

Spectra of Monadic Second-Order Formulas with One Unary Function*

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Abstract

We establish the eventual periodicity of the spectrum of any monadic second-order formula where

- (i) all relation symbols, except equality, are unary, and
- (ii) there is only one function symbol and that symbol is unary.

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1 Introduction

Durand, Fagin and Loescher established the eventual periodicity of the spectrum of any monadic first-order formula with one unary function symbol [3]. (They also discuss the cases of two unary function symbols and one binary function symbol; in either of those cases the spectra are more complicated.) In this paper, we are interested in monadic second-order logic (MSO).

Let us recall the relevant definitions. The *spectrum* of a first-order or second-order formula is the set of the cardinalities of its finite models. A set S of natural numbers is *eventually periodic* if there exist natural numbers θ and $p > 0$ such that for every $n \geq \theta$, if S contains n then it contains $n + p$.

Monadic first-order formulas with one unary function symbol are first-order formulas subject to the following two restrictions:

- All predicate symbols, with the exception of equality, are unary.
- There is only one function symbol. It is unary.

MSO formulas with one unary function symbol are defined similarly.

Upon learning the result of Durand, Fagin and Loescher, we noticed that the monadic second-order composition method gives the following generalization.

Theorem 1.1 (Main Theorem) *The spectrum of any monadic second-order formula with one unary function symbol is eventually periodic.*

Durand, Fagin and Loescher kindly mentioned our generalization at the end of their paper. For no reason, a proof sketch of the generalization was lying idle all these years. In this paper we prove the Main Theorem. Additional results will appear in [17]. We give also a direct proof of the decidability of the finite satisfiability of MSO formulas with one unary function symbol. The fact of decidability is known [12, 1].

The paper is organized as follows. Section 2 presents the basics of the composition method. Sections 3–6 prepare the ground for the proof of the Main Theorem; the proof itself is given in Section 7. Specifically, Section 3 gives

simple facts on eventual periodicity, Section 4 introduces structures of relevance to the proof of the Main Theorem, Section 5 reduces the Main Theorem to a similar theorem where the role of formulas is played by special finite fragments of the theories of function graphs, and Section 6 introduces the relevant operations on the relevant structures. The direct decidability result is proven in Section 8.

2 The Composition Method

Under certain circumstances, a composition of structures gives rise to a composition of their appropriately defined types. That observation lies in the heart of the composition method. We explain a simple version of the method that goes a long way and that is sufficient for our purposes in this paper. At the end, we mention some further developments. Lower your expectation of the feasibility of algorithms.

Let L be a purely relational language of finite vocabulary, X, Y be models for L , and x_1, x_2, \dots (resp. y_1, y_2, \dots) be elements of X (resp. elements of Y). In the running example of this section and in the rest of this paper, X and Y are finite, but in general they may be infinite. We write $\varphi(v_1, \dots, v_j)$ to indicate that φ is a formula with free variables among v_1, \dots, v_j .

Definition 2.1 The 0 -theory $th^0(X, x_1, \dots, x_j)$ is the set of atomic formulas $\varphi(v_1, \dots, v_j)$ such that $X \models \varphi(x_1, \dots, x_j)$, and the $(d+1)$ -theory $th^{d+1}(X, x_1, \dots, x_j)$ is the set of d -theories $th^d(X, x_1, \dots, x_j, x_{j+1})$ where x_{j+1} ranges over the elements of X . \square

Let $\text{Atom}(j)$ be the set of atomic formulas with variables among v_1, \dots, v_j , $\text{Box}^0(j)$ be the powerset of $\text{Atom}(j)$, and $\text{Box}^{d+1}(j)$ be the powerset of $\text{Box}^d(j+1)$. Every $\text{Box}^d(j)$ is hereditarily finite.

Lemma 2.2 *Every d -theory $th^d(X, x_1, \dots, x_j)$ belongs to $\text{Box}^d(j)$.*

Proof Induction on d . \square

It follows that every $th^d(X, x_1, \dots, x_j)$ is finite. A bound on the cardinality of $th^d(X, x_1, \dots, x_j)$ is computable from d and j .

The d -theory $th^d(X, x_1, \dots, x_j)$ is closely associated with the set of formulas $\varphi(v_1, \dots, v_j)$ of quantifier depth d such that $X \models \varphi(x_1, \dots, x_j)$.

Lemma 2.3 *For every $t \in \text{Box}^d(j)$, there is a formula $\varepsilon_t(v_1, \dots, v_j)$ of quantifier depth d such that the following hold for all X and all x_1, \dots, x_j .*

1. $th^d(X, x_1, \dots, x_j) = t$ if and only if $X \models \varepsilon_t(x_1, \dots, x_j)$.
2. If $th^d(X, x_1, \dots, x_j) = t$ then, for every formula $\varphi = \varphi(v_1, \dots, v_j)$ of quantifier depth $\leq d$, we have
 - if $X \models \varphi(x_1, \dots, x_j)$ then ε_t implies φ , and
 - if $X \models \neg\varphi(x_1, \dots, x_j)$ then ε_t implies $\neg\varphi$.

Proof Both claims are proved by induction on d , and both times the case $d = 0$ is obvious. Let $d = c + 1 > 0$.

1. The desired $\varepsilon_t(v_1, \dots, v_j)$ is the conjunction of formulas $\exists v_{j+1} \varepsilon_s(v_1, \dots, v_{j+1})$ where $s \in t$ and formulas $\neg \exists v_{j+1} \varepsilon_s(v_1, \dots, v_{j+1})$ where $s \in \text{Box}^c(j+1) - t$.

