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THE DECOUPLING OF MULTIVARIABLE SYSTEMS BY STATE FEEDBACK*

ELMER G. GILBERT†

1. Introduction. The objective of this paper is to develop a comprehensive theory for the decoupling of multivariable systems by state feedback. We begin by giving a preliminary formulation of the decoupling problem, discussing certain aspects of its solution, reviewing previous research, and indicating the contributions of this paper.

Consider the linear dynamical system with input u, output y, and state x:

$$\frac{dx}{dt} = Ax + Bu(t), \qquad y(t) = Cx. \tag{1.1}$$

Here t is time, $u(t)$ and $y(t)$ are real m-vectors, x is a real n-vector, and A, B and C are real, constant matrices of appropriate size. Often one is interested in applying feedback control in order to implement certain control objectives. For example, one might use the control law $u = \mathscr{L}_F y + \mathscr{L}_I v$, where $v(t)$, a real m-vector, is the input to the closed-loop system and $\mathscr{L}_F$ and $\mathscr{L}_I$ are linear operators. With suitable assumptions on initial conditions this leads to $y = \mathscr{L}_C v$, where $\mathscr{L}_C$ is a linear operator which represents the closed-loop system. A common control objective is to "decouple" the closed-loop system by making $\mathscr{L}_C$ be diagonal, i.e., causing $y_i = \mathscr{L}_C^i v_i$, $i = 1, \cdots, m$, where y_i and v_i are respectively the ith components of y and v. Early efforts in this direction relied on transfer-function descriptions for $\mathscr{L}_F$ and $\mathscr{L}_I$ and were characterized by a lack of rigor and of solid results. In this paper we consider control laws of the form originally proposed by Morgan [1]:

$$u(t) = Fx + Gv(t), \tag{1.2}$$

where F and G are real, constant matrices of appropriate size. This control law (state feedback) admits a precise problem formulation and is of real interest in applications.

The desire to decouple raises four questions: (a) Is decoupling possible? (b) What is the class of control laws which decouple? (c) What is the class of decoupled closed-loop systems? (d) What is the correspondence between elements of the classes mentioned in (b) and (c)? These four questions constitute the decoupling problem as it is treated in this paper.

Partial answers to the decoupling problem have been obtained. Morgan [1] gave a sufficient condition for decoupling (CB nonsingular) and under this con-

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† Computer, Information and Control Engineering, University of Michigan, Ann Arbor, Michigan 48104. This research was supported by the United States Air Force under Grant AF-AFOSR 814-66.

dition defined a rather restrictive class of control laws which decouple. These results were extended somewhat by Rekasius [2]. More recently Falb and Wolovich [3] gave necessary and sufficient conditions for decoupling, thus answering question (a). They also described a (restricted) class of control laws which decouple, which subsumes the classes introduced in [1] and [2]. Still more recently they obtained necessary and sufficient conditions on F and G for decoupling [4]. While this answers question (b), their conditions are in a cumbersome algebraic form which makes them difficult to apply when n is large. For several simple examples they have also characterized the class of decoupled closed-loop systems.

This paper extends the results outlined above to obtain more or less complete answers to questions (a), (b), (c) and (d). In addition the method of attack makes clearer the general nature of the decoupling problem and should lead to the solution of other interesting problems in nonlinear control and optimal control.

The paper is organized as follows. In § 2 we introduce notation, give a precise problem formulation, and state some important formulas. In § 3 it is shown that certain closed-loop-system properties are invariant with respect to F. These invariants lead naturally to necessary conditions for decoupling and an important matrix discovered by Falb and Wolovich [3]. The general approach to the decoupling problem is to treat an equivalent problem of simple structure. The required equivalence is introduced in § 4, along with the notion of an integrator decoupled system. The material in § 3 and § 4 yields an alternate proof of a theorem of Falb and Wolovich, which appears in § 5. Section 6 establishes a canonical form for integrator-decoupled systems which is the key to the main results, which are summarized in the theorems of § 7. In § 8 we discuss briefly the significance of the main results.

2. Problem formulation and basic formulas. Matrices, which we generally denote by capital letters, have real elements unless explicit dependence on the complex variable s is indicated. The notation $A = [\ ]$ will denote a partitioning of the matrix A into matrices or elements. We use I_n for the $n \times n$ identity matrix, E_i for the ith row of I_m, and 0 for the number zero or any null matrix.

An *m-input, m-output, n-th order system* S is the triple $\{A, B, C\}$, where A, B, C are respectively matrices of size $n \times n$, $n \times m$, $m \times n$. Although it is not really essential, we shall assume as is usual in the literature that $m \leqq n$. The *transfer function* of S is

$$H(s) = C(I_n s - A)^{-1}B. \tag{2.1}$$

Clearly $H(s)$ is an $m \times m$ rational matrix in the complex variable s. If s is interpreted as the Laplace transform variable, the relation of $H(s)$ to the Laplace transform solution of (1.1) is obvious.

In a similar way we introduce notation appropriate to the description of the closed-loop system arising from (1.2). A *control law* is the pair $\{F, G\}$, where the matrices F, G are respectively $m \times n$, $m \times m$. We say $S(F, G) = \{A + BF, BG, C\}$ is the system S with the control law $\{F, G\}$. The transfer function of $S(F, G)$ is

$$H(s, F, G) = C(I_n s - A - BF)^{-1}BG. \tag{2.2}$$

DEFINITION 1. The system $S(F, G)$ is *decoupled* if $H(\cdot, F, G)$ is diagonal and nonsingular.

This definition of decoupling is equivalent to the one given by Falb and Wolovich. By using it we give precise meaning to the questions raised in § 1.

