleftrightarrow x \in X))) \\ \vee (y < x \wedge \forall z(y \leqslant z \leqslant x \rightarrow (z \in X \longleftrightarrow x \in X))) \ .$$

Note that if $A \subseteq T$ and we define $a \equiv_A b$ to hold iff $\mathfrak{T} \models \tau(a, b, A)$, then $\equiv_A$ is a congruence. Let $\sigma_0(X)$ be the conjunction of:

(1) $\forall x \exists y(x \neq y \wedge \tau(x, y, X))$
(2) $\forall x \exists y \sim \tau(x, y, X)$
(3) $\forall x \forall y(x < y \wedge \sim\tau(x, y, X) \rightarrow \exists z(x < z < y \wedge \sim\tau(x, z, X) \wedge \sim\tau(z, y, X)))$.

Note that if $\mathfrak{T} \models \sigma_0[A]$, and $\equiv_A$ is as above, then (1) tells us that each $\equiv_A$-congruence class is nondegenerate, (2) tells us that the quotient $\mathfrak{T}/{\equiv_A}$ is nondegenerate, and (3) tells us that that quotient is dense.

Let $\sigma_{u,v}(X)$ be the result of replacing each quantification $\forall t$ or $\exists t$ in $\sigma_0(X)$ by $\forall t(u < t < v \rightarrow \ldots)$ or $\exists t(u < t < v \wedge \ldots)$. Let $\rho_0(X) = \exists u \exists v(\exists t(u < t < v) \wedge \sigma_{u,v})$. Let $\sigma = \forall X \sim \sigma_0(X)$, $\rho = \forall X \sim \rho_0(X)$. So ρ says that σ holds relative to any nonempty open interval.

2.11 Proposition *Let $\mathfrak{T} = (T, <_T)$ be a linear order. Then $\mathfrak{T} \models \sim\sigma$ iff there exists a congruence $\equiv$ on $\mathfrak{T}$ such that:*

(a) *Each $\equiv$-congruence class $\bar{a}$ is nondegenerate*
(b) *The quotient $\mathfrak{T}/\equiv$ is nondegenerate*
(c) *The quotient $\mathfrak{T}/\equiv$ is densely ordered.*

Proof: We have already seen one direction. For the other, suppose that the congruence $\equiv$ satisfies (a), (b), (c). We claim it is of form $\equiv_A$ for some $A \subseteq T$ with $\mathfrak{T} \models \sigma_0[A]$. Indeed, if U is dense in and has dense complement in the quotient $\mathfrak{T}/\equiv$, then $A = \{a: \bar{a} \in U\}$ will do.

2.12 Proposition *The following hold for ρ as well as for σ:*

(a) *If $\mathfrak{T}$ is a separable complete linear order, then $\mathfrak{T} \models \sigma$. Thus σ belongs to the universal monadic theory of* **R**.
(b) $\mathfrak{S} \models \sim\sigma$. *Thus σ does not belong to the universal monadic theory of continuous orders.*

Proof: (a) If $\mathfrak{T}$ is a complete linear order and $\equiv$ a congruence, then the quotient $\mathfrak{T}/\equiv$ is also complete. If (b) and (c) of Proposition 2.11 are assumed, then the quotient is uncountable. If (a) of Proposition 2.11 is also assumed, then the set of $\equiv$-equivalence classes forms an uncountable antichain in $\mathfrak{T}$. In that case, $\mathfrak{T}$ cannot be separable.

(b) We have $\mathfrak{S} \models \sigma_0[A]$ where $A = \{(\alpha, \beta) \in S$: α is rational$\}$. In that case $(\alpha, \beta) \equiv_A (\alpha', \beta')$ iff $\alpha = \alpha'$.

2.13 Proof of Theorem 2.9(a) We refer the reader to the Appendix of [6]. An *n-chain* is a structure $\mathfrak{A} = (A, <_A, P_0^A, \ldots, P_n^A)$ consisting of a linear order with $n+1$ designated subsets. To prove the decidability of the universal monadic theory of complete linear orders, it suffices to prove for all n uniformly in n the decidability of the first-order theory of n-chains. Fix n. [7] is concerned with *weak second-order* logic (monadic logic with set-variables restricted to range only over finite sets). In Section 14 decidability is proved for: (i) the weak second-order theory of complete linear orders. In Section 15 decidability is deduced for: (ii) the weak second-order theory of complete n-chains. This proves Theorem 2.9(a). Actually, the method of (i) applies directly to the first-order theory of complete n-chains, and can be adapted to separable complete n-chains.

