EXPANDED THEORY OF ORDERED ABELIAN GROUPS

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The theory of ordered Abelian groups with quantification over convex subgroups is studied. An elimination of the elementary quantifiers is presented and a primitive recursive decision procedure for this theory is constructed.

0. Introduction

For the sake of brevity the terms "group", "o-group" and "chain" will be used for "Abelian group", "linearly ordered Abelian group" and ""linearly ordered set" respectively.

Algebraically speaking an o-group $G$ is a group and a chain, and for every $x, y, z \in G$, $x < y$ implies $x + z < y + z$.

Let $G$ be an o-group. It is easy to check that $G$ is torsion free, has neither minimal nor maximal element and is either discretely or densely ordered. A subset $X \subseteq G$ is called a convex subgroup of $G$ iff $X$ is a subgroup and $X$ is convex (the latter means that for every $x_1, x_2 \in X$ and each $y \in G$, if $x_1 < y < x_2$ then $y \in X$). Convex subgroups play a fundamental role in non-formalized theory of o-groups (see [3]). It is easy to check that convex subgroups of $G$ are linearly ordered by inclusion.

Let us review the history.

$G$ is called Archimedean iff for every positive $x, y \in G$ there exists a natural $n$ such that $x < ny$. $G$ is Archimedean iff $[0]$ and $G$ are the only convex subgroups of $G$. Archimedean o-group is embeddable into the naturally ordered additive group of reals (see Holder’s Theorem in [3]). The elementary theory of Archimedean o-groups was studied by Robinson and Zakon (see [15]). Here are their main results. $G$ is called $n$-regular iff for every $x_1, \ldots, x_n \in G$ there exists $y \in G$ such that $x_1 < \ldots < x_n$ implies $x_1 \leq ny \leq x_n$. $G$ is called regular iff it is $n$-regular for each positive integer $n$. Each Archimedean o-group is regular. Each regular o-group is elementarily equivalent to some Archimedean o-group. Two discrete (respectively
dense) regular $o$-group are elementarily equivalent iff they are elementarily equivalent as groups.

The main result of Kargapolov's paper [10] is a classification of the $o$-groups of finite rank by their elementary properties (a torsion free group has a finite rank iff it is embeddable in a finite dimensional vector space over the rational field). According to [4], every two $o$-groups are universally equivalent.

In [5], all $o$-groups were classified by their elementary properties, and the elementary theory of $o$-groups was algorithmically reduced to the elementarily theory of chains. Together with [11] it gives a decision procedure for the elementary theory of $o$-groups. An elimination of quantifiers in the elementary theory of $o$-groups was presented in [6]. Together with [12] it gives a primitively recursive decision procedure for the elementary theory of $o$-groups. The part of [6] relating to $o$-groups was never published.

Here we study the theory of $o$-groups with quantification over convex subgroups. We eliminate the elementary quantifiers, and construct a primitively recursive decision procedure for this theory. The main results of the present paper were announced in [7]. An earlier version of a part of this paper may be found in the Soviet Institute of Scientific and Technical Information (Moscow), number 6708–73, [8] is the corresponding abstract.

Let us summarize the contents of the present paper.

Part 1 of the present paper is purely algebraic. The key notion here is the functor $F(s,x)$ (called the $s$-fundament of $x$).

In Part 2 we define the Expanded Theory of $o$-groups, and eliminate the elementary quantifiers. The Expanded Theory is the theory of $o$-groups with quantification over convex subgroup enriched by some definable predicates. The elimination of elementary quantifiers reduces the Expanded Theory to so-called Convex Subgroups Theory. The latter is an elementary theory of the chains of convex subgroups with some surplus one-place predicates (each $o$-group provides us with a model of the Convex Subgroups Theory).

In Part 3 we axiomatize the Convex Subgroups Theory in the elementary theory of complete chains with surplus one-place predicates in such a way that for each sentence $\alpha$ in the language of the Convex Subgroups Theory one can easily select a finite number of axioms deciding $\alpha$.

In the Appendix we prove that the weak monadic second order theory of complete chains with surplus one-place predicates is primitively recursive (this strengthens the result of [13], but was obtained simultaneously and independently). Together with the previous parts it gives a primitively recursive decision procedure for the Expanded Theory of $o$-groups.

Some words about possible generalizations. The theorem about elimination of elementary quantifiers can be easily generalized by enriching the part of the language concerning convex subgroups. Generalizations of the decidability result are restricted by undecidability results in theory of chains. For example allowing quantification over arbitrary subsets of convex subgroups leads to undecidable...
theory if the Continuum Hypothesis holds. This follows from undecidability of the monadic theory of the real line, see Shelah’s paper [16]. One of the possible generalizations is obtained by allowing quantification over finite subsets of convex subgroups. The elementary quantifiers can be eliminated, and the enriched theory remains primitively recursive. One can have some generalizations of the form: this specific theory of o-group is recursive modulo that specific theory of chains. About decision problem for lattice ordered Abelian groups see [9].

Some words about notation. It a group $H$ is the (internal) direct sum of its subgroups $H, \ i \in I$, we write $H = \Sigma\{H, \ i \in I\}$ In this case each $h \in H$ is equal to some finite sum $h_n + \cdots + h_n$ where $h_n \in H_n$, $\ n \in H_n$. The external direct sum of groups $H, \ i \in I$, is also denoted by $\Sigma\{H, \ i \in I\}$. The elements of the external direct sum are functions $f : I \to \bigcup\{H, \ i \in I\}$ such that $f(i) \in H_i$ and $\{i : f(i) \neq 0\}$ is finite. We can write $f = \Sigma f(i)$. Now let $I$ be a chain and $H, s(i) \in I$ be $o$-groups. By $L \Sigma\{H, \ i \in I\}$ we denote the lexicographic (or $\omega$-lexicographic) sum of $H, s$. It is the direct sum $\Sigma H, s$ ordered as follows $\Sigma h, s > 0 \iff \Sigma h, \neq 0$ and $h, s > 0$ where $j = \max\{i : h, \neq 0\}$. The lexicographic multiple $H^I$ of an o-group $H$ is $\Sigma\{H, \ i \in I\}$ and $H, s = H$.

"w l o g " is an abbreviation for "without loss of generality."

A I Kokorin persuaded me (after [5] was published) to continue to work on algorithmic problems for ordered groups (I have returned to o-groups after Cohen’s preprint [1] demonstrating potentialities of the method of elimination of quantifiers). Jonathan Levin corrected the English of an earlier version of this paper. The referee found some places which had to be corrected. I am grateful to all these people.

PART 1. ALGEBRA

1. Fundamental subgroups

Throughout this section $G$ is an o-group, $x, y, z \in G$ and $X, Y$ are convex subgroups of $G$. Here and below $p$ is a prime number.

Definition 1.1 ([5]) For an integer $s \neq 0$ we define $F(s, x) = \bigcup\{X \ : \ \forall y (x + sy \notin X)\}$, $F(p, x) = F(p', x)$.

$F(s, x)$ is called the $s$-fundamental of $x$.

Corollary 1.1. $F(s, x)$ is a convex subgroup or $\emptyset$, $F(s, x) = \emptyset \iff x \equiv 0 \pmod{s}$.

Corollary 1.2. Let $a, b \neq 0$ be integers. Then $F(ab, bx) = F(a, x)$. $F(a, x) \subseteq F(ab, x)$, $F(a, bx) \subseteq F(a, x)$. If $a \text{ and } b$ are relatively prime, then $F(a, bx) = F(a, x)$ and $F(ab, x) = F(a, x) \cup F(b, x)$. 
Corollary 1.3. Let \( a = p'b \) where \( b \not\equiv 0 \pmod{p} \). Then \( F(p, k, x) = F(p, k + t, ax) \)

Corollary 1.4. Let \( s = p'^i \quad p'^k \). Then \( F(s, x) = F(p, i, x) \cup F(p, k, x) \)

Corollary 1.5. \( F(s, x + y) \subseteq F(s, x) \cup F(s, y) \) and if \( F(s, x) \neq F(s, y) \), then \( F(s, x + y) = F(s, x) \cup F(s, y) \)

A proof is easy. For example, we will show that \( F(ab, x) \subseteq F(a, x) \cup F(b, x) \) if \( a \) and \( b \) are relatively prime. Suppose it is not true. Then \( F(ab, x) \) contains some \( x + ay \) and \( x + bz \). Because \( a \) and \( b \) are relatively prime, \( ca + db = \) for some integers \( c \) and \( d \), and so

\[
ca(x + bz) + db(x + ay) = x + ab(cz + dy) \in F(ab, x)
\]

However, this is impossible.

Definition 1.2. \( \Gamma_1(s, X) \) is the subgroup \( \{ x \mid F(s, x) \subseteq X \} \)

\( \Gamma_2(s, X) \) is the subgroup \( \{ x \mid F(s, x) \subseteq X \} \),

\( \Gamma(s, X) \) is the factor group \( \Gamma_2(s, X)/\Gamma_1(s, X) \),

\( \Gamma(p, k, X) = \Gamma(p^k, X) \) and the same for \( \Gamma_1 \) and \( \Gamma_2 \)

It is more precise to write \( \Gamma(s, X, G) \) instead of \( \Gamma(s, X) \) and the same for \( F, \Gamma_1, \Gamma_2 \).

This more precise notation is used in the following two lemmas.

Lemma 1.1. Let \( \bar{a} \) denote the natural homomorphism \( G \rightarrow G/X \). Then \( \Gamma(s, X, G) \) is isomorphic to \( \Gamma(s, \bar{X}, \bar{G}) \).

Proof. For every \( Y \supseteq X \), \( F(s, x) \subseteq Y \) iff \( F(s, \bar{x}) \subseteq \bar{Y} \). Now it is easy to check that the correspondence \( x + \Gamma_1(s, X) \rightarrow \bar{x} + \Gamma_1(s, \bar{X}) \) is a required isomorphism.

Lemma 1.2. Let \( X \subseteq Y \). Then \( \Gamma(s, X, C) \) is isomorphic to \( \Gamma(s, X, Y) \).

Proof. The correspondence \( x + \Gamma_1(s, X, Y) \rightarrow x + \Gamma_1(s, X, G) \) is an isomorphism from \( \Gamma(s, X, Y) \) onto \( \Gamma(s, X, G) \).

A group \( \Gamma(s, X) \) satisfies the axiom \( \forall v (st = 0) \) and so it has a representation as a direct sum of cyclic groups.

Definition 1.3. \( p(s, k, X) \) is the cardinal number of cyclic direct summands of the order \( p^k \) in a representation of \( \Gamma(p, s, X) \) as a direct sum of cyclic groups.

Definition 1.4 (cf. [17]) Elements \( v_1, \ldots, v_n \) of a group \( H \) are independent (strongly independent) modulo \( p^k \) if, for every integer \( a_1, \ldots, a_n, \Sigma a_i v_i = 0 \) \( (\Sigma a_i v_i = 0 \pmod{p^k}) \) implies \( a_1 \equiv \ldots \equiv a_n \equiv 0 \pmod{p^k} \). A subset \( M \subseteq H \) is inde-
pendent (strongly independent) modulo $p^k$ if every finite subset of $M$ is so $p^i(p, k, H)$ ($p^j(p, k, H)$) is the power of a maximal independent (strongly independent) modulo $p^k$ subset $M \subseteq H$ such that every element of $M$ has the order $p^k$.

**Corollary 1.6.** Let $H = \Gamma(p, s, X)$ and $1 \leq k \leq s$. Then $p^i(p, k, \Gamma) = p(s, k, X)$ and $p^j(p, k, H) = \Sigma(p(s, t, X))$ $k \leq i \leq s$.

A proof is easy.

**Lemma 1.3.** Let $F(p, s, x) = \{0\}$ and $\bar{x} = x + \Gamma(p, s, \{0\}) \in \Gamma(p, s, \{0\})$.

1. The order of $\bar{x}$ is $p^k$ iff $F(p, s, p^k x) \subset F(p, s, p^k - 1 x) = \{0\}$.
2. If the order of $\bar{x}$ is $p^k$ then $x \equiv 0 \pmod{p^{k-1}}$.

A proof is easy.

**Theorem 1.1.** $p(s, k, X) = p(s + 1, k, X)$ for $k < s$ and $p(s, s, X) = p(s + 1, s, X) + p(s + 1, s + 1, X)$.

**Proof.** WLOG $X = \{0\}$ (see Lemma 1.1). To simplify notation we sometimes omit $p$ and $\{0\}$. Let a bar (respectively prime) denote the natural homomorphism from $\Gamma_2(s)$ onto $\Gamma(s)$ (resp. from $\Gamma_2(s + 1)$ onto $\Gamma(s + 1)$). Note that $x \in \Gamma(s)$ iff $px \in F_2(s + 1)$ and

(i) $x \in \Gamma_2(s)$ iff $px \in F_2(s + 1)$ and

(ii) if $x \in \Gamma_2(s)$, then $\bar{x}$ and $(px)'$ have the same order.

**Case** $k < s$. Let $U$ (resp. $V$) be the family of strongly independent modulo $p^k$ subsets of $\Gamma(s)$ (resp. $\Gamma(s + 1)$) consisting of elements of order $p^k$. $U$ and $V$ are partially ordered by inclusion. Let $\{x_i, i \in I\}$ be maximal in $U$. It is enough to check that $\{(px)_i, i \in I\}$ belongs to $V$ and is maximal there. First we check the strong independence. Suppose that $\Sigma_{\alpha} (px)_i = p^k u_i$ where $\Sigma_{\alpha} px_i = p^k u + p^{k+1} v$ for some $\alpha$. Then $\Sigma_{\alpha} p\bar{x}_i = p^k \bar{u}$ hence there exist $b_i$ such that $a_i \equiv p^k b_i$. By Lemma 1.3, $x_i = p^{-k} y_i$ for some $y_i$. So $p^k u = \Sigma p^k b_i p^k y_i + p^{k+1} v$ and $u = pw$ for some $w$. Then $\Sigma_{\alpha} px_i = p^k pw + p^{k+1} v$ and $\Sigma_{\alpha} \bar{x}_i = p^k \bar{w}$ which implies $a_i \equiv 0 \pmod{p^k}$ for every $i$. Now we check the maximality. Let $y'$ be of order $p^k$. By Lemma 1.3, $y = p^k z$ for some $z$. There exist $a_i$, $b$ and $u$ such that $b \equiv 0 \pmod{p^k}$ and $b \bar{z} = \Sigma_{\alpha} \bar{x}_i + p^k \bar{u}$, since $\{\bar{x}_i, i \in I\}$ is maximal. Then $b\bar{x} = \Sigma_{\alpha} \bar{x}_i + p^k \bar{u}$ (mod $p^k$), $by = \Sigma_{\alpha} px_i + p^k pu$ (mod $p^{k+1}$) and $by' = \Sigma_{\alpha} (px)_i + p^k (pu)$.