The following formulas and notation are basic to our subsequent developments. By extending the well-known expansion for $(I_n s - A)^{-1}$, cf. [5, pp. 82–85], to $(I_n s - A - BF)^{-1}$, we have

$$H(s, F, G) = q(s, F)^{-1}(CBs^{n-1} + CR_1(F)Bs^{n-2} + \cdots + CR_{n-1}(F)B)G, \tag{2.3}$$

where

$$q(s, F) = s^n - q_1(F)s^{n-1} - \cdots - q_n(F) = \det(I_n s - A - BF), \tag{2.4}$$

$$R_0(F) = I_n, \qquad R_i(F) = (A + BF)R_{i-1}(F) - q_i(F)I_n, \quad i = 1, \cdots, n-1. \tag{2.5}$$

Alternatively (2.5) may be replaced by

$$\begin{aligned} R_0(F) &= I_n, \\ R_1(F) &= (A + BF) - q_1(F)I_n, \\ R_2(F) &= (A + BF)^2 - q_1(F)(A + BF) - q_2(F)I_n, \\ &\vdots \\ R_{n-1}(F) &= (A + BF)^{n-1} - q_1(F)(A + BF)^{n-2} - \cdots - q_{n-1}(F)I_n. \end{aligned} \tag{2.6}$$

We adapt the above formulas to S by writing $H(s) = H(s, 0, I_m)$ and using the notations $q(s) = q(s, 0)$, $q_i = q_i(0)$, $R_i = R_i(0)$.

Occasionally it will be necessary to work with several systems concurrently, say S and $\bar{S}$. In these cases the notation developed above is extended in the obvious way, e.g., $\bar{q}(s, \bar{F}) = \det(I_n s - \bar{A} - \bar{B}\bar{F})$.

3. F-invariants. In this section we study properties of $S(F, G)$ which are not affected by changes in F.

DEFINITION 2. An *F-invariant* of S is any property of $S(F, G)$ which for any fixed G does not depend on F.

Denote the ith row of $H(\cdot, F, G)$ by $H_i(\cdot, F, G)$ and define the integer $d_i(F, G)$ and the $1 \times m$ row matrix $D_i(F, G)$ as follows: if $H_i(\cdot, F, G) = 0$, $d_i(F, G) = n - 1$ and $D_i(F, G) = 0$; if $H(\cdot, F, G) \neq 0$, $d_i(F, G)$ is the integer j such that $\lim_{s\to\infty} s^{j+1}H_i(s, F, G)$ is nonzero and finite and $D_i(F, G) = \lim_{s\to\infty} s^{j+1}H_i(s, F, G)$. From (2.3) it is clear that $0 \leqq d_i(F, G) \leqq n - 1$.

PROPOSITION 1. *For $i = 1, \cdots, m$, $d_i(F, G)$ and $D_i(F, G)$ are F-invariants of S. In particular: $D_i(F, G) = D_iG$ and, for G nonsingular, $d_i(F, G) = d_i$, where $d_i = d_i(0, I_m)$ and $D_i = D_i(0, I_m)$.*

Proof. Let C_i be the ith row of C. From (2.3) and (2.6) with $F = 0$, $G = I_m$, it follows that

$$D_i = C_iA^{d_i}B \tag{3.1}$$

and

$$d_i = \begin{cases} 0, & C_iB \neq 0, \\ j, & C_iB = 0, \end{cases} \tag{3.2}$$

where j is the largest integer from $\{1, \cdots, n-1\}$ such that $C_iA^kB = 0$ for $k = 0, 1, \cdots, j-1$. Then by (2.6), $C_iR_k(F)B = 0, k = 0, 1, \cdots, d_i - 1$ and $C_iR_{d_i}(F)B = D_i$. From this the proposition is true by (2.3).

In engineering terms Proposition 1 says that certain "high frequency" gain properties of the closed-loop system are F-invariants. Falb and Wolovich introduce d_i and D_i (which they call B_i^*) via (3.1) and (3.2). However, they do not bring up the notion of invariance or attach physical meaning to these quantities. For future use we form the $m \times m$ matrix

$$D = \begin{bmatrix} D_1 \\ \vdots \\ D_m \end{bmatrix}. \tag{3.3}$$

The general question of F-invariants will not be developed here, although additional invariants are known. For instance, it is possible to prove the following.

PROPOSITION 2. *Let $h(s) = q(s) \det H(s)$. Then $h(s)$ is a polynomial in s of degree not greater than $n - m$ and*

$$h(s, F, G) = q(s, F) \det H(s, F, G) = h(s) \det G.$$

4. Integrator decoupled systems and control law equivalence. The key to the solution of the decoupling problem is a canonical representation of integrator decoupled systems. In this section integrator decoupled systems are defined and it is shown how they are related to the decoupling problem.

DEFINITION 3. $S = \{A, B, C\}$ is *integrator decoupled* (ID) if $D = \Gamma$, where Γ is diagonal and nonsingular, and $C_iA^{d_i+1} = 0, i = 1, \cdots, m$.

Denote the diagonal elements of Γ by $\gamma_1, \cdots, \gamma_m$. Then we have the following result.

PROPOSITION 3. *If S is* ID, *then $H(\cdot)$ is diagonal and has diagonal elements*

$$h_i(s) = \gamma_i s^{-d_i-1}, \qquad i = 1, \cdots, m.$$

Proof. Write

$$H_i(s) = q(s)^{-1}(C_iBs^{n-1} + C_iR_1Bs^{n-2} + \cdots + C_iR_{n-1}B). \tag{4.1}$$

Application of $C_iA^{d_i}B = D_i = \gamma_iE_i, C_iA^kB = 0$ for $k \neq d_i$ and (2.6) with $F = 0$ then gives

$$H_i(s) = q(s)^{-1}(s^{n-1-d_i} - q_1s^{n-2-d_i} - \cdots - q_{n-1-d_i})\gamma_iE_i. \tag{4.2}$$

Now from the Cayley–Hamilton theorem $C_iA^{n+j}B - q_1C_iA^{n+j-1}B - \cdots - q_nC_iA^jB = 0$, where j is any nonnegative integer. Taking $j = 0$ and using $C_iA^kB = 0$, we see that $k \neq d_i$ and $C_iA^{d_i}B \neq 0$ imply $q_{n-d_i} = 0$. Similarly, by taking $j = 1, \cdots, d_i$, we have $q_{n-d_i+1}, \cdots, q_n = 0$. Thus $q(s) = s^n - q_1 s^{n-1} - \cdots - q_{n-d_i-1} s^{d_i+1}$ and by (4.2) the proof is complete.

