A DECISION PROBLEM FOR DECISION PROBLEMS

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1. Question A: does there exist an algorithm which, with respect to a given theory T, will detect whether or not it is decidable?

Let us restrict and refine question A. Let L be the ordinary language of first-degree predicate logic. For each formula α of L let L_α be a sublanguage of L in which the predicate symbols are predicate symbols of α. Further, let T_α be a theory in L_α with the unique axiom α. We put $S = \{\alpha : T_\alpha \text{ decidable}\}$, $\bar{S} = \{\alpha : T_\alpha \text{ undecidable}\}$.

Question B1: is S recursively countable?

Question B2: is $\bar{S}$ recursively countable?

With the help of the brilliant study by Hanf [1] we shall give below negative answers to questions B1 and B2.

The uncountability of $\bar{S}$ can be shown directly without difficulty. It is just as easy to establish the uncountability of the set $\{\alpha : T_\alpha \text{ not complete}\}$. The author does not know whether the set $\{\alpha : T_\alpha \text{ complete}\}$ is countable.

2. Let T be a theory in the language L with a recursive set of axioms, and let M_T be a Turing machine, which, for an arbitrary formula α of L, checks whether α is an axiom of T or not. In [1] Hanf constructs a Turing machine M which has M_T as a detachable attachment, and a finitely axiomatizable theory $F(T)$ which describes the work of the machine M with the attachment M_T. Here T and $F(T)$ have much in common, and, in particular, $F(T)$ is decidable if and only if T is decidable. Let the machine M_T' do only half the work of M_T. If α is an axiom of the theory T, the machine M_T' will stop after a finite number of steps when it has the formula α on its tape at the initial moment. But if α is not an axiom of T, M_T' will work forever. It is possible to modify the Hanf machine M somewhat so that it can use M_T' as its detachable attachment. For this purpose we introduce the following function $f_T(n,\alpha)$: if M_T', having formula α on its tape at the initial moment, stops after not more than n steps, then $f_T(n,\alpha) = 1$, otherwise $f_T(n,\alpha) = 0$. Let Γ be a natural [2] and recursive numeration of all the formulas of L. Let α_0 be a universally significant formula of L, and let Δ be a natural and recurisve numeration of all formal proofs of the formula $\neg\alpha_0$. Finally, let E be a natural and recursive numeration of all the pairs (n,α). We replace the block-diagram of the machine M in [1] by the block-diagram represented in Fig. 1. Here T' has the same meaning as in [1]. As a result we obtain a new Hanf machine M'. Let $F'(T)$ be a finitely axiomatizable theory describing the work of M' with the attachment M_T' in a manner analogous to the way in which the theory $F(T)$ describes the work of the machine M with the attachment M_T. It is not difficult to see that, as before, $F'(T)$ is decidable if and only if T is decidable.

3. Let π_n be the n-th recursively countable set of natural numbers in the Post numeration [2]. Let us denote by T_n the theory in the language L, the set of whose axioms is $\{\Gamma(m) : m \in \pi_n\}$. Let us put $X = \{n : T_n \text{ decidable}\}$. We shall assume that the set S of Section 1 is countable. By Section 2 the set X here is also countable. By p. 166 of [2] the family $\{\pi_n : n \in X\}$ here is the family of all supersets of some countable system of finite sets, so that each solvable T_n is an extension of some solvable and

Translated from Algebra i Logika, Vol. 8, No. 6, pp. 640-642, November-December, 1969. Original article submitted July 11, 1969.

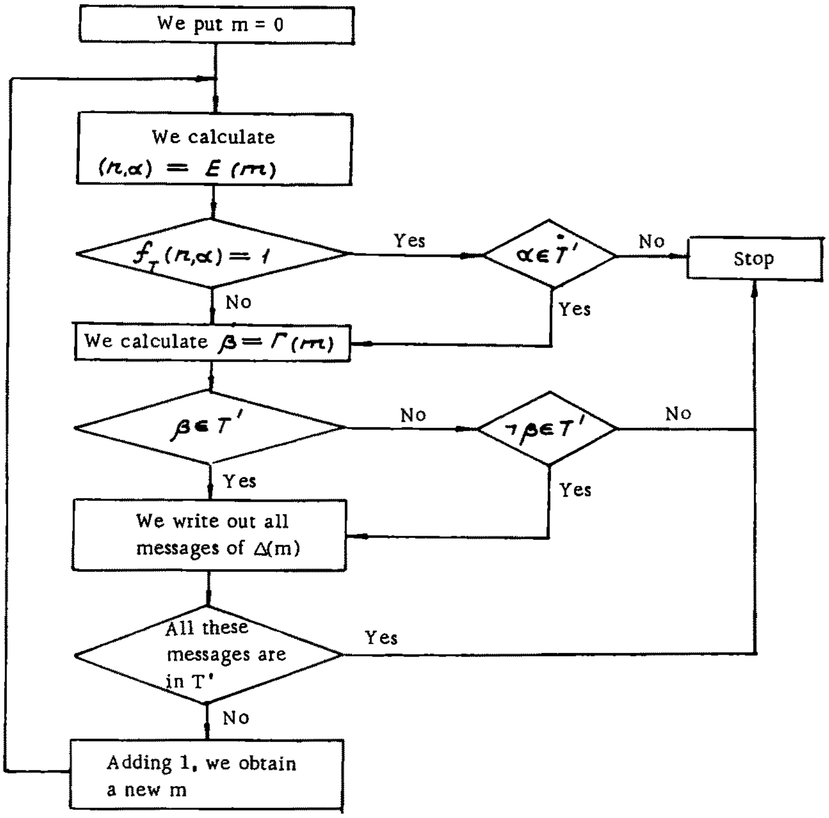

Fig. 1

finitely axiomatizable T_m . But this latter theory is inconsistent (consider the richness of the language L). Thus each decidable T_n is inconsistent. We have obtained a contradiction; thus, the set $\mathcal{S}$ is not recursively countable. The uncountability of $\bar{S}$ can be established in an analogous manner.

LITERATURE CITED

1. W. Hanf, "Model-theoretic methods in the study of elementary logic," Proceedings of 1963 International Symposium on Theory of Models at Berkeley, North Holland Publ. Co., Amsterdam (1965).
2. A. I. Mal'tsev, Algorithms and Recursive Functions, Nauka (1965).