**Case** $k = s$. By Corollary 1.6 it is enough to check that $p^i(p, s, \Gamma(s)) = p^j(p, s, \Gamma(s + 1))$. Let $U$ (resp. $V$) be the family of independent modulo $p^k$ subsets of $\Gamma(s)$ (resp. $\Gamma(s + 1)$) consisting of elements of order $p^k$. Let $\{x_i, i \in I\}$ be maximal in $U$. We check that $\{(px)_i, i \in I\}$ belongs to $V$ and is maximal there. If $\Sigma_{\alpha} (px)_i = 0$ then $\Sigma_{\alpha} px_i = 0$ (mod $p^{k+1}$), $\Sigma_{\alpha} \bar{x}_i = 0$ (mod $p^k$), $\Sigma_{\alpha} \bar{x}_i = 0$, and $a_i \equiv 0 \pmod{p^k}$ for every $i$. The maximality is checked as above (but $u = 0$).
Definition 1.5. (i) \( A(x) = \bigcup \{X \mid x \not\in X\} \),
(ii) \( X = \bigcap \{Y \mid X \subseteq Y\} \) if \( X \neq G \) and \( X^+ = G \) if \( X = G \).

Corollary 1.7. \( A(x) \) is a convex subgroup or \( \emptyset \), \( A(x) = \emptyset \) iff \( x = 0 \)

Corollary 1.8. For any integer \( c \), \( A(cx) \subseteq A(x) \) If \( c \neq 0 \), then \( A(cx) = A(x) \)

Corollary 1.9. \( A(x + y) \subseteq A(x) \cup A(y) \) If \( A(x) \neq A(y) \), then \( A(x + y) = A(x) \cup A(y) \)

Corollary 1.10. \( Y \subseteq X^+ \) iff \( \exists x (X = A(x)) \)

A proof is easy

Note Cf the definition of \( A(x) \) and Definition 1.1

Definition 1.6. Let a bar denote the natural homomorphism \( G \to G/X \), \( R \in \{<, \leq, =, \geq, >\} \) and \( a \neq 0 \) be an integer

1. \( x \bar{R} 0 \pmod{X} \equiv \bar{x}R0 \),
2. \([x = 1 \pmod{X}] \equiv 0 < \bar{x} \) and \( \neg \exists y (0 < y < \bar{x}) \),
3. \( E(X) \equiv \exists x (x = 1 \pmod{X}) \),
4. \([x = a \pmod{X}] \equiv \exists y (y = 1 \pmod{X} \) and \( \bar{a} \bar{y} \),
5. \( xRy \pmod{X} \equiv \bar{x}R\bar{y} \)

Let \( a \) and \( b \) be integers and \( b > 0 \). Then

Corollary 1.11. \( xRy \pmod{X} = bxRhy \pmod{X} = (-by)R(-bx) \pmod{X} \)

Corollary 1.12. \( xRa \pmod{X} = bxRba \pmod{X} = (-ba)R(-bx) \pmod{X} \)

Theorem 1.2. Let \( X \subseteq Y \), \( k < s \) and

\[ \forall Z (X \subseteq Z \subseteq Y \implies \land \{p(s, t, Z) = 0 : t \leq s\}) \]

Then \( p(s, k, X) = 0 \)

Proof. By Lemmas 1.1 and 1.2 it can be assumed that \( X = \{0\} \) and \( Y = G \). Let a bar denote the natural homomorphism \( G \to G/p^s G \). Clearly every \( F(p, s, x) \subseteq \{0\} \). Let (reductio ad absurdum) \( p(s, k, \{0\}) > 0 \). Then there exists \( x \) such that \( \bar{x} \not\equiv t \pmod{p} \) and \( p^s \bar{x} = 0 \). The last means \( p^s x = p^s y \) for some \( y \). But \( F(p, s, y) \subseteq \{0\} \) and \( \bar{y} \in \bar{F}(p, s, \{0\}) \) and \( \bar{x} = p^{-s} \bar{y} \), which contradicts \( \bar{x} \not\equiv 0 \pmod{p} \).

Theorem 1.3. \( E(X) \) implies \( X \subseteq X^+ \) and \( p(s, s, X) = 1 \)

Proof. Let \( E(X) \) \( X \subseteq X^+ \) follows clearly from the definition of \( E \). In order to
prove that \( p(s, s X) = 1 \) it can be assumed that \( X = \{0\} \) and \( X = G \), see Lemmas 11 and 12. Then \( G \) is isomorphic to the naturally ordered additive group of natural numbers and \( G/p'G \) is the cyclic group of the order \( p' \).

**Definition 1.7.**

\[
D(p, s, k, x) = \{ x \equiv 0 \pmod{p'} \lor \exists y [F(p, s, x - p^ky) \subset F(p, s, y) = F(p, s, y)]
\]

**Corollary 1.13.** Let \( \emptyset \neq F(p, s, x) = X \) and a bar denote the natural homomorphism \( \GammaZ(p, s, X) \to \Gamma(p, s, X) \). Then \( D(p, s, k, \cdot) \) is equivalent to \( \bar{x} \equiv 0 \pmod{p^s} \).

**A proof is easy.**

**Lemma 1.4.** Let \( F(p, s, x) = \{0\} \). Then \( D(p, s, k, x) \equiv \exists y [F(p, s, y) = \{0\} \text{ and } x = p^ky + p'z] \)

**A proof is easy.**

**Lemma 1.5.** Let \( \emptyset \neq F(p, s, x) = X \) and \( \cdot \equiv p'd \) where \( d \equiv 0 \pmod{p} \). Then \( D(p, s, k, x) \equiv [F(p, k, x) \subset X \text{ and } D(p, s + \lambda, k, cx)] \)

**Proof.** According to Corollary 1.13 and Lemma 1.1 we can assume that \( d = 1 \) and \( X = \{0\} \). We also use Lemma 1.4.

\( 1 \) Let \( F(p, s, y) = \{0\} \) and \( x = p^ky + p'z \). Then \( F(p^k, x) = \emptyset \) and \( F(p, s + \lambda, p'y) = \{0\} \) and \( p'x = p^k(p'y) + p'z \).

\( 2 \) Let \( F(p, k, x) = \emptyset \), \( F(p, s + \lambda, y) = \{0\} \) and \( p'x = p^ky + p'z \). Then \( x \equiv 0 \pmod{p^k} \), \( y = p'y \) for some \( y' \), \( F(p, s, y') = F(p, s + \lambda, y') = \{0\} \) and \( x = p^ky' + p'z \).

**Definition 1.8.** For any integer \( c \), \( E(p, s, c, x) \equiv \exists X \exists y [X = F(p, s, x) \text{ and } y \equiv 1 \pmod{X} \text{ and } F(p, s, x - c) \subset X] \)

**Corollary 1.14.** Let \( X = F(p, s, x) \), \( y \equiv 1 \pmod{X} \) and a bar denote the natural isomorphism \( \GammaZ(p, s, X) \to \Gamma(p, s, X) \). Then \( E(p, s, c, x) \) is equivalent to \( \bar{x} = c\bar{y} \).

**Corollary 1.15.** If \( x \equiv 0 \pmod{p} \), or \( X = F(p, s, x) \) and \( \neg E(X) \), or \( c = 0 \), then \( \neg E(p, s, c, x) \)

**Corollary 1.16.** If \( c \equiv d \pmod{p} \), then \( E(p, s, c, x) \equiv E(p, s, d, x) \)

**A proof is easy.**

**Lemma 1.6.** Let \( c = p'd \) where \( d \equiv 0 \pmod{p} \). Then \( E(p, s, k, x) \equiv E(p, s + \lambda, ck, cx) \).
**Proof.** \( F(p, s, \lambda - ky) = F(p, s + \lambda, cx - cky) \)

### 2. First special \( o \)-group

Let \( H \) be an \( o \)-group, \( h \in H \) and \( h \not\equiv 0 \) (mod \( p \)) in \( H \). Let \( k \) and \( s \) be integers and \( 1 \leq k \leq s \). \( \omega(\omega^*) \) is the chain (the inverse chain) of naturals.

**Lemma 2.1.** There exists a subgroup \( H' \subset H \) such that \( p^k h \in H' \) and \( p^k h \not\equiv 0 \) (mod \( p \)) in \( H' \) and the factor group \( H/H' \) has the power \( p^k \).

**Proof.** By reasons of induction it is enough to prove the lemma for \( k = 1 \). The factor group \( H/pH \) is a vector space over the field of power \( p \). Let \( S \) be a maximal subspace in \( H/pH \) such that \( h + pH \not\in S \). The full pre-image of \( S \) in \( H \) is a required subgroup.

For every \( n \in \omega^* \), let \( f_n : H_n \to H \) be an \( o \)-group isomorphism, \( H_n' = f_n^{-1}(H') \) and \( h_n = f^{-1}(h) \). Let \( G_0 = L \Sigma \{ H_n' : n \in \omega^* \} \). \( G_1 = L \Sigma \{ H_n : n \in \omega^* \} \) and \( G \) be the least subgroup of \( G_1 \) containing \( G_0 \cup \{ h_n - h_{n+1} : n \in \omega \} \). Let \( X_m \) be the subgroup \( L \Sigma \{ H_n : n \geq m \} \) of \( G_1 \).

**Lemma 2.2.** Every factor group \( X_m \cap G/X_m+1 \cap G \) is \( o \)-isomorphic to \( H \).

A proof is easy.

**Lemma 2.3.** \( p^k h_0 \not\equiv 0 \) (mod \( p \)) in \( G \).

**Proof.** Let \( f : G_1 \to H \) be defined as follows: \( f(x_n + x_0) = f_x x_n + f_{x_0} \). It is easy to check that \( f \) is a group isomorphism, \( fG = fG_0 = H' \) and \( f(p^k h_0) = p^k h \). But \( p^k h \not\equiv 0 \) (mod \( p \)) in \( H' \).

**Lemma 2.4.** \( F(p, k, p^k h_0) \subseteq \{0\} \) in \( G \).

**Proof.** It is enough to prove that every \( X_m \cap G \supset F(p, k, p^k h_0) \). But \( p^k h_0 \equiv p^k h_0 \) (mod \( p^k \)) in \( G \) and \( p^k h_0 \in X_m \cap G \).

**Lemma 2.5.** \( F(p, 1, p^k h_0) = F(p, k, p^k h_0) = \{0\} \) in \( G \).

**Proof.** \( \{0\} \subseteq \{0\} \) (by the Lemma 2.3) \( F(p, 1, p^k h_0) \subseteq F(p, k, p^k h_0) \subseteq \{0\} \) (by the Lemma 4).

Let an asterisk denote the natural group homomorphism \( G \to G/p'G \) and \( \Gamma \) be the group \( \Gamma(p, s, \{0\}) \) of \( G \). Clearly \( \Gamma \subseteq G^* \) and \((p'h_0)^* \in \Gamma\).
Lemma 2.6. \((p' h_0)^*\) has the order \(p'\) in \(\Gamma\)

**Proof.** \(F(p', p^k p' h_0) = F(1, p^k h_0) = \emptyset\) in \(G\) and \(F(p', p^k 'p' h_0) = F(p, p' h_0) = \{0\}\) in \(G\) by Lemma 25. By Lemma 13 in Section 1 the order of \((p' h_0)^*\) in \(\Gamma\) is \(p^k\).

Lemma 2.7. Every \(x^* \in \Gamma\) is a multiple of \((p' h_0)^*\)

**Proof.** Let \(x^* \in \Gamma\) in \(G_1\), \(x = x_n + \ldots + x_0\) for some \(n\) and \(x_i \in H_i\). It is easy to see that \(x \equiv 0 \pmod{p'}\) in \(G_1\). Let \(y_i = p'y_i\) and \(y_i = ah_i + h_i'\) where \(h_i' \in H_i'\). Then,

\[
X = \sum p'(a, h_i + h_i') \equiv \sum p'a, h_i = \left(\sum a_i\right) p' h_0 (\mod p') \text{ in } G
\]

Lemma 2.8. In \(G\), \(p(s, k, \{0\}) = 1\) and \(p(s, \ast, \{0\}) = 1\) for every \(i \neq k\)

**Proof.** See Lemmas 26 and 27.

3. Second special \(\omega\)-group

Let \(Q\) be the naturally ordered additive group of rational numbers

Lemma 3.1. Every \(p(s, k, \{0\}) = 0\) in \(Q\)

**Proof.** Clear

Fix an integer \(s \geq 1\) and a prime \(p\). Let \(Q_p\) be the least subgroup of \(Q\) containing all quotients \(a/b\) where \(a\) and \(b\) are integers and \(b \neq 0 (\mod p)\)

Lemma 3.2. In \(Q_p\), \(p(s, s, \{0\}) = 1\), and \(p(s, k, \{0\}) = 0\) for \(k <\)

**Proof.** Clear

For every \(n \in \omega\) let \(f_n\) : \(H_n \to Q\) be an isomorphism of \(\omega\)-groups and \(H_n = f^{-1}(Q_n)\) and \(h_n = f^{-1}(1)\). Let \(G_0 = L \Sigma\{H_n \ n \in \omega^*\}\) and \(G_1 = L \Sigma\{H_n \ n \in \omega^*\}\) where \(\omega^*\) is the inverse ordered set of natural numbers. Let \(G\) be the least subgroup of \(G_1\) containing \(G_0 \cup \{(h_m - h_{m+1})/p^m \ m \in \omega\}\), and \(X_m\) be the subgroup \(L \Sigma\{H_n \ n \geq m\}\) of \(G_1\).

Lemma 3.3. For every \(m\), \(G/X_m \cap G\) is divisible and \(X_m \cap G/X_{m+1} \cap G\) is isomorphic to \(Q\).

**Proof.** Clear
Lemma 3.4. \( h_0 \not\equiv 0 \pmod{p} \) in \( G \)

**Proof.** Let \( f : G \to \mathbb{Q} \) be defined as follows: \( f(x_n + x_0) = f_n x_n + \cdots + f_0 x_0 \). It is easy to check that \( f \) is a group isomorphism, \( fG = fG_0 = \mathbb{Q}_p \) and \( fh_0 = 1 \). But \( 1 \not\equiv 0 \pmod{p} \) in \( \mathbb{Q}_p \).