By Proposition 3 the transfer properties of an ID system are such that the ith output is the $(d_i + 1)$-fold integral of the ith input. This justifies the terminology, integrator decoupled. The example

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

shows that the converse of Proposition 3 is not true.

To establish the connection between ID systems and the decoupling problem we introduce the following definition.

DEFINITION 4. $S = \{A, B, C\}$ and $\bar{S} = \{\bar{A}, \bar{B}, \bar{C}\}$ are *control law equivalent* (CLE) if a one-to-one correspondence between $\{F, G\}$ and $\{\bar{F}, \bar{G}\}$ can be established such that, for this correspondence, $H(\cdot, F, G) = \bar{H}(\cdot, \bar{F}, \bar{G})$.

Remark 1. If the decoupling problem has been solved for S, it has been solved for $\bar{S}$.

Remark 2. Control law equivalence is transitive, i.e., if S and $\bar{S}$ are CLE and $\bar{S}$ and $\tilde{S}$ are CLE, then S and $\tilde{S}$ are CLE.

PROPOSITION 4. *Consider the system $S = \{A, B, C\}$, where D is nonsingular. Let A^* denote the $m \times n$ matrix*

$$A^* = \begin{bmatrix} C_1 A^{d_1+1} \\ \vdots \\ C_m A^{d_m+1} \end{bmatrix}. \tag{4.3}$$

Then the systems S and $\bar{S} = S(-D^{-1}A^*, D^{-1})$ are CLE. Furthermore $\bar{S}$ is ID and $\bar{d}_i = d_i, \bar{D}_i = E_i$ for $i = 1, \cdots, m$.

Proof. The one-to-one correspondence between $\{F, G\}$ and $\{\bar{F}, \bar{G}\}$, $DF + A^* = \bar{F}$ and $DG = \bar{G}$, proves the CLE property since then $A + BF = \bar{A} + \bar{B}\bar{F}$, $BG = \bar{B}\bar{G}$ and $C = \bar{C}$. The last part of the proposition follows by direct calculation of $\bar{d}_i$, $\bar{C}_i \bar{A}^{\bar{d}_i+1}$ and $\bar{D}_i$.

5. Necessary and sufficient conditions for decoupling. From the results of § 3 and § 4 we obtain by different means the theorem of Falb and Wolovich [3].

THEOREM 1. *S can be decoupled if and only if D is nonsingular. If $\{F, G\}$ decouples $S(F, G)$, $G = D^{-1}\Lambda$, where the $m \times m$ matrix Λ is diagonal and nonsingular.*

Proof. If $H(\cdot, F, G)$ is decoupled, then $H_i(\cdot, F, G) = h_i(\cdot, F, G)E_i$, where $h_i(\cdot, F, G) \neq 0$. This together with Proposition 1 implies $D_i G = \lambda_i E_i$, $i = 1, \cdots, m$. The numbers $\lambda_1, \cdots, \lambda_m$ are all nonzero. Suppose to the contrary. Then for some i, $D_i G = 0$. But if $H(\cdot, F, G)$ is to be nonsingular, G must be nonsingular. This implies $D_i = 0$ and $d_i = n - 1$, and from (3.1), (3.2), (2.6) and (2.3) we obtain $H_i(\cdot, F, G) = 0$, which contradicts the nonsingularity of $H(\cdot, F, G)$. From $D_i G = \lambda_i E_i$, $\lambda_i \neq 0, i = 1, \cdots, m$, we have $DG = \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$, Λ nonsingular, which proves the necessary conditions. Sufficiency follows from Propositions 3 and 4 and the control law $\{-D^{-1}A^*, D^{-1}\}$.

Falb and Wolovich [3] prove Theorem 1 by manipulating some rather involved algebraic expressions. Besides being simpler, our proof has the advantage that it makes clear that the necessary conditions have their origin in the F-invariants of Proposition 1. The matrix A^* was used also by Falb and Wolovich in their sufficiency proof.

Since the nonsingularity of D plays such an important role in the decoupling problem, it deserves some special comment. It is easy to see that $\det H(\cdot) = 0$ implies $\det D = 0$. In this case we say S has *strong inherent coupling* and it is obvious that no control law can effect decoupling. If $\det D \neq 0$, we say S has *no inherent coupling*. If $\det D = 0$ and $\det H(\cdot) \neq 0$, we say S has *weak inherent coupling*. Systems which have weak inherent coupling cannot be decoupled by state feedback, but other control laws can achieve decoupling. We shall not pursue this issue in depth here, but the following indicates one path which can be taken.