2. Assume that $th^d(X, x_1, \dots, x_j) = t$. Every formula $\varphi = \varphi(v_1, \dots, v_j)$ of quantifier depth $\leq d$ is equivalent to a Boolean combination of formulas $\exists v_{j+1} \varphi_i(v_1, \dots, v_{j+1})$ where the quantifier depth of every φ_i is $\leq c$. Therefore it suffices to prove the second claim for the case when φ has the form $\exists v_{j+1} \psi(v_1, \dots, v_{j+1})$.

First suppose that $X \models \varphi(x_1, \dots, x_j)$. Then there exists x_{j+1} such that $X \models \psi(x_1, \dots, x_{j+1})$. Let $s = th^c(X, x_1, \dots, x_{j+1})$. Clearly, $s \in t$. By the induction hypothesis, ε_s implies ψ . It follows that $\exists v_{j+1} \varepsilon_s(v_1, \dots, v_{j+1})$ implies φ , and therefore ε_t implies φ .

Second suppose that $X \models \neg\varphi(x_1, \dots, x_j)$. By contradiction suppose that ε_t is consistent with φ . Then there exist Y and y_1, \dots, y_{j+1} such that $Y \models \varepsilon_t(y_1, \dots, y_j) \wedge \psi(y_1, \dots, y_{j+1})$. By the first claim, $th^d(Y, y_1, \dots, y_j) = t$, and so the c -theory $s = th^c(Y, y_1, \dots, y_{j+1})$ belongs to t . By the induction hypothesis, ε_s implies ψ . Hence $\exists v_{j+1} \varepsilon_s$ implies φ . Since $s \in t = th^d(X, x_1, \dots, x_j)$,

we have $X \models \varepsilon_s(x_1, \dots, x_j, x_{j+1})$ for some x_{j+1} . Hence $X \models \varphi(x_1, \dots, x_j)$ which is impossible. \square

Corollary 2.4 *There is an algorithm that, given a d -theory $th^d(X)$ and a sentence φ of quantifier depth d decides whether $X \models \varphi$.*

Corollary 2.5 *For every sentence φ of quantifier depth d , there are d -theories t_1, \dots, t_k such that any $X \models \varphi$ if and only if $th^d(X) \in \{t_1, \dots, t_k\}$.*

There is a close connection between finite theories and Ehrenfeucht-Fraïssé games.

Lemma 2.6 *The following are equivalent:*

1. $th^d(X, x_1, \dots, x_j) = th^d(Y, y_1, \dots, y_j)$,
2. the duplicator has a winning strategy in $EF^d((X, x_1, \dots, x_j), (Y, y_1, \dots, y_j))$.

Proof Induction on d . The case $d = 0$ is obvious. Suppose that $d = c + 1 > 0$.

$1 \rightarrow 2$. Without loss of generality, on the first step, the spoiler chooses some element x_{j+1} in X . The c -theory $th^c(x_1, \dots, x_{j+1})$ belongs to $th^d(X, x_1, \dots, x_j)$ and therefore to $th^d(Y, y_1, \dots, y_j)$. So the duplicator can find an element $y_{j+1} \in Y$ such that $th^c(X, x_1, \dots, x_{j+1}) = th^c(Y, y_1, \dots, y_{j+1})$. In the rest of the game, the duplicator uses the winning strategy guaranteed by the induction hypothesis.

$2 \rightarrow 1$. By the virtue of symmetry, it suffices to prove that every member $th^c(X, x_1, \dots, x_{j+1})$ of $th^d(X, x_1, \dots, x_j)$ equals some member $th^c(Y, y_1, \dots, y_{j+1})$ of $th^d(Y, y_1, \dots, y_j)$. Consider a play where the spoiler starts by choosing an element x_{j+1} in X and the duplicator chooses some y_{j+1} in Y such that he has a winning strategy in the remainder game $EF^d((X, x_1, \dots, x_{j+1}), (Y, y_1, \dots, y_{j+1}))$. By the induction hypothesis, $th^c(X, x_1, \dots, x_{j+1}) = th^c(Y, y_1, \dots, y_{j+1})$. \square

To illustrate the most basic version of the composition method, we give a simple example. Later in this paper we define a finite forest as a finite acyclic directed graph, where every vertex has at most one outgoing edge, together with a unary relation Orphan that consists of the vertices without outgoing edges, the *orphans*. (Actually we will use finite forests that may be endowed with additional unary relations, but for now let's forgo that expansion.)

Every finite forest is obtained from a singleton forest by means of the disjoint union operation $X + Y$ and the unary operation X' that attaches a root to the given forest X and thus turns it into a tree.

Lemma 2.7

1. $th^d(X, x_1, \dots, x_j)$ and $th^d(Y, y_1, \dots, y_k)$ uniquely determine $th^d(X + Y, x_1, \dots, x_j, y_1, \dots, y_k)$.
2. $th^d(X, x_1, \dots, x_j)$ uniquely determines $th^d(X', x_1, \dots, x_j)$.