Lemma 3.5. \( F(p, 1, h_0) = \phi = F(p, s, h_0) = \{0\} \) in \( G \)

**Proof.** See the proofs of Lemmas 2.4 and 2.5 in Section 2.

Let an asterisk denote the natural homomorphism \( G \to G/pG \) and \( \Gamma \) be the group \( \Gamma(p, s, \{0\}) \) of \( G \). It is easy to see that \( \Gamma \) is a subgroup of \( G^* \) and \( h_0^* \in \Gamma \).

Lemma 3.6. \( h_0^* \) has order \( p^s \) in \( \Gamma \)

**Proof.** We need only prove that \( p^{s-1}h_0^* 
ot\equiv 0 \). But, \( F(p, s, p^{s-1}h_0) = F(p, 1, h_0) \not\equiv 0 \) by Lemma 3.5.

Lemma 3.7. Every \( x^* \in \Gamma \) is a multiple of \( h_0^* \)

**Proof.** Let \( x^* \in \Gamma \) In \( G \), \( x = x_n + \cdots + x_0 \) for some \( n \) and \( x_i \in H_i \). There exist a natural number \( m \), integers \( a_m, \ldots, a_0 \) and elements \( y_m \in H_m, \ldots, y_0 \in H_0 \) such that

\[
p^{-m}x_n = a_m h_n + \cdots + a_0 h_0 + p^{-m}y_0
\]

Now we count in \( G \): \( p^{-m}x = \Sigma a_i h_i = (\Sigma a_i) h_0 \pmod{p^{-m}} \). By Lemma 3.4, \( \Sigma a_i = p^{-m}b \) for some integer \( b \). Then \( x = bh_0 \pmod{p^s} \), i.e. \( x^* = bh_0^* \).

Lemma 3.8. In \( G \), \( p(s, s, \{0\}) = 1 \) and \( p(s, k, \{0\}) = 0 \) for every \( k < s \)

**Proof.** See Lemmas 3.6 and 3.7.

4. Third special \( \omega \)-group

**Definition 4.1.** A succession is a function \( \alpha : \omega \to \omega \) such that for some \( n \), \( \alpha_n \neq 0 \) and (\( \forall i > n \)) \( \alpha_i = 0 \). That \( n \) is called the length of succession \( \alpha \).
In this section α, β, γ and δ are successions. The restriction of α to \( n = \{1 \leq n \leq m \} \) is denoted by \( \alpha \mid n \).

**Definition 4.2.** \( S \) is the set of successions ordered as follows: \( \alpha < \beta \) if there exists \( n \) such that \( \alpha \mid n = \beta \mid n \) and \( \alpha n > \beta n \).

**Corollary 4.1.** The chain \( S \) is dense and has neither maximal nor minimal succession.

**Corollary 4.2.** If \( \alpha < \beta < \gamma \) and \( \alpha \mid n = \gamma \mid n \), then \( \beta \mid n = \alpha \mid n \).

**Corollary 4.3.** If the length of \( \alpha \) is \( n \) and \( \alpha \mid n + 1 = \beta \mid n + 1 \), then \( \beta \leq \alpha \).

**Definition 4.3.** \( \beta = \phi(\alpha, n) \) if \( \beta \mid n = \alpha \mid n \), \( \beta n = (\alpha n) + 1 \), \( \beta(n + 1) = \beta(n + 2) = \).

**Corollary 4.4.** \( \forall \alpha \forall n \exists \beta [\beta = \phi(\alpha, n)] \) and \( \forall \beta \exists \alpha \exists n [\beta = \phi(\alpha, n)] \).

**Corollary 4.5.** \( \phi(\alpha, n) = \inf\{\beta \mid n + 1 = \alpha \mid n + 1 \} \) and \( \alpha = \sup\{\phi(\alpha, n) \mid n \leq \omega\} \).

**Corollary 4.6.** \( \phi(\alpha, m) = \phi(\beta, n) \) is equivalent to \( m = n \) and \( \alpha \mid n + 1 = \beta \mid n + 1 \).

Fix a natural \( k > 0 \). About \( Q \) and \( Q_p \) see Section 3. For every succession \( \alpha \) let \( f_\alpha \colon H_\alpha \to Q \) be an isomorphism of \( \alpha \)-groups and \( f_\alpha'(Q_p) = H_\alpha^* \), \( f_\alpha'(1) = h_\alpha \). Let \( G_0 = L \Sigma(H_\alpha^* \alpha \in S), G_1 = L \Sigma(H_\alpha \alpha \in S) \) and \( G \) be the least subgroup of \( G_1 \) containing \( G_0 \) and such that \( \alpha \mid n = \beta \mid n \) always implies \( (h_\alpha - h_\beta)/p^{\delta k} \in G \). In other words if \( \alpha \mid n = \beta \mid n \) then \( h_\alpha \equiv h_\beta \mod p^{\delta k} \) in \( G \). Let \( X_\alpha \) be the subgroup \( G \cap L \Sigma(H_\beta^* \beta \leq \alpha) \) and \( Y_\alpha = X_\alpha^* \) in \( G \).

It is evident that \( X_\alpha = A(h_\alpha) \) in \( G \) and that for every \( x \in G \) there exists \( \alpha \) such that \( A(x) = X_\alpha \) in \( G \).

**Lemma 4.1.** \( Y_\alpha/X_\alpha \) is isomorphic to \( Q \).

**Proof.** Clear.

**Lemma 4.2.** If \( \beta = \phi(\alpha, n) \) then \( Y_\beta \subseteq F(p, nk + 1, h_\alpha) \).

**Proof.** Let \( f \colon G_1 \to Q \) be defined as follows: \( f(\Sigma a, h_\alpha) = \Sigma a \beta < \gamma \}. \) Clearly \( f \) is a group homomorphism and \( fY_\beta = \{0\} \). \( fG_0 = Q_p, fh_\alpha = 1 \). If \( F(p, nk + 1, h_\alpha) \subseteq Y_\beta \), then \( fh_\alpha \equiv 0 \mod p^{\delta k + 1} \) in \( fG \). So it is enough to prove that if \( x \in G_0 \) then \( fx \) is a multiple of \( 1/p^{\delta k} \). \( G \) is constructed from \( G_0 \) and elements \( (h_\alpha - h_\delta)/p^k \) where \( \gamma \mid 1 = \delta \mid 1 \). If \( x \in G_0 \) then \( fx \) is a multiple of 1. Let \( x = (h_\gamma - h_\delta)/p^k \) where
Let $\beta = \phi(\alpha, n)$ and $s \neq k$. Let an asterisk denote the natural homomorphism $G \to G/\Gamma(p, s, Y)$. By Lemma 4.5 it can be assumed that $(p^*x)^* = \Sigma(b_n)h_n$ where $\alpha \neq \beta$ and $a_0 \neq 0$ implies $F(p, s + n, b_n h_n) \neq F(p, s + n, b_n h_n)$. Let $F(p, s + n, b_n h_n) = F(p, s + n, b_n h_n)$. Then $(p^*x)^* = (h_n)$. 

Corollary 4.8. If $\emptyset \neq F(r, s, x) = Y$ in $G$, then $Y = Y_\beta$ for some $\beta$.

Proof. See Lemma 4.6 and Corollary 4.7.
Lemma 4.7. $g^*$ is an element of $I'$ of order $p^k$

**Proof.** $F(p, s, g^*) = (by$ Corollary 13 in Section 1)  $F(p, k + nk, h^*_n \cdots h^*_k) = F(p, k + nk, h^*_n) = Y^*_n$ by Corollary 47. So $g^* \in I'$

Clearly, $F(p, s, p^kg^*) = 0$. But

$F(p, s, p^kg^*) = F(p, 1 + nk, h^*_n \cdots h^*_k) = F(p, 1 + nk, h^*_n) = Y^*_n$

by Corollary 47. So the order of $g^*$ is $p^k$ by Lemma 13 in Section 1.

Lemma 4.8. Every $\tau^* \in I'$ is a multiple of $g^*$

**Proof.** By Lemma 4.6 $p^*x = ah, + y$ and $F(p, s + n, y) \subseteq Y$ for some $n, a, \gamma$ and $y$. Let $a = \alpha' \beta$ where $b \not\equiv 0 (mod p)$. By Lemma 4.5 $ah, \equiv ah, (mod p'kk')$. Wlog $\gamma = \alpha'$. By Corollary 4.7 $nk + 1 \leq s + n - i \leq nk + k$. Let $m = (nk + k) - (s + n - i)$ So $p^*x = p'^*b(h, a - h_b)^*$ and $p'^*x = p'^*b(p''b)^* = p'^*b(h, a - h_b)^*$ and $x^* = p'^*b\gamma^*$

Corollary 4.9. $p(s, k, Y_b) = 1$ and $p(s, i, Y_b) = 0$ for $i \neq k \in G$

5. Gluing and interlacement

**Definition 5.1.** $\Delta G$ is the set of all convex subgroups of an o-group $G$ ordered by inclusion.

**Lemma 5.1.** Let $H$ be a subgroup of an o-group $G$ and $\sigma : \Delta G \to \Delta G$ be defined as follows. $\sigma Y = \bigcup \{X \in \Delta G \mid X \cap H \subseteq Y\}$. Then (1) $\sigma Y \cap H = Y$ and (2) $\sigma$ is a monomorphism.

**Proof.** (1) Clearly $\sigma Y \cap H \subseteq Y$. Let $h \in Y$ and $Z = A(h, G)$. The latter means that $Z$ is $A(h)$ calculated in $G$. Then $Z \cap H \subseteq Y$ and $h \in Z \subseteq \sigma Y$.

(2) $Y_1 \subseteq Y_2$ implies $\sigma Y_1 \subseteq \sigma Y_2$.

Indeed, $\sigma Y_2 - \sigma Y_1 \subseteq (\sigma Y_2 - \sigma Y_1) \cap H = Y_2 - Y_1$.

**Definition 5.2.** The monomorphism $\sigma$ of Lemma 5.1 will be called **canonical**.

**Theorem 5.1.** Let a direct sum $G = \Sigma H$, be linearly ordered and every $\sigma_i : \Delta^* H_i \to \Delta^* G$ be the canonical monomorphism. Let $X \in \Delta G$ and $Y_i = X \cap H_i$. Then every

$p(s, k, X, G) = \Sigma(p(s, k, Y_i, H_i)) \sigma, Y_i = X$

**Proof.** Let $r = p^*$.

(1) $\Gamma_1(r, X) = \Sigma \Gamma_1(r, Y_i, H_i)$.
Indeed, let \( h \in \Gamma_1(r, Y, H) \), i.e. some \( h + rh' \in Y \). Then \( h + rh' \in X \) and \( h \in \Gamma_1(r, X, G) \).

Conversely, let \( \Sigma h_i \in \Gamma_1(r, X) \), i.e. some \( \Sigma h_i + r\Sigma h_i' \in X \). Then \( h_i + rh_i' \in Y_i \) and \( h_i \in \Gamma_1(r, Y_i, H_i) \).

(2) \( \Gamma_2(r, X) = \Sigma \{ \Gamma_1(i, Y_i, H_i) \mid X \subseteq \sigma Y_i \} + \Sigma \{ \Gamma_2(r, Y_i, H_i) \mid X = \sigma Y_i \} \)

Indeed, let \( h \in \Gamma_1(r, Y_i, H_i) \), i.e. for every \( Y_i \subseteq Z_i \in \Delta^*H_i \) some \( h + rh' \in Z_i \).

And let \( \sigma Y_i = X \subseteq Z_i \in \Delta^*G \) Then \( Y_i \subseteq Z \cap H_i \) and some \( h + rh' \in Z \cap H_i \), i.e. \( h \in \Gamma_2(r, X) \).

Conversely, let \( \Sigma h_i \in \Gamma_2(r, X) \), i.e. for every \( X \subseteq Z \in \Delta^*G \) some \( \Sigma h_i + r\Sigma h_i' \in Z \).

Case 1. Let \( X \subseteq \sigma Y_i = Z \) Then some \( \Sigma h_i + r\Sigma h_i' \in Z \) and \( h_i + rh_i' \in Z \cap H_i = Y_i \), i.e. \( h_i \in \Gamma_1(r, Y_i, H_i) \).

Case 2. Let \( X = \sigma Y_i \) and \( Y_i \subseteq Z_i \in \Delta^*H_i \) and \( \sigma Z_i = Z \). Then \( X \subseteq Z \) and some \( \Sigma h_i + r\Sigma h_i' \in Z_i, h_i + rh_i' \in Z_i \), i.e. \( h_i \in \Gamma_2(r, Y_i, H_i) \).

(3) \( \Gamma(r, X) = \Sigma \{ \Gamma(r, Y_i, H_i) \mid X = \sigma Y_i \} \)

Statement (3) follows from statements (1) and (2) and implies the statement of the Theorem 5.1.

