Some Efficient Algorithms for a Class of Abstract Optimization Problems Arising in Optimal Control

ROBERT O. BARR, MEMBER, IEEE, AND ELMER G. GILBERT, MEMBER, IEEE

Abstract—Three abstract optimization problems are presented along with doubly iterative algorithms for their numerical solution. These algorithms are generalizations of particular algorithms described by Barr and Gilbert [19], [21] and Fujisawa and Yasuda [22]. The supporting theory is fully developed along with proofs of convergence. Practical aspects of computations are considered and procedures which insure rapid convergence are discussed. Two applications to discrete-time optimal control problems are described.

I. Introduction

MANY computational procedures for optimal control problems have appeared in the literature. The basic methods employed in these procedures vary widely. They include, for example, gradient methods in function space [1], [2], Newton–Raphson techniques [3], [4], nonlinear programming [5], [6], dynamic programming [7], schemes for iterating on boundary conditions [8], [9], and methods based on the convexity of the reachable set of system states [10]–[17]. The emphasis in this paper is on extending the methods of this last class, which will be referred to as convexity methods.

Most of the convexity methods are based on a general idea which was described first by Neustadt [10]–[12] and later refined by Eaton [13]. These methods as well as others [17] may appear to be quite different in their approach but they can be viewed in a common setting [14]. In particular they involve determining the smallest real number ω^* such that $0 \in K(\omega^*)$ where $K(\omega)$ is a convex set in R^n parameterized by the real number ω. The algorithms are iterative procedures which lead to a monotonically increasing sequence $\{\omega_i\}$ such that $\omega_i \rightarrow \omega^*$. The determination of ω_{k+1} from ω_k corresponds geometrically to determining a support hyperplane of $K(\omega_k)$ which separates $K(\omega_k)$ strictly from the origin. An assumption essential to the determination of this hyperplane is that $K(\omega_k)$ be (in a local sense) strictly convex. While the Neustadt procedures are known to converge and are numerically efficient in the sense that they reduce an infinite-dimensional problem to a finite-dimensional problem, they tend to have poor convergence rates unless rather complex conditioning procedures are employed [14], [15], [18].

More recently Barr and Gilbert [19]–[21] and Fujisawa and Yasuda [22] have proposed doubly iterative procedures which also can be shown to involve parameterized convex sets. These procedures use a projection algorithm [which minimizes $\|z\|$ on $K(\omega_i)$] [23]–[25] as a subalgorithm for determining the separating hyperplane. These procedures, while retaining the advantages of other convexity methods, eliminate strict convexity requirements and allow for rigorous treatment of singular and uncontrollable problems. Moreover, if sufficiently sophisticated projection algorithms are employed [21], [26], convergence rates are rapid and do not seem to depend on problem conditioning.

It is the purpose of this paper to treat these latter procedures in a comprehensive and generalized framework. Three abstract problems having wide applicability are formulated, and algorithms for their solution are described and shown to converge. The required theoretical background material is presented as well as a modified version of Barr's projection algorithm [26] which is a particularly effective subalgorithm.

This paper does not emphasize applications to a wide variety of optimal control problems. These matters will be discussed more fully in a subsequent paper. If the algorithms are to be applied it is essential that schemes for evaluating "contact functions" of the reachable set and the terminal set be available. This is generally the case if the equations of motion are linear in state [25] and the initial and terminal sets are convex. Within these requirements on the optimal control problem it is possible to allow treatment of both discrete-time and continuous-time equations of motion, nonlinearities with respect to the vector control variable, and very general terminal conditions, indices of performance, and control constraints. For example, practical algorithms for all the problems considered in [10]–[17] may be developed. The basic details for two discrete-time control problems are worked out in this paper. Gilbert and Harasty [27] consider a minimum-fuel impulsive control problem, and include a description

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R. O. Barr is with the Department of Electrical Engineering and Systems Science, Michigan State University, East Lansing, Mich. 48823.

E. G. Gilbert is with the Computer, Information, and Control Engineering Program, University of Michigan, Ann Arbor, Mich.

of the resulting algorithm along with numerical results for problems of order 6. For problems where the equations of motion are nonlinear in state, good results have been obtained by methods of successive linearization. This work is still in progress and will be reported at a later date.

The outline of the paper is as follows: Section II contains some notation and definitions; Section III reviews contact and support functions and gives some of their properties essential to the computing procedures; Section IV treats a basic quadratic programming problem and the projection algorithm for its solution; Section V gives some required technical results on the continuity of parameterized sets; Sections VI–VIII present the algorithms for the three abstract problems; finally, in Section IX applications to several discrete-time optimal control problems are given.

A reader interested in the main ideas and not the proofs may jump from the second paragraph in Section III to Section VI. If a concrete interpretation of the abstract problems is desired, they may be viewed as minimum-time regulator problems where $K(\omega)$ is the reachable set and ω is time. Alternative interpretations may be obtained by reading Section IX concurrently with Section VI.

II. Some Notation and Definitions

Let $x = (x^1,\cdots,x^n)$, $y = (y^1,\cdots,y^n)$, $z = (z^1,\cdots,z^n)$ be elements in the space of real n-tuples R^n and $\omega \in R^1$. The following notation is employed: $\langle x,y\rangle = \sum_{i=1}^n x^iy^i$; $\|x\| = \langle x,x\rangle^{1/2}$; $N(x;\omega) = \{z: \|z - x\| < \omega\}, \omega > 0$, the open sphere at x with radius ω; $\bar{N}(x;\omega) = \{z: \|z - x\| \leq \omega\}$, the corresponding closed sphere; $S = \{z: \|z\| = 1\}$, the surface of the unit sphere $\bar{N}(0;1)$; $L(x;y) = \{z: z = x + \omega(y - x), -\infty < \omega < \infty\}, x \neq y$, the line passing through x and y; $Q(x;y) = \{z: \langle z,y\rangle = \langle x,y\rangle\}, y \neq 0$, the hyperplane (dimension $n - 1$) through x with normal y. If $x = \sum_{i=1}^m \lambda_i y_i$ where $y_i \in R^n, \lambda_i \in R^1, \lambda_i \geq 0$ for $i = 1,2,\cdots,m$ and $\sum_{i=1}^m \lambda_i = 1$, then x is said to be a convex combination of the vectors $y_1,\cdots,y_m$.

Now let $X,X_1,X_2,\cdots,X_m$ be sets in R^n, H be an $m \times n$ matrix, and f be a function mapping R^n into R^m. Then ∂X denotes the boundary of X; the set $-X = \{z: z = -x, x \in X\}$; $X_1 - X_2 = \{z: z = x_1 - x_2, x_1 \in X_1, x_2 \in X_2\}$; $\sum_{i=1}^m X_i = \{z: z = \sum_{i=1}^m x_i, x_i \in X_i\}$; $HX = \{z: z = Hx, x \in X\}$; $f(X) = \{z: z = f(x), x \in X\}$; H' = the transpose of H. For arbitrary sets $X_1,\cdots,X_m$ the product set $X_1 \times X_2 \times \cdots \times X_m = \{z: z = (x_1,x_2,\cdots,x_m), x_i \in X_i\}$. The *convex hull* of X, written co X, is the intersection of all convex sets which contain X. By the Caratheodory theorem [28], [29] it is always possible to write co $X = \{z: z =$ convex combination of $x_1,\cdots,x_q; x_i \in X, i = 1,\cdots,q\}$, where $q = n + 1$.

III. Contact Functions and Support Functions

In this section several elementary results for contact and support functions of compact sets in R^n are reviewed. The evaluation of these functions is essential to the computing procedures.

Consider a set $X \subset R^n$ which is compact, but not necessarily convex. The function h_X which maps R^n into R^1 such that

$$h_X(\eta) = \max_{z \in X} \langle z,\eta\rangle$$

is called the *support function* of X. Since X is compact, $h_X(\eta)$ is defined for all $\eta \in R^n$. For each $\eta \in R^n$ the set $S_X(\eta) = \{x: \langle x,\eta\rangle = h_X(\eta), x \in X\} \subset R^n$ is nonempty and compact. Elements of $S_X(\eta)$ are called *contact points*. A function s_X which maps R^n into X so that $s_X(\eta) \in S_X(\eta)$ for all $\eta \in R^n$ is called a *contact function* of X. The terminology is suggestive, since of all hyperplanes with normal $\eta \neq 0$ that meet $X, Q(x;\eta)$ where x is a contact point is "farthest" in the direction of η. If for some $\eta \in R^n$, $S_X(\eta)$ contains more than one contact point, more than one contact function of X exists. In what follows, this possible lack of uniqueness causes no difficulty, and s_X will denote a specific but arbitrary element of the set of all contact functions.

Clearly $h_X(\eta) = \langle s_X(\eta),\eta\rangle$ and s_X is bounded. It also follows that for $\eta \neq 0$: $s_X(\eta) \in S_X(\eta) \subset \partial X$; $S_X(\eta) = S_X(\omega\eta), \omega > 0$; and $Q(s_X(\eta);\eta)$ is the support hyperplane of X with outward normal η.

Remark 1: If X is strictly convex, i.e., $S_X(\eta)$ contains only one point for each $\eta \neq 0$, then $s_X(\cdot)$ is uniquely determined away from the origin. Moreover, it is known [30] that $s_X(\eta)$ is continuous at all $\eta \neq 0$ and grad $h_X(\eta) = s_X(\eta)$.

In spite of the fact that X need not be convex, the following theorem shows the close relationship of support and contact functions to convexity.

Theorem 1

Let s_X and h_X be contact and support functions of X, a compact set in R^n. Then s_X and h_X are contact and support functions of co X.

The theorem follows directly from the known fact [29] that $h_X = h_{\text{co}X}$. With respect to the contact function the converse of Theorem 1 is not true, i.e., $s_{\text{co}X}$ is not necessarily a contact function of X.

Theorem 1 is sometimes useful in developing formulas for the evaluation of contact functions. For example, a contact function of a convex polyhedron may be evaluated by evaluating a contact function of its set of vertices.

Since the support function of a convex set is convex [28], [29], Theorem 1 implies that the support function of any compact set in R^n is convex. Another property of support functions, which will be needed in the sequel, is contained in the following theorem.

Theorem 2

Let X be a compact set in R^n such that $X \subset \bar{N}(0;r)$. Then the support function h_X is Lipschitz continuous on R^n with Lipschitz constant r, i.e., $|h_X(\eta) - h_X(\nu)| \leq r\|\eta - \nu\|$ for all η and ν in R^n.