Proof Use Ehrenfeucht-Fraïssé games (and Lemma 2.6). □

This lemma can be called the first-order composition lemma for finite forests. It allows us to define two operation $th^d(X, x_1, \dots, x_j) + th^d(Y, y_1, \dots, y_k) = th^d(X + Y)$ and $(th^d(X, x_1, \dots, x_j))' = th^d(X', x_1, \dots, x_j)$. For technical reasons, we generalize the second operation: $(th^d(X, x_1, \dots, x_j))'_k = th^d(X', x_1, \dots, x_j, y_1, \dots, y_k)$ where each y_i is the orphan of X' . The case $d = 0$ is obvious. By the induction hypothesis, we have the following.

$$\begin{aligned}
& th^{d+1}(X + Y, x_1, \dots, x_j, y_1, \dots, y_k) \\
&= \{th^d(X + Y, x_1, \dots, x_j, x_{j+1}, y_1, \dots, y_k) : x_{j+1} \in X\} \\
&\quad \cup \{th^d(X + Y, x_1, \dots, x_j, y_1, \dots, y_k, y_{k+1}) : y_{k+1} \in Y\} \\
&= \{th^d(X, x_1, \dots, x_{j+1}) + th^d(Y, y_1, \dots, y_k) : x_{j+1} \in X\} \\
&\quad \cup \{th^d(X, x_1, \dots, x_j) + th^d(Y, y_1, \dots, y_{k+1}) : y_{k+1} \in Y\}.
\end{aligned}$$

$$\begin{aligned}
& th^{d+1}(X', x_1, \dots, x_j, y_1, \dots, y_k) \\
&= \{th^d(X', x_1, \dots, x_j, x_{j+1}, y_1, \dots, y_k) : x_{j+1} \in X\} \\
&\quad \cup \{th^d(X', x_1, \dots, x_j, y_1, \dots, y_k, y_{k+1}) : y_{k+1} \text{ is an orphan of } X'\} \\
&= \{(th^d(X, x_1, \dots, x_j, x_{j+1})'_k : x_{j+1} \in X\} \\
&\quad \cup \{(th^d(X, x_1, \dots, x_j))'_{k+1}\}
\end{aligned}$$

Obviously, these operations on d -theories are computable.

Proposition 2.8 *The first-order theory of finite forests is decidable.*

Proof By Corollary 2.4, it suffices to show that the set S_d of the d -theories of finite forests is computable. Every member of S_d is obtained from the d -theory of the singleton forest by means of the two operations on d -theories. Since S_d is finite, this allows us to compute S_d . \square

Now we turn to MSO logic. It is convenient to represent second-order structures as special first-order structures.

Definition 2.9 The *Boolean associate* $\mathfrak{B}(X)$ of a structure X is the atomic Boolean algebra whose atoms are the elements of X together with

- the relations of X as relations of $\mathfrak{B}(X)$, and
- an additional unary relation Atomic that consists of the atoms. \square

Definition 2.10 The MSO d -theory $Th^d(X)$ is the first-order d -theory $th^d(\mathfrak{B}(X))$. \square

Remark 2.11 The definition of first-order d -theories above had a prerequisite: the first-order language is purely relational. The point of the prerequisite is to ensure that every $\text{Box}^0(j)$ is finite. In the presence of function symbols, the set $\text{Atom}(1)$ is already infinite. The prerequisite can be waived for Boolean associates because, in the case of Boolean algebras, we have only finitely many inequivalent terms. But then the definition of $\text{Atom}(j)$ should be modified so that only terms in an appropriate normal form are used in atomic formulas. \square

Our example generalizes to MSO.

Lemma 2.12

1. $Th^d(X + Y)$ is uniquely determined by $Th^d(X)$ and $Th^d(Y)$.
2. $Th^d(X')$ is uniquely determined by $Th^d(X)$.

This lemma can be called the MSO composition lemma for finite forests. It allows us to define the operations $Th^d(X) + Th^d(Y) = Th^d(X + Y)$ and $(Th^d(X))' = Th^d(X')$ on d -theories. Both operations on MSO d -theories are computable.

Proposition 2.8 and its proof generalize to the MSO case but we give an alternative proof that bounds the size of the minimal forest with a given MSO d -theory.

Proposition 2.13 *There is a computable function $F(d)$ such that, for every d and every finite forest X , there is a forest Y of cardinality $\leq F(d)$ with $Th^d(X) = Th^d(Y)$.*

Proof Given a number d , build a finite sequence $S = (X_1, X_2, \dots)$ of finite forests as follows. X_1 is the singleton forest. Suppose that X_1, \dots, X_k have been constructed and let i, j range over $\{1, \dots, k\}$. If there is a finite forest Y of the form X'_i or $X_i + X_j$ such that $Th^d(Y)$ differs from any $Th^d(X_i)$, choose any such Y and set $X_{k+1} = Y$; otherwise halt.

The sequence S contains the MSO d -theory of every finite forest. The length of S is bounded by the cardinality of the $\text{Box}^d(0)$ for the first-order language of the Boolean associates of finite forests. The cardinality of any X_{k+1} is bounded by the double of the maximal cardinality of the forests in X_1, \dots, X_k . This gives the desired $F(d)$. \square

The proposition implies that the MSO theory of finite forests is decidable. The decision algorithm is non-elementary, but there is no elementary algorithm for that decision problem. This follows from Stockmeyer's result that the decision problem for the first-order theory of words is known to be non-elementary [18, 14]. Words can be seen as special trees with additional

unary relations. The order relation (on the positions) which is given in Stockmeyer's theorem is MSO definable from the successor relation which is given in our case.

Remark 2.14 One subtlety of the composition method is that it is sensitive to the precise choice of language. We explain this on the example of finite forests where, to simplify notation, we ignore the difference between subsets of a forest X and the elements of $\mathfrak{B}(X)$.