**Definition 5.3.** A chain \( C \) is compact if

\[ (\forall X \subseteq C)(\exists y, z \in C) (X \neq \emptyset \cup y = \inf X \text{ and } z = \sup X) \]

**Definition 5.4.** Let \( C \) be a chain and \( x \in C \). Then

\[ x^* = \begin{cases} \inf \{ y \mid x < y \} & \text{if } x \neq \max C \text{ and } \\ x & \text{if } x = \max C \end{cases} \]

**Definition 5.5.** Let \( H \) be an o-group and \( C \) be a compact chain. Monomorphism \( \sigma : \Delta^*H \rightarrow C \) is regular if

1. \( Y \subseteq Y^* \) implies \( \sigma Y \subseteq (\sigma Y)^* \) and
2. every \( \sigma Y = \inf \{ \sigma Z \mid Y \subseteq Z \subseteq Z^* \} \)

**Theorem 5.2** (Interlacement Theorem) Let

1. \( C \) be a compact chain and \( C \models \forall x \forall y \exists z \) \((x < y \cup x < z < z^* \leq y)\),
2. \( \{ H_i, i \in I \} \) be a family of o-groups and \( \psi_i : \Delta^*H_i \rightarrow C \) be a regular monomorphism and
3. \((C \models x < x^*) \) implies \((\exists ! i)(x \in \text{rng} \psi_i)\)

Then there exist an o-group \( G \) and an isomorphism \( \phi : C \rightarrow \Delta^*G \) such that every

\[ p(s, k, X, G) = \Sigma \{ p_i (\psi_i Y_i, H_i) \mid \phi \psi_i Y_i = X \} \]

and

\[ H_i \models E(Y) \text{ implies } G \models E(\phi \psi_i Y). \]

**Proof.** Let \( \psi_i h_i \) be an abbreviation for \( \psi_i A(h_i, H_i) \).
Lemma 5.2. \( h_i, h_j \neq 0 \) and \( \psi_i h_i = \psi_j h_j \) implies \( i = j \)

Proof. Let \( h_i, h_j \neq 0 \) and \( x = \psi_i h_i = \psi_j h_j \) Then \( x < \Lambda^+ \) because of regularity of \( \psi \) Now use (3) from Theorem 5.2

Let \( G \) be the direct sum \( \Sigma H_i \) ordered as follows If \( g = \Sigma h_i \) and \( \psi_i h_i = \max\{\psi_i h_i, h_i \neq 0\} \) then \( g > 0 \) if \( h_i > 0 \)

Let \( \phi x = \{\Sigma h_i \text{ every } \psi_i h_i < x\} \)

Lemma 5.3. \( \phi \) is an isomorphism from \( C \) onto \( \Delta^* G \)

Proof. (1) Evidently \( \phi x \in \Delta^* G \)

(2) Let \( x < y \) Then (see (1) Theorem 5.2) \( x \leq z < z' \leq y \) for some \( z = \psi_i h_i \) and \( h_i \in \phi y - \phi x \)

(3) Let \( X \in \Delta^* G \) and \( x = \sup\{\psi_i h_i \text{ every } h_i \in X\} \) We state that \( \phi x = X \) It is enough to prove that always \( X \cap H_i = \phi x \cap H_i \) Clearly \( X \cap H_i \subseteq \phi x \) Conversely, let \( h_i \in \phi x \) Then \( \psi_i h_i < (\psi_i h_i)^* \) for some \( h_i \in X \) Therefore, \( \psi_i h_i \leq \psi_i h_i \) and \( |h_i| < n |h_i| \) in \( G \) Because \( X \) is convex, \( h_i \in X \)

Lemma 5.4. The monomorphism \( \phi \psi_i : \Delta^* H_i \rightarrow \Delta^* G \) is canonical

Proof. (1) \( \phi \psi_i Y \cap H_i = Y \) Indeed, \( h \in Y \Rightarrow A(h, H_i) \subseteq Y \Rightarrow \psi_i h < \psi_i Y \Rightarrow h \in \phi \psi_i Y \)

(2) \( X \in \Delta^* G \) and \( X \cap H_i \subseteq Y \) imply \( X \subseteq \phi \psi_i Y \) Indeed, let \( X = \phi x \in \Delta^* G \) and \( X \cap H_i \subseteq Y \) And let (reductio ad absurdum) \( \phi \psi_i Y \subseteq \phi x \), i.e. \( \psi_i Y < \psi_i X \) Because of regularity of \( \psi_i \), there exists \( h_i \in H_i \) such that \( \psi_i Y \leq \psi_i h_i < x \) Then \( h \in (\phi x \cap H_i) - Y \) which contradicts \( X \cap H_i \subseteq Y \)

Proof of Theorem 5.2. Now the first statement of Theorem 5.2 follows from Theorem 5.1 The second statement is evident Theorem 5.2 is proved

Lemma 5.5. Let \( H_1 \) and \( H_2 \) be countable Archimedean \( o \)-groups Then there exists an Archimedean ordering of the direct sum \( H_1 + H_2 \) preserving the orderings of the summands

Proof. By Holder’s Theorem (see [3]) it can be assumed that \( H_1 \) and \( H_2 \) are subgroups of the naturally ordered additive group \( R \) of reals For any real \( r \neq 0 \), \( H_1 \) is isomorphic to the subgroup \( \{rx \mid x \in H_1\} \) So it can be assumed that \( H_1 \cap H_2 = \{0\} \) But in that case the statement of the Lemma 5.5 is clear

Let \( G \) be an \( o \)-group, \( x \in G \) and \( X = A(x) \) in \( G \) Then \( X \subseteq X^+ \) and the Archimedean \( o \)-group \( X^+/X \) is called an Archimedean factor of \( G \)
Theorem 5.3 (Gurevich Theorem) Let $G_i$ be an o-group and all Archimedean factors of $G_i$ are countable $i = 1, 2$. Let $\psi \Delta^* G_1 \to \Delta^* G_2$ be an isomorphism of the chains. Then there exists an o-group $G_0$ (a gluing of $G_1$ and $G_2$) and chain isomorphisms $\phi_i \Delta^* G_0 \to \Delta^* G_i$ such that $\phi_2 = \psi \phi_1$ and every $p(s, k, x, G_0) = p(s, k, \phi_1 x, G_1) + p(s, k, \phi_2 x, G_2)$.

Proof. For $x \in G_i$, let $H_i(x)$ be the Archimedean factor of $G_i$ corresponding to $x$. If $H_i(x) = Y_i^* / Y_i$ and $\psi Y_i = Y_i$, fix an Archimedean order of $H_i(x) + H_i(x_2)$ preserving the orders of the summands.

Let $G_0$ be direct sum $G_1 + G_2$ ordered as follows. Let $g_0 = g_1 + g_2 \neq 0$ and $H_i(g_0) = Y_i^* / Y_i$. If $\psi Y_i \subset Y_i$ (respectively $Y_i \subset \psi Y_i$) then $g_0 > 0$ if $g_2 > 0$ (respectively $g_1 > 0$). If $\psi Y_i = Y_i$, then $g_0 > 0$ if the element $(g_1 + Y_i) + (g_2 + Y_i)$ of the Archimedean o-group $H_1(g_1) + H_2(g_2)$ is positive. $G_0$ is a desired o-group.

6. Fourth special o-group

Let us fix $p$, $s$ and $k$ where $1 \leq k \leq s$. Let $Q$ and $Q_p$ be as in Section 3. We build a countable o-group $G$ satisfying the following conditions:

(i) if $x \neq 0$ then $A^*(x) / A(x)$ is isomorphic to $Q$,
(ii) the chain $\{A(x) x \neq 0\}$ is order isomorphic to $Q$,
(iii) $G$ is $q$-divisible for each prime $q \neq p$,
(iv) if $p(s, t, X) = r > 0$ then $t = k$, $r = 1$ and $X$ is different from any $A(x)$,
(v) for each $x$, $p(s, k, A^*(x)) = 0$,
(vi) if $x^* \subset Y$ then $\exists z (X^* \subset z \subset Y$ and $p(s, k, z) = 1)$

Here is the idea of the construction. Let $G'$ be a copy of the third special o-group $G''$ satisfies conditions (i)-(iv) and (vi). For each non-zero $x \in G'$, "move" another copy of the third special o-group "between" $A^*(x)$ and $G'/A^*(x)$. Do the same for the new copies of the third special o-group. Repeat the process.

Now we construct the desired o-group. Let $\alpha$, $\beta$ range over the successions of Section 4 and $S$ be the chain of successions. $S$ is the set of functions $t \in \omega$. We order $S$ as follows $t_1 < t_2$ if $t_1 \subset t_2$. Or $\exists m (t_1 m = t_2 m$ and $t_1 m < t_2 m)$. We imagine elements of $S$ as sequences, hence it is clear what $t^\alpha$ means.

For each $t \in S$ let $H_t$ be an o-group, isomorphic to $Q, f_t H_t \to Q$ be an isomorphism, $H_t^* = f_t^{-1}(Q)$ and $h_t = f_t^{-1}(1)$. Let $U_t = L(\Sigma[h_t \alpha \in S])$, $V_t = L(\Sigma[H_t \alpha \in S])$, $V_t = (h_{t, n} - h_{t, \beta}^x) / p^{\alpha n} \alpha = \beta, n$ and $G_t$ be the least subgroup of $W_t$ containing $U_t \cup V_t$. Clearly, $G_t$ is a copy of the third special o-group. Let $W = L(\Sigma[H_t t \subset S])$ and $G$ be the least subgroup of $W$ containing $\bigcup\{G_t t \neq 0\}$. It is not difficult to check that $G$ is the desired o-group.
PART 2. ELIMINATION OF QUANTIFIERS

7. Elimination theorem

The elementary language of o-groups ELL is the first order language with an equality sign whose non-logical constants are the individual constant "0" the symbol "−" of one-place operation the symbol "+" of two-place operation and the symbol "<" of two-place predicate The elementary theory of o-groups ELT is given in ELL by axioms of (Abelian) groups, axioms of chains (i.e. linear order axioms) and by the following axiom

\[ \forall x \forall y \forall z \ (x < y \supset x + z < y + z) \]

Terms of ELL are called elementary terms \( t_1 - t_2 \) is the abbreviation for \( t_1 + (-t_2) \)

An Expanded Theory of o-groups EXT is now defined Let \( L_2 \) be the monadic second order language corresponding to ELL Every o-group \( G \) gives us a natural model of \( L_2 \) by the following definition second order variables range over the set \( A^*G \cup \{0\} \) (the convex subgroups of \( G \) and the empty set) Let \( T_2 \) be the set of \( L_2 \)-formulas which are true in all these natural models We shall essentially be studying the theory \( T_2 \) but in order to eliminate quantifiers some essential extension of \( T_2 \) is more conveniently used An Expanded Language of o-groups EXL is obtained from \( L_2 \) by adding some non-logical constants

Definition 7.1 (of second order terms (superterms) of EXL)

1. Second order variables of EXL (i.e. second order variables of \( L_2 \)) are superterms,
2. \( 0 \) is a superterm, and for each elementary term \( t \), \( A(t) \) is a superterm,
3. \( F(p, s, t) \) is a superterm for every elementary term \( t \), prime \( p \) and natural \( s \geq 1 \),
4. If \( T \) is a superterm, then so is \( T^* \)

Definition 7.2 (Of atoms (atom formulas) of EXL) Here \( t \) is an elementary term, \( T, T_1, T_2 \) are superterms, \( p \) is a prime number, \( k, s, r \) are naturals and \( 1 \leq k \leq s \) and \( l \) is an integer)

1. \( D(p, s, k, t), E(p, s, l \ t) \) are atoms,
2. \( T_1 = T_2, T_1 \subset T_2, E(T) \) and \( p(s, k, T) > r \) are atoms,
3. \( t \in T \) is an atom and
4. \( t = l \ (\text{mod} \ T), t < l \ (\text{mod} \ T), t > l \ (\text{mod} \ T) \) are atoms

A natural model of EXL is obtained from a natural model of \( L_2 \) by means of definitions of Section 1 and the following definition
**Definition 7.3.** (1) $\emptyset^*$ is the zero-subgroup \{0\},
(2) $T_i = T_2$ and $T_i \subset T_2$ are defined naturally,
(3) $E(\emptyset)$ is false,
(4) $p(i, k, 0) > r$ is always false and $t = l (\text{mod } 0)$, $t < l (\text{mod } 0)$, $t > l (\text{mod } 0)$ are always false

So every o-group gives us one natural model of EXL. An Expanded Theory of o-groups EXT is the set of EXL-formulas which are true in all these natural models.

The atoms of $T_2$ are expressible in EXT

$|t_i = t_j| = [t_i - t_j = 0 (\text{mod } 0^*)].$

$|t_i < t_j| = [t_i - t_j < 0 (\text{mod } 0^*)].$

The inverse statement is also true but we do not need it and we do not prove it.

**Theorem 7.1** (Elimination Theorem) For every EXL-formula $\alpha$ there exists an EXL-formula $\alpha^*$ such that $\alpha^*$ has no bound elementary variables and $\alpha \equiv \alpha^*$ in EXT.

The Elimination Theorem is the object of Part 2 (Sections 7–10). The proof below gives a primitive recursive procedure for building $\alpha^*$ from $\alpha$. And of course $\alpha^*$ has the same free variables as $\alpha$.

The Convex Subgroups Theory, CST, is defined in Section 11. As a corollary of Theorem 7.1 we have the following.

**Theorem 7.2.** There exists a primitive recursive algorithm which for every EXL-sentence $\alpha$ builds a CSL-sentence $\alpha^*$ such that $\alpha \in \text{EXT} \iff \alpha^* \in \text{CST}$

**Proof.** Let $\alpha$ be an EXL-sentence $\alpha$ does not contain free elementary variables. By Theorem 7.1, $\alpha$ does not contain elementary variables at all.

We log the individual constant 0 does not occur in $\alpha$. Indeed $D(p, s, k, 0)$ is always true, $E(p, s, k, 0)$ is always false, $0 \in T \equiv \emptyset \subset T$ and it is easy to eliminate 0 from atoms $0 = l (\text{mod } T)$, $0 > l (\text{mod } T)$, $0 < l (\text{mod } T)$.

Furthermore,

$$(Y = X^*) \equiv (X \subset Y \land \neg \exists Z (X \subset Z \subset Y))$$

$$\lor (Y = X \land (\forall U \subset X) \exists Z (X \subset Z \subset U)).$$

So it can be assumed that every superterm of $\alpha$ is a variable or 0. We also admit a new individual constant $U$ which denotes the maximal (non-proper) convex subgroup.

We log all quantifications in $\alpha$ are restricted by $\emptyset \subset X \subset U$. Indeed, $\exists X \beta(X) \equiv \beta(\emptyset) \lor \beta(U) \lor \exists X (\beta(X) \land (\emptyset \subset X \subset U)).$
Wlog the individual constants \( \emptyset \) and \( U \) do not occur in \( \alpha \). Indeed, \( E(\emptyset), E(U) \), \( p(s,k,\emptyset) > r, p(s,k,U) > r, U \subset \emptyset \) are false \( \emptyset = \emptyset, \emptyset \subset U, U = U \) are true. And because every variable in \( \alpha \) is bounded by the open interval \((\emptyset, U)\) we can replace \( \emptyset = X \) by the propositional constant "false", \( \emptyset \subset X \) by the "true" and so on.

As a matter of fact we now have a desired formula \( \alpha^* \).

An EXL-formula \( \alpha \) is called open if \( \alpha \) has no bound elementary variables. Below, we write "\( \alpha \equiv \beta \)" instead of "\( \alpha \equiv \beta \) in EXT", "\( \alpha \) implies \( \beta \)" instead of "\( \alpha \) implies \( \beta \) in EXT" and so on.

The Elimination Theorem is proved by induction on \( \alpha \). The only non-trivial case is the following.