The system $S = \{A, B, C\}$,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

has weak inherent coupling because $\det H(\cdot) \neq 0$. We now form a new system $\tilde{S} = \{\tilde{A}, \tilde{B}, \tilde{C}\}$ which is related to S in the following way:

$$\tilde{A} = \begin{bmatrix} A & BK_2 \\ 0 & \bar{A} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} BK_1 \\ \bar{B} \end{bmatrix}, \qquad \tilde{C} = [C \quad 0],$$

where $\bar{A}, \bar{B}, K_1, K_2$ are respectively $\bar{n} \times \bar{n}, \bar{n} \times m, m \times m, m \times \bar{n}$ matrices. $\tilde{S}$ may be interpreted as the dynamical system (state $\tilde{x} = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}$, input $\bar{u}(t)$) arising from the interconnection of (1.1) and

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{A}\bar{x} + \bar{B}\bar{u}(t), \\ u(t) &= K_1\bar{u}(t) + K_2\bar{x}. \end{aligned} \tag{5.1}$$

Thus (5.1) acts as a precompensator for (1.1). If we choose $\bar{n} = 1$ and

$$\bar{A} = [0], \qquad \bar{B} = [0 \quad 1], \qquad K_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

it is easily verified that $\tilde{S}$ has no inherent coupling $\left(\tilde{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$. In general it is always possible to precompensate a system S which has weak inherent coupling so as to obtain a system $\tilde{S}$ with no inherent coupling. The development which shows this is tedious but straightforward.

6. Canonically decoupled systems. If in (1.1) we change the coordinate system by writing $\bar{x} = Tx$, where T is a nonsingular matrix, we obtain a new system $\bar{S} = \{\bar{A}, \bar{B}, \bar{C}\} = \{TAT^{-1}, TB, CT^{-1}\}$. This motivates what follows.

DEFINITION 5. S and $\bar{S}$ are *similar* if there exists a nonsingular $n \times n$ matrix T such that $TA = \bar{A}T$, $TB = \bar{B}$ and $C = \bar{C}T$.

Remark 3. If S and $\bar{S}$ are similar, $q(\cdot) = \bar{q}(\cdot)$ and $H(\cdot) = \bar{H}(\cdot)$. Thus $D = \bar{D}$ and $d_i = \bar{d}_i$, $i = 1, \cdots, m$. Furthermore if S is ID, $\bar{S}$ is ID.

Remark 4. Similar systems are CLE. This is a consequence of the correspondence $F = \bar{F}T$ and $G = \bar{G}$.

When S and $\bar{S}$ satisfy the conditions in Definition 5 we shall use the terminology that *T carries S into $\bar{S}$*.

For a canonically decoupled system (to be defined shortly) the decoupling problem has a particularly simple form. The main result of this section (Theorem 2) is to show that every ID system is similar to a canonically decoupled system. By Remarks 2 and 4 and Proposition 4, this means that if S can be decoupled ($\det D \neq 0$) it is possible to find a canonically decoupled system which is CLE to S. Thus by Remark 1 the treatment of the decoupling problem for S is simplified.

DEFINITION 6. $S = \{A, B, C\}$ is *canonically decoupled* (CD) if the following conditions are satisfied:

(i) The matrices A, B and C have the partitioned form:

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 & 0 & A_1^u \\ 0 & A_2 & 0 & \cdots & 0 & 0 & A_2^u \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_m & 0 & A_m^u \\ A_1^c & A_2^c & A_3^c & \cdots & A_m^c & A_{m+1} & A_{m+1}^u \\ 0 & 0 & 0 & \cdots & 0 & 0 & A_{m+2} \end{bmatrix}, \quad \begin{array}{l} A_i \text{ is } p_i \times p_i, \\ A_i^c \text{ is } p_{m+1} \times p_i, \\ A_i^u \text{ is } p_i \times p_{m+2}, \end{array}$$

$$B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_m \\ b_1^c & b_2^c & \cdots & b_m^c \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \begin{array}{l} b_i \text{ is } p_i \times 1, \\ b_i^c \text{ is } p_{m+1} \times 1, \end{array}$$

$$C = \begin{bmatrix} c_1 & 0 & \cdots & 0 & 0 & c_1^u \\ 0 & c_2 & \cdots & 0 & 0 & c_2^u \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_m & 0 & c_m^u \end{bmatrix}, \quad \begin{array}{l} c_i \text{ is } 1 \times p_i, \\ c_i^u \text{ is } 1 \times p_{m+2}, \end{array}$$

where $p_i \geqq d_i + 1$, $i = 1, \cdots, m$.

(ii) For $i = 1, \cdots, m$ the matrices A_i, b_i and c_i have the partitioned form:

$$A_i = \begin{bmatrix} \begin{bmatrix} 0 & I_{d_i} \\ 0 & 0 \end{bmatrix} & 0 \\ \Upsilon_i & \Phi_i \end{bmatrix}, \qquad \begin{array}{l} \Upsilon_i \text{ is } r_i \times (d_i + 1), \\ \Phi_i \text{ is } r_i \times r_i, \end{array}$$

$$b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \gamma_i \\ \beta_i \end{bmatrix}, \qquad \beta_i = \begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \vdots \\ \beta_{ir_i} \end{bmatrix},$$

$$c_i = [1 \quad 0 \quad \cdots \quad 0],$$

where $r_i = p_i - 1 - d_i$.

(iii) For $i = 1, \cdots, m$ the p_i column matrices $b_i, Ab_i, \cdots, A^{p_i - 1}b_i$ are linearly independent.

(iv) Let $p = \sum_{i=1}^{m} p_i$. If $p_{m+1} \neq 0$ and the n-row $\eta = [\eta_1 \quad \eta_2 \quad \cdots \quad \eta_n]$ is such that $\eta_{p+1}, \cdots, \eta_{p+p_{m+1}}$ are not all zero, then the row matrix function $\eta(I_n s - A)^{-1}B$ has at least two nonzero elements.

To proceed we need some additional notation and terminology. Let $\mathcal{Q}$ denote the n-dimensional space of n element row matrices. For $i = 1, \cdots, m$ define

(6.1) $\mathcal{Q}_i = \{\eta | \eta \in \mathcal{Q}; \eta A^j B_k = 0 \text{ for } k = 1, \cdots, m, \ k \neq i \text{ and } j = 0, \cdots, n-1\}$,

where B_i is the ith column of B. According to accepted practice we say $S = \{A, B, C\}$ is controllable if the nm columns $A^j B_k, j = 0, \cdots, n-1, k = 1, \cdots, m$, span the n-dimensional linear space of n columns.