We mentioned above that $Th^d(X')$ is determined by $Th^d(X)$. Let r be the root of X' , let R be the singleton forest containing r , let A_1, \dots, A_j be arbitrary subsets of X' , let $B_i = A_i \cap \{r\}$, and let $C_i = A_i - B_i$. More generally, $Th^d(X', A_1, \dots, A_j)$ is determined by $Th^d(R, B_1, \dots, B_j)$ and $Th^d(X, C_1, \dots, C_j)$. But this claim of unique determination fails if we abandon the unary relation Orphan which might have seemed to play no role until now. Orphan is needed already in the case $d = 0$ and $j \geq 2$. Indeed, let x be any element of X , $A_1 = B_1 = \{r\}$ and $A_2 = C_2 = \{x\}$. In order to determine whether there is an edge from x to r , we need to know whether x is an orphan in X .

This little exercise shows also another subtlety of the method. Notice how neatly each set A_i splits into subsets B_i and C_i . It would be less convenient to work with first-order d -theories $th^d(X, x_1, \dots, x_j)$ instead of second-order d -theories $Th^d(X, A_1, \dots, A_j)$. Elements are not splittable. The method fits MSO better than first-order logic. \square

The origins of the composition method can be traced back to the Feferman-Vaught article on "the first-order properties of products of algebraic systems" [4]. Läuchli introduced d -theories in the context of weak monadic second-order logic; he proved that the weak MSO theory of linear order is decidable [11]. (The weak MSO is the version of MSO where second-order quantification is restricted to finite sets.)

Shelah generalized the method to full MSO and used it in particular to prove in a uniform way all known decidability results for the MSO theories of various classes of linear orders [15]. He introduced (k_1, \dots, k_d) -theories comprising sentences whose prenex form has d blocks of quantifiers: k_1 quantifiers of one kind (say, existential quantifiers), followed by k_2 quantifiers of the other kind, followed by k_3 quantifiers of the first kind, and so on. A d -theory is

a (k_1, \dots, k_d) -theory where every $k_i = 1$. In the theory of linear orders, the main composition lemma is about the addition (that is concatenation) of linear orders. In [15], an important role was played by two generalizations of the classical Ramsey theorem (one for dense orders and another for uncountable well orderings) that take advantage of the fact that, for each d , the d -theories of linear orders together with the derived addition operation form a semigroup.

MSO logic is arguably the right paradigm for the composition method. Composition theorems reduce the theory of a composition of structures to a composition of their finite theories, but — even in the case of first-order composition theorems — the reduction depends on the MSO theory of the index structure. MSO composition theorems reduce the MSO (k_1, \dots, k_d) -theory of a composition of structures to some MSO (ℓ_1, \dots, ℓ_d) -theory of the index structure with disjoint unary relations t where $t(i)$ means that t is the (k_1, \dots, k_d) -theory of the i^{th} component.

The authors used the method on numerous occasions. Article [5] lays a technical foundation for more advanced applications of the method. Section 3 of the survey [6] describes the composition method (calling it the model-theoretic decidability technique) and Section 5 mentions various applications of the method. See also the dissertation [20] and exposition [19]. A recent sophisticated use of the composition method over finite structures is found in [16].

The power of the method is under-appreciated. Throughout the years we saw various problems that could be solved by the composition method. This claim was put to the test a couple of years ago when Alexander Rabinovich posed a conjecture to the first coauthor who insisted that the method will confirm or refute the conjecture. The conjecture was confirmed [7], and Rabinovich went on to use the method [13].

The present paper uses the simple form of the composition method explained and exemplified above.

Remark 2.15 One of the reviewers asked how does the composition method compare with the automata-theoretic method. The application domains of the two methods intersect. For example, the decidability of S1S, established first by means of Büchi automata [2], has also a simple model-theoretic proof [15, 6]. Rabin used automata to prove the decidability of the MSO theory,

known as S2S, of the infinite binary tree [12, 1]. It is not clear whether the composition theory can be used for the purpose. One of the consequences of Rabin’s result is the decidability of the MSO theory of rational order. This consequence was proved, more directly and naturally, by the composition method [18]. This alternative proof generalizes to a class of dense linear orders (so-called short modest linear orders) [8] to which the automata method does not seem to apply. At least on one occasion, the two methods were used in a complimentary way [9]. The automata method has been used much more for establishing complexity results. Lemma 8.1 is one modest example of the use of the composition method for that purpose. \square

The composition method is inherently model-theoretic, like that of Ehrenfeucht-Fraïssé; it may be a useful addition to your toolbox.

3 Eventually Periodic Sets

Definition 3.1 A set S of natural numbers is *eventually periodic* if there exist natural numbers $p > 0$ (a *period* of S) and θ (a *p -threshold* for S) such that $n \in S$ implies $n + p \in S$ for all $n > \theta$. \square

The definition of eventual periodicity in [3] is similar except that “implies” is replaced with “is equivalent to”. The following lemma shows that this alteration makes no real difference.

Lemma 3.2 *Assume that a set S is eventually periodic with a period p and a p -threshold θ . There exists a natural number θ' such that*

$$n \in S \text{ is equivalent to } n + p \in S \text{ for all } n \geq \theta'.$$

The number θ' could be called a *strict p -threshold* for S .

Proof The lemma is trivial if S is finite, so we assume that S is infinite. For each $i = 0, \dots, p - 1$, let

$$A_i = \{n \in S : n \geq \theta \text{ and } n = i \pmod{p}\}.$$

The desired strict p -threshold $\theta' = \max\{\min(A_i) : A_i \neq \emptyset\}$. \square

Corollary 3.3

Every eventually periodic set S has a least period, and the least period divides any other period of S .