**Lemma 7.1 (Main Lemma)** For every open EXL-formula \( \alpha(x) \) there exists an open EXL-formula \( \alpha^* \) such that \( \exists x \alpha(x) \equiv \alpha^* \).

The following simple statements are used often.

**Lemma 7.2 (Cases Lemma)** If \( \alpha \) implies \( \forall \beta, \) then \( \exists x \alpha \equiv \forall x (\alpha \& \beta) \).

**Lemma 7.3.** Let \( \alpha \) be an EXL-formula and \( \beta \) be a subformula of \( \alpha \) such that any free occurrence in \( \beta \) of any variable is never bound in \( \alpha \). Let \( \alpha_i \) (respectively \( \alpha_t \)) be obtained from \( \alpha \) by replacing \( \beta \) by the propositional constant "false" (respectively "true") Then \( \exists x \alpha \equiv \exists x (\beta \& \alpha_t) v \exists x (\neg \beta \& \alpha_i) \).

**Lemma 7.4.** Let \( \alpha \) be an EXL-formula and \( T \) be a superterm in \( \alpha \) such that any occurrence in \( T \) of any variable is never bound in \( \alpha \). Let \( \alpha' \) be obtained from \( \alpha \) by replacing \( T \) by a new second order variable \( X \). Then \( \exists x \alpha \equiv \exists X \exists x (X = T \& \alpha') \).

8. Primary case

An EXL-formula \( \alpha(x) \) is called a p-formula if \( x \) can occur in \( \alpha(x) \) only through \( F(p,r,t), D(p,r,c,t) \) or \( E(p,r,c,t) \). In other words a p-formula contains neither \( A(t), t \mod T \) nor \( F(q,r,t), D(q,r,c,t), E(q,r,c,t) \) where \( q \neq p \).

**Theorem 8.1.** Let \( \alpha(x) \) be an open p-formula. There exists an open p-formula \( \alpha \) such that \( \exists x \alpha(x) \equiv \alpha^* \).

Theorem 8.1 is the object of this section. Let \( R \) be the set of numbers \( r \) occurring in \( \alpha \) through \( F(p,r,t(x)), D(p,r,c,t(x)) \) or \( E(p,r,c,t(x)) \). Let \( s = \max R \). It can be assumed that \( s \) is the only element of \( R \), see Corollary 13 and Lemmas 17 and 6 in Section 1. Below we write \( F(t), D(c,t), E(c,t) \) instead of \( F(p,s,t), D(p,s,c,t), E(p,s,c,t) \) respectively. \( t_1 = t_2 \) is an abbreviation for \( F(t_1 \& t_2) = \emptyset \).
Note that $x = y$ implies $\alpha (x) \equiv \alpha (y)$. Every elementary term $t$ of $\alpha$ can be represented in a form $ax + by_1 + \ldots + by_m$ where $0 \leq a, b < p$. Moreover, it can be assumed that $a = p^k$, see Corollary 1.3 and Lemmas 1.5 and 1.6 in Section 1.

Below $\tau$ is an elementary term without $x$ and $M$ is the set of terms $\tau$ occurring in $\alpha$ through $F(ax + \tau)$, $D(c, ax + \tau)$ or $E(c, ax + \tau)$. It can be assumed that if $\tau \in M$ then $p\tau \in M$.

By the Cases Lemma it can be assumed that for $k = 1, \ldots, s$, $\alpha$ has conjuncts $F(p^{-k}x + \tau_k) \subseteq F(p^{-k}x + \tau)$ for some $\tau_k$ and every $\tau \in M$. Let $t_k = p^{-k}x + \tau_k$ and $t_0 = 0$. Then $\alpha$ implies $F(t_0) \subseteq F(t_1) \subseteq \ldots \subseteq F(t_s)$. Indeed, $F(t_k) \subseteq F(p^{-k}x + p\tau_{k+1}) = F(p \cdot t_{k+1}) \subseteq F(t_{k+1})$.

By the Cases Lemma 7.2 it can be assumed that $(F(t_k) \subseteq F(t_{k+1}))$ or $(F(t_k) = F(t_{k+1}))$ is a conjunct in $\alpha$. In order to avoid using indices we assume that $\emptyset = F(t_0) = F(t_1) \subseteq F(t_{k+1}) = F(t_s)$.

Evidently

$$
p^{-k}x = \begin{cases} p^{-k}p^i x = p^{-k}(t_i - \tau_i) = -p^{-k}\tau_i, & \text{if } k \leq i, \\
p^{-k}p^j x = p^{-k}(t_j - \tau_j), & \text{if } i < k \leq j, \\
p^{-k}x = p^{-k}(t_s - \tau), & \text{if } j < k \leq s.
\end{cases}
$$

Let $\alpha'(x, y)$ be a formula such that $\alpha(x) \equiv \alpha'(t, s)$.

**Lemma 8.1.**

$$\exists x \alpha(x) \equiv \exists x, \exists y, \left[ \alpha'(x, y) \wedge p^{-i}(x_i + \tau) = -\tau_i \wedge p^{-j}(y_i + \tau) = \tau_i \right]$$

**Proof.** $\alpha(x)$ implies $\alpha'(t, s)$. Conversely $\alpha'(x, y)$ implies $\alpha(x, \tau)$. 

**Corollary 8.1.** It is enough to prove Theorem 8.1 for $\alpha(x)$ such that $\alpha(x)$ has conjuncts $p^i x = \tau_0$ and $\emptyset \subset F(x) \subseteq F(p^i x + \tau)$ for some $1 \leq i < s$ and $\tau_0$ and every $0 \leq i < k$ and $\tau \in M$.

Let $N = \{p^i x + \tau \mid 0 \leq i < k \wedge \tau \in M\}$. Wlog $N$ is the set of all elementary terms $t(x)$ in $\alpha$.

Wlog $(X = F(x))$ is a conjunct in $\alpha$, see Lemma 7.4 in Section 7.

**Lemma 8.2.** Wlog $\alpha$ has conjuncts $F(\tau) \subseteq X$, $\tau \in M$.

**Proof.** Let $\tau \in M$. By the Cases Lemma it can be assumed that $F(\tau) \subseteq X$ or $X \subseteq F(\tau)$ is a conjunct of $\alpha$. But $F(x) = X \subseteq F(\tau)$ implies $F(p^i x + \tau) = F(\tau)$, $D(c, p^i x + \tau) = D(c, \tau)$ and $E(c, p^i x + \tau) = E(c, \tau)$. So we can cancel $\tau$ from $M$.

**Lemma 8.3.** The conjunct $p^i x = \tau_0$ can be replaced in $\alpha$ by $p^i x = \tau_0$ and $F(p^i x - \tau_0) \subseteq X$. 
Proof. Let $\alpha'(x)$ be the result of the replacement $\alpha(x)$ implies $\alpha'(x)$. Conversely, suppose $\alpha'(x)$ Then $p^x x - \tau_0 = \gamma$ for some $y \in X$ and $p^{x + \gamma} y = 0$, i.e. $y = p^z x$ for some $z$. It is easy to check that $\alpha(x - z)$ is true.

Evidently $\alpha$ implies $F(t) = X$ for $t \in N$.

Corollary 8.2. Wolog $\alpha = \alpha_0 \land \alpha_i(x) \land \beta(x)$ where $\alpha_o$ is $p^{-i} \tau_0 = \gamma$ and $\beta \subset X$ \land $\{F(t) \subseteq X \land \tau \in M\}$. $\alpha_i$ is $F(p^{-i} x - \tau_0) \subseteq X$ \land $\{F(t) = X \land \tau \in N\}$ and $x$ can occur in $\beta(x)$ only through atoms $D(c, t)$ and $E(c, t)$.

Lemma 8.4. Let $\alpha_0 \land E(X)$ imply $\exists x (\alpha, \beta) \equiv \gamma$ and $\alpha_0$ and $\neg E(X)$ imply $\exists x (\alpha, \beta) \equiv \delta$. Then, $\exists x \alpha \equiv \alpha_0 \land E(X) \land \gamma \lor \alpha_0 \land \neg E(X) \land \delta$.

Proof. Clear.

So it is enough to find the corresponding $\gamma$ and $\delta$. Suppose $\alpha_0$ and $E(X)$ By the Cases Lemma it can be assumed that $\beta$ has a conjunct $E(a, x)$, $1 \leq a < p^i$. Therefore we can replace $D(t, t(x))$ by $\{E(b, p^i, t) \mid 0 < b < p^{-i}\}$, $E(b, p^i x + \tau)$ by $E(b - a p^i, \tau)$, $F(p^i x + \tau) = X$ by $\neg E(-a p^i m)$.

As a result $\alpha_i(x) \land \beta(x) = F(x) = X \land E(a, x) \land \alpha'$ for some open $\alpha'$ without $x$. And,

$\exists x (\alpha, \beta) \equiv \alpha' \land \exists x (F(x) = X \land E(a, x)) \equiv \alpha' \land E(X)$

Suppose $\alpha_0$ and $\neg E(X)$

Wolog every atom in $\beta$ has a form $D(t, t(x))$. Indeed, let $\beta_0$ be an atom of $\beta(x)$. If $\beta_0 = E(t, t(x))$ then $\beta_0$ can be replaced by "false." Let $\beta_0$ not contain $x$. By the Cases Lemma it can be assumed that $\beta_0$ or $\neg \beta_0$ is a conjunct of $\beta(x)$. It can be assumed that $\beta_0$ occurs only once in $\beta(x)$. Let $\alpha_i$, and $t = \pm \beta_0 \land \alpha'$, then $\exists x (\alpha, \beta) \equiv \beta_0 \land \exists x \alpha'(x)$.

Let a bar denote the natural isomorphism $\Gamma(p, s, X) \rightarrow \Gamma(p, s, X)$ Let $\alpha$ be $p^i x = \bar{x}$ and $\Lambda\{t \neq 0 \mid t \in N\}$ and $\beta'$ be obtained from $\beta$ by replacement of $D(t, t(x))$ by $t \equiv 0 (\mod p^i)$. Evidently $\exists x (\alpha_i(x) \land \beta(x)) \equiv \exists \bar{x} (\alpha_i(\bar{x}) \land \beta'(\bar{x}))$

Let $K(p, s)$ be the class of (Abelian) groups satisfying the axiom $p^iv = 0$. Let a first order language $L(p, s)$ be obtained from the elementary language of groups by adding the atoms $t \equiv 0 (\mod p^i)$. Let $T(p, s)$ be the theory of $K(p, s)$ in $L(p, s)$.

Lemma 8.5. $T(p, s)$ admits a quantifier elimination.

Proof. It is easy to check Lemma 8.5 with the aid of [17] or even without it.

According to Lemma 8.5 the formula $\exists \bar{x} (\alpha_i(\bar{x}) \land \beta'(\bar{x}))$ is equivalent in $T(p, s)$ to some Boolean combination of atoms $\bar{\tau} = 0$ and $\bar{\tau} \equiv 0 (\mod p^i)$. Then
\[ \exists x (\alpha(x) \land \beta(x)) \] is equivalent to the corresponding Boolean combination of atoms \( F(\tau) \subseteq X \) and \( D(j, \tau) \). Theorem 8.1 is proved.

9. Without exiles

Superterms \( A(t(x)) \) and atoms \( t(x)Rk \mod T \) will be called exiles.

**Theorem 9.1.** Let \( \alpha(x) \) be an open EXL-formula without exiles. There exists an open EXL-formula \( \alpha^* \) such that \( \exists x \alpha(x) \equiv \alpha^* \).

**Proof.** Let \( \sigma \) be the set of pairs \((p, r)\) occurring in \( \alpha \) through \( F(p, r, t(x)), D(p, r, c, t(x)), E(p, r, c, t(x)) \). Let \( \pi = \{p \mid \exists r (p, r) \in \sigma\} \) and \( s_{\pi} = \max\{p \mid (p, r) \in \sigma\} \).

**Lemma 9.1.** Without \( \alpha(x) = \beta \land \{\alpha_p(x) \mid p \in \pi\} \) where every \( \alpha_p(x) \) is a \( p \)-formula and \( \beta \) does not contain \( x \).

**Proof.** See Lemmas 7.3 and 7.4.

**Lemma 9.2.** \( \exists x \alpha(x) \equiv \beta \land \{\exists x \alpha_p(x) \mid p \in \pi\} \)

**Proof.** Suppose \( \beta \) and \( \alpha_p(x_p), p \in \pi \). There exist integers \( a_p \) such that \( a_p \equiv 1 \mod s_p \) and \( a_p \equiv 0 \mod s_q \) for \( q \in \pi - \{p\} \). It is easy to check that \( \alpha(\sum a_p x_p) \) holds.

Now Theorem 8.1 implies Theorem 9.1.

10. Banishment

Superterms \( A(t(x)) \) and atoms \( t(x)Rk \mod T \) are called exiles.

**Theorem 10.1.** Let \( \alpha(x) \) be an open EXL-formula. There exists an open EXL-formula \( \alpha^*(x) \) without exiles such that \( \exists x \alpha(x) = \exists x \alpha^*(x) \).

Theorem 10.1 is the object of this section. The Main Lemma of Section 7 follows from Theorem 10.1 and Theorem 9.1.

Below \( \tau \) is an elementary term without \( x \).

**Lemma 10.1.** W.l.o.g. every elementary term \( t(x) \) in \( \alpha \) has a form \( x + \tau \).

**Proof.** Every \( t(x) \) can be represented in a form \( ax + \tau \) for some integer \( a \). Let
S = \{a \neq 0 \text{ ax + r occurs in } a \} \text{ Let } b \text{ be the least common multiple of numbers in } S \text{. It can be assumed that } b \text{ is the only element of } S \text{, see Corollaries 12, 18, 112 and Lemmas 15, 16. Let } \alpha'(x) \text{ be the formula such that } \alpha(x) = \alpha'(bx) \text{. Then } \exists \alpha(x) \equiv \exists x [\alpha'(x) & F(b, x) = \emptyset] \text{. Now Corollary 14 is used.}

Lemma 10.2. \text{ Wlog } \alpha = (A(x) = X) & \beta & \gamma \text{ where}
1. \beta \text{ is a conjunction of exile atoms } (x + \tau)Rk \text{ (mod X)},
2. \beta \text{ has no conjuncts } x + \tau = 0 \text{ (mod X)},
3. \gamma \text{ has no exiles at all},
4. for every conjunct } (x + \tau)Rk \text{ (mod X) in } \beta \text{ there exists a conjunct } A(\tau) \subseteq X \text{ in } \gamma \text{ and}
5. X \neq \emptyset \text{ is a conjunct in } \gamma.