LEMMA 1. *Assume $S = \{A, B, C\}$ is ID and controllable. Then for $i = 1, \cdots, m$ the following conditions are satisfied*:

(i) *$\mathcal{Q}_i$ is a row invariant subspace of A, i.e., $\eta \in \mathcal{Q}_i$ implies $\eta A \in \mathcal{Q}_i$*;

(ii) *$\mathcal{Q}_i \cap \mathcal{Q}_j = \{0\}$ for $j = 1, \cdots, m, j \neq i$*;

(iii) *$C_i, C_iA, \cdots, C_iA^{d_i}$ are linearly independent elements of $\mathcal{Q}_i$.*

Proof. To prove (i) we need only to show that $\eta \in \mathcal{Q}_i$ implies $\eta A^n B_k = 0$ for $k = 1, \cdots, m, \ k \neq i$. But this follows from the Cayley-Hamilton theorem $(A^n = q_1 A^{n-1} + \cdots + q_n I_n)$ and the definition of $\mathcal{Q}_i$. Assume $\eta \in \mathcal{Q}_i \cap \mathcal{Q}_j$ for $i \neq j$. Then from the definition of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ it is apparent that $\eta A^j B_k = 0$ for $j = 0, \cdots, n-1, \ k = 1, \cdots, m$. By the controllability of S this implies $\eta = 0$ and (ii) is true. From (3.1) and Definition 3, $C_iA^{d_i} \neq 0$ and $C_iA^{d_i+k} = 0$ for $k = 1, 2, \cdots$. Now assume $\rho_0 C_i + \rho_1 C_i A + \cdots + \rho_{d_i} C_i A^{d_i} = 0$, where $\rho_0, \cdots, \rho_{d_i}$ are scalars. Postmultiply this equation by A^{d_i} and obtain $\rho_0 C_i A^{d_i} = 0$ which implies $\rho_0 = 0$. By multiplying by successively lower powers of A we obtain $\rho_0, \rho_1, \cdots, \rho_{d_i} = 0$, which implies $C_i, \cdots, C_iA^{d_i}$ are linearly independent. From (3.1) and Definition 3 it follows that $C_iA^jB = 0$ for $j \geqq 0, \ j \neq d_i$ and

$C_iA^{d_i}B = \gamma_i E_i$. These conditions show that $C_iA^j \in \mathcal{Q}_i$, where j is any nonnegative integer. Thus (iii) is proved.

Because of Lemma 1 there exists a linear space $\mathcal{Q}_{m+1} \subset \mathcal{Q}$ such that the direct sum $\mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \cdots \oplus \mathcal{Q}_{m+1} = \mathcal{Q}$. We adopt the notation

$$p_i = \dim \mathcal{Q}_i, \qquad i = 1, \cdots, m+1, \qquad p = \sum_{i=1}^{m} p_i. \tag{6.2}$$

Clearly $p_{m+1} = n - p$ is uniquely defined by S although $\mathcal{Q}_{m+1}$ is not (unless $p_{m+1} = 0$). Moreover from part (iii) of Lemma 1 it is clear that $p_i \geqq d_i + 1$, $i = 1, \cdots, m$. Now we can state the following lemma.

LEMMA 2. *Assume $S = \{A, B, C\}$ is ID, controllable, and $p_{m+1} \neq 0$. Let $\eta \in \mathcal{Q}$ and write $\eta \in \sum_{i=1}^{m+1} \xi_i$, where $\xi_i \in \mathcal{Q}_i$ for $i = 1, \cdots, m+1$. If $\xi_{m+1} \neq 0$, then there exist at least two integers from the set $\{1, \cdots, m\}$, say q and r, such that $\eta A^j B_q \neq 0$ for at least one $j \in \{0, \cdots, n-1\}$ and $\eta A^j B_r \neq 0$ for at least one $j \in \{0, \cdots, n-1\}$.*

Proof. If $\eta A^j B_k = 0$ for all $j = 0, \cdots, n-1$ and $k = 1, \cdots, m$, the controllability of S would imply $\eta = 0$. Thus there is at least one integer from $\{1, \cdots, m\}$, say q, such that $\eta A^j B_q \neq 0$ for all $j = 0, \cdots, n-1$. If q were the only such integer then $\eta \in \mathcal{Q}_q$. But this would imply $\xi_{m+1} = 0$ and thus the lemma is proved.

PROPOSITION 5. *Assume S is ID and controllable. Then S is similar to a CD system $\bar{S}$, where $\bar{p}_i = p_i$, $i = 1, \cdots, m+1$ and $\bar{p}_{m+2} = 0$.*

Proof. We use the results of Lemmas 1 and 2 and in Definition 6 replace S by $\bar{S}$. First we form the matrix

$$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_{m+1} \end{bmatrix}, \tag{6.3}$$

where the rows of the $p_i \times n$ matrix Q_i are a basis for $\mathcal{Q}_i$. Because of the definition of $Q_1, \cdots, Q_{m+1}$, the rows of Q, which we denote by $q_1^*, \cdots, q_n^*$, are a basis for $\mathcal{Q}$. If we define $\bar{A}$ by $\bar{A}Q = QA$, the elements of the ith row of $\bar{A}$ are the components of q_i^*A with respect to the basis $q_1^*, \cdots, q_n^*$. Using this and part (i) of Lemma 1, we see that $\bar{A}$ has the structure of Definition 6, part (i). To be more specific define

$$Q_i = \begin{bmatrix} C_i \\ C_iA \\ \vdots \\ C_iA^{d_i} \\ q_{\sigma_i}^* \\ \vdots \\ q_{\rho_i}^* \end{bmatrix}, \tag{6.4}$$

where $q_{\sigma_i}^*, \cdots, q_{\rho_i}^*$ are any rows which extend $C_i, C_iA, \cdots, C_iA^{d_i}$ to form a basis for $\mathcal{Q}_i$. This, together with $C_iA^{d_i+1} = 0$, gives $\bar{A}_i$ the structure of Definition 6, part (ii). Moreover, $\bar{d}_i = d_i$ and $\bar{p}_i = p_i$.