Proof It suffices to prove that the greatest common divisor p of periods p_1, p_2 is a period. Let θ_i be a strict p_i -threshold and $\theta = \max(\theta_1, \theta_2)$. We show that θ is a p -threshold for S .

Since $p = \gcd(p_1, p_2)$, there exist integers a_1, a_2 such that $a_1 p_1 + a_2 p_2 = p$. Without loss of generality, $a_1 > 0$ and $a_2 < 0$. Suppose that $n \geq \theta$ and $n \in S$. Since θ_1 is a p_1 -threshold, $n + a_1 p_1 \in S$. Since θ_2 is a strict p_2 -threshold, $n + p = (n + a_1 p_1) - |a_2| \cdot p_2 \in S$. \square

Recall that an arithmetic progression is a set of integers of the form

$$\{b + jp : j = 0, 1, \dots\}.$$

where $p > 0$.

Lemma 3.4 ([3]) *A set of natural numbers is eventually periodic if and only if it is a finite union of arithmetic progressions and singleton sets.*

Corollary 3.5 *The union of finitely many eventually periodic sets is eventually periodic.*

Lemma 3.6 *If the sets S_1, \dots, S_m are eventually periodic then so is the set*

$$\{n_1 + \dots + n_m : \text{every } n_i \in S_i\}$$

Proof Suppose that S_i is eventually periodic with period p_i and p_i -threshold θ_i , and let p be the least common multiple of p_1, \dots, p_m . Then S is eventually periodic with period p and a p -threshold $\theta = \theta_1 + \dots + \theta_m$. Indeed suppose that $n \in S$ and $n \geq \theta$. Then $n = n_1 + \dots + n_m$ where $n_i \in S_i$. There is an index i such that $n_i \geq \theta_i$. We have

$$n + p = n_1 + \dots + n_{i-1} + (n_i + p) + n_{i+1} + \dots + n_m \in S. \quad \square$$

4 Structures of Interest

In the rest of the paper, all structures are finite. It is often convenient to view unary relations as sets.

Definition 4.1 A *partial-function graph* is a directed graph where every vertex has at most one outgoing edge together with the unary relation Orphan that consists of the vertices without outgoing edges. The expression “partial-function graph” may be abbreviated to “PF-graph”. \square

The relation Orphan is added for technical reasons; we saw already in Remark 2.14 that the composition method is sensitive to the language. Orphan allows us to express the absence of orphans in a quantifier-free way: Orphan = \emptyset .

Definition 4.2 A *function graph* is a PF-graph where every vertex has exactly one outgoing edge. \square

The edge relation of a PF-graph will be denoted by E . If $E(x, y)$ holds (so that there is an edge from x to y), then y is the *parent* of x , and x is a *child* of y . Children of the same parent are *siblings*. A vertex without a parent is an *orphan*.

A nonempty sequence x_0, \dots, x_n of vertices such that $E(x_i, x_{i+1})$ holds for every $i = 0, \dots, n - 1$ is a *path of length n from x_0 to x_n* . A path is *trivial* if its length is zero; otherwise the path is *proper*. If there is a path from x to y then x is a *descendant* of y and y is an *ancestor* of x . If there is a proper path from x to y then x is a *proper descendant* of y and y is a *proper ancestor* of x .

An n -vertex substructure of a PF-graph is a *cycle* if the n vertices form a proper path x_0, \dots, x_{n-1}, x_0 . A vertex is *cyclic* if it belongs to a *cycle*. A PF-graph without a cycle is *acyclic*.

A *colored PF-graph* X is a PF-graph together with a finite collection of unary relations, the *colors*. The set of the names of the unary relations is the *palette* of X . It is convenient to think of colors as sets of vertices.

Proviso 4.3 *In the rest of this paper, a PF-graph is always finite and colored.*

A PF-graph is *connected* if every two vertices have a common ancestor. A connected PF-graph has at most one orphan.

Definition 4.4 A *tree* is a connected PF-graph with an orphan; the orphan is the *root* of the tree. \square

A connected function graph is a connected PF-graph without an orphan. It has a unique cycle and is formed by the cyclic vertices and their descendants.

In any PF-graph, the relation “ x and y have a common ancestor” is an equivalence relation. It partitions the PF-graph into connected components.

Definition 4.5 A *forest* is an acyclic PF-graph. \square

A PF-graph X is a forest if and only if every component of X is a tree if and only if every component of X has an orphan. A PF-graph X is a function graph if and only if no component of it has an orphan.

5 Finite Theories

An *MSO PF-graph formula* χ is an MSO formula in the vocabulary of PF-graphs. The *spectrum* of χ is the set of the cardinalities of PF-graphs satisfying χ .

Lemma 5.1 *For every MSO formula φ with one unary function symbol, there exists an MSO PF-graph formula χ such that (i) every PF-graph satisfying χ is a function graph and (ii) the spectrum of φ equals the spectrum of χ .*

Proof We start with a construction. First, replace in φ the unary function with its graph. For example, a subformula $f(x) = y$ becomes $E(x, y)$, and a subformula $f(f(x)) = f(y)$ may become

$$\exists x' \exists z (E(x, x') \wedge E(x', z) \wedge E(y, z))$$

Second, augment the resulting PF-graph formula with a conjunct $\text{Orphan} = \emptyset$. This gives the desired χ .