Proof. Let } M \text{ be the set of terms } \tau \text{ occurring in } \alpha \text{ through } A(x + \tau) \text{ or } (x + \tau)Rk \text{ (mod T)}. \text{ By the Cases Lemma it can be assumed that } (A(x + \tau_{n}) = A(x + \tau)) \text{ or } (A(x + \tau_{0}) \subseteq A(x + \tau)) \text{ is a conjunct in } \alpha \text{ for some fixed } \tau_{n} \text{ and every } \tau \in M \text{. Because of } \exists \alpha(x) \equiv \exists x \alpha(x - \tau_{n}) \text{ it can be assumed } \tau_{n} = 0 \text{. Moreover it can be assumed that } (A(x) = A(x + \tau)) \text{ is a conjunct in } \alpha, \tau \in M \text{. Indeed, } A(x) \subseteq A(x + \tau) \text{ is equivalent to } A(x) \subseteq A(\tau) \text{ and implies } (x + \tau)Rk \text{ (mod T)} \equiv \tau Rk \text{ (mod T)}. \text{ So } \alpha = \alpha_{1} \& \alpha_{2}, \text{ where } \alpha_{1} = \land \{A(x) = A(x + \tau), \tau \in M\}.

Wlog } \alpha_{2} = (A(x) = X) & \alpha_{1}, \text{ see Lemma 74. Wlog } \alpha_{1} \text{ has no exile superterms. Let } \beta = (x + \tau)Rk \text{ (mod T)} \text{ be an exile atom in } \alpha_{1} \text{. Wlog } T = X \text{. Indeed, it can be assumed that } T \subseteq X, T = X \text{ or } X \subseteq T \text{ is a conjunct in } \alpha_{1}. \text{ In the case } X \subseteq T \text{ we can replace } \beta \text{ by } 0Rk \text{ (mod T)} \text{. In the case } T \subseteq X \text{ and } k \neq 0 \text{ we can replace } \beta \text{ by } E(T) & (x + \tau)R0 \text{ (mod X)} \text{. In the case } T \subseteq X \text{ and } k = 0 \text{ we can replace } \beta \text{ by } (x + \tau)R0 \text{ (mod X)} \text{. Wlog } \beta \text{ or } \neg \beta \text{ is a conjunct in } \alpha_{1}, \text{ see Lemma 73}\text{. It can be assumed that } \beta \text{ is a conjunct because of}

\neg(x + \tau < k \text{ (mod X)}) \equiv (x + \tau = k \text{ (mod X)}) \vee (x + \tau > k \text{ (mod X)})

and similarly for other cases.

A conjunct } A(x) = A(x + \tau) \text{ in } \alpha_{1} \text{ can be replaced by}

A(\tau) \subseteq X \& x + \tau < 0 \text{ (mod X)} \text{ or}

A(\tau) \subseteq X \& x + \tau > 0 \text{ (mod X)}

If } \alpha_{1} \text{ has a conjunct } x + \tau = 0 \text{ (mod X)} \text{ then } \alpha \text{ is false.}

By the Cases Lemma } X = \emptyset \text{ or } X \neq \emptyset \text{ is a conjunct in } \alpha \text{. In the case } X = \emptyset \text{, } \alpha(x) \equiv \alpha(0) \text{. Q E D.}

Let } \sigma \text{ be the set of pairs } (p, r) \text{ occurring in } \gamma \text{ through } F(p, r, t(x)), D(p, r, c, t(x)) \text{ or } E(p, r, c, t(x)) \text{. Let } s \text{ be the least common multiple of the numbers } p' \text{ (p, r) } \in \sigma

Lemma 10.3. \text{ If } \tau \equiv y \text{ (mod s), then } \gamma(x) \equiv \gamma(y)
Proof. Clear

Below \( F(s, t) \subseteq X \) and \( E(s, c, t) \) are used as abbreviations for
\[
\land \{ F(p, r, t) \subseteq X \ (p, r) \in \sigma \} \quad \text{and} \\
\land \{ E(p, r, c, t) \ (p, r) \in \sigma \} \quad \text{respectively}
\]
(cf, Corollary 1.4)

Suppose \( (x + \tau_0 = k \ (\text{mod } X)) \) is a conjunct in \( \beta \) W log it is the only conjunct in \( \beta \) Indeed it implies that
\[
(x + \tau)Rl \ (\text{mod } X) \equiv (\tau - \tau_0)R(l - k) \ (\text{mod } X)
\]
W log \( \tau_0 = 0 \) Indeed,
\[
\exists x [A(x) = X \ & \ x + \tau_0 = k \ (\text{mod } X) \ & \ \gamma(x)] = \\
\equiv \exists x [A(x) = X \ & \ x = k \ (\text{mod } X) \ & \ \gamma(x - \tau_0)]
\]

Let \( \alpha_0(x) = E(X) \ & \ \gamma(x) \ & \ F(s, x) \subseteq X \ & \ E(s, k, x) \)

Lemma 10.4. \( \exists x \ & \ (\gamma) = \exists x \ & \ \alpha_0(x) \)

Proof. \( \alpha(x) \) implies \( \alpha_0(x) \) Conversely, suppose \( \alpha_0(x) \) Because of \( F(s, x) \subseteq X \) there exists \( y \in X^+ - X \) such that \( y = \nu \ (\text{mod } s) \) Clearly \( \alpha_0(y) \ & \ A(y) = X \) holds Therefore \( y = k + ns \ (\text{mod } X) \) for some \( n \) Let \( u = 1 \ (\text{mod } X) \) Then \( \alpha(y - nsu) \) holds

Let every conjunct in \( \beta \) be an inequality Note that
\[
x + \tau_1 < k_1 (\text{mod } X) \ & \ x + \tau_2 < k_2 (\text{mod } X) = \\
\equiv x + \tau_1 < k_1 (\text{mod } X) \ & \ \tau_2 - \tau_1 \leq k_2 - k_1 (\text{mod } X) \lor \\
x + \tau_2 < k_2 (\text{mod } X) \ & \ \tau_1 - \tau_2 \leq k_2 - k_1 (\text{mod } X)
\]

So it can be assumed that \( \beta \) has at most one conjunct of a form \( x + \tau < k \ (\text{mod } X) \) and (similarly) at most one conjunct of a form \( x + \tau > k \ (\text{mod } X) \) It also can be assumed that \( E(X) \) or \( \neg E(X) \) is a conjunct in \( \gamma \)

Case 1 \( \beta \) has at most one conjunct Let \( \alpha_1 \) be \( F(s, x) \subseteq X \ & \ X \subseteq X^+ \ & \ \gamma(x) \)

Lemma 10.5. \( \exists x \ & \ \alpha(x) = \exists x \ & \ \alpha_1(x) \)

Proof. \( \alpha(x) \) implies \( \alpha_1(x) \) Conversely, suppose \( \alpha_1(x) \) W log \( A(x) = X \), see the proof of Lemma 10.4 If \( \alpha \) has no exile-atoms then \( \alpha(x) \) holds Let \( \beta = x + \tau < k \ (\text{mod } X) \) (respectively \( \beta = x + \tau > k \ (\text{mod } X) \)) Let \( y > 0 \ (\text{mod } X) \) Then \( \alpha(x - nsy) \) (respectively \( \alpha(x + nsy) \)) holds for sufficiently large \( n \)
Case 2. \( \beta = \tau_1 + k_1 < x < \tau_2 + k_2 \mod X \) and \( \neg E(X) \) is a conjunct in \( \gamma \). It can be assumed that \( k_1 = k_2 = 0 \). If \( k_1 \neq 0 \) or \( k_2 \neq 0 \) then \( \alpha \) is false. Let \( \alpha = \alpha_1 \land \tau_1 < \tau_2 \mod X \).

**Lemma 10.6.** \( \exists \lambda \alpha(\lambda) = \exists \lambda \alpha_3(\lambda) \)

**Proof.** \( \alpha(x) \) implies \( \alpha_3(x) \). Conversely, suppose \( \alpha_3(x) \) and \( \log A(x) = X \). The Archimedean o-group \( X^+ \mod X \) is isomorphic to some dense ordered subgroup of the o-group of reals. So there exists \( y > 0 \mod X \) such that \( sy < \tau_2 - \tau_1 \mod X \). Then \( \alpha(x + nsy) \) holds for some \( n \).

Case 3. \( \beta = \tau_1 + k_1 < x < \tau_2 + k_2 \mod X \) and \( E(X) \) is a conjunct in \( \gamma \). Let \( \delta \) be \( \tau_1 + k_1 + 2s < \tau_2 + k_2 \mod X \). It can be assumed that \( \delta \) or \( \neg \delta \) is a conjunct in \( \gamma \). But in the case \( \neg \delta \) we can replace \( \beta \) by one of the atoms \( \tau = \tau_1 + k_1 - l, 1 \leq l \leq 2s \) and use Lemma 10.4. So it can be assumed that \( \delta \) is a conjunct in \( \gamma \). Let \( \alpha_3 = F(s, x) \subseteq X \land \gamma \).

**Lemma 10.7.** \( \exists x \alpha(x) \equiv \exists x \alpha_3(x) \)

**Proof.** \( \alpha(x) \) implies \( \alpha_3(x) \). Conversely, suppose \( \alpha_3(x) \) and \( \log A(x) = X \). The Archimedean o-group \( X^+ \mod X \) is isomorphic to the o-group of integers. Let \( y = 1 \mod X \). Then \( \alpha(x + nsy) \) holds for some \( n \).

**PART 3. CONVEX SUBGROUPS THEORY**

11. Decidability theorem

The Convex Subgroups Language CSL is a first order language whose non-logical constants are \( \langle \cdot \rangle \), the one-place predicate symbol \( E \) and the one-place predicate symbols \( p(s, k, X) > r \) where \( p, s, k \) and \( r \) are naturals, \( p \) is prime and \( 1 \leq k \leq s \). Every o-group \( G \) gives the natural model \( \Delta G \) of CSL as follows. Elements of \( \Delta G \) are proper convex subgroups of \( G \) (a convex subgroup \( X \subseteq G \) is proper if \( X \neq G \)) and \( X < Y \equiv X \subseteq Y \). The predicates \( E(X) \) and \( p(s, k, X) > r \) are defined according to Section 1. The Convex Subgroups Theory CST is the set of CSL-formulas holding in all \( \Delta G \).

**Theorem 11.1.** CST is decidable

Let \( \sigma \) be a finite set of quadruples \( (p, s, k, r) \) of naturals where \( p \) is prime and \( 1 \leq k \leq s \). Let \( L_\sigma \) be a sublanguage of CSL whose non-logical symbols are \( \langle \cdot \rangle, E \) and \( p(s, k) > r \) where \( (p, s, k, r) \in \sigma \). Let \( T_\sigma = L_\sigma \cap \text{CST} \).
Theorem 11.2. \( T_\sigma \) is uniformly decidable on \( \sigma \)

Clearly Theorem 11.2 implies Theorem 11.1

Let \( \Delta_\sigma G \) be the corresponding \( L_\sigma \) projection of the natural CSL-model \( \Delta G \)

Evidently \( T_\sigma \) is the theory of all \( \Delta_\sigma G \)

Let \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) be the corresponding projections of \( \sigma \) and \( s = \max \sigma_2 \). According to Theorem 11 it can be assumed that \( s \) is the only element of \( \sigma_2 \) and it can be assumed also that if \( p \in \sigma_1, 1 \leq k \leq s \) and \( 0 \leq r \leq \max \sigma_3 \), then \( (p, s, k, r) \in \sigma \)

The following abbreviations are used

\[
\begin{align*}
    p(k, x) > r & \quad \text{for} \quad p(s, k, x) > r, \\
    p(k, x) = 0 & \quad \text{for} \quad \neg (p(k, x) > 0), \\
    p(k, x) = r + 1 & \quad \text{for} \quad p(k, x) > r & \neg (p(k, x) > r + 1), \\
    p(x) = 0 & \quad \text{for} \quad \{p(k, x) = 0 \mid 1 \leq k \leq s\}, \\
    p'(x) = 0 & \quad \text{for} \quad \{p(k, x) = 0 \mid 0 \leq k < s\}, \\
    y = x^+ & \quad \text{for} \quad x < y & \exists z (x < z < y) \lor x = y & \forall u > x \exists z (x < z < u)
\end{align*}
\]

A model \( A \) of \( L_\sigma \) is called a \( \sigma \)-chain (a complete \( \sigma \)-chain) if \( A \) is a chain (a complete chain). The definition of complete chains is found in Section 14

\( K_\sigma \) is the class of complete \( \sigma \)-chains satisfying the following axioms (where \( p \in \sigma_1 \) and \( r, r + 1 \in \sigma_4 \))

K1. \( \exists x \forall y (x \leq y) \),

K2. \( x < y \supset \exists z (x \leq z < z^+ \leq y) \),

K3. \( p(k, x) > r + 1 \supset p(k, x) > r \),

K4. \( p'(x) \neq 0 \supset \exists y (x < y) \land (\forall y > x) \exists z (x < z < y \land p(z) \neq 0) \),

K5. \( E (x) \supset (x < x^+ \lor \forall y (y \leq x)) \land p(s, x) = 1 \)

\( \text{Th}K_\sigma \) is the theory of \( K_\sigma \) in \( L_\sigma \).

Lemma 11.1. \( \text{Th}K_\sigma \) is uniformly decidable on \( \sigma \)

Proof. Let \( C_\sigma \) be the theory of all complete \( \sigma \)-chains in \( L_\sigma \). By Theorem 15.2 \( C_\sigma \) is uniformly decidable on \( \sigma \). But \( \text{Th}K_\sigma \) is finitely axiomatizable in \( T_\sigma \).

Lemma 11.2. \( \Delta_\sigma G \in K_\sigma \)

Proof. Evidently \( \Delta_\sigma G \) is a complete \( \sigma \)-chain and satisfies axioms (K1)-(K3). For axioms (K4) and (K5) see Theorems 12 and 13.