Proof: Let s_X be a contact function of X and observe that

$$\begin{aligned} h_X(\eta) - h_X(\nu) &= \langle s_X(\eta) - s_X(\nu),\nu\rangle + \langle s_X(\eta),\eta - \nu\rangle \\ &\leq \langle s_X(\eta),\eta - \nu\rangle \leq \| s_X(\eta) \|\cdot\| \eta - \nu \| \\ &\leq r \| \eta - \nu \|. \end{aligned}$$

The first inequality follows from the fact that

$$\langle x - s_X(\nu),\nu\rangle \leq 0, \quad \text{for all} \quad x \in X.$$

Reversing the role of η and ν completes the proof.

Remark 2: If $X \subset R^n$ is compact, it follows directly that h_X is Lipschitz continuous on S.

When a set is characterized in certain special ways it is possible to utilize the characterization in the evaluation of the contact and support functions. As will be seen, the characterizations considered in the following theorem will have applications in the numerical solution of control problems.

Theorem 3

Assume that $X,X_1,\cdots,X_N$ are compact sets in R^n and that $Y = Y_1 \times \cdots \times Y_N$ where Y_i, $i = 1,\cdots,N$, are compact sets in R^{n_i} and $n = \sum_{i=1}^N n_i$. Let $Z = \sum_{i=1}^N X_i$, $W = \bigcup_{i=1}^N X_i$, and let H denote an arbitrary $m \times n$ matrix. Then

1) $h_{-X}(\eta) = h_X(-\eta)$, $-s_X(-\eta)$ is a contact function of $-X$
2) $h_{HX}(\eta) = h_X(H'\eta)$, $Hs_X(H'\eta)$ is a contact function of HX
3) $h_Z(\eta) = \sum_{i=1}^N h_{X_i}(\eta)$, $\sum_{i=1}^N s_{X_i}(\eta)$ is a contact function of Z
4) $h_Y(\eta) = \sum_{i=1}^N h_{Y_i}(\eta_i)$, $(s_{Y_1}(\eta_1),\cdots,s_{Y_N}(\eta_N))$ is a contact function of Y, where $\eta_i \in R^{n_i}$, $i = 1,\cdots,N$, and $\eta = (\eta_1,\cdots,\eta_N) \in R^n$
5) $h_W(\eta) = h_{X_j}(\eta),s_{X_j}(\eta)$ is a contact function of W, where the integer j, $1 \leq j \leq N$, is such that $h_{X_j}(\eta) = \max \{h_{X_1}(\eta),h_{X_2}(\eta),\cdots,h_{X_N}(\eta)\}$.

The results of this theorem are straightforward consequences of the definitions of the contact functions and support functions of the sets involved. When a set is characterized as an intersection of compact sets, simple formulas like those of Theorem 3 cannot in general be obtained.

IV. Basic Problem

The computing procedures to be developed in the sequel involve on each iteration an approximate numerical solution of the following basic problem.

Basic Problem (*BP*)

Given: K a compact, convex set in R^n. Find: a point $z^* \in K$ such that

$$\| z^* \|^2 = \min_{z \in K} \| z \|^2.$$

The constraint set K in this quadratic programming problem, in contrast to the quadratic programming problems usually treated in the literature [31]–[33], need not be specified by some set of functional inequalities. To develop an efficient computational algorithm it is required only that there be a known contact function s_K of K. This requirement is satisfied for many optimal control applications (for examples see Section IX and [25]).

Since K is compact and $\| z \|$ is continuous, a solution z^* exists. Moreover [25], the following holds.

Remark 3: z^* is unique; $\| z^* \| = 0$, if and only if $0 \in K$; for $\| z^* \| > 0$, $z^* \in \partial K$; for $\| z^* \| > 0, z = z^*$, if and only if $z \in S_K(-z)$.

Gilbert [25] describes an algorithm (based on s_K), for solving BP, which is of the Frank and Wolfe [31] type. When K is a convex polyhedron this algorithm is known [25], [34] to yield very slow convergence (essentially $\| z_k \|^2 - \| z^* \|^2$ decreases as k^{-1} for large k). When K is not a convex polyhedron, convergence may be much faster [25], e.g., geometric. Still, poor conditioning of ∂K in the neighborhood of z^* may lead to poor performance [25], [26]. An extension of Gilbert's procedure, which in certain forms yields much more rapid convergence, is the following one due to Barr [20], [21], [26].

*Algorithm BP*1

Let $s_K(\cdot)$ be an arbitrary contact function of K, choose a positive integer p, and take $z_0 \in K$. Then a sequence of vectors $\{z_{k+1}\}$, $k = 0,1,2,\cdots$, is generated as follows.

Step 1: Select any p vectors $y_1(k),y_2(k),\cdots,y_p(k)$ in K and let

$$H_k = \text{co}\,\{y_1(k),y_2(k),\cdots,y_p(k),s_K(-z_k),z_k\}.$$

Step 2: Find $z_{k+1} \in H_k$ such that

$$\| z_{k+1} \|^2 = \min_{z \in H_k} \| z \|^2.$$

Increment k by one and return to Step 1.

Note that the quadratic programming problem in Step 2 is much simpler than BP because the constraint set H_k is the convex hull of $p + 2$ known points. The problem can be solved by standard quadratic programming techniques as is illustrated by the following. Let $s_K(-z_k) = y_{p+1}(k)$, $z_k = y_{p+2}(k)$, and the $(p + 2) \times (p + 2)$ symmetric matrix $D = [d_{ij}]$ where $d_{ij} = \langle y_i(k),y_j(k)\rangle$. Each $z \in H_k$ has the representation

$$z = \sum_{i=1}^{p+2} x^i y_i(k)$$

where

$$\sum_{i=1}^{p+2} x^i = 1, \quad x^i \geq 0, \quad (i = 1,2,\cdots,p + 2).$$

Thus if $x = (x^1,x^2,\cdots,x^{p+2})$,

$$\| z \|^2 = \sum_{i=1}^{p+2} \sum_{j=1}^{p+2} x^i x^j \langle y_i(k),y_j(k)\rangle = \langle x,Dx\rangle. \tag{1}$$

Defining

$$X = \{x: \sum_{i=1}^{p+2} x^i = 1, \quad x^i \geq 0, \quad (i = 1,2,\cdots,p+2)\} \subset R^{p+2} \tag{2}$$

it follows from (1) that minimization of $\|z\|^2$ on H_k is equivalent to minimization of $\langle x,Dx\rangle$ on X. Standard quadratic programming techniques [32], [33] can be applied to the latter minimization problem to yield a solution x^* and then

$$z_{k+1} = \sum_{i=1}^{p+2} x^{*i} y_i(k). \tag{3}$$

Barr [26] has shown that for $k \geq 0$ and $k \to \infty : z_k \in K$; $\|z_k\| \geq \|z_{k+1}\|$, $\|z_k\| \to \|z^*\|$, and $\|z_k\| = \|z_{k+1}\|$ implies $z_k = z^*$; $z_k \to z^*$. In addition, error bounds on $\|z_k - z^*\|$ and $\|z_k\| - \|z^*\|$ are available as the computation proceeds. For rapid convergence it is essential to select the points $y_1(k),\cdots,y_p(k)$ so that ∂H_k will closely approximate ∂K in the vicinity of z_{k+1}. This requires that $p \geq n$ and that the points be in the vicinity of z^* (when $z^* \in \partial K$). One procedure which works well is the following special case of Algorithm BP1, which is a modification of selection rule C in [26]. In [35] this algorithm is discussed for K a convex polyhedron.

Algorithm BP2

Let $s_K(\cdot)$ be an arbitrary contact function of K, choose a positive integer $p \geq n$, select nonzero $\eta_1,\cdots,\eta_p \in R^n$, and take $z_0 \in \text{co}\{s_K(\eta_1),s_K(\eta_2),\cdots,s_K(\eta_p)\}$. Then generate $\{z_{k+1}\}$, $k = 0,1,2,\cdots$, as follows.

Step 1: If $k = 0$, set $y_i(k) = s_K(\eta_i)$ for $i = 1,2,\cdots,p$. If $k > 0$, express z_k as a convex combination

$$\sum_{i=1}^{p} \lambda_i y_i(k-1) + \lambda_{p+1} s_K(-z_{k-1})$$

such that at most p scalars from $\{\lambda_1,\lambda_2,\cdots,\lambda_{p+1}\}$ are greater than 0; now let $\eta_1,\eta_2,\cdots,\eta_{p+1}$ be the elements in R^n which generated the contact points $y_1(k-1),\cdots,y_p(k-1)$, $s_K(-z_{k-1})$, respectively [i.e., $y_i(k-1) = s_K(\eta_i)$, $i = 1,2,\cdots,p$, and $\eta_{p+1} = -z_{k-1}$]; considering only those contact points corresponding to $\lambda_i = 0$, reject one contact point for which $-\|\eta_i\|^{-1} h_K(\eta_i)$ is a minimum; set the remaining p contact points equal to $y_1(k),\cdots,y_p(k)$. Thus, by relabelling the λ_i and η_i, $z_k = \sum_{i=1}^{p} \lambda_i y_i(k)$ where $\sum_{i=1}^{p} \lambda_i = 1$, $\lambda_i \geq 0$, and $y_i(k) = s_K(\eta_i)$, $i = 1,2,\cdots,p$.

Step 2: Let $\bar{H}_k = \text{co}\{y_1(k),y_2(k),\cdots,y_p(k),s_K(-z_k)\}$. Find $z_{k+1} \in \bar{H}_k$ such that

$$\|z_{k+1}\|^2 = \min_{z \in \bar{H}_k} \|z\|^2.$$

Increase k by one and return to Step 1.

Algorithm BP2 is a specific realization of Algorithm BP1, and therefore the convergence results quoted above also apply here. The requirement $p \geq n$ guarantees the existence of the convex combination expression for z_k in terms of at most p points from $\{y_1(k-1),\cdots,y_p(k-1), s_K(-z_{k-1})\}$. This expression will be readily available on each iteration if the quadratic programming technique used for Step 2 yields a solution

$$z_{k+1} = \sum_{i=1}^{p+1} x^{*i} y_i(k)$$

[cf. (3)] in which x^* has a maximum number of zero elements.

Since in Step 2 of Algorithm BP2 $z_k \in \text{co}\{y_1(k),\cdots, y_p(k)\}$, for all $k = 0,1,2,\cdots$, it follows that $\bar{H}_k$ in Algorithm BP2 equals H_k in Algorithm BP1. The use of $\bar{H}_k$ reduces by one the dimensionality of the quadratic programming problem in Step 2.

In Gilbert's algorithm and other versions of BP1, z_k may be the convex combination of as many as k contact points. An important advantage of Algorithm BP2 is that for all k, z_k is a known convex combination of only p contact points of K (see end of Step 1). That is, vectors $\eta_1,\cdots,\eta_p$ are known such that

$$z_k = \sum_{i=1}^{p} \lambda_i s_K(\eta_i) \tag{4}$$

where $\sum_{i=1}^{p} \lambda_i = 1$, $\lambda_i \geq 0$.