Claim (i) is obvious. To prove (ii), notice that every model X of φ gives rise to a function graph Y satisfying χ such that $|X| = |Y|$; in fact, the underlying set of Y is that of X . Let φ' be the formula obtained from χ by replacing every atomic formula $E(x, y)$ with equation $f(x) = y$; clearly φ' is equivalent to φ . Every function graph Y satisfying χ gives rise to a model X of φ' , and therefore of φ , such that $|Y| = |X|$; in fact, the underlying set of X is that of Y . \square

Fix an arbitrary palette π .

Proviso 5.2 *In the rest of the paper, PF-graphs are of palette π . Accordingly, MSO PF-graph formulas use only colors of palette π .*

According to Definition 2.9, the *Boolean associate* $\mathfrak{B}(X)$ of a PF-graph X is the finite Boolean algebra with the vertices of X as the atoms together with the edge relation of X as a binary relation of $\mathfrak{B}(X)$, the colors of X as unary relations of $\mathfrak{B}(X)$, and an additional unary relation Atomic that consists of the atoms. Let Υ be the vocabulary of $\mathfrak{B}(X)$. Notice that the vocabulary Υ does not depend on the choice of X .

Lemma 5.3 *For every MSO PF-graph formula χ , there is a first-order Υ -formula ψ such that, for every PF-graph X , we have*

$$X \models \chi \quad \text{if and only if} \quad \mathfrak{B}(X) \models \psi$$

Proof Obvious. \square

According to Definition 2.10, the MSO d -theory $Th^d(X)$ of a PF-graph X is the first-order d -theory $th^d(\mathfrak{B}(X))$ of the Boolean associate $\mathfrak{B}(X)$ of X . In the rest of the paper, we deal only with MSO d -theories $Th^d(X)$ of PF-graphs X and never with first-order d -theories $th^d(X)$. And so the d -theory of a PF-graph X will mean $Th^d(X)$.

By Lemma 2.6, two PF-graphs X and Y have the same d -theory if and only if the duplicator has a winning strategy in the d -step Ehrenfeucht-Fraïssé

game $\text{EF}^d(\mathfrak{B}(X), \mathfrak{B}(Y))$. Every d -theory $t = \text{Th}^d(X)$ will be called a *finite theory* of PF-graphs of depth d . If X is a function graph then t contains the formula $\text{Orphan} = \emptyset$ and so every PF-graph Y with $\text{Th}^d(Y) = t$ is a function graph; in this case we say that t is a *finite theory* of function graphs.

Definition 5.4 The *spectrum* of a finite theory t of PF-graphs of depth d is the set of the cardinalities of the PF-graphs X with $\text{Th}^d(X) = t$. The *spectrum* of a first-order Υ -formula ψ is the set of the cardinalities of PF-graphs X such that $\mathfrak{B}(X) \models \psi$. \square

Proposition 5.5 *In order to prove the Main Theorem, it suffices to prove that the spectrum of any finite theory of function graphs is eventually periodic.*

Proof Assume that the spectrum of any finite theory of function graphs is eventually periodic, let φ be any MSO formula with one unary function symbol, and let χ be as in Lemma 5.1. Every PF-graph satisfying χ is a function graph. It suffices to prove that the spectrum of χ is eventually periodic.

Let ψ be as in Lemma 5.3. Every PF-graph satisfying χ is a function graph. It suffices to prove that the spectrum of ψ is eventually periodic.

Let d be the quantifier depth of ψ . By Corollary 2.5, there are d -theories t_1, \dots, t_j of function graphs such that any function graph $X \models \psi$ if and only if $\text{Th}^d(X) \in \{t_1, \dots, t_j\}$. By Corollary 3.5, the spectrum of ψ is eventually periodic. \square

Remark 5.6 Since the definable relation Orphan is present in the vocabulary of PF-graphs, we should be careful with the notion of a substructure of a PF-graph X . Consider the following requirement on a subset A of X : for every edge $E(x, y)$ of X , if A contains x then it contains y . If the requirement is satisfied then the normal definition of a substructure of X generated by subset A works correctly; otherwise the normal definition does not work correctly. We will use the notion of substructure only in cases when the normal definition works. \square

6 Composing Graphs and their Theories

Definition 6.1 The *sum* $X + Y$ of PF-graphs X, Y is the PF-graph that is the disjoint union of X and Y . \square

Every PF-graph is the sum of its connected components. If $X = X_1 + \dots + X_m$ and $Y = Y_1 + \dots + Y_n$ then $X + Y = X_1 + \dots + X_m + Y_1 + \dots + Y_n$. If X, Y are function graphs then so is $X + Y$.

If X and Y are disjoint they will be presumed to be substructures of $X + Y$. If X and Y are not disjoint, one or both of them should be replaced with isomorphic copies in some canonic way. (Of course, we are interested in structures only up to their isomorphism type, so it does not make any difference how the isomorphic copies are chosen. It is convenient though to work with structures rather than with their isomorphism types. In particular, it is convenient that the components of a PF-graph form a set rather than a multiset.)

Lemma 6.2 $Th^d(X + Y)$ is uniquely determined by $Th^d(X)$ and $Th^d(Y)$.

Proof Use the Ehrenfeucht-Fraïssé games. \square

The lemma justifies the following operation on finite theories of the same depth.

Definition 6.3 $Th^d(X) + Th^d(Y) = Th^d(X + Y)$. \square

Obviously the operation is commutative and associative. For technical reasons, we introduce the notion of dotted PF-graphs.

Definition 6.4 A *dotted PF-graph* (X, a) is a PF-graph X with a distinguished element a , the *dot* of (X, a) . \square

The *Boolean associate* $\mathfrak{B}(X, a)$ of (X, a) is the Boolean associate $\mathfrak{B}(X)$ of X expanded with a distinguished element a . The MSO d -theory $Th^d(X, a)$ of a dotted PF-graph (X, a) is the first-order d -theory $th^d(\mathfrak{B}(X, a))$ of the Boolean associate of (X, a) .