In Section 12 we build a class \( M_\sigma \) of \( \sigma \)-chains such that \( \text{Th}M_\sigma \subseteq \text{Th}K_\sigma \). According to Section 13 for every \( C \in M_\sigma \) there exists an \( \sigma \)-group \( G \) such that \( \Delta_\sigma G \cong C \). So \( T_\sigma \subseteq (\text{according to Section 13}) \text{Th}M_\sigma \subseteq (\text{according to Section 12}) \).
ThK, \subseteq \text{(by the Lemma 11.2)} \subseteq T_\omega. \text{ So, } T_\omega = \text{ThK, and Lemma 11.1 imply Theorem 11.2)}

### 12. \(\sigma\)-chains

**Definition 12.1.** A \(\sigma\)-chain \(S\) is the internal ordinal sum \(\Sigma\{A_i, i \in I\}\) of its convex submodels \(A_i\) on a chain \(I\) if

1. \(S = \bigcup\{A_i, i \in I\}\) and
2. \(i < j\) \(\forall x \in A_i, v \in A_j\) imply \(x < v\)

**Definition 12.2.** An external ordinal sum \(S = \Sigma\{A, i \in I\}\) of \(\sigma\)-chains \(A_i\) on a chain \(I\) is defined as follows. Elements of \(S\) are pairs \((i, x)\) where \(i \in I\) and \(x \in A_i\), \((i, x) < (j, y)\) iff \(i < j\) or \(i = j\) and \(x < y\). And for every one-place predicate symbol \(P\) in \(L_\omega\), \(S \models P(i, x)\) iff \(A_i \models P(x)\). An ordinal multiple \(A \cdot I = \Sigma\{A_i, i \in I\}\) and \(A_i = A\).

**Notations.** Let \(A\) and \(B\) be \(\sigma\)-chains, \(B\) is one-element and \(b \in B\). The following abbreviations are used:

- \(p(k, B) = r\) for \(B \models p(k, b) = r\)
- \(E(B)\) for \(B \models E(b)\)
- \(p'A = 0\) for \(\forall p(a) = 0\) \(a \in A\) and \(k < s\)
- \(pA = 0\) for \(\forall \{p(a) = 0\} a \in A\) and \(k \leq s\)

Let \(U_\omega\) be the class of such one-element \(\sigma\)-chains \(B\) that \(\neg E(B)\). Let \(0_\omega\) denote every \(\sigma\)-chain \(B \in U_\omega\) such that \(\forall p \in \sigma)pB = 0\). Let \(\omega\) (respectively \(\omega^\uparrow\)) be the naturally (resp. inversely) ordered set of natural numbers. Let \(R\) be the chain of reals.

**Definition 12.3.** Let \(F\) be a finite set of \(\sigma\)-chains. An ordinal sum \(\Sigma\{A, i \in I\}\) is called \(F\)-dense if

1. \(\forall i \exists B \in F \setminus A_i B \equiv B\),
2. \(\forall B \in F\{i A, i \equiv B\}\) is dense in \(I\) and
3. \(I\) has neither minimal nor maximal elements

**Lemma 12.1.** Every two \(F\)-dense \(\sigma\)-chains are elementary equivalent

**Proof.** By the Ehrenfeucht Criterion [2]

**Definition 12.4.** An ordinal sum \(S = \Sigma\{A_r, r \in R\}\) is called an \(F\)-shuffling and is denoted by \(\tau F\) if

1. \(S\) is \(F\)-dense,
2. \(\exists B \in F\) \(B\) is not one-element,
3. \(\exists B \in F\{r A, r \neq B\}\) is countable and
(4) if \( \{ r \ A, \neq B \} \) is countable and \( B \) is one-element then \( B = 0, \)

**Definition 12.5.** Let \( \mathcal{M} \) be the least class of \( \sigma \)-chains such that

1. If \( \sigma \)-chain \( A \) is one-element and \( p'A = 0 \) then \( A \in \mathcal{M} \),
2. If \( A, B \in \mathcal{M} \) then \( A + B \in \mathcal{M} \),
3. If \( A \in \mathcal{M} \) then \( A \omega \in \mathcal{M} \),
4. If \( A \in \mathcal{M} \), \( B \in U_\sigma \) and \( (\forall p \in \sigma)(pA = 0 \supset p'B = 0) \), then \( B + A \omega^* \in \mathcal{M} \).
5. \( C + F \in \mathcal{M} \) if \( C \in U_\sigma \) and finite \( F = F_1 \cup F_2 \) where non-zero \( F_1 \subseteq \{ A + B \ A \in M_\sigma \text{ and } B \in U_\sigma \} \) and \( F_2 \subseteq U_\sigma \), and
\[
(\forall p \in \sigma)[((\forall D \in F)pD = 0) \supset p'C = 0]
\]

**Theorem 12.1.** \( \mathcal{M} \subseteq \text{Th} K_\sigma \)

**Proof.** It is enough to prove that for every \( n = 1, 2 \), every \( A \in \mathcal{K} \), is \( n \)-equivalent to some \( B \in \mathcal{M} \). Fix \( n \)

**Definition 12.6.** \( \sigma \)-chain \( A \) will be called **good** if it satisfies one of the following requirements

- (G1) \( A \) is \( n \)-equivalent to some \( B \in \mathcal{M} \),
- (G2) \( A \) does not have the minimal element and \( B + A \) satisfies (G1) for every \( B \in U_\sigma \) such that, for every \( p \in \sigma \), and \( a \in A \), if \( p'B \neq 0 \) then \( (\exists c \in A)(c < a \text{ and } pc \neq 0) \),
- (G3) \( A \) is one-element and \( (\exists p \in \sigma)p'A \neq 0 \) and
- (G4) \( A \equiv A' + B \) where \( A' \) satisfies (G1) or (G2) and \( B \) satisfies (G3)

**Lemma 12.2.** If a good \( \sigma \)-chain \( A \in \mathcal{K}_\sigma \), then \( A \) satisfies (G1)

**Proof.** Clear

**Definition 12.7.** \( \sigma \)-chain \( A \) is called **quasi-good** if every non-void half-closed interval \( [x, y) = \{ z \mid x \leq z < y \} \) in \( A \) is good

**Lemma 12.3.** Every quasi-good \( \sigma \)-chain is good

**Proof.** See the proof of Lemma 14.3

**Lemma 12.4.** Every \( \sigma \)-chain in \( K_\sigma \) is good

**Proof.** See the proof of Lemma 14.4

Theorem 12.1 is proved
13. Constructing o groups

**Theorem 13.1.** For every \( \sigma \)-chain \( A \in M_\sigma \) there exists an o-group \( G \) such that \( \Delta_\sigma G \) is isomorphic to \( A \)

**Proof.** By an induction on \( A \) Desired o-groups will be constructed as subgroups of lexicographic sums of countable Archimedean o-groups. The operations of gluing and interlacement of Section 5 preserve this property.

Let \( A \) be one-element Then \( (\forall p \in \sigma) pA = 0 \) If the only element of \( A \) satisfies the predicate \( E \) then the naturally ordered group of natural numbers is a desired o-group. Let \( A \in U_\sigma \). By the Gluing Theorem of Section 5 it can be assumed that \( \forall p (pA = 0) \) or \( pA = 1 \) for some \( p \) and \( qA = 0 \) for every \( q \neq p \). So \( Q \) or \( Q_p \) (see Section 3) is a desired o-group.

Let \( A = B_1 + B_2 \) and \( B_i = \Delta_\sigma H_i, \ i = 1, 2 \). Then the lexicographic sum \( H_1 + H_2 \) is a desired o-group.

Let \( A = B \) and \( B \approx \Delta_\sigma H \). Then \( H \omega = L \Sigma \{ H, \ t \in \omega \} \) is a desired o-group. Let \( A = C + B \). \( B \approx \Delta_\sigma H \) and \( C \in U_\sigma \). If \( C \equiv 0_\sigma \), then \( H \omega^* = L \Sigma \{ H, \ t \in \omega^* \} \) is a desired group. Let \( p(k, C) \neq 0 \) for some \( p \) and \( k \).

It can be assumed that \( p(k, C) = 1 \) and \( q(l, C) = 0 \) if \( q \neq p \) or \( l \neq k \). Indeed let \( p(k, C_p) = 1 \) and \( q(l, C_p) = 0 \) if \( q \neq p \) or \( l \neq k \) and let \( \Delta_\sigma H_p \equiv C_p + B \omega^* \). Then a suitable interlacement of o-groups \( H_p \) (see the Interlacement Theorem in Section 5) is a desired o-group.

If \( H \) is not \( p \)-divisible then the o-group \( G \) of Section 2 is a desired group. Let \( H \) be \( p \)-divisible. It can be assumed that \( H \) is a lexicographic multiple of \( Q \). Indeed, let \( H' = L \Sigma \{ H, \ t \in \Delta_\sigma H \} \) and \( \Delta_\sigma G' \equiv C + \Delta_\sigma H' \omega^* \). Then a gluing of \( H \omega^* \) and \( G' \) is a desired o-group. It can be assumed that \( H = Q \). Indeed if \( H \neq Q \) then \( H \) is an interlacement of \( Q \) and some \( H' \). Let \( \Delta_\sigma G' \equiv C + Q \omega^* \). Then a suitable interlacement of \( H' \omega^* \) and \( G' \) is a desired o-group.

Now the group \( G \) of Section 3 is a desired o-group.

Suppose

\[
A = D + \tau(F_1 \cup F_2), \quad D \in U_\sigma,
\]

\[
F_1 = \{ B, + C, \ 1 \leq i \leq m \ \text{and} \ B_i = \Delta_\sigma H_i, \ \text{and} \ C_i \in U_\sigma \}
\]

and

\[
F_2 = \{ C, \ m < i \leq n \ \text{and} \ C_i \in U_\sigma \}
\]

W log \( D = 0_\sigma \). Indeed let \( \Delta_\sigma G' \equiv 0_\sigma + \tau(F_1 \cup F_2) \). Then \( \Delta_\sigma (G' / X) \equiv B_1 + C_1 + \tau(F_1 \cup F_2) \) for some convex subgroup \( X \subset G' \) and \( A \equiv D + \Delta_\sigma (G' / X) \omega^* \). See the previous case.

Let \( A = 0_\sigma + \{ D, \ r \in \mathbb{R} \} \) and
\[ R_i = \begin{cases} \{t \in \mathbb{R} \mid D_t = B_i + C_i\}, & \text{if } i \leq m, \\ \{t \in \mathbb{R} \mid D_t = C_i\}, & \text{if } m < i \end{cases} \]

Wlog \( m = 1 \) If \( m > 1 \) let \( A_i = 0_o + \Sigma\{F_r \mid r \in \mathbb{R}\} \) where
\[ F_r = \begin{cases} B_i + C_i, & \text{if } r \in R_i, \\ C_i, & \text{if } r \in R_i \text{ and } m < j, \\ 0_o, & \text{in other cases} \end{cases} \]

and for \( 1 < i \leq m \) let \( A'_i = 0_o + \Sigma\{F'_r \mid r \in \mathbb{R}\} \) where
\[ F'_r = \begin{cases} B_i + C_i, & \text{if } r \in R_i, \\ C_i, & \text{if } i = u, r \in R_i \text{ and } i > 1, \\ 0_o, & \text{in other cases} \end{cases} \]

Let \( \Delta, G_i \equiv A_i, i = 1, m \) Then the corresponding interlacement of o-groups \( G_1, \ldots, G_m \) is a desired o-group.

Below \( B = B_1 \) and \( H = H_1 \).

Wlog \( C_1 = 0_o \) and \( R - R_1 \) is countable for some \( i > 1 \). Indeed, by the definition of shuffling in Section 12, some \( R - R_1 \) is countable and if \( i > 1 \) then \( C_i = 0_o \). Let \( R - R_1 \) be countable. There exists a representation \( R_i = \bigcup\{R_{i,t} \mid t \in I\} \) where summands \( R_{i,t} \) are countable, dense in \( R \) and disjoint. Let \( u \in I \) and \( A_i = 0_o + \Sigma\{F'_r \mid r \in \mathbb{R}\} \) where
\[ F'_r = \begin{cases} B + C_1, & \text{if } r \in R_{i,u}, \\ C_i, & \text{if } i = u, r \in R_i \text{ and } i > 1, \\ 0_o, & \text{in other cases} \end{cases} \]

Let \( \Delta_s G_i \equiv A_i \). The corresponding interlacement of o-groups \( G_i \) is a desired o-group.

Wlog \( C_1 = 0_o \). If \( C_1 \neq 0_o \) let \( R_i = S_1 \cup S_2 \) where summands \( S_i \) are dense in \( R \) and disjoint. Let \( A_i = 0_o + \Sigma\{F'_r \mid r \in \mathbb{R}\} \) where
\[ F'_r = \begin{cases} B + 0_o, & \text{if } r \in S_1, \\ C_i, & \text{if } r \in S_2, \\ 0_o, & \text{in other cases} \end{cases} \]
\[ F''_r = \begin{cases} C_i, & \text{if } r \in S_1 \text{ and } i > 1, \\ B + 0_o, & \text{if } r \in S_2, \\ 0_o, & \text{in other cases} \end{cases} \]

Let \( \Delta_s G_i \equiv A_i \). Then the corresponding interlacement of \( G_1 \) and \( G_2 \) is a desired o-group.

Let \( s = \Sigma\{p(k, C_i) \mid p \in \sigma_1, k \in \sigma_3, 1 < i < n\} \).

Wlog \( s \leq 1 \) The statement is proved by induction on \( s \). Let \( C_1, C_2, \in U_o \) and every \( p(k, C_i) = p(k, C_{i,1}) + p(k, C_{i,2}) \). Let \( R_1 = S_1 \cup S_2 \) and the summands \( S_i \) are dense in \( R \) and disjunctive. Let \( A_i = 0_o + \Sigma\{F'_r \mid r \in \mathbb{R}\} \) where
Let $\Delta, G_i \equiv A_i$. Then the corresponding interlacement of $G_1$ and $G_2$ is a desired o-group.

If $s = 0$ then the lexicographic multiple $H \ R_1$ is a desired o-group.

Suppose $s = p(k, C_2) = 1$ (and so $F_1 = \{B_1, 0_o\}$, $F_2 = \{C_2, 0_o\}$).

Case 1 $H$ is not $p$-divisible. There exists a representation $R_i = \bigcup \{R_i, \ t \in R_i \}$ such that $R_i \cap R_{1,i} = \emptyset$ if $t \neq u$ and every chain $R_i$ is isomorphic to $\omega^+$ and $\lim R_i = t$. For every $t \in R_2$ let $A_t = C_2 + B_1 R_i$ and $G_i$ be the o-group $G$ of Section 2. Then $\Delta, G_i \equiv A_i$ and the interlacement of o-groups $G_i$ is a desired o-group.