Barr [26] gives data which indicate that $\{z_k\}$ generated by Algorithm BP2 with $p = n$ converges rapidly to z^* regardless of the curvature of ∂K at z^*. Roughly speaking, the rate of decrease of $\|z_k\| - \|z^*\|$ is dependent on n alone. The number of iterations per decade of decrease, after a few initial iterations, is approximately 2 for $n = 2$, 4 for $n = 3$, 6 for $n = 4$, 9 for $n = 5$, and 13 for $n = 6$. This reference also shows that in some cases convergence is faster. For example, when the range of s_K is a finite set of points, the sequence $\{z_k\}$ generated by Algorithm BP2 converges in a finite number of iterations. Such a situation may arise when K is a convex polyhedron (see CP1 in Section IX). Taking $p > n$ does not seem to offer any advantage [26].

V. Parameterized Sets

The abstract optimization problems of the next section involve sets which depend on a parameter. Conditions for the continuity of these parameterized sets, and the relationship to the support function and certain distance functions, are important in the sequel and are developed here.

Consider first the following definitions:

$$r(x,X) = \min_{y \in X} \|x - y\|, \quad x \in R^n, \quad X \subset R^n \text{ is compact} \tag{5}$$

$$d(X_1,X_2) = \max\{d_1,d_2\}, \qquad X_1, \quad X_2 \subset R^n \text{ are compact} \tag{6}$$

$$d_1 = \max_{x \in X_1} r(x,X_2)$$

$$d_2 = \max_{y \in X_2} r(y,X_1).$$

The function $d(X_1,X_2)$ is a Hausdorff metric for compact sets in R^n.

Now let $\Omega = [\omega_L,\omega_U]$, $\omega_L < \omega_U$, be a compact interval in R^1. For each $\omega \in \Omega$, let the set $K(\omega) \subset R^n$ be nonempty and compact. Then $K(\cdot)$ is continuous at $\omega \in \Omega$, if for any $\epsilon > 0$, there exists $\delta(\epsilon,\omega) > 0$ such that $d(K(\omega),K(\bar\omega)) < \epsilon$ when $|\omega - \bar\omega| < \delta(\epsilon,\omega)$, $\bar\omega \in \Omega$. Also, $K(\cdot)$ is continuous on Ω if it is continuous for all $\omega \in \Omega$; $K(\cdot)$ is uniformly continuous on Ω if $\delta(\epsilon,\omega)$ is independent of ω; $K(\cdot)$ is Lipschitz continuous on Ω if there exists $\alpha > 0$ such that $d(K(\omega),K(\bar\omega)) \le \alpha|\omega - \bar\omega|$, for all $\omega,\bar\omega \in \Omega$.

Let $h(\omega,\eta) = h_{K(\omega)}(\eta)$ and $s(\omega,\eta) = s_{K(\omega)}(\eta)$ denote support and contact functions of $K(\omega)$. Note that $h(\omega,\eta)$ and $s(\omega,\eta)$ are defined on $\Omega \times R^n$.

Theorem 2 shows that for fixed $\omega \in \Omega$, $h(\omega,\cdot)$ is Lipschitz continuous on R^n or on S (Remark 2). Specifically, if $K(\omega) \subset \bar N(0;r)$, $r > 0$, for all $\omega \in \Omega$, then

$$|h(\omega,\eta) - h(\omega,\bar\eta)| \le r\,\|\eta - \bar\eta\|, \quad \text{for all } \eta,\ \ \bar\eta \in S. \tag{7}$$

The following theorem concerns the continuity of $h(\cdot,\eta)$ for fixed $\eta \in S$.

Theorem 4

Let $K(\omega)$ be a nonempty, compact convex set in R^n defined for each $\omega \in \Omega = [\omega_L,\omega_U]$. $K(\cdot)$ is continuous on Ω if and only if $h(\cdot,\eta)$ is continuous on Ω for every fixed $\eta \in S$.

Proof: First assume $K(\cdot)$ is continuous on Ω. Let η be an arbitrary element of S and $\omega,\bar\omega$ be arbitrary elements of Ω. From (6) $d(K(\omega),K(\bar\omega)) < \epsilon$ implies $K(\omega) \subset K(\bar\omega) + N(0;\epsilon)$ and $K(\bar\omega) \subset K(\omega) + N(0;\epsilon)$. These results yield $h(\omega,\eta) < h(\bar\omega,\eta) + \epsilon$ and $h(\bar\omega,\eta) < h(\omega,\eta) + \epsilon$. From these inequalities and the continuity of $K(\cdot)$

$$|h(\omega,\eta) - h(\bar\omega,\eta)| < \epsilon, \quad \text{for all } \bar\omega \in \Omega, \quad |\omega - \bar\omega| < \delta(\epsilon,\omega). \tag{8}$$

Thus the continuity of $h(\cdot,\eta)$ on Ω is established. Note that the continuity is uniform with respect to η. Because Ω is compact it follows [36] that $h(\cdot,\eta)$ is uniformly continuous on Ω.

Now assume $h(\cdot,\eta)$ is continuous on Ω for each fixed $\eta \in S$; i.e., for any $\epsilon > 0$, there exists $\tilde\delta(\epsilon,\omega,\eta) > 0$, such that $|h(\omega,\eta) - h(\bar\omega,\eta)| < \epsilon$ when $\omega,\bar\omega \in \Omega$, $|\omega - \bar\omega| < \tilde\delta(\epsilon,\omega,\eta)$. From (7) and the triangle inequality it follows that

$$|h(\bar\omega,\bar\eta) - h(\omega,\eta)| < \epsilon, \quad \bar\omega \in \Omega, \quad \bar\eta \in S$$
$$|\omega - \bar\omega| < \tilde\delta(\epsilon/2,\omega,\eta), \quad \|\eta - \bar\eta\| < \epsilon/2r. \tag{9}$$

Thus $h(\cdot,\cdot)$ is continuous on $\Omega \times S$. Moreover, it is uniformly continuous on $\Omega \times S$ because $\Omega \times S$ is compact. Thus given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$|h(\bar\omega,\bar\eta) - h(\omega,\eta)| < \epsilon, \quad \omega,\bar\omega \in \Omega$$
$$\eta,\bar\eta \in S, \quad |\omega - \bar\omega| + \|\eta - \bar\eta\| < \delta(\epsilon). \tag{10}$$

Suppose that $K(\cdot)$ is not continuous on Ω. It will now be shown that this leads to a contradiction. Since $K(\cdot)$ is not continuous there exists $\omega \in \Omega$ and $\epsilon > 0$ such that

$$d(K(\bar\omega),K(\omega)) < \epsilon$$
$$\text{cannot hold for all } \bar\omega \in \Omega, \quad 0 < |\omega - \bar\omega| < a \tag{11}$$

no matter what the choice of a. Now consider the set $\hat\Omega = \{\bar\omega : d(K(\bar\omega),K(\omega)) \ge \epsilon, \bar\omega \ne \omega, \bar\omega \in \Omega\}$. There exists an (infinite) sequence $\{\omega_i\}$ contained in $\hat\Omega$ such that $\omega_i \to \omega$. If this were not true there would exist $\hat a$, $0 < \hat a$, such that $\{\bar\omega : 0 < |\omega - \bar\omega| < \hat a, \bar\omega \in \Omega\} \subset \hat\Omega$. But this would imply $d(K(\omega),K(\bar\omega)) < \epsilon$ for all $|\omega - \bar\omega| < \hat a$, $\bar\omega \in \Omega$, which contradicts (11). Thus for the sequence $\{\omega_i\}$, $d(K(\omega_i), K(\omega)) \ge \epsilon$. This result leads directly to the desired contradiction. It implies either one or both of the following results [see (6)]: 1) there exists $x_i \in K(\omega_i)$ such that $r(x_i,K(\omega)) \ge \epsilon$; 2) there exists $x_i \in K(\omega)$ such that $r(x_i,K(\omega_i)) \ge \epsilon$. For result 1) let s_i denote the point in $K(\omega)$ which is closest to x_i. Then $\|x_i - s_i\| \ge \epsilon$. It follows from the convexity of $K(\omega)$ that if

$$\eta_i = \|x_i - s_i\|^{-1}(x_i - s_i)$$

then $\eta_i \in S$ and η_i is an (outward) normal to a support hyperplane of $K(\omega)$ passing through s_i. Thus

$$\begin{aligned} -h(\omega,\eta_i) + h(\omega_i,\eta_i) &= -\langle\eta_i,s_i\rangle + \max_{x \in K(\omega_i)} \langle\eta_i,x\rangle \\ &\ge -\langle\eta_i,s_i\rangle + \langle\eta_i,x_i\rangle = \|x_i - s_i\| \ge \epsilon. \end{aligned} \tag{12}$$

Using 2) and a similar argument, $h(\omega,\eta_i) - h(\omega_i,\eta_i) \ge \epsilon$. Thus for any ω_i there exists η_i such that

$$|h(\omega,\eta_i) - h(\omega_i,\eta_i)| \ge \epsilon.$$

Since $(\omega_i,\eta_i) \in \Omega \times S$ and $\Omega \times S$ is compact, there exists a convergent subsequence $\{(\hat\omega_i,\hat\eta_i)\}$ such that $\hat\omega_i \to \omega$, $\hat\eta_i \to \eta$, and $|h(\omega,\hat\eta_i) - h(\hat\omega_i,\hat\eta_i)| \ge \epsilon$. But this contradicts (10) and completes the proof.

Theorem 5

Let $K(\omega)$ be defined as in Theorem 4. $K(\cdot)$ is Lipschitz continuous on Ω if and only if $h(\cdot,\eta)$ is Lipschitz continuous on Ω for every fixed $\eta \in S$.

The proof is omitted since the arguments are similar to those used in the proof of Theorem 4.

Remark 4: From the proof of the theorems it is clear that if $K(\omega) \subset R^n$ is nonempty, compact, and convex on Ω, the following conditions are equivalent: 1) $K(\cdot)$ continuous (Lipschitz continuous) on Ω; 2) $h(\cdot,\eta)$ continuous (Lipschitz continuous) on Ω for any $\eta \in S$; 3) $h(\cdot,\cdot)$ continuous (Lipschitz continuous) on $\Omega \times S$.

In certain computational situations it is necessary to work with sets $K(\omega)$ which are not convex. Then the following theorem is useful.

Theorem 6

Let $K(\omega)$ be a nonempty, compact set in R^n defined for each $\omega \in \Omega$ and let $\bar K(\omega) = \text{co}\, K(\omega)$. Then if $K(\cdot)$ is continuous (Lipschitz continuous) on Ω, $\bar K(\cdot)$ is continuous (Lipschitz continuous) on Ω.