Definition 6.5 Let (X, a) and (Y, b) be dotted PF-graphs.

1. The *dotted sum* $X \dot{+} (Y, b)$ is the non-dotted PF-graph Z obtained from $X + Y$ by making b the parent of every orphan in X .
2. The *dotted sum* $(X, a) \dot{+} (Y, b)$ is the dotted PF-graph $(X \dot{+} (Y, b), a)$.
3. The *circular sum* $(X, a) \oplus (Y, b)$ is the non-dotted PF-graph Z obtained from $X + Y$ by making b the parent of every orphan in X and making a the parent of every orphan in Y . \square

Lemma 6.6

1. $Th^d(X \dot{+} (Y, b))$ is uniquely determined by $Th^d(X)$ and $Th^d(Y, b)$.
2. $Th^d((X, a) \dot{+} (Y, b))$ is uniquely determined by $Th^d(X, a)$ and $Th^d(Y, b)$.
3. $Th^d((X, a) \oplus (Y, b))$ is uniquely determined by $Th^d(X, a)$ and $Th^d(Y, b)$.

Proof Use the Ehrenfeucht-Fraïssé games. \square

The lemma justifies the following operations on finite theories of the same depth.

Definition 6.7 Let (X, a) and (Y, b) be dotted PF-graphs.

1. $Th^d(X) \dot{+} Th^d(Y, b) = Th^d(X \dot{+} (Y, b))$
2. $Th^d(X, a) \dot{+} Th^d(Y, b) = Th^d((X, a) \dot{+} (Y, b))$
3. $Th^d(X, a) \oplus Th^d(Y, b) = Th^d((X, a) \oplus (Y, b))$ \square

The second operation is associative; we will use that fact.

7 Proof of the Main Theorem

Let d be an arbitrary natural number. For each d -theory σ of function graphs, let p_σ be the cardinality of the smallest function graph with d -theory σ . For each d -theory τ of dotted PF-graphs, let q_τ be the cardinality of the smallest dotted PF-graph with d -theory τ . Let p be the least common multiple of the numbers p_σ and q_τ for σ and τ as above. We prove that the spectrum of any d -theory s of function graphs is eventually periodic with period p . By Proposition 5.5, this implies the Main Theorem.

So let s be an arbitrary d -theory of function graphs and let X be a sufficiently large function graph with d -theory s . We construct a function graph Y of cardinality $|Y| = |X| + p$ such that $Th^d(Y) = s$. The meaning of sufficiently large will be clarified in the course of the construction. We consider several cases.

Case 1: The number of components of X exceeds the number of d -theories of function graphs.

Let X_1, \dots, X_m be the components of X in some order, and let Y_i be the sum $X_1 + \dots + X_i$ of the first i components. We use the fact that the sum operation on PF-graphs is associative. By the definition of Case 1, there exist $i < j$ such that $Th^d(Y_i) = Th^d(Y_j)$. Let $t = Th^d(Y_i)$, $u = Th^d(X_{i+1} + \dots + X_j)$, and $v = Th^d(X_{j+1} + \dots + X_m)$. Then $t + u = t$ and $t + v = s$. Furthermore, $t + k \cdot u = t$ for every positive k . By the definition of p_u , there is a function graph Z such that $Th^d(Z) = u$ and $|Z| = p_u$. To obtain the desired Y , augment X with $k = p/p_u$ components isomorphic to Z . Clearly, $|Y| = |X| + k \cdot p_u = |X| + p$. The order of the components does not matter of course but it is convenient to imagine that the new components come after X_j and before X_{j+1} . Then

$$Th^d(Y) = t + k \cdot u + v = t + v = s$$

Case 2: X has a vertex a such that the number of noncyclic children of a exceeds the number of d -theories of forests.

Each noncyclic child y of a , together with its descendants, forms a tree; let us call it T_y . Let F be the forest formed by the trees T_y , so that the number of the components of F exceeds the number of the d -theories for forests. By

the argument of Case 1, applied to forests rather than function graphs, there exists a forest G of cardinality $|G| = |F| + p$ such that $Th^d(G) = Th^d(F)$.

Let Q be the function graph obtained from X by removing the trees T_y so that $X = F \dot{+} (Q, a)$. The desired function graph $Y = G \dot{+} (Q, a)$.

Case 3: X has a cycle of cardinality m that exceeds the number of d -theories of dotted trees.

Arrange the vertices of the cycle into a path a_0, \dots, a_{m-1}, a_0 . Let T_i be the tree formed by the vertex a_i and its non-cyclic descendants. We use the fact that the dotted sum operation on dotted PF-graphs is associative. For each $i \in \{1, \dots, m\}$, let $(X_i, a_0) = (T_0, a_0) \dot{+} \dots \dot{+} (T_{i-1}, a_{i-1})$, so that (X_i, a_0) is a dotted tree with root a_{i-1} .

By the definition of Case 3, there exist $i < j$ such that $Th^d(X_i, a_0) = Th^d(X_j, a_0)$. Let $(U, a_i) = (T_i, a_i) \dot{+} \dots \dot{+} (T_{j-1}, a_{j-1})$ and let $t = Th^d(X_i, a_0)$, $u = Th^d(U, a_i)$ so that $t = t \dot{+} u$. Let V be the PF-graph obtained from X by removing the tree X_j . (V, a_j) is a dotted tree with root a_{m-1} , and $X = (X_j, a_0) \oplus (V, a_j)$. Let $v = Th^d(V, a_j)$ so that $s = t \oplus v$.