Now define $\bar{B} = QB$. Using (6.3), (6.4), and the definition of $\mathcal{Q}_i$ gives $\bar{B}$ the structure of Definition 6, part (i). The further structure indicated in part (ii) follows from (6.4) and the fact that $C_i A^{d_i} B_i = \gamma_i$. Define $\bar{C}$ by $C = \bar{C}Q$. Then (6.3) and (6.4) give $\bar{C}$ the structure of Definition 6, parts (i) and (ii).

From the foregoing it is obvious that Q carries S into $\bar{S}$. It remains to show that parts (iii) and (iv) of Definition 6 are true for $\bar{S}$. Suppose that (iii) is not true. Then from the form of $\bar{A}$ and $\bar{B}$ indicated in (i), the columns $\bar{A}^j\bar{B}_k, j = 0, \cdots, n-1$, $k = 1, \cdots m$, do not span the n-dimensional column space. Because $\bar{A}^j\bar{B}_k = QA^jB_k$ this implies S is not controllable. This contradiction proves that (iii) holds for $\bar{S}$. To prove that $\bar{S}$ satisfies part (iv) of Definition 6 we note that

$$\eta(I_n s - A)^{-1}B = \bar{\eta}(I_n s - \bar{A})^{-1}\bar{B},$$

where $\eta = \bar{\eta}Q$. By the definition of Q, η satisfies the condition $\xi_{m+1} \neq 0$ (see Lemma 2 for notation) if and only if $\bar{\eta}_{p+1}, \cdots, \bar{\eta}_n$ are not all zero. Thus we need only to show that $\eta(I_n s - A)^{-1}B$ has at least two nonzero elements if $\xi_{m+1} \neq 0$. Using

$$\eta(I_n s - A)^{-1}B_k = q(s)(s^{n-1}\eta B_k + s^{n-2}\eta R_1 B_k + \cdots + \eta R_{n-1}B_k),$$

we easily see that if $\eta A^j B_k \neq 0$ for at least one $j \in \{0, \cdots, n-1\}$, then $\eta(I_n s - A)^{-1}B_k \not\equiv 0$. Since the kth element of $\eta(I_n s - A)^{-1}B$ is $\eta(I_n s - A)^{-1}B_k$, Lemma 2 gives the desired result.

Proposition 5 requires S to be controllable. To remove this restriction we need the following lemma.

LEMMA 3. *For the n-th order system* $S = \{A, B, C\}$ *let* $n_c = \dim \mathscr{C}$, *where* $\mathscr{C}$ *is the subspace spanned by the mn columns* $A^jB_k, j = 0, \cdots, n-1, k = 1, \cdots, m$. *Then S is similar to* $\tilde{S} = \{\tilde{A}, \tilde{B}, \tilde{C}\}$, *where* :

(i)

$$\tilde{A} = \begin{bmatrix} A^c & A^u \\ & \\ 0 & A^U \end{bmatrix}, \qquad \begin{array}{l} A^c \text{ is } n_c \times n_c, \\ A^u \text{ is } n_c \times (n - n_c), \\ A^U \text{ is } (n - n_c) \times (n - n_c), \end{array}$$

$$\tilde{B} = \begin{bmatrix} B^c \\ 0 \end{bmatrix}, \qquad B^c \text{ is } n_c \times m,$$

$$\tilde{C} = [C^c \quad C^u], \qquad C^c \text{ is } m \times n_c, \quad C^u \text{ is } m \times (n - n_c),$$

(ii) $S^c = \{A^c, B^c, C^c\}$ *is controllable*,

(iii) *if S is ID*, S^c *is ID and* $\Gamma^c = \Gamma$.

Proof. Parts (i) and (ii) are well known [6], [7] and may be established by taking the first n_c columns of a nonsingular matrix L to be a basis for $\mathscr{C}$. Then $T_1 = L^{-1}$ carries S into $\tilde{S}$. Part (iii) follows from Remark 3 and direct calculation of $\tilde{C}_i\tilde{A}^{\tilde{d}_i}\tilde{B}$ and $\tilde{C}_i\tilde{A}^{\tilde{d}_i+1}$ in terms of A^c, B^c and C^c.

The steps required to construct a CD representation of an ID system can now be summarized. Let S be an nth order ID system. Apply Lemma 3 obtaining $\mathscr{C}$ and thence a matrix T_1 which carries S into $\tilde{S}$. Since S^c is ID and controllable, Proposition 5 is applicable with S^c taking the role of S in Proposition 5. The matrix

Q which appears in the proof of Proposition 5 will in this case be $n_c \times n_c$. Define $\hat{p}_{m+2} = n - n_c$ and

$$T_2 = \begin{bmatrix} Q & 0 \\ 0 & I_{\hat{p}_{m+2}} \end{bmatrix}.$$

Then direct calculation shows that T_2 carries $\tilde{S}$ into the CD system $\hat{S}$, where $\hat{p}_i = \dim \mathcal{Q}_i^c$, $i = 1, \cdots, m + 1$. Thus $T_2 T_1$ carries S into $\hat{S}$ and we have a constructive proof of the promised result.

THEOREM 2. *Every ID system is similar to a CD system.*

7. Principal results. In this section we characterize the solution of the decoupling problem for CD systems and then by means of Theorem 2 extend these results to general systems.