Case 2 $H$ is $p$-divisible. Wlog $H$ is a lexicographic multiple of the rational o-group $Q$. Indeed, let subchain $I = \{X \in A^+ H \ X \subseteq X\}$, $H' = Q \ I$ and $A_i = 0_o + \Sigma F_i, \ r \in \mathbb{R}$ such that

$$F_i = \begin{cases} 
\Delta_o H' + 0_o, & \text{if } r \in R_i, \\
C_2, & \text{if } r \in R_2, \\
0_o & \text{in other cases}
\end{cases}$$

Let $\Delta, G_i \equiv A_i$ and $G_2 = H_1 \ R_1$. Then the corresponding gluing of $G_1$ and $G_2$ is desired o-group.

Wlog $H \equiv Q$. Suppose that $H$ is not isomorphic to $Q$. Then $H$ is isomorphic to the lexicographic sum $H_1 + Q + H_2$ where $H_1$ or $H_2$ can be zero-group. Let $A_i = 0_o + \Sigma F_i, \ r \in \mathbb{R}$ such that

$$F_i = \begin{cases} 
\Delta_o Q + 0_o = 0_o + 0_o, & \text{if } r \in R_i, \\
C_2, & \text{if } r \in R_2, \\
0_o & \text{in other cases}
\end{cases}$$

Let $\Delta, G_i \equiv A_i$ and $G_2 = (H_1 + H_2) \ R_1$. Then the corresponding interlacement of $G_1$ and $G_2$ is a desired o-group.

**Lemma 13.1.** Let $X_1, X_2, Y_1, Y_2$ be countable dense subsets of the chain $\mathbb{R}$ of reals and $X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset$. There exists an automorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi X_i = Y_i$, $i = 1, 2$.

**Proof.** Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. It is enough to construct an isomorphism $\phi : X \to Y$ such that $\phi X_i = Y_i$. Indeed, this isomorphism can be extended as follows: $\phi (\lim x_n) = \lim \phi x_n$.

Fix a numeration of $X \cup Y$ by naturals. A 1-1-function $f$ is called admissible if $\text{dom} f$ is finite and $\text{rng}(f | X_i) \subseteq Y_i$. A sequence $f_0, f_1, \ldots$ of admissible functions is constructed as follows: $f_0 = \emptyset$ if $n = 2k$ and $x$ is the element in $X - \text{dom} f_n$ of the minimal number then $f_{n+1}$ is an admissible extension of $f_n$ such that $x \in \text{dom} f_{n+1}$. If $n = 2k + 1$ and $y$ is the element in $Y - \text{rng} f_n$ of the minimal number then $f_{n+1}$ is an
admissible extension of \( f \), such that \( y \in \text{rng} f_{n+1} \). Evidently \( \lim f_n \) is a desired isomorphism.

Now it is clear that the o-group \( G \) of Section 6 is a desired o-group. Theorem 13.1 is proved.

APPENDIX. COMPLETE CHAINS WITH ONE-PLACE PREDICATES

A chain is a linear ordered set. A chain \( A \) is complete if \( A \) satisfies the following second order axiom:

\[
(\forall X \subseteq A)(\forall Y \subseteq A)[X \neq \emptyset \land Y \neq \emptyset \land (\forall x \in X)(\forall y \in Y)x < y \\
\supset \exists z (\forall x \in X)(\forall y \in Y)x \leq z \leq y]
\]

The decidability of the weak monadic second order theory of complete chains is proved in Section 14. The proof uses [11] and [12]. The decidability of the weak monadic second order theory of complete chains with one-place predicates is proved in Section 15, where this theory is reduced to the predecessor theory. A similar reduction was used in [5]. The decision procedures are primitively recursive.

14. Complete chains

\( L_0 \) is the weak monadic second order language whose only non-logical constant is "\(<"." \( K_0 \) is the class of complete chains. \( \text{Th} K_0 \) is the \( L_0 \)-theory of \( K_0 \).

**Definition 14.1.** A chain \( S \) is the internal ordinal sum \( \Sigma \{A_i \mid i \in I\} \) of its convex subchains \( A_i \) on a chain \( I \) if:

1. \( S = \bigcup \{A_i \mid i \in I\} \) and
2. \( i > j, x \in A_i, y \in A_j \) imply \( x < y \)

**Definition 14.2.** An external ordinal sum \( S = \Sigma \{A_i \mid i \in I\} \) of chains \( A_i \) on a chain \( I \) is defined as follows. Elements of \( S \) are pairs \((i, x)\) where \( i \in I \) and \( x \in A_i \), \((i, x) \prec (j, y)\) iff \( i < j \) or \( i = j \) and \( x < y \). In particular \( A + B = \Sigma \{A_i \mid i \in \{0, 1\}, 0 < 1, A_0 = A \quad B_0 = B\} \). The ordinal product \( A \times I = \Sigma \{A_i \mid i \in I \text{ and } A_i = A\} \).

Below \( \omega \) (respectively \( \omega^* \)) is the naturally (respectively inversely) ordered set of natural numbers and \( Q \) is the chain of rationals.

**Definition 14.3.** Let \( F \) be a finite set of chains. An ordinal sum \( \Sigma \{A_i \mid i \in I\} \) is called \( F \)-dense if:

1. every \( A_i \in F \);
2. for every \( B \in F \) the subset \( \{i \mid A_i = B\} \) is dense in \( A \) and
3. \( I \) has neither minimal nor maximal elements.
**Definition 14.4.** In the case $I \subseteq \mathbb{Q}$ an $F$-dense sum is called a *shuffling* of $F$ and is denoted by $\tau F$.

Every two shufflings of $F$ are isomorphic.

**Lemma 14.1.** Every two $F$-dense chains are $L_\omega$-equivalent

**Proof.** By the Ehrenfeucht Criterion [2]

Let $M$ be the minimal class of chains such that:

1. $M$ contains all one-element chains.
2. if $A, B \in M$ and either $A$ contains the last element or $B$ contains the first element then $A + B \in M$.
3. if $A \in M$ and $A$ contains either the first or the last element then $A, \omega$ and $A, \omega^*$ belong to $U$.
4. if a finite $F \subseteq M$ and every member of $F$ contains the first and the last elements then $\tau F \in M$.

Let $Th M$ be the $L_\omega$-theory of $M$.

**Lemma 14.2.** $Th K_n \subseteq Th M$

**Proof.** It is enough to prove that every $A \in M$ is $L_\omega$-equivalent to some $A', \in K_n$. An induction on $A$ and the Ehrenfeucht Criterion [2] are used. The case of one-element $A$ is trivial $(A + B)' = A' + B'$, $(A, \omega)' = A', \omega$ and $(A, \omega)^* = A', \omega^*$. Let $A = \tau F$ and $F' = \{B' \mid B \in F\}$. Then $A$ is $L_\omega$-equivalent to every $F'$-dense sum $\sum \{A_i, i \in R\}$ where $R$ is the chain of reals.

**Theorem 14.1.** $Th M \subseteq Th K_n$

**Proof.** It is enough to prove that for every $n = 1, 2$, every $A \in K_n$ is $n$-equivalent to some $B \in M$. Fix $n$. Chain $A$ will be called good if it is equivalent to some $B \in M$. Chain $A$ will be called quasi-good if every non-void half-closed interval $[x, y) = \{z \mid x \leq z < y\}$ of $A$ is good.

**Lemma 14.3.** Every quasi-good chain is good

**Proof.** There exists $L_\omega$-sentence $\alpha$ such that a chain $A$ is good iff it satisfies $\alpha$. Let $\beta(x, y)$ be obtained from $\alpha$ by the restriction of the quantifiers to the interval $[x, y)$. Lemma 14.3 states that $\forall x(y(x < y \supset \beta(x, y)) \implies \alpha$. So it is enough to prove Lemma 14.3 only for countable chains. Let $A$ be a countable quasi-good chain.

**Case 1** $A$ has the minimal element $a$. If $A = \{a, b\} = \{a, b\} \cup \{b\}$ then $A$ is good. Suppose $A$ does not contain the maximal element and $B$ be a subset of $A$ such that $B \equiv \omega$ and $(\forall x \in A)(\exists y \in B)x < y$. Let $\{x, y\} \rightarrow \{a, b\}$ iff the intervals
[x,y) ∪ [y,x) and [u,v) ∪ [v,u) are non-void and ρ-equivalent. By the Ramsey Theorem [14] there exists an infinite C ⊆ B such that every pair of different elements of C are equivalent. Let b, c ∈ C and b < c. By means of the Ehrenfeucht Criterion [2] it is easy to check that A is n-equivalent to [a,b) ∪ [b,c)ω. So A is good.

Case 2 A does not contain the minimal element. Similarly it is proved that there exists infinite C ⊆ A such that (∀x ∈ A)(∃y ∈ C)y < x and if x, y, u, v ∈ C and x < y, u < v then [x,y) and [u,v) are n-equivalent. Let b, c ∈ C and b < c. Then A is n-equivalent to [b,c)ω ∪ {x < c ≤ x} and A is good.

Lemma 14.4. Every complete chain is good.

Proof. Let A be a complete chain. For x,y ∈ A let x ∼ y iff x = y or x ≠ y and [x,y) ∪ [y,x) is quasi-good. The introduced relation is an equivalence relation. Every ̃x = {y | x ∼ y} is convex, quasi-good and good. Let ̃A = {x | x ∈ A} be ordered as follows ̃x < ̃y ≡ x < y ̃A is a dense chain. If ̃A is one-element Lemma 14.4 is proved. Suppose (reductio ad absurdum) ̃A is not one-element. For ̃x < ̃y let F(̃x, ̃y) be a minimal subset of M such that every ̃z ∈ (̃x, ̃y) is n-equivalent to some B ∈ F(̃x, ̃y). Let F = F(̃α, ̃β) have the minimal possible power. Then ∪{̃z | ̃u < ̃z < ̃v} is n-equivalent to an F-dense chain and is quasi-good. This contradicts to density of ̃A. Lemma 14.4 is proved.

Theorem 14.1 is proved.

Theorem 14.2. Th M is decidable.

Proof. We assume the knowledge of [12]. Let n ≥ 2. We say that n-type t is l-good (r-good) if t_n(A) = t implies A ⊨ ∃x ∀y (x ≤ y) (A ⊨ ∃x ∀y (y ≤ x)) The predicates "l-good" and "r-good" are effective. Let S_n be the least set of n-types such that

1. n-type of one-element chains belongs to S_n,
2. if s,t ∈ S_n and either s is r-good or t is l-good then s + n t ∈ S_n,
3. if s ∈ S_n and s is either l-good or r-good thenω_n(s), ω^*_n(s) ∈ S_n,
4. if X ⊆ S_n and every s ∈ X is l-good and r-good then σ_n(X) ∈ S_n.

It's easy to see that S_n is the set of n-types of M and S_n effectively depends on n. So Th M is decidable.

Lemma 14.2 and Theorems 14.1 and 14.2 imply

Theorem 14.3. Th K_e is decidable.
15. Adding one-place predicates

Let \( L_m \) be the weak monadic second order language whose non-logical constants are \( "<" \) and the one-place predicate symbols \( P_1, \ldots, P_m \). Let \( K_m \) be the class of such \( L_m \)-models \( A \) that \( L_m \)-reduction of \( A \) is a complete chain. Let \( K_n \) be the class of such models \( A \in K_m \) that \( A \) satisfies the following axioms

\[
\forall {P_i(x) \mid 1 \leq i \leq m} \quad \text{and} \\
P_i(x) \supset \neg P_j(x) \quad \text{where} \ 1 \leq i < j \leq m
\]

(In other words \( A \models \forall x \exists ! i P_i(x) \))

Let \( \text{Th}_{K_m} \) (respectively \( \text{Th}_{K_n} \)) be the \( L_m \)-theory of \( K_m \) (resp \( K_n \))

**Lemma 15.1.** \( \text{Th}_{K_m} \) is uniformly on \( m \) reducible to \( \text{Th}_{K_n} \) where \( n = 2^m \)

**Proof.** Clear

**Theorem 15.1.** \( \text{Th}_{K_n} \) is uniformly on \( n \) reduced to \( \text{Th}_{K_m} \)

**Proof.** The following abbreviations are used:

(i) \( y = x^- \) for \( y < y \) \& \( \neg \exists z \ (x < z < y) \)

\[
\forall x = y \ (\forall u > x) \exists z \ (x < z < u).
\]

(ii) \( y = x \) for \( y < x \) \& \( \neg \exists z \ (y < z < x) \)

\[
\forall x = y \ (\forall u < x) \exists z \ (u < z < x).
\]

(iii) \( R_i(x) \) for \( x \in \mathbb{N} \)

\[
\begin{align*}
\exists x_2 & \ x_i \ [\{ x_i < x_i^+ = x_{i-1}, 1 \leq i < j \ \&\ x_j = x_j^+ \}, \text{if} \ j = 1, \\
\exists x_2 & \ x_n \ [\{ x_i < x_i^+ = x_{i-1}, 1 \leq i < n \}, \text{if} \ j = n.
\end{align*}
\]

Let \( \beta \) be an \( L_n \)-sentence and \( \beta' \) be obtained from \( \beta \) by

(i) the restriction of quantifiers by \( x = x^- \) \&

(ii) the replacing of every \( P_i \) by \( R_i \).

Let \( \alpha = (\beta' \& \exists x (x = x^-)) \)

**Lemma 15.2.** \( \beta \) has a model in \( K_n \) iff \( \alpha \) has a model in \( K_m \)

**Proof.** Let \( A \in K_n \) and \( A \models \alpha \). Let \( A' = \{ x \in A \mid x = x^- \} \) \( A' \) is complete. The definitions \( P_i(x) \equiv R_i(x) \) turn \( A' \) to an \( L_n \)-model satisfying \( \beta \)

Let \( B \in K_n \) and \( B \models \beta \). Let \( A \) be the ordinal sum \( \sum \{ C_b \mid b \in B \} \) of chains \( C_b \) which are defined as follows. Let \( B \models \beta \) (b) If \( x < x' \) let \( C_b = t + \omega^{x'} + \omega \) (where \( t \) denotes a chain containing exactly \( t \) elements). If \( x = x' \) let \( C_b = t + \omega^{x} \). It is easy to check that \( A \) is complete and \( A \models \alpha \).
Because $\beta$ is an arbitrary formula of $L_\omega$, Lemma 15.2 implies Theorem 15.1.

From Theorem 14.3, Theorem 15.1 and Lemma 15.1 we obtain

**Theorem 15.2.** The $K_\omega$ is uniformly decidable on $m$.

**References**