There exists a dotted tree (Z_1, c_1) such that $Th^d(Z_1, c_1) = u$ and $|Z_1| = q_u$. Let (Z, c) be the dotted sum of p/q_u copies of (Z_1, c_1) so that $|Z| = p$. The desired $Y = ((X_j, a_0) \dot{+} (Z, c)) \oplus (V, a_j)$. Clearly, $|Y| = |X| + p$. Since $t \dot{+} u = t$ and $t \oplus v = s$, we have

$$Th^d(Y) = (t \dot{+} (p/q_u) \cdot u) \oplus v = t \oplus v = s$$

Case 4: X has a path a_0, \dots, a_{m-1} composed of non-cyclic elements such that m exceeds the number of d -theories of trees.

This case is similar to Case 3. Trees T_i and dotted trees (X_i, a_0) are defined as above but this time we are interested in undotted trees X_i . By the definition of Case 4, there exist $i < j$ such that $Th^d(X_i) = Th^d(X_j)$. Let $(U, a_i) = (T_i, a_i) \dot{+} \dots \dot{+} (T_{j-1}, a_{j-1})$ and let $t = Th^d(X_i)$, $u = Th^d(U, a_i)$ so that $t = t \dot{+} u$. Let V be the function graph obtained from X by removing the tree X_j . Clearly $X = X_j \dot{+} (V, a_j)$. Let $v = Th^d(V, a_j)$ so that $s = t \dot{+} v$.

As in Case 3, there is a dotted tree (Z, c) such that $|Z| = p$ and $t \dot{+} Th^d(Z, c) = t$. The desired $Y = X_j \dot{+} (Z, c) \dot{+} (V, a_j)$. Clearly, $|Y| = |X| + p$. Further,

$$Th^d(Y) = (t \dot{+} Th^d(Z, c)) \dot{+} v = t \dot{+} v = s$$

This completes Case 4.

Finally, let K be the class of function graphs X that do not fall into any of the four cases. Since X does not fall into Case 1, it has only so many components. Since X does not fall into Case 3, the cycle of any component of X is only so long. Since X does not fall into cases 2 and 4, every cyclic element of X has only so many descendants. It follows that there is a bound θ on the cardinality of any member of K . Thus, the spectrum of s is eventually periodic with period p and p -threshold $\theta + 1$. The main theorem is proved.

Remark 7.1 In the original proof sketch, we used the finite Ramsey theorem in Cases 3 and 4. Writing up the proof we realized that these cases are similar to Case 1 where only the associativity of the sum operation was used. \square

8 Finite Satisfiability

Lemma 8.1 *There exists a computable function $F(d)$ such that, for every d and every function graph X , there is a function graph Y of cardinality $\leq F(d)$ such that $Th^d(X) = Th^d(Y)$.*

Proof The proof is similar to the proof of Proposition 2.13. For the reader's convenience, we make it independent from Section 7. All we have to do is to show that every function graph can be constructed from singleton function graphs by means of the tree operations of Section 6. Unfortunately this is not quite true. We need to revise two aspects of this plan.

First, it will be easier to deal with a larger class K of finite structures: function graphs, forests, dotted forests and singleton dotted function graphs.

Second, we need to extend Definitions 6.1 and 6.5.

Definition 8.2 Let (X, a) and (Y, b) be dotted PF-graphs.

1. $X + (Y, b) = ((X + Y), b)$.
2. $(Y, b) \dot{+} X = ((X \dot{+} (Y, b)), b)$.

For any d , the d -theories $Th^d(X + (Y, b))$ and $Th^d((Y, b) \dot{+} X)$ are uniquely determined by $Th^d(X)$ and $Th^d(Y, b)$. Now we can carry out our plan.

Call a K structure *good* if it is obtained from singleton K structures by means of the tree operations. It suffices to prove that all K structures are good. By contradiction assume that there is a bad K structure and let \mathcal{X} be a bad K structure of minimal cardinality. Clearly \mathcal{X} cannot be singleton. To get the desired contradiction, we show that \mathcal{X} is a composition of K structures of smaller cardinality. Clearly X is connected; otherwise it is the sum of its components. We have three cases.

1. \mathcal{X} is a function graph X . The cyclic elements of X form a path a_0, \dots, a_{m-1}, a_0 . If $m = 1$, so that there is a unique cyclic element, then $X = Y \dot{+} (Z, a_0)$ where Y is the forest of the non-cyclic elements of X and Z is the singleton function graph containing a_0 . So $m > 1$. Let Y be the tree formed by a_0 and its non-cyclic descendants, and let Z be the remaining part of X . Then $X = (Y, a_0) \oplus (Z, a_1)$.

2. \mathcal{X} is a tree X . Let b be the root of X , Y be the forest of the non-root elements of X , and Z be the singleton forest formed by b . Then $X = Y \dot{+} (Z, b)$.

3. \mathcal{X} is a dotted tree (X, a) . Let b be the root of X , Y be the forest of the non-root elements of X and Z be the singleton forest formed by b . If $a = b$ then $(X, a) = (Z, a) \dot{+} Y$. Otherwise $(X, a) = (Y, a) \dot{+} (Z, b)$ where b is the root of X . \square

Theorem 8.3 *The finite satisfiability problem for MSO formulas with one unary function symbol is decidable.*

Proof Use Lemma 5.1 and the lemma above. \square

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