THEOREM 3. *If S is CD, the control law $\{F, G\}$ decouples S if and only if*

$$F = \begin{bmatrix} \theta_1 & 0 & 0 & \cdots & 0 & 0 & \theta_1^u \\ 0 & \theta_2 & 0 & \cdots & 0 & 0 & \theta_2^u \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_m & 0 & \theta_m^u \end{bmatrix},$$

where θ_i is $1 \times p_i$ and θ_i^u is $1 \times p_{m+2}$, and

$$G = \operatorname{diag}(\lambda_1, \cdots, \lambda_m), \qquad \lambda_i \neq 0, \quad i = 1, \cdots, m.$$

Proof. Sufficiency follows by substitution which shows $H(\cdot, F, G)$ is diagonal. In fact, the ith diagonal element of $H(s, F, G)$ is given by

$$h_i(s, F, G) = c_i(I_{p_i}s - A_i - b_i\theta_i)^{-1}b_i\gamma_i\lambda_i. \tag{7.1}$$

The necessity of $G = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$ and $\lambda_i \neq 0$ is an obvious consequence of Theorem 1. To prove the necessity of the condition on F we write

$$H(s, F, G) = H(s)(I_m - F(I_n s - A)^{-1}B)^{-1}G, \tag{7.2}$$

an identity which is derived by straightforward manipulation of the two obvious identities: $(I_n s - A)^{-1}(I_n - BF(I_n s - A)^{-1})^{-1}B = (I_n s - A - BF)^{-1}B$ and $(I_n - BF(I_n s - A)^{-1})B = B(I_m - F(I_n s - A)^{-1}B)$. Since $H(\cdot)$ is diagonal, (7.2) implies $F(I_n s - A)^{-1}B$ must be diagonal if $S(F, G)$ is to be decoupled. By partitioning F into rows and using Definition 6, this leads to the required conditions on F.

THEOREM 4. *Assume S is CD and the control law $\{F, G\}$ has the form indicated in Theorem 3. Then:*

(i) $$h_i(s, F, G) = \frac{\alpha_i(s)\gamma_i\lambda_i}{\psi_i(s, \sigma_i)}, \qquad i = 1, \cdots, m,$$

where $\alpha_i(s) = s^{r_i} - \alpha_{i1}s^{r_i - 1} - \cdots - \alpha_{ir_i}$ and

$$\psi_i(s, \sigma_i) = s^{p_i} - \sigma_{i1}s^{p_i - 1} - \cdots - \sigma_{ip_i}, \qquad \sigma_i = [\sigma_{ip_i} \cdots \sigma_{i1}];$$

(ii) $\alpha_i(s) = \det(I_{r_i}s - \Phi_i)$;

(iii) $\theta_i = (\sigma_i - \pi_i)V_i$, *where* V_i *is a* $p_i \times p_i$ *nonsingular matrix which depends only on* A_i *and* b_i, *and the* $1 \times p_i$ *matrix* $\pi_i = [0 \cdots 0\ \alpha_{ir_i} \cdots \alpha_{i1}]$;

(iv) $q(s, F) = \alpha_{m+1}(s)\alpha_{m+2}(s) \prod_{i=1}^{m} \psi_i(s, \sigma_i)$, *where* $\alpha_i(s) = \det(I_{p_i}s - A_i)$, $i = m + 1, m + 2$.

Proof. From (7.1) it is apparent that $h_i(\cdot, F, G)$ may be interpreted as the transfer function of the system $S^i(\theta_i, \lambda_i)$, where $S^i = \{A_i, b_i, c_i\}$. Since S^i is a controllable single-input, single-output system, the theory developed by Bass and others (see, e.g., Morgan [8], [9]) may be applied. We summarize this theory in the following lemma.

LEMMA 4. *Assume* $S = \{A, b, c\}$ *is single-input, single-output, order* n *and controllable. Then the transfer function of* $S(\theta, \lambda)$ *has the form*

$$H(s, \theta, \lambda) = \frac{\omega(s)\lambda}{\psi(s, \sigma)},$$

where $\omega(s)$ *is a polynomial of degree* $n - 1$ *or less and* $\psi(s, \sigma) = s^n - \sigma_1 s^{n-1} - \cdots - \sigma_n$. *Let* $\sigma = [\sigma_n \cdots \sigma_1]$, $\pi = [q_n \cdots q_1]$ *and define the matrix* $K = [k_1 \cdots k_n]$, *where the columns* $k_i = R_{n-i}b, i = 1, \cdots, n$. *Then* K *is nonsingular and* $\theta K = \sigma - \pi$.

Except for the form of $\alpha_i(s)$, application of Lemma 4 to S^i proves part (i) of the theorem. From the form of A_i it is clear that $q_i(s) = \det(I_{p_i}s - A_i) = s^{d_i+1} \cdot \det(I_{r_i}s - \Phi_i) = s^{d_i+1}\alpha_i(s) = s^{p_i} - \alpha_{i1}s^{p_i-1} - \cdots - \alpha_{ir_i}s^{p_i-r_i}$, where we have used the notation of (ii). Letting V_i correspond to K^{-1} of Lemma 4 verifies (iii). The remaining part of (i) follows by noting that

$$h_{ii}(s, 0, 1) = \omega_i(s)\psi_i^{-1}(s, 0) = \omega_i(s)(s^{d_i-1}\alpha_i(s))^{-1} = \gamma_i s^{-d_i-1}.$$

Part (iv) is obtained by observing that $I_n s - A - BF = W(s)$ can be written

$$W = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix},$$

where $W_{22} = I_{p_{m+2}}s - A_{m+2}$. Thus $q(s, F, G) = \det W_{11} \det W_{22}$. Finally, W_{11} is quasi-triangular and is easily expanded to give

$$\det W_{11} = \det(I_{p_{m+1}}s - A_{m+1}) \prod_{i=1}^{m} \psi_i(s, \sigma_i).$$