2.14 Proof of Theorem 2.9(c) We outline this adaptation: Let $\mathfrak{T} = (T, <_T)$ be a linear order, $(\mathfrak{A}_t\colon t \in T)$ a family of n-chains $\mathfrak{A}_t = (A_t, <_t, P_i^t)$ indexed by $\mathfrak{T}$. Its *sum* is the n-chain $\mathfrak{B} = (B, <_B, P_i^B)$ given by:

$$\begin{aligned} B &= \{(t, a)\colon t \in T \wedge a \in A_t\} \\ (t, a) <_B (t', a') &\text{ iff } (t <_T t' \vee (t = t' \wedge a <_t a')) \\ P_i^B &= \{(t, a)\colon t \in T \wedge a \in P_i^t\}. \end{aligned}$$

When $\mathfrak{T}$ is the two-element order $\{0, 1\}$ we simply write $\mathfrak{A}_0 + \mathfrak{A}_1$ for the sum. When all $\mathfrak{A}_t$ are the same structure $\mathfrak{A}$, we write $\mathfrak{A} \cdot \mathfrak{T}$ for the sum. The family $(\mathfrak{A}_t\colon t \in T)$ and its sum are called $\mathfrak{F}$*-dense* (respectively, an $\mathfrak{F}$*-shuffling*) if $\mathfrak{T}$ is nondegenerate and dense (respectively, $\mathfrak{T} = \mathbf{Q}$) and $\mathfrak{F}$ is a finite set of n-chains with:

$$\forall t \in T(\mathfrak{A}_t \in \mathfrak{F})$$
$$\forall \mathfrak{A} \in \mathfrak{F}(\{t \in T\colon \mathfrak{A}_t = \mathfrak{A}\} \text{ is dense in } \mathfrak{T}).$$

We claim any two $\mathfrak{F}$-dense sums have the same first-order theory, and any two $\mathfrak{F}$-shufflings are isomorphic. The proof uses the *back-and-forth* method (cf. [7], Lemma 14.1).

Let $\mathfrak{M}$ be the smallest class of n-chains containing the degenerate ones and closed under isomorphism and the following conditions:

(1) If $\mathfrak{A}, \mathfrak{B} \in \mathfrak{M}$ and $\mathfrak{A}$ has a maximum or $\mathfrak{B}$ has a minimum, then $\mathfrak{A} + \mathfrak{B} \in \mathfrak{M}$
(2) If $\mathfrak{A} \in \mathfrak{M}$ has either a minimum or a maximum, and $\mathfrak{T} = \mathbf{Z}^+$ or $\mathbf{Z}^-$, then $\mathfrak{A} \cdot \mathfrak{T} \in \mathfrak{M}$
(3) If $\mathfrak{F} \subseteq \mathfrak{M}$ is finite, and each $\mathfrak{A} \in \mathfrak{M}$ has both a minimum and a maximum, *and some* $\mathfrak{A} \in \mathfrak{M}$ *is degenerate,* and $\mathfrak{B}$ is an $\mathfrak{F}$-shuffling, then $\mathfrak{B} \in \mathfrak{M}$.

Let $\mathfrak{K}_0$ be the class of all separable complete n-chains, $\mathfrak{K}_1$ the class of all complete n-chains in which ρ holds. Write "$\theta(\ )$" for "first-order theory of". Since $\mathfrak{K}_0 \subseteq \mathfrak{K}_1$ by Proposition 2.12(a), $\theta(\mathfrak{K}_1) \subseteq \theta(\mathfrak{K}_0)$. To prove Theorem 2.9(c) it will thus suffice to show:

(4) $\theta(\mathfrak{M})$ is decidable
(5) $\theta(\mathfrak{K}_0) \subseteq \theta(\mathfrak{M})$
(6) $\theta(\mathfrak{M}) \subseteq \theta(\mathfrak{K}_1)$.

The proof of (4) uses an analysis of *types* realized in elements of $\mathfrak{M}$, borrowed (like so much of this proof) from [12] (cf. [7], Theorem 14.2).