Proof: Let $\bar{h}(\omega,\eta)$ denote the support function of the compact set $\bar{K}(\omega)$. Using arguments similar to those in the proofs of Theorems 4 and 5, it is clear that continuity (Lipschitz continuity) of $K(\cdot)$ implies continuity (Lipschitz continuity) of $h(\cdot,\eta)$, for all $\eta \in S$. Theorem 1 implies that $\bar{h}(\omega,\eta) = h(\omega,\eta)$, the support function of $\bar{K}(\omega)$. This fact together with Theorems 4 and 5 gives the desired result.

Now consider $K(\omega)$, a nonempty, compact, convex set in R^n defined for each $\omega \in \Omega$, and define

$$\rho(\omega) = \min_{z \in K(\omega)} \| z \| \tag{13}$$

$$\zeta(\omega) = z \in K(\omega), \quad \text{such that } \| z \| = \rho(\omega) \tag{14}$$

$$P(\omega,\eta) = \{z \in R^n : \langle z,\eta \rangle = h(\omega,\eta)\}, \quad \eta \neq 0. \tag{15}$$

Clearly ρ maps Ω into R^1 and ζ maps Ω into R^n. Moreover, $P(\omega,\eta)$ is the support hyperplane of $K(\omega)$ with outward normal η. Thus by Remark 3, $\rho(\omega) = 0$ if and only if $0 \in K(\omega)$; for $\rho(\omega) > 0$, $\zeta(\omega) \in \partial K(\omega)$; for $\rho(\omega) > 0$, $z = \zeta(\omega)$ if and only if $z \in P(\omega,-z) \cap K(\omega) = S_{K(\omega)}(-z)$. Furthermore, straightforward arguments [20] show that the following result holds.

Theorem 7

Let $K(\omega)$ be a nonempty, compact, convex set in R^n defined for each $\omega \in \Omega$. If $K(\cdot)$ is continuous (Lipschitz continuous) on Ω, then $\rho(\cdot)$ is continuous (Lipschitz continuous) on Ω.

VI. Abstract Problem 1

In this section the first of three abstract problems which have application to wide classes of optimization problems is presented. An algorithm for solving the problem is described and conditions for convergence are given.

The first abstract problem is the following.

Abstract Problem 1 (AP1)

Given: $\Omega = [\omega_L,\omega_U]$, $\omega_L < \omega_U$, a compact interval in R^1; $K(\omega)$ a nonempty set in R^n defined for each $\omega \in \Omega$ and satisfying assumptions

A1: $K(\omega)$ compact for each $\omega \in \Omega$,
A2: $K(\omega)$ convex for each $\omega \in \Omega$,
A3: $K(\cdot)$ Lipschitz continuous on Ω.

Find:

$$\omega^* = \min_{\omega \in \Omega, 0 \in K(\omega)} \omega.$$

A solution ω^* of AP1 may or may not exist since $0 \notin K(\omega)$, for all $\omega \in \Omega$, is a possibility. If $0 \in K(\omega)$, for some $\omega \in \Omega$, then A1 and A3 imply that ω^* does exist, i.e., $0 \in K(\omega^*)$, and $0 \notin K(\omega)$, for all $\omega < \omega^*, \omega \in \Omega$. Moreover if $\omega^* \neq \omega_L$, $0 \in \partial K(\omega^*)$.

In many applications of AP1 a contact function $s(\omega,\eta)$ of $K(\omega)$ provides the only means of computing points in $K(\omega)$. Thus an iterative method must be employed, which usually means that an exact solution ω^* with $0 \in K(\omega^*)$ cannot be obtained. The goal is to find an ϵ-approximate solution of AP1: a pair (ω^+,z^+) in $\Omega \times R^n$ such that $\omega^+ \leq \omega^*$, $z^+ \in K(\omega^+)$, and $\| z^+ \| < \epsilon$.

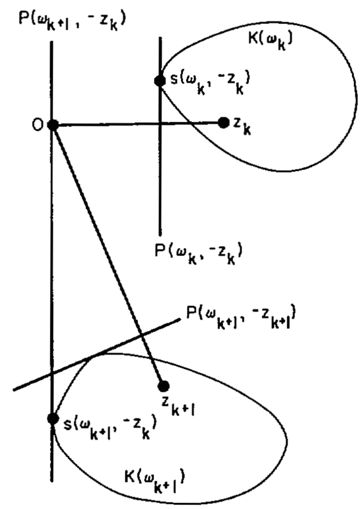

Fig. 1. Geometric interpretation of Algorithm AP1.

It should be emphasized that if $\epsilon > 0$ is small, this does not necessarily imply that $\omega^* - \omega^+$ is small. In most applications if $\omega^* - \omega^+$ is large, this would represent a desirable tradeoff in the "accuracy" of satisfying $0 \in K(\omega)$ and minimizing ω.

The basic idea of the algorithm for solving AP1 is indicated geometrically in Fig. 1. Suppose $0 \notin K(\omega)$ for $\omega_L \leq \omega \leq \omega_k$ and it is desired to obtain ω_{k+1} such that $\omega_k < \omega_{k+1} \leq \omega^*$ and $0 \notin K(\omega)$ for $\omega_L \leq \omega < \omega_{k+1}$. The first step is to apply a projection algorithm such as BP2 to the minimization of $\| z \|$ on $K(\omega_k)$. When a sufficiently good approximation to this subproblem is obtained, say z_k, the hyperplane $P(\omega_k,-z_k)$, which is the support hyperplane of $K(\omega_k)$ having outward normal $-z_k$, strictly separates $K(\omega_k)$ from the origin. Thus if ω is allowed to increase until $P(\omega,-z_k)$ just touches the origin, $\omega = \omega_{k+1} > \omega_k$. Clearly $\omega_{k+1} \leq \omega^*$ because $P(\omega,-z_k)$ strictly separates $K(\omega)$ from the origin for $\omega_k \leq \omega < \omega_{k+1}$. The algorithm obtained by repeating this process is doubly iterative, the inner loop involving a projection algorithm such as described in Section IV and the outer loop producing an increasing sequence $\{\omega_k\}$, which will be shown to converge upwards to ω^*.

Before giving the details of the algorithm and proving convergence, some additional facts and notation are needed. Let ω be an element of Ω and $s(\omega,\cdot)$ a specific contact function of $K(\omega)$. Consider

$$\begin{aligned}\gamma(\omega,\eta) &= \| \eta \|^{-2}(-h(\omega,-\eta)), && h(\omega,-\eta) < 0\\ &= 0, && h(\omega,-\eta) \geq 0.\end{aligned} \tag{16}$$

Thus $\gamma(\omega,\cdot)$ is a function defined on R^n. Geometrically, for $\omega \in \Omega$, $\omega < \omega^*$, $z \in K(\omega)$, the point $\gamma(\omega,z)z$ is either the point $L(0;z) \cap P(\omega,-z)$ or the origin, depending on whether or not $L(0;z) \cap P(\omega,-z)$ is on the line segment connecting z and the origin. This along with the functions introduced in (13) and (14) is shown in Fig. 2. Notice that $\gamma(\omega,z) > 0$ implies $P(\omega,-z)$ strictly separates $K(\omega)$ from

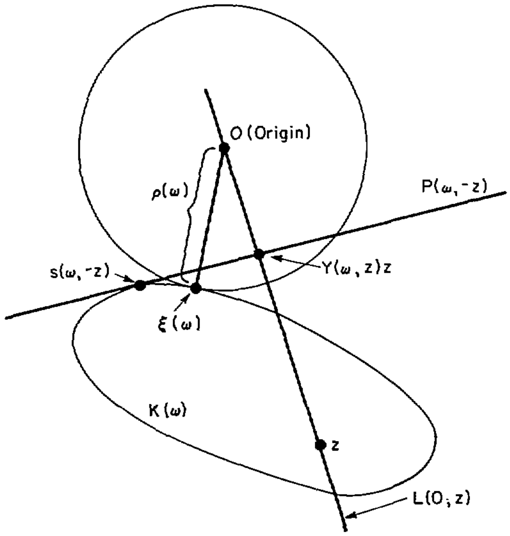

Fig. 2. Geometric interpretation of functional notation used in Algorithm AP1.

the origin. The following theorem summarizes some required properties of these functions.

Theorem 8

Let $K(\cdot)$ satisfy assumptions A1 and A2. For $\omega \in \Omega$ let $\gamma(\omega,\cdot)$ be defined by (16) and let $h(\omega,\cdot)$ denote the support function of $K(\omega)$. Then: 1) $0 \leq \gamma(\omega,z) \leq 1$, $z \in K(\omega)$; 2) $\gamma(\omega^*,z) = 0, z \in K(\omega^*)$; 3) for $\omega < \omega^*, \gamma(\omega,z) = 1$ if and only if $z = \zeta(\omega)$; 4) for $\omega < \omega^*$ and $z \in K(\omega)$, $\|z\| \gamma(\omega,z) \leq \rho(\omega)$; 5) if $\omega < \omega^*, z \in K(\omega)$, and $\gamma(\omega,z) > 0$, then $h(\omega,-z) = -\|z\|^2\gamma(\omega,z) < 0$; 6) $h(\omega^*,\eta) \geq 0$, $\eta \in R^n$.

The results of the theorem are geometrically obvious and can be proved by modifying arguments used in [25] and [26].

Remark 5: Given $\omega \in \Omega, \omega \leq \omega^*$, Gilbert's algorithm and Algorithms BP1 and BP2 generate sequences $\{z_i\}$ such that for $i \geq 0$ and $i \to \infty$: 1) $z_i \in K(\omega)$; 2) $\|z_i\| \geq \|z_{i+1}\|$, $\|z_i\| \to \|\zeta(\omega)\|$, and $\|z_i\| = \|z_{i+1}\|$ implies $z_i = \zeta(\omega)$; 3) $z_i \to \zeta(\omega)$; 4) $\|z_i\| \gamma(\omega,z_i) \to \rho(\omega)$; 5) if $\omega < \omega^*$, $\gamma(\omega,z_i) \to 1$. These results are proved in [25] and [26].

Now it is possible to consider the procedure for finding an ϵ-approximate solution of AP1.

*Algorithm AP*1

Assume $0 \in K(\omega)$ for some $\omega \in \Omega$. Let $\epsilon > 0$ be given, select θ, $0 < \theta < 1$, and set $\omega_0 = \omega_L$, $k = 0$.