THEOREM 5. *Assume* $S = \{A, B, C\}$ *can be decoupled. If* $\{F, G\}$ *decouples* S, *the diagonal elements of* $H(\cdot, F, G)$ *have the form given in part* (i) *of Theorem* 4 *where the integers* p_i *and* r_i *and the polynomials* $\alpha_i(s)$ *are uniquely determined by* S, *and* $\gamma_i = 1, i = 1, \cdots, m$. *Furthermore,* $q(s, F)$ *has the form given in part* (iv) *of Theorem* 4, *where* $\alpha_{m+1}(s)$ *and* $\alpha_{m+2}(s)$ *are polynomials of degree* p_{m+1} *and* p_{m+2} *uniquely determined by* S. *The class of control laws which decouple* S *can be characterized by* $G \in \mathscr{G}$ *and* $F \in \mathscr{F}$, *where* $\mathscr{G}$ *is an* m*-dimensional linear space and* $\mathscr{F}$ *is a* $(\sum_{i=1}^{m} p_i + mp_{m+2})$*-dimensional linear manifold. More specifically, there exist matrices* $G_i, J_1^i, \cdots, J_{p_i}^i$, $i = 1, \cdots, m$, *and an* (mp_{m+2})*-dimensional linear space*

$\mathscr{F}^u$, *which are uniquely determined by S, such that*

$$G = \sum_{i=1}^{m} \lambda_i G_i, \tag{7.3}$$

$$F = -D^{-1}A^* + \sum_{i=1}^{m} \sum_{k=1}^{p_i} (\sigma_{ik} - \pi_{ik})J_k^i + F^u, \tag{7.4}$$

where $F^u \in \mathscr{F}^u$, $\pi_{ik} = \alpha_{ik}$, $k = 1, \cdots, r_i$, *and* $\pi_{ik} = 0$, $k = r_i + 1, \cdots, p_i$.

Proof. The form of $H(\cdot, F, G)$ follows immediately from the CLE property between S and a CD system (Proposition 4 and Theorem 2) and Theorem 4. The one-to-one control law correspondence associated with this CLE property yields

$$G = D^{-1}\Lambda, \qquad \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_m), \tag{7.5}$$

and

$$F = -D^{-1}A^* + D^{-1}\begin{bmatrix} \theta_1 & 0 & 0 & \cdots & 0 & 0 & \theta_1^u \\ 0 & \theta_2 & 0 & \cdots & 0 & 0 & \theta_2^u \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_m & 0 & \theta_m^u \end{bmatrix} T_2 T_1, \tag{7.6}$$

where T_1 and T_2 are the nonsingular matrices which arise in the proof of Theorem 2. Results (7.3) and (7.4) are a direct consequence of (7.5) and (7.6), Theorem 3 and Theorem 4. Substitution of (7.6) into $\det(I_n s - A - BF)$ leads to the expression for $q(s, F)$.

8. Discussion. Theorem 5 establishes all the data needed for the design of a decoupled multivariable system. The class of decoupled systems is given and convenient formulas for computing F and G for an arbitrarily specified decoupled system within the class exist. The general approach for obtaining the data for these formulas (p_i, r_i, α_{ij}, π_{ij}, A^*, D, G_i, J_j^i) should be clear from the foregoing developments. But for nontrivial cases of S, hand calculations are not practical. For this reason a computer program is now being written. Given A, B and C, it will generate all the necessary data. This program, along with some example applications, will be reported in a subsequent paper.

It is also possible to determine stability and decide when decoupling by output feedback [4] is possible. We say the system $S(F, G)$ is *stable* if $q(s, F)$ is Hurwitz, i.e., all roots of $q(s, F) = 0$ have negative real parts. Clearly the design of a stable decoupled system is impossible if either $\alpha_{m+1}(s)$ or $\alpha_{m+2}(s)$ is not Hurwitz. If both $\alpha_{m+1}(s)$ and $\alpha_{m+2}(s)$ are Hurwitz, $S(F, G)$ can be made stable by appropriate choice of $\sigma_1, \cdots, \sigma_m$. Using the design formulas of Falb and Wolovich [3], we see that the stability question is more critical. It can be shown that these formulas lead to

$$h_i(s, F, G) = \frac{\lambda_i}{s^{d_i+1} + m_{i1}s^{d_i} + \cdots + m_{i(d_i+1)}}, \qquad i = 1, \cdots, m.$$

Thus $S(F, G)$ can be stable if and only if the $\alpha_i(s), i = 1, \cdots, m + 2$, are Hurwitz.

We say S is decoupled by *output feedback* if there exists a pair of $m \times m$ matrices $\{K, G\}$ such that $S(KC, G)$ is decoupled. The motivation for decoupling by output feedback is clear since it corresponds to replacing (1.2) by

$$u(t) = Ky(t) + Gv(t).$$

If $\{K, G\}$ is to output decouple S, KC must have the form of F in Theorem 5. This means it may not be possible to output decouple S even if D is nonsingular. If $\{K, G\}$ output decouples S, linear constraint equations on $\sigma_1, \cdots, \sigma_m$ may be imposed. The details of the analysis which gives these results are straightforward and are therefore omitted.

Still other questions arise: what is the effect of parameter variations and disturbance inputs, what should be done if D is singular or $\alpha_{m+1}(s)$ and $\alpha_{m+2}(s)$ are not Hurwitz, can dynamic estimators of x be used to supply x when only y is available, how are constraints on control effort imposed, what happens when (1.2) is replaced by a sampled-data version, what form does the theory take if S is time varying, can the σ_i be chosen by solving an optimization problem for S_i, what is the effect of using nonlinear feedback on the system S_i. Some of these questions will be explored in later papers.

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