Doklady 1966
Tom 166, No. 5

ON THE DECISION PROBLEM FOR THE PURE RESTRICTED PREDICATE CALCULUS

Ju. Š. GUREVIČ

1. A résumé of the results concerning the decision problem for satisfiability in the pure restricted predicate calculus may be found in [1]. It turns out that if one excludes classes of formulas of the form

$$\exists^a\exists^b\exists^n M(F_1, \ldots, F_c), \quad \exists^a \forall^b \exists^n M(F_1, \ldots, F_c)$$

not containing one-place predicates exclusively where a, b, and c are bounded in a given class and $b > 0$, then for any of the remaining classes of prenex formulas without free variables and with a fixed collection of predicates the problem has been solved.

Theorem. *Let σ contain one two-place and $(9 + k)$ one-place predicates, where $2^k \geq m^*$ and m^* is the number of states of a universal Turing machine. The sets of refutable and finitely satisfiable σ-formulas of the form $\forall\exists\forall^{\&} \exists^n$ are not recursively separable.*

The present article is devoted to the proof of this theorem.

2. By a Turing machine we understand here a variant of the kind of machines described in [2]. M^* will be used to designate a universal Turing machine. Let X_1 (X_2) be the set of all n such that M^* terminates with a tape (cycles), given the initial situation $\overbrace{11\ldots1}^{n+1}00\ldots$ M^* scans the left-most square. X_1 and X_2 are not recursively separable.

3. In [2] there is constructed for each Turing machine M a formula $\alpha_M(x, x'y)$ containing two-place predicates, S, K, and H, and a certain number of one-place predicates which is such that if M scans initially the zero square of the empty tape, it terminates with a tape at some time or goes into cycles) if and only if $\forall xy \alpha_M(x, x+1, y)$. $Z0$ is unsatisfiable (is satisfiable with periodic predicates) in the domain of natural numbers. Here Zy is interpreted to mean: y is zero; and Syx to mean: at moment y the square of number x is nonempty. In α_{M^*} we replace $(Zy \supset \urcorner Syx')$ by $(Zy \supset (Syx' \supset Syx))$. We obtain the formula $\alpha(x, x', y)$.

Lemma 1. $n \overline{\in} X_1$ $(n \in X_2)$ *if and only if* $\forall xy\alpha(x, x+1, y)$. $Z0$. $Son \urcorner So(n+1)$ *is satisfiable (satisfiable with periodic predicates) in the domain of nautral numbers.*

4. Let $\beta(x, x', y)$ be the conjunction of the following five formulas:

4.1. $\urcorner Ax.\ \urcorner Ay.\ Cxy \supset Dxy.$

4.2. $Dyx \supset \urcorner Ax'.\ Cyx'.$

4.3. $Ay.\ Cyx \supset Ax'.$

4.4. $Ax.\ Bx.\ Zy \supset Syx.\ \urcorner Syx'.$

4.5. $Bx \supset Bx'.$

Let

$$\varphi_n = \forall x \exists x' \forall y\, (\alpha(x, x', y) . \beta(x, x', y))\, \&$$
$$\& \exists x_0 x_1 \ldots x_n\, [Ax_0 . \bigwedge_{i>0} \urcorner Ax_i . \bigwedge_{i<n} Cx_i x_{i+1} .$$
$$. \bigwedge_{1+i \neq j} \urcorner Cx_i x_j . Zx_n . Bx_n].$$

Lemma 2. $\phi = [\forall xy\alpha(x, x+1, y).\ Z0.\ Son.\ \urcorner So(n+1)]$ *is satisfiable (satisfiable with periodic predicates) in the domain of natural numbers if and only if* ϕ_n *is satisfiable (finitely satisfiable).*

We will show how to construct a model for ϕ_n if there exists a model $\mathfrak{N}$ for ϕ. Let $|\mathfrak{M}_i| = \{(i, k) | k = 0, 1, \cdots\}$ and let $(i, k) \longleftrightarrow k$ be an isomorphism of $\mathfrak{M}_i$ and $\mathfrak{N}$, $i = 0, \cdots, n$.

For $i \neq j$ we set $\urcorner S(i, k)(j, l)$, $\urcorner K(i, k)(j, l)$, $\urcorner H(i, k)(j, l)$. We assume further:

$A(i, k)$ is equivalent to $i = k$,

$B(i, k)$ is equivalent to $i = n$,

$C(i, k)(j, l)$ is equivalent to $j = i + 1$ and $k = l \leq i$,

$D(i, k)(j, l)$ is equivalent to $j = i + 1$ and $k = l + 1 \leq i$.