Step 1: Let ω_k be such that $\omega_L \leq \omega_k \leq \omega^*$. Apply a projection algorithm to the minimization of $\|z\|$ on $K(\omega_k)$. At each iteration determine $\|z\|$ and $\gamma(\omega_k,z)$. If $\|z\| \leq \epsilon$ or $\gamma(\omega_k,z) \geq \theta$, end iterations and set $z = z_k$. If $\|z\| \leq \epsilon$, the desired result is obtained: $(\omega^+,z^+) = (\omega_k,z_k)$. If $\|z\| > \epsilon$, go to Step 2. [Observe that if $\omega_k = \omega^*$ Step 2 will not be entered because $\gamma(\omega^*,z) = 0$, for all $z \in K(\omega^*)$.]

Step 2: Since $\gamma(\omega_k,z_k) > 0$, it follows from Theorem 8, A3, and Theorem 4 that there exists $\omega_{k+1} \in \Omega$ such that $\omega_{k+1} > \omega_k$, $h(\omega_{k+1},-z_k) = 0$, and $h(\omega,-z_k) < 0$, for $\omega_k \leq \omega < \omega_{k+1}$. (Observe $0 \notin K(\omega)$ for $\omega_L \leq \omega < \omega_{k+1}$.) Determine ω_{k+1} by a root finding procedure applied to $h(\omega,-z_k) = 0$. Increase k by one and return to Step 1.

This algorithm clearly generates sequences $\{\omega_k\}$ and $\{z_k\}$ where $\omega_k \in \Omega$, $\omega_L \leq \omega_k \leq \omega^*$, and $z_k \in K(\omega_k)$. The following theorem describes more fully the behavior of these approximating sequences.

Theorem 9

Let $K(\cdot)$ satisfy assumptions A1, A2, and A3, let ω^* defined in AP1 exist, let $s(\omega,\cdot)$ be an arbitrary contact function of $K(\omega)$, and consider Algorithm AP1. Then: 1) for any $\epsilon > 0$ there exists an integer $\tilde{k} = \tilde{k}(\epsilon) \geq 0$ such that $\|z_{\tilde{k}}\| \leq \epsilon$, $\omega_{\tilde{k}} \leq \omega^*$, $z_{\tilde{k}} \in K(\omega_{\tilde{k}})$, and if $\tilde{k} > 0$, $\omega_0 < \omega_1 < \cdots < \omega_{\tilde{k}}$ and $\|z_k\| > \epsilon$, for $k = 0,1,\cdots,\tilde{k}-1$; 2) as $\epsilon \to 0$, $\omega_{\tilde{k}(\epsilon)} \to \omega^*$ and $z_{\tilde{k}(\epsilon)} \to 0$.

Proof: If $\omega^* = \omega_L$, then $\tilde{k} = 0$ and the proof follows trivially from Remark 5. Consider $\omega^* > \omega_L$. If for some finite k, $\omega_k = \omega^*$, then $k = \tilde{k}$ and again the proof follows trivially from Remark 5. Now assume $\omega_L \leq \omega_k < \omega^*$ for all $k \geq 0$; i.e., $\omega_{k+1} > \omega_k$ for all $k \geq 0$.

Define

$$\bar{v} = \sup_{\substack{\omega_a,\omega_b \in \Omega \\ \omega_a > \omega_b \\ z \in \bigcup K(\omega) \\ \omega_L \leqq \omega < \omega^*}} \frac{h(\omega_a,-z) - h(\omega_b,-z)}{(\omega_a - \omega_b)\|z\|}. \tag{17}$$

Noting that $h(\omega,-z/\|z\|) = (1/\|z\|)h(\omega,-z), z \neq 0$, the existence of $\bar{v}$ follows from A3 and Theorem 5. Since $h(\omega_0,-z_0) < 0$ and $h(\omega_1,-z_0) = 0$, $\bar{v} > 0$. Thus (17) implies

$$\begin{aligned} \omega_{k+1} - \omega_k &\geq [1/(\bar{v}\|z_k\|)][h(\omega_{k+1},-z_k) - h(\omega_k,-z_k)] \\ &= (1/\bar{v})\|z_k\|\gamma(\omega_k,z_k) \\ &\geq (1/\bar{v})\|z_k\|\theta. \end{aligned} \tag{18}$$

Part 1) of the theorem can fail only if for all $k \geq 0$, a) $< \omega_k\, \omega_{k+1} < \omega^*$ and b) $\|z_k\| > \epsilon$. But a) implies $\omega_k \to \bar{\omega} \leq \omega^*$ and $\omega_{k+1} - \omega_k \to 0$ which by (18) yields $\|z_k\| \to 0$. This contradicts b) and proves 1). Moreover, by (13) $\|z_{\tilde{k}(\epsilon)}\| \geq \rho(\omega_{\tilde{k}(\epsilon)})$ and thus $\rho(\omega_{\tilde{k}(\epsilon)}) \to 0$ as $\epsilon \to 0$. This, Theorem 7, $\rho(\omega) > 0$ for $\omega_L \leq \omega < \omega^*$, and $\rho(\omega^*) = 0$ imply $\omega_{k(\epsilon)} \to \omega^*$ as $\epsilon \to 0$, which completes the proof of 2).

Remark 6—*Relaxation of A*1: The boundedness requirement on $K(\omega)$ is considered in the next section. If $K(\omega)$ is bounded but not closed, Algorithm AP1 can be applied by replacing $K(\omega)$ by its closure $\bar{K}(\omega)$. Let contact and support functions of $\bar{K}$, $\bar{s}$ and $\bar{h}$, be defined, and let $\bar{K}$ satisfy A1. If K has a contact function s and a support function h it is possible to set $\bar{s} = s$ and $\bar{h} = h$. An ϵ-approximate solution for AP1 with $\bar{K}$ is often an acceptable approximate solution for AP1 with K. An application of this idea is where it is difficult or impossible to prove that K is closed and yet it is feasible to determine s and h.

Remark 7—*Relaxation of A*2: Even if $K(\omega)$ is not convex, functions s and h exist so that the steps of the algorithm may be followed. If this is done it is clear form Theorem 1 that $z_k \in \bar{K}(\omega_k)$ where $\bar{K}(\omega) = \text{co } K(\omega)$. Moreover $\omega_k \to \bar{\omega}^*$ where $\bar{\omega}^*$ is the smallest $\omega \in \Omega$ such that $0 \in \bar{K}(\omega)$. Clearly $\bar{\omega}^* \leq \omega^*$. If an ϵ-approximate solution for $\bar{K}$, $(\bar{\omega}^+,\bar{z}^+)$, is obtained such that $\bar{z}^+ \in K(\bar{\omega}^+)$, then this ϵ-approximate solution is also an ϵ-approximate solution for the nonconvex problem involving K. There are applications where this circumstance occurs. If it does not occur $(\bar{\omega}^+,\bar{z}^+)$ may still lead to an acceptable solution of the nonconvex problem (see Section IX).

Remark 8—*Relaxation of* A3: In all applications which have been considered by the authors, A3 is satisfied. Actually Theorem 9 is true under weaker smoothness conditions, e.g., that

$$[h(\omega_a,\eta) - h(\omega_b,\eta)]/(\omega_a - \omega_b) \leq \bar{\nu} < +\infty$$

for all ω_a, $\omega_b \in \Omega$, $\omega_a > \omega_b$, $\eta \in S$. By arguments similar to those used in the proof of Theorem 4 this condition also implies that $K(\omega)$ is upper semi-continuous with respect to inclusion. This guarantees that the abstract problem has a solution if $0 \in K(\omega)$ for some $\omega \in \Omega$.

Remark 9: It is assumed in the statement of the algorithm that $0 \in K(\omega)$ for some $\omega \in \Omega$. In applications this fact may be difficult or impossible to verify a priori. Suppose the algorithm is applied and $0 \notin K(\omega)$ for all $\omega \in \Omega$. By considering the steps in the proof of Theorem 9 it follows that for ϵ sufficiently small there is an integer $\hat{k}$ such that $h(\omega,-z_{\hat{k}}) < 0$ for $\omega_{\hat{k}} \leq \omega \leq \omega_U$. Thus in Step 2 it is impossible to determine ω_{k+1} when $k = \hat{k}$. Hence the failure of the algorithm to continue implies that AP1 does not have a solution. Note that AP1 may have an ϵ-approximate solution even when it does not have a solution. Thus if the algorithm stops in the normal way this does not imply that AP1 has a solution.

Remark 10: Suppose Algorithm BP2 is used in Algorithm AP1. Each time Algorithm AP1 returns to Step 1, Algorithm BP2 is applied to a new BP [minimize $\|z\|$ on $K(\omega_k)$ rather than on $K(\omega_{k-1})$]. Initialization of Algorithm BP2 requires the stipulation of $\eta_1,\cdots,\eta_p$, and if these directions are chosen so that $s_{K(\omega_k)}(\eta_1),\cdots,s_{K(\omega_k)}(\eta_p)$ are boundary points of $K(\omega_k)$ "near" $\zeta(\omega_k)$, it would be expected that $z_k \cong \zeta(\omega_k)$ would be obtained in a small number of iterations of Algorithm BP2. Fortunately a natural good choice for $\eta_1,\cdots,\eta_p$ is available. On the previous application of Algorithm BP2 [to $K(\omega_{k-1})$], Algorithm BP2 ends with directions $\eta_1,\cdots,\eta_p$ such that $z_{k-1} = \sum_{i=1}^p \lambda_i s_{K(\omega_{k-1})}(\eta_i)$, $\sum_{i=1}^p \lambda_i = 1$, $\lambda_i \geq 0$, $i = 1,\cdots,p$ [see (4)]. The directions $\eta_1,\cdots,\eta_p$ are "good" directions for $K(\omega_{k-1})$. Since $K(\omega_k)$ and $K(\omega_{k-1})$ are not apt to differ greatly, these directions should also be "good" initializing directions for Algorithm BP2 applied to $K(\omega_k)$. That this is indeed the case has been confirmed in numerical experiments [27].

Remark 11: For the determination of ω_{k+1} in Step 2 of Algorithm AP1 it is necessary to require that

$$h(\omega_{k+1},-z_k) = 0.$$

Sometimes this equation can be solved analytically (see the examples in Section IX). If it is necessary to use a root finding procedure, it would be highly desirable to relax the requirement $h(\omega_{k+1},-z_k) = 0$. This can be done in the following way. Let $0 < \phi < 1$ and assume ω_{k+1} is selected so that $h(\omega,-z_k) < 0$ for $\omega_k \leq \omega < \omega_{k+1}$ and $h(\omega_{k+1},-z_k) \geq \phi h(\omega_k,-z_k)$. Then (18) is replaced by

$$\begin{aligned} \omega_{k+1} - \omega_k &\geq [(1-\phi)/(\bar{\nu}\|z_k\|)][-h(\omega_k,-z_k)] \\ &\geq [(1-\phi)/\bar{\nu}]\|z_k\|\theta \end{aligned} \tag{19}$$

and the results of Theorem 9 are unchanged.