To prove (5) it suffices to show that for every $\mathfrak{B} \in \mathfrak{M}$ there exists a $\mathfrak{B}' \in \mathfrak{K}_0$ with $\theta(\mathfrak{B}) = \theta(\mathfrak{B}')$. The proof is inductive like the definition of $\mathfrak{M}$ (cf. [7], Lemma 14.2). The key step occurs when $\mathfrak{B}$ is an $\mathfrak{F}$-shuffling for some $\mathfrak{F}$ satisfying the hypotheses of (3) above. As induction hypothesis we assume that to each $\mathfrak{A} \in \mathfrak{F}$ has been associated an $\mathfrak{A}' \in \mathfrak{K}_0$ with $\theta(\mathfrak{A}) = \theta(\mathfrak{A}')$. Let $\mathfrak{F}' =$

$\{\mathfrak{A}' : \mathfrak{A} \in \mathfrak{F}\}$; it contains a degenerate element. Let $(\mathfrak{A}_t : t \in \mathbf{R})$ be an $\mathfrak{F}$-dense family indexed by $\mathbf{R}, \mathfrak{B}'$ its sum. Then $\mathfrak{B}'$ is complete, *and by taking* $\mathfrak{A}_t$ *to be degenerate except when* $t \in \mathbf{Q}$ *we can guarantee that it will be separable.* That $\theta(\mathfrak{B}) = \theta(\mathfrak{B}')$ is again proved by the back-and-forth method.

To prove (6): Let $\theta_k(\mathfrak{K})$ be the set of sentences in $\theta(\mathfrak{K})$ having no nesting of quantifiers of depth $> k$. It suffices to prove for all k that for every $\mathfrak{A} \in \mathfrak{K}_1$ there exists a $\mathfrak{B} \in \mathfrak{M}$ with $\theta_k(\mathfrak{A}) = \theta_k(\mathfrak{B})$. Fix k. Call $\mathfrak{A} \in \mathfrak{K}_1$ *good* if such a $\mathfrak{B} \in \mathfrak{M}$ exists; call $\mathfrak{A}$ *quasi-good* if for every nonempty half-open interval $[a, b[$ the restriction $\mathfrak{A} | [a, b[$ of $\mathfrak{A}$ to that interval is good. We claim quasi-goodness implies goodness. The proof uses *Ramsey's Theorem* from combinatorics (cf. [7], Lemma 14.3). To complete the proof it suffices to derive a contradiction from the assumption that some $\mathfrak{A} = (A, <_A, P_i^A) \in \mathfrak{K}_1$ is bad. Define a relation on A by setting:

$$a \equiv b \text{ iff } (a = b) \vee (a < b \wedge \mathfrak{A} | [a, b[\text{ is quasi-good}) \\ \vee (b < a \wedge \mathfrak{A} | [b, a[\text{ is quasi-good}).$$

We claim $\equiv$ is a congruence; in fact, each equivalence class $\bar{a}$ is a closed interval, and the restriction $\mathfrak{A} | \bar{a}$ quasi-good and hence good; moreover the quotient $(A, <_A)/\equiv$ is nondegenerate and dense (cf. [7], Proof of Lemma 14.4). *Since* $\mathfrak{A} \models \rho$, *any nonempty open interval I in the quotient must contain a degenerate equivalence class.* For each such I, quasi-goodness provides a finite $\mathfrak{F}_I \subseteq \mathfrak{M}$ such that for each $\bar{a} \in I$ we have $\theta_k(\mathfrak{A} | \bar{a}) = \theta_k(\mathfrak{B})$ for some $\mathfrak{B} \in \mathfrak{F}_I$. Take I for which $\mathfrak{F}_I$ has the least possible cardinality. We claim a contradiction can be derived by comparing any $\mathfrak{F}_I$-shuffling, which belongs to $\mathfrak{M}$ by (3), with the restriction of $\mathfrak{A}$ to $\{a : \bar{a} \in I\}$ (again cf. [7]).

This concludes our sketch of the proof. The italicized passages are the main points of difference from [7]. Let us make explicit a bonus result:

2.15 Corollary *Let $\mathfrak{T}$ be a continuous order in which ρ holds. Then $\mathfrak{T}$ is a model of the universal monadic theory of* $\mathbf{R}$.

2.16 Examples Here are some inseparable continuous orders in which ρ holds:

The Long Line: This results from the countable ordinals when each point is replaced by a copy of the half-open unit interval $[0, 1[$.

Suslin Lines: The hypothesis that they exist has the same status as the Continuum Hypothesis; see [10].

NOTE

1. Roughly, the material in Section 1 is due to Burgess, that in Section 2 to Gurevich.

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