The pairs (i, k) with the defined relations form a model for ϕ_n.

Now if there exists a model $\mathfrak{M}$ for ϕ_n, then, without loss of generality, one may assume that $|\mathfrak{M}|$ consists of the pairs (i, k), where $i = 0, \cdots, n$ and $k = 0, 1, \cdots$. The element $(i, 0)$ plays the role of the x_i in ϕ_n. We prove $A(i, i)$ by induction on i. $S(n, 0)(n, n)$ and $\urcorner S(n, 0)(n, n+1)$ follow from $A(n, n)$ and $B(n, 0)$ by means of 4.5 and 4.4. The elements (n, k), $k = 0, 1, \cdots$, with the corresponding relations form a model for ϕ.

5. Let $\mathfrak{M}$ be a model for ϕ_n. We replace each element a of $\mathfrak{M}$ by eleven new elements $a^0, \cdots \cdots, a^{10}$. Let h be any one-place predicate of ϕ_n; if ha is true ($\urcorner ha$ is true), we make ha^i to be true (correspondingly, $\urcorner ha^i$); $i = 0, \cdots, 10$. We now define over the set $\{a^i | a \in \mathfrak{M}, i = 0, \cdots, 10\}$ a new predicate Fxy (or simply xy):

5.1. $Cab \rightarrow \bigwedge_i (a^i b^0 . b^i a^2) . b^1 a^{10}$.

5.2. $Dab \rightarrow \bigwedge_i (a^i b^1 . b^i a^3)$.

5.3; 5.4; 5.5 are analogous to 5.1 and 5.2, but for S, K, and H respectively.

5.6. $a^i b^j$ is true only when its truth follows from 5.1–5.5. As a result we obtain a model for the formulas ψ_n defined below.

6. Let $\gamma(x, x', y)$ be the conjunction of the following formulas:

6.1. $\bigvee_{i=0}^{10} f^i x . \bigwedge_{i \neq j} \urcorner (f^i x . f^j x)$.

6.2. $\bigwedge_{i<10} (f^i x \sim f^{i+1} x')$.

6.3. $\bigwedge_{h, i<10} f^i x \supset (hx \sim hx')$ (here h runs through the one-place predicates of ϕ_n).

6.4. $\bigwedge_{i<10} [f^i x . (f^0 y \vee f^2 y) \supset (xy \sim x'y)] . [f^{10} y . f^0 x \supset (yx \sim x'y)]$.

6.5–6.8. Formulas relating to D, S, K, and H in the same way that 6.4 relates to C.

6.9. $f^{10} x . f^1 y . \urcorner Ax . \urcorner Ay . yx \subset x'y$.

6.10 and so on are formulas relating to 4.2, $\cdots$, 4.5 and to the conjunctions of $\alpha(x, x', y)$ in the same way that 6.9 relates to 4.1.

Let

$$\psi_n = \forall x \exists x' \forall y\ \gamma(x, x', y)\ \& \\ \& \exists x_0 x_1 \ldots x_n [\bigwedge_i f^0 x_i . Ax_0 . \bigwedge_{i>0} \urcorner Ax_i . \\ . \bigwedge_{i<n} x_i x_{i+1} . \bigwedge_{1+i \neq j} \urcorner x_i x_j . Zx_n . Bx_n].$$

Lemma 3. ϕ_n *is satisfiable (finitely satisfiable) if and only if* ψ_n *is satisfiable (finitely satisfiable).*

7. ψ_n contains one two-place predicate and $m^* + 17$ one-place ones. The individual groups of one-place predicates satisfy a property similar to 6.1 which permits us to reduce the necessary number of one-place predicates to $9 + k$, where $2^k \geq m^*$.

Thus the statement of the theorem follows from Lemmas 1, 2, and 3.

I would like to express my gratitude to Academician A. I. Mal′cev for having proposed the problem to me.

Ural State University

Received 21/MAY/65

BIBLIOGRAPHY

[1] V. F. Kostyrko, Algebra i Logika Sem. 3(1964), no. 5–6, 45. MR 31 #27.

[2] J. R. Büchi, Math. Ann. 148(1962), 201. MR 29 #5719.

Translated by:
S. Walker