If the determination of ω_{k+1} is to be made by evaluating $h(\omega,-z_k)$ on a grid of points for $\omega_k \leq \omega$, it is necessary to take the spacing of grid points, $\Delta\omega$, sufficiently small so that $h(\omega,-z_k)$ remains negative between the grid points. The bound (17) permits such a spacing to be determined: $\Delta\omega < -\bar{\nu}^{-1}\|z\|^{-1}h(\omega_k,-z_k)$.

Remark 12: In general $\|z_k\|$ does not decrease monotonically. An illustration is given in Fig. 1.

Theorem 9 gives no information on the rate of convergence of $\{\omega_k\}$ and $\{z_k\}$. By introducing an additional assumption A4, which assumes that $K(\omega)$ sweeps into the origin sufficiently fast, it is possible to prove geometric convergence of $\{\omega_k\}$ and $\{z_k\}$.

A4: There exist ω^* defined in AP1, $\alpha \in R^1$, $\alpha > 0$, such that $\rho(\omega) \geq \alpha(\omega^* - \omega)$ for all $\omega_L \leq \omega \leq \omega^*$.

Observe that if the constant α in A4 is available, then $\|z_k\| \geq \rho(\omega_k) \geq \alpha(\omega^* - \omega_k)$ for $\omega_L \leq \omega_k \leq \omega^*$ implies that the stopping condition $\|z_k\| \leq \epsilon$ may be chosen to achieve any desired accuracy of ω_k. The main consequence of A4 is the following theorem.

Theorem 10

Let $K(\cdot)$ satisfy assumptions A1, A2, A3, and A4, and consider Algorithm AP1. If $\omega_k < \omega^*$, for all $k \geq 0$, and the stopping condition $\|z_k\| \leq \epsilon$ is not imposed, then geometric convergence is obtained, i.e., for $k \geq 0$ and $k \to \infty$:

1) $\omega^* - \omega_k \leq (\omega^* - \omega_L)\mu^k$, $\quad 0 < \mu < 1$

2) $\|z_k\| \leq \beta\mu^k$, $\quad \beta > 0$.

Proof: From A4, $\|z_k\| \geq \rho(\omega_k)$, and (19)

$$\omega_{k+1} - \omega_k \geq [(1-\phi)/\bar{\nu}]\theta\alpha(\omega^* - \omega_k). \tag{20}$$

Letting $q = (1-\phi)\theta\alpha/\bar{\nu}$ and $\Delta_k = \omega^* - \omega_k$, (20) gives $\Delta_k - \Delta_{k+1} \geq q\Delta_k$ which yields

$$(1-q)\Delta_k \geq \Delta_{k+1}. \tag{21}$$

Theorem 9 implies $\Delta_k > 0$ and $\Delta_k \to 0$. This and (21) yields $(1-q) > 0$. Since $q > 0$, $\mu = 1 - q$ satisfies $0 < \mu < 1$ which together with the definition of Δ_k proves 1). By using (19) and observing that $\omega_{k+1} - \omega_k \leq \omega^* - \omega_k \leq (\omega^* - \omega_L)\mu^k$, 2) follows if $\beta = (\omega^* - \omega_L)\bar{\nu}\theta^{-1}(1-\phi)^{-1}$.

Since A4 is often difficult to verify directly, the following theorem is useful.

Theorem 11

Suppose there exist ω^* defined in AP1, $c \in R^1$, $c > 0$, $\bar{\omega} \in \Omega$, $\bar{\omega} < \omega^*$, $\eta^* \in R^n$, $\eta^* \neq 0$ such that $h(\omega,\eta^*)/\|\eta^*\| \leq -c(\omega^* - \omega)$ for all $\bar{\omega} \leq \omega \leq \omega^*$. Then: 1) A4 is true; 2) η^* is the (outward) normal to a support hyperplane of $K(\omega^*)$ passing through the origin.

Proof: Consider 2) first. From the condition on h,

$$h(\omega^*,\eta^*) = \max_{z \in K(\omega^*)} \langle z,\eta^* \rangle \leq 0. \tag{22}$$

But since $z = 0 \in K(\omega^*)$, (22) yields

$$\langle z,\eta^* \rangle \leq h(\omega^*,\eta^*) = 0, \quad \text{for all } z \in K(\omega^*) \tag{23}$$

which implies $\{z: \langle z,\eta^* \rangle = 0\}$ is a support hyperplane of $K(\omega^*)$ passing through 0. Now consider 1). By the definition of h, $\langle z,\eta^* \rangle \leq h(\omega,\eta^*)$ for all $z \in K(\omega)$. Thus for $\bar{\omega} \leq \omega \leq \omega^*$

$$\rho(\omega) = \min_{z \in K(\omega)} \|z\| \geq \min_{\langle z,\eta^* \rangle \leq h(\omega,\eta^*)} \|z\|$$

$$= -h(\omega,\eta^*)/\|\eta^*\| \geq c(\omega^* - \omega). \tag{24}$$

Now let

$$\rho_m = \min_{\omega_L \leq \omega \leq \bar{\omega}} \rho(\omega).$$

Clearly $\rho_m > 0$ and $\rho(\omega) \geq [\rho_m/(\omega^* - \omega_L)](\omega^* - \omega)$ for $\omega_L \leq \omega \leq \bar{\omega}$. Thus by defining $\alpha = \min\{c, \rho_m/(\omega^* - \omega_L)\}$, A4 results.

A condition similar to that in Theorem 11 has been used by Fujisawa and Yasuda [22] in the special context of a minimum-time optimal control problem.

VII. Abstract Problem 2

Without complicating the Algorithm for AP1 considerably it does not appear possible to eliminate the requirement that $K(\omega)$ be bounded on Ω. There are, however, several situations of practical interest where the boundedness requirement cannot be satisfied. Fortunately, the most important of these can be treated indirectly by considering an equivalent problem. The case of concern is the second abstract problem.

Abstract Problem 2 (*AP*2)

Given: $\Omega = [\omega_L,\omega_U]$, $\omega_L < \omega_U$, a compact interval in R^1; for each $\omega \in \Omega$

$$\bar{K}(\omega) = K_C(\omega) + L(\omega) \tag{25}$$

where $K_C(\omega)$ is a nonempty set in R^n defined for all $\omega \in \Omega$ and satisfying assumptions A1, A2, and A3; $L(\omega)$ is the linear subspace of R^n given by

$$L(\omega) = \{z: H(\omega)z = 0\}; \tag{26}$$

$H(\omega)$ is an $m \times n$ matrix function, Lipschitz continuous on Ω; and (to avoid trivialities) rank $H(\omega) > 0$ for all $\omega \in \Omega$, rank $H(\omega) < n$ for some $\omega \in \Omega$. Find:

$$\bar{\omega}^* = \min_{\omega \in \Omega,\, 0 \in \bar{K}(\omega)} \omega.$$

Now consider the set function

$$K(\omega) = H(\omega)\bar{K}(\omega) \tag{27}$$

which maps Ω into subsets of R^m. Clearly

$$K(\omega) = H(\omega)K_C(\omega). \tag{28}$$

From (28) and the assumptions in AP2 it follows easily that $K(\omega)$ satisfies A1, A2, and A3. Moreover, if Algorithm AP1 is applied to $K(\omega)$ it leads to a solution of AP2. This follows from Theorem 12.

Theorem 12

Consider AP1 with $K(\omega) \subset R^m$ defined by (27) and AP2 with $\bar{K}(\omega) \subset R^n$ defined by (25). Then: 1) $0 \in K(\omega)$ if and only if $0 \in \bar{K}(\omega)$; 2) a solution ω^* of AP1 exists if and only if a solution $\bar{\omega}^*$ of AP2 exists; 3) if ω^* exists, $\omega^* = \bar{\omega}^*$.

Proof: Parts 2) and 3) follow directly from 1). Consider 1). The implication that $0 \in K(\omega)$ if $0 \in \bar{K}(\omega)$ is trivially true from (27). Now suppose $0 \notin \bar{K}(\omega)$ and there exists a $\bar{z} \in \bar{K}(\omega)$ such that $H(\omega)\bar{z} = 0$. Since $\bar{z} \in \bar{K}(\omega)$, (25) implies $\bar{z} = \bar{z}_1 + \bar{z}_2$ where $\bar{z}_1 \in K_C(\omega)$, $\bar{z}_2 \in L(\omega)$. But $H(\omega)\bar{z} = 0 = H(\omega)\bar{z}_1$ gives $-\bar{z}_1 \in L(\omega)$. Then (25) yields $0 = \bar{z}_1 + (-\bar{z}_1) \in \bar{K}(\omega)$, a contradiction. Thus $0 \notin \bar{K}(\omega)$ implies $0 \notin H(\omega)\bar{K}(\omega) = K(\omega)$, which completes the proof.

In order to apply Algorithm AP1 to $K(\omega) = H(\omega)K_C(\omega)$ it is necessary to make evaluations of $h(\omega,\eta)$ and $s(\omega,\eta)$. These evaluations can be made in terms of the data of AP2 by using part 2) of Theorem 3. In particular,

$$h(\omega,\eta) = h_{K_C(\omega)}(H'(\omega)\eta) \tag{29}$$

and a realization for the contact function of $K(\omega)$ is

$$s(\omega,\eta) = H(\omega)s_{K_C(\omega)}(H'(\omega)\eta). \tag{30}$$

Another question remains to be answered. If an ϵ-approximate solution (ω^+,z^+) of AP1 is obtained with $K(\omega) = H(\omega)K_C(\omega)$, is there a corresponding $\bar{\epsilon}$-approximate solution $(\bar{\omega}^+,\bar{z}^+)$ of AP2 where $\bar{\epsilon}$ can be made small by making ϵ small? To investigate this question suppose that z^+, $\|z^+\| \leq \epsilon$, is obtained by an application of Algorithm AP1. Then [see (4)]

$$z^+ = \sum_{i=1}^{p} \lambda_i s(\omega^+,\eta_i) \tag{31}$$

where $\sum_{i=1}^{p} \lambda_i = 1$ and $\lambda_i \geq 0$, $i = 1,\cdots,p$. Because of (30) it is clear that

$$x = \sum_{i=1}^{p} \lambda_i s_{K_C(\omega^+)}(H'(\omega^+)\eta_i) \tag{32}$$

satisfies $x \in K_C(\omega^+)$ and $H(\omega^+)x = z^+$. Furthermore, $H(\omega^+)(x + y) = z^+$ for all $y \in L(\omega^+)$. If $y \in L(\omega^+)$ can be chosen so that $\|z^+\| \leq \epsilon$ implies $\|x + y\| \leq \alpha\epsilon$ where $\alpha > 0$ is a fixed constant, then $\bar{\omega}^+ = \omega^+$ and $\bar{z}^+ = x + y$ will form an $\alpha\epsilon$-approximate solution of AP2.

The best choice for y satisfies: minimize $\|x + y\|$ subject to $H(\omega^+)(x + y) = z^+$. The solution of this minimization problem can be expressed conveniently by

$$\bar{z}^+ = x + y = H\dagger(\omega^+)z^+ \tag{33}$$

where $H\dagger(\omega)$ denotes the pseudoinverse of $H(\omega)$. Let

$$\| H\dagger(\omega) \| = \max_{\eta \in S} \| H\dagger(\omega)\eta \|$$

denote the norm of $H\dagger(\omega)$. Then if

$$\| H\dagger(\omega) \| \leq \alpha \quad \text{for all } \omega \in \Omega \tag{34}$$

it is clear that

$$\| \bar{z}^+ \| \leq \alpha \| z^+ \| = \alpha\epsilon. \tag{35}$$

Hence if (34) is satisfied the question is answered affirmatively. Specifically, $\bar{\omega}^+ = \omega^+$, $\bar{\epsilon} = \alpha\epsilon$, and $\bar{z}^+$ is given by (33) with $x \in K_C(\omega^+)$ given by (32) and $y \in L(\omega^+)$ given by (33).

Condition (34) often may be verified directly in applications. Alternatively, it is sometimes possible to give explicit conditions on $H(\omega)$ which imply (34). For example, if $H(\omega)$ is continuous on Ω (required by the hypotheses of AP2) and rank, $H(\omega) = m$ for all $\omega \in \Omega$, then it is not difficult to show that there exists an $\alpha > 0$ such that (34) is satisfied.

Finally, it should be remarked that one does not always care if an ϵ-approximate solution of AP2 is obtained. There are applications where it is sufficient to find $\bar{z}^+ \in \bar{K}(\omega^+)$, $\omega^+ \leq \bar{\omega}^*$, such that $\| H(\omega^+)\bar{z}^+ \| \leq \epsilon$.

VIII. Abstract Problem 3

There are a number of applications in which Ω is a finite set of integers rather than a compact interval in R^1. In this case the statement of the abstract problem and the algorithm for its solution are considerably simpler than in Section VI. Let I_q denote the integer set $\{0,1,\cdots,q\}$ and consider the third abstract problem.

Abstract Problem 3(*AP*3)

Given: $\Omega = I_N$, where N is a positive integer; $K(\omega)$ a nonempty set in R^n defined for each $\omega \in \Omega$ and satisfying assumptions

A1: $K(\omega)$ compact for each $\omega \in \Omega$
A2: $K(\omega)$ convex for each $\omega \in \Omega$.

Find:

$$\omega^* = \min_{\omega \in \Omega,\, 0 \in K(\omega)} \omega.$$

The problem may or may not have a solution. If ω^* exists, then $0 \in K(\omega^*)$ and either $\omega^* = 0$ or $0 \notin K(\omega)$ for all $\omega \in I_{\omega^*-1}$.

If a contact function $s(\omega,\cdot)$ of $K(\omega)$ is available, let $\gamma(\omega,\cdot)$ be defined by (16) and consider the following.

*Algorithm AP*3

Let $\epsilon > 0$ be given, select θ, $0 < \theta < 1$, and set $\omega_0 = 0$, $k = 0$.

Step 1: For $\omega_k \in \Omega$ apply Algorithm BP1 or BP2 to the minimization of $\| z \|$ on $K(\omega_k)$. At each iteration determine $\| z \|$ and $\gamma(\omega_k,z)$. If $\| z \| \leq \epsilon$ or $\gamma(\omega_k,z) \geq \theta$, end iterations and set $z = z_k$. If $\| z \| \leq \epsilon$, the desired result is obtained $(\omega^+,z^+) = (\omega_k,z_k)$. If $\| z \| > \epsilon$, go to Step 2. [Observe that if $\omega_k = \omega^*$, Step 2 will not be entered because $\gamma(\omega^*,z) = 0$, for all $z \in K(\omega^*)$.]

Step 2: Since $\gamma(\omega_k,z_k) > 0$, Theorem 8 implies

$$h(\omega_k,-z_k) < 0.$$

Let $\omega_{k+1} \in \Omega$ be such that $h(\omega,-z_k) < 0$, for $\omega_k \leq \omega < \omega_{k+1}$, and $h(\omega_{k+1},-z_k) \geq 0$. If ω_{k+1} does not exist, i.e., $h(\omega,-z_k) < 0$ for $\omega_k \leq \omega \leq N$, then a solution ω^* of AP3 does not exist and the procedure terminates (see Remark 9). If ω_{k+1} does exist, determine it, increase k by one, and return to Step 1.

This algorithm either indicates that ω^* does not exist or generates an ϵ-approximate solution of AP3. The following result is obtained.

Theorem 13

Let $K(\cdot)$ satisfy assumptions A1 and A2, let ω^* defined in AP3 exist, and let $s(\omega,\cdot)$ be an arbitrary contact function of $K(\omega)$. Then the Algorithm for AP3 generates finite sequences $\{\omega_k\}$ and $\{z_k\}$ where $\omega_k \in I_N$, $0 \leq \omega_k \leq \omega^*$, and $z_k \in K(\omega_k)$. Moreover: 1) for any $\epsilon > 0$ there exists $\tilde{k} = \tilde{k}(\epsilon)$, $0 \leq \tilde{k} \leq N$, such that $\| z_{\tilde{k}} \| \leq \epsilon$, $z_{\tilde{k}} \in K(\omega_{\tilde{k}})$, and if $\tilde{k} > 0$, $\omega_0 < \omega_1 < \cdots < \omega_{\tilde{k}}$ and $\| z_k \| > \epsilon$, for $k = 0,1,\cdots,\tilde{k}-1$; 2), for $\epsilon > 0$ sufficiently small, $\omega_{\tilde{k}(\epsilon)} = \omega^*$ and $z_{k(\epsilon)} \to 0$ as $\epsilon \to 0$.

Proof: If $\omega^* = 0$, then $\tilde{k} = 0$ and the proof follows trivially from Remark 5. If $\omega^* > 0$, let

$$\alpha = \min_{\omega \in I_{\omega^*-1}} \rho(\omega).$$

Clearly $\alpha > 0$. Furthermore, if ϵ is chosen $<\alpha$, then $\| z_k \| \leq \epsilon < \alpha$ cannot be satisfied for $\omega_k \in I_{\omega^*-1}$. Thus $\omega_{\tilde{k}(\epsilon)} = \omega^*$, $\epsilon < \alpha$, and $z_{\tilde{k}(\epsilon)} \to 0$ as $\epsilon \to 0$ follows from Remark 5.

It should be observed that Remarks 6–9 can be extended to the present situation. Also it is possible to consider AP2 when Ω is an integer set.

IX. Application of Algorithm AP1 to Discrete-Time Optimal Control Problems

To illustrate the application of Algorithm AP1 to optimum control problems, two minimum-effort discrete-time control problems are now considered. Suppose the terminal state of a discrete-time dynamical system is given by

$$x(u) = d + \sum_{j=0}^{N-1} r_j u(j) \tag{36}$$

where x is an n vector and u denotes the control sequence $\{u(0),u(1),\cdots,u(N-1)\}$, $u(j) \in R^1$, $j \in I_{N-1} = \{0,1,\cdots,N-1\}$. The peak amplitude of the control u is given by

$$\| u \|_\infty = \max_{j \in I_{N-1}} | u(j) | \tag{37}$$

and the control "energy" by

$$\| u \|_2 = \left(\sum_{j=0}^{N-1} (u(j))^2 \right)^{1/2}. \tag{38}$$

The two control problems considered are as follows.

Control Problem 1 (*CP*1) (*Minimum Peak Amplitude Control*): Find u such that $x(u) = 0$ and $\| u \|_\infty$ is minimum.

Control Problem 2 (*CP*2) (*Minimum Energy with Peak Amplitude Constraint*): Find u such that $x(u) = 0$, $\| u \|_\infty \leq 1$, and $\| u \|_2$ is minimum.

A version of CP2 has been described by Polak and Deparis [17]. His numerical procedure makes use of parameterized convex sets, but is much closer in philosophy to the Neustadt procedures (see Section I) than it is to the procedures of this paper.

Control problem CP1 can be formulated as AP1 by defining

$$K(\omega) = \{z : z = x(u), \| u \|_\infty \leq \omega\}. \tag{39}$$

To apply Algorithm AP1 assumptions A1, A2, and A3 must be verified and expressions for $h(\omega,\eta)$ and a contact function must be obtained. It is easy to verify A1 and A2. Forming

$$\langle \eta, x(u) \rangle = \sum_{j=0}^{N-1} \langle \eta, r_j \rangle u(j) + \langle \eta, d \rangle \tag{40}$$

it is seen that $\langle \eta, x(u) \rangle = \langle \eta, z \rangle$ is maximized for $z \in K(\omega)$ if $u = u_{\omega,\eta}$ where

$$u_{\omega,\eta}(j) = \omega \operatorname{sgn} \langle r_j, \eta \rangle \tag{41}$$

(to be specific take sgn $\alpha = 0$ for $\alpha = 0$). Thus

$$s(\omega,\eta) = x(u_{\omega,\eta}), \quad h(\omega,\eta) = \langle \eta, d \rangle + \omega \sum_{j=0}^{N-1} | \langle \eta, r_j \rangle |. \tag{42}$$

The linearity of h in ω and Theorem 5 shows that A3 is satisfied. Also it allows ω_{k+1} to be obtained by a simple formula rather than a root finding procedure. Thus Algorithm AP1 is applicable. An ϵ-approximate solution (ω^+, z^+) of AP1 yields an ϵ-approximate control in the following sense: if $\eta_1, \cdots, \eta_p$ are obtained such that $z^+ = \sum_{i=1}^p \lambda_i s(\omega^+, \eta_i)$, $\lambda_i \geq 0$, $\sum_{i=1}^p \lambda_i = 1$, then

$$u^+ = \sum_{i=1}^{p} \lambda_i u_{\omega^+,\eta_i} \tag{43}$$

gives $\| x(u) \| \leq \epsilon$ and $\| u^+ \|_\infty \leq \omega^+ \leq \omega^*$. These results follow from $z^+ = x(u^+)$ and $\| u_{\omega^+,\eta_i} \|_\infty = \omega^+$.

It is of interest to note that

$$\bar{\nu} = \max_{j \in I_{N-1}} \| r_j \|.$$

In general it is not possible to obtain the η^* mentioned in Theorem 11 without knowing the solution CP1. However, if the r_j span R^n, for $j = 0, \cdots, N-1$, the conditions in Theorem 11 can be verified. This follows from the fact that $\| \bar{\eta} \|^{-1} h(\omega, \bar{\eta}) = -(\omega^* - \omega) \| \bar{\eta} \|^{-1} \sum_{j=0}^{N-1} | \langle \bar{\eta}, r_j \rangle |$ when $\bar{\eta}$ is an outward normal to a support hyperplane of $K(\omega^*)$ passing through the origin. Thus it is possible to take $\eta^* = \bar{\eta}$ and $c = \| \bar{\eta} \|^{-1} \sum_{j=0}^{N-1} | \langle \bar{\eta}, r_j \rangle |$. Clearly $c \neq 0$ since $\langle \bar{\eta}, r_j \rangle = 0$ for all $j \in I_{N-1}$ is not possible. Of course geometric convergence may still take place even if the r_j do not span R^n.

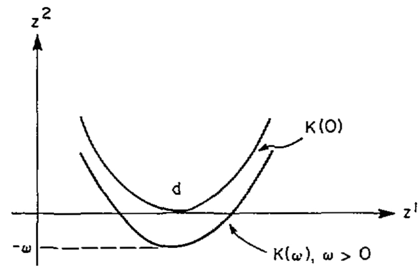

Fig. 3. $K(\omega)$ for CP2 with $n = 1$, $N = 1$.

There are several ways of approaching CP2. One is to define

$$K(\omega) = \{z : z = x(u), \| u \|_\infty \leq 1, \| u \|_2 \leq \omega\}. \tag{44}$$

This approach leads to a numerically feasible realization of Algorithm AP1. However, the numerical evaluation of $h(\omega,\eta)$ and a contact function is rather difficult.

An alternative approach is the following. Write $z = (\hat{z}, z^{n+1})$ where $z \in R^{n+1}$, $\hat{z} \in R^n$, $z^{n+1} \in R^1$ and define

$$K(\omega) = \{z : \hat{z} = x(u), z^{n+1} = \| u \|_2 - \omega, \| u \|_\infty \leq 1\}. \tag{45}$$

Let $K(\omega)$ be used to form AP1 (dimension $n+1$ rather than n). Then it is clear that u is optimal for CP2 if and only if $\| u \|_2 = \omega^*$. Unfortunately $K(\omega)$ is not convex (see Fig. 3 which shows $K(\omega)$ for $n = 1$ and $N = 1$). Thus it is necessary to follow the suggestion of Remark 7 and consider AP1 with $\bar{K}(\omega) = \text{co}\, K(\omega)$ taking the place of $K(\omega)$. It will now be shown that this leads to an acceptable numerical procedure for solving CP2.

Clearly $\bar{K}(\omega)$ satisfies A1 and A2. Moreover it is not difficult to show that a contact function $\bar{K}(\omega)$ is

$$\bar{s}(\omega,\eta) = (x(u_\eta), \| u_\eta \|_2 - \omega) \tag{46}$$

and that

$$\bar{h}(\omega,\eta) = \langle \hat{\eta}, x(u_\eta) \rangle + \eta^{n+1} \| u_\eta \|_2 - \eta^{n+1}\omega \tag{47}$$

where $\eta = (\hat{\eta}, \eta^{n+1}) \in R^{n+1}$ and u_η is defined by

$$\begin{aligned} u_\eta(j) &= -\text{sat}\,(\langle \hat{\eta}, r_j \rangle / 2\eta^{n+1}), \quad && \eta^{n+1} \neq 0 \\ &= \operatorname{sgn} \langle \hat{\eta}, r_j \rangle, && \eta^{n+1} = 0. \end{aligned} \tag{48}$$

The linearity of $\bar{h}$ in ω shows that $\bar{K}(\omega)$ satisfies A3 and again a formula for ω_{k+1} may be written. Thus it is possible to apply Algorithm AP1. Suppose an ϵ-approximate solution is obtained, i.e., $(\bar{\omega}^+, \bar{z}^+)$ where $\bar{\omega}^+ \leq \bar{\omega}^*$, $\bar{z}^+ \in \bar{K}(\bar{\omega}^+)$, and $\| \bar{z}^+ \| \leq \epsilon$.

Now it will be shown that this ϵ-approximate solution leads to an acceptable solution of CP2. Suppose $\eta_1, \cdots, \eta_p$ have been found such that $\bar{z}^+ = \sum_{i=1}^p \lambda_i \bar{s}(\bar{\omega}^+, \eta_i)$, $\lambda_i \geq 0$, $\sum_{i=1}^p \lambda_i = 1$. For

$$u^+ = \sum_{i=1}^{p} \lambda_i u_{\eta_i} \tag{49}$$

it is clear that $\bar{z}^+ = (x(u^+), \sum_{j=1}^p \lambda_i \| u_{\eta_i} \|_2 - \bar{\omega}^+)$. Since $\| \bar{z}^+ \| \leq \epsilon$ it follows that

$$\| x(u^+) \| \leq \epsilon \tag{50}$$

and $\sum_{i=1}^{p} \lambda_i \| u_{\eta_i} \|_2 \leq \bar{\omega}^+ + \epsilon$. But this, the triangle inequality, and $\bar{\omega}^* \leq \omega^*$ yield

$$\| u^+ \|_2 = \| \sum_{i=1}^{p} \lambda_i u_{\eta_i} \|_2 \leq \sum_{i=1}^{p} \lambda_i \| u_{\eta_i} \|_2 \leq \bar{\omega}^+ + \epsilon \leq \omega^* + \epsilon; \tag{51}$$

(50) and (51) show that u^+ satisfies the requirements of CP2 with terminal and energy error not greater than ϵ.

In this problem $\bar{\nu} = 1$. It is also possible to apply Theorem 11 to prove geometric convergence. Generalizations of the terminal condition $x(u) = 0$ in CP1 and CP2 can be handled without great difficulty. These and other matters pertaining to the way in which numerical calculations can be carried out will be considered in a future paper.

In most applications of CP1 and CP2, N is significantly greater than n. This shows the advantage of Algorithm AP1 as compared with methods in mathematical programming. Using programming methods $x(u) = 0$ would be treated as n constraints on u, and iterations would be made in the N-dimensional u-space. With Algorithm AP1 effective use of the structure of the equations of motion is made. This allows the calculations to be carried out in n-dimensional space (CP1) or $(n + 1)$-dimensional space (CP2). Since computer time per iteration and convergence rates tend to be degraded appreciably by dimensionality, computer time should be greatly reduced by using AP1.

X. Conclusions

Three abstract optimization problems have been presented along with algorithms for their numerical solution. When used with the BP2 subalgorithm these algorithms appear to be efficient and have a number of important advantages when applied to the solution of optimal control problems: their utilization of contact functions makes it possible to solve (infinite dimensional) optimal control problems as finite dimensional problems; their generality admits the solution of a wide variety of control problems involving many different indices of performance, terminal conditions, and equations of motion; the required hypotheses are relatively weak compared to the Neustadt procedures, e.g., it is possible to treat systems with singular controls; problem conditioning does not have a strong effect on convergence rate; the algorithms are completely determinant, e.g., there is no need to use empirical procedures for step size evaluation. Many of these advantages have been demonstrated conclusively in specific applications not discussed in this paper (e.g. [27]).

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Robert O. Barr (S'64–M'66) was born in Lima, Ohio, on January 4, 1940. He received the B.S.E., M.S., and Ph.D. degrees from the University of Michigan, Ann Arbor, in 1961, 1962, and 1966, respectively.

Since 1966 he has been associated with Michigan State University, East Lansing, in the Department of Electrical Engineering and Systems Science. He was appointed Assistant Professor in 1966 and Associate Professor in 1969. He also has had summer appointments with TRW Systems, Redondo Beach, Calif., and Sandia Laboratories, Livermore, Calif. His research interests include systems and control with emphasis on computational methods of optimization.

Dr. Barr is a member of Tau Beta Pi, Sigma Xi, and Phi Kappa Phi.

Elmer G. Gilbert (S'51–A'52–M'57) was born in Joliet, Ill., on March 29, 1930. He received the B.S.E. and M.S.E. degrees in electrical engineering in 1952 and 1953, and the Ph.D. degree in instrumentation engineering in 1957, all from the University of Michigan, Ann Arbor.

Since 1957 he has been on the faculty of the College of Engineering of the University of Michigan, and he is presently a Professor in the Computer, Information, and Control Engineering Program. His main interests are in the areas of control theory, computational algorithms for optimal control problems, and analog-hybrid computer technology. Presently he is an associate editor for the *SIAM Journal on Control*. He is an active consultant and holds several patents on computer devices.

Dr. Gilbert is a member of a number of technical and honorary societies and serves several of them as a committee member.

A Computer Program for the Synthesis of Decoupled Multivariable Feedback Systems

ELMER G. GILBERT, MEMBER, IEEE, AND JOHN R. PIVNICHNY, MEMBER, IEEE

Abstract—**Recently Gilbert [6] obtained general solution results on the decoupling of multivariable systems by state feedback. This paper presents a general-purpose computer program which carries out all of the calculations necessary for reducing these results to a useful synthesis procedure. The program is described in general terms and several examples of its application are given.**

I. Introduction

During the last few years there has been a renewed interest in the problem of decoupling multivariable systems by state feedback. While the decoupling problem was first posed by Morgan [7] in 1964, it was not until 1967 that a necessary and sufficient condition for decoupling was obtained by Falb and Wolovich [3], [4]. Falb and Wolovich also made definite contributions to the synthesis problem, but the complete structure of the solution did not appear until the paper by Gilbert [6]. Since the general solution results in [6] were proved by constructive arguments, they should in principle form the basis for a useful synthesis procedure.

The main purpose of this paper is to show that this is indeed the case. The proofs in [6] are reduced to an algorithmic form and are then mechanized by a general purpose computer program which yields all the necessary structural results and synthesis data. Computer mechanization is essential because the calculations are unwieldy for systems of order greater than three.

The paper is organized as follows. In Section II the decoupling problem is stated and key results of [6] are given. Section III describes in a step-by-step way the basic calculations which must be carried out. To make the presentation reasonably easy to follow and to give some insight into the arguments used in [6], justifications are given when it is easy to do so; a full understanding of some steps requires reference to [6]. Section IV indicates specific features of the computer program. Several design examples (including the control of a distillation column) are given in Section V. They illustrate how the computer output is used and give some idea of the potentialities of the synthesis procedure. Concluding remarks appear in Section VI.

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E. G. Gilbert is with the Computer, Information, and Control Engineering Program, University of Michigan, Ann Arbor, Mich.

J. R. Pivnichny is with the Computer, Information, and Control Engineering Program, University of Michigan, Ann Arbor, Mich., on leave from IBM Corp., Endicott, N